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An exact sequence for the Brauer group of a finite quantum group

Yinhuo Zhang [1]

School of Mathematics and Computing Science, Victoria University of Wellington, PO Box 600, Wellington, New Zealand

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Abstract

In this paper, we give a general method to compute the Brauer group of a finite quantum group, i.e., a faithfully projective coquasitriangular Hopf algebra over a commutative ring with unity. Let (H, R) be a finite quantum group with an R-matrix R on $H \otimes H$. There exists a braided Hopf algebra $\mathcal{H}_R$ in the braided monoidal category of right H-comodules [S. Majid, J. Pure Appl. Algebra 86 (1993) 187–221]. We construct a group $\mathrm{Gal}(\mathcal{H}_R)$ consisting of quantum commutative $\mathcal{H}_R^*$-bigalois objects and show that there is an exact sequence of group homomorphisms:

$$1 \to \mathrm{Br}(k) \to \mathrm{BC}(k, H, R) \to \mathrm{Gal}(\mathcal{H}_R),$$

where $\mathrm{Br}(k)$ is the usual Brauer group of k and $\mathrm{BC}(k, H, R)$ is the Brauer group of (H, R) with respect to the R-matrix R.

Introduction

Let k be a commutative ring with unity, H a Hopf algebra over k with a bijective antipode. In [6], we introduced H-Azumaya algebras and the Brauer group $\mathrm{BQ}(k, H)$ classifying the H-Azumaya algebras. When H is a finite commutative and cocommutative Hopf algebra, the Brauer group $\mathrm{BQ}(k, H)$ turns out to be the Brauer–Long group introduced by F.W. Long in [15,16] which in turn is the generalization of the Brauer–Wall

E-mail address: yinhuo.zhang@mcs.vuw.ac.nz.
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group $\mathrm{BW}(k)$ (see [38]). In [7], we made further investigation on the basic properties of H-Azumaya algebras and studied in detail the split part of the Brauer group $\mathrm{BQ}(k, H)$ in order to find a non-abelian cohomological interpretation of some subgroups of $\mathrm{BQ}(k, H)$. This approach turns out to be difficulty as later on we found that the Hopf automorphism group can be embedded into the Brauer group $\mathrm{BQ}(k, H)$ (see [33]), which showed that $\mathrm{BQ}(k, H)$ is not necessarily a torsion group. In [35] we calculated the Brauer group of the Sweedler's 4-dimensional Hopf algebra and found that the Hopf automorphism group is not the only non-torsion part of $\mathrm{BQ}(k, H)$, the group of Galois objects plays the non-torsion role as well in the Brauer group $\mathrm{BQ}(k, H)$. Nevertheless, when H is commutative and cocommutative, the group of Galois objects and the group of Hopf automorphisms generate a subgroup that is isomorphic to a factor group of the group of bigalois objects, cf. [4,5]. A similar situation occurs for the Brauer group of a triangular Hopf algebra (see [36]). This fact indicates that the group of bigalois objects plays the vital role in the computation of the Brauer group of a finite quantum group. The indication was further strengthened by a beautiful exact sequence of the Brauer–Long group due to K.-H. Ulbrich in [31] where the group of bigalois objects appears in the picture. However, when (H, R) is no longer commutative and cocommutative, the group of bigalois objects of H or H^* does not fit into an exact sequence of the Brauer group of H. The solution found in this paper is the deformation (or transmutation in the sense of Majid [20]) $\mathcal{H}_R$ of a finite quantum group (H, R), which is no longer a Hopf algebra, but a left coideal subalgebra of the quantum double $D(H)$. The main idea of this paper is to embed the quotient group $\mathrm{BC}(k, H, R)/\mathrm{Br}(k)$ of a finite quantum group (H, R) into a suitable group of 'bigalois objects' of the $\mathcal{H}_R$, which is easier to compute (or to estimate). Since the full Brauer group $\mathrm{BQ}(k, H)$ of any finite Hopf algebra H is equal to $\mathrm{BC}(k, D(H)^*, R')$, where $(D(H)^*, R')$ is the dual of the Drinfel'd quantum double group, it is sufficient to consider the general case $\mathrm{BC}(k, H, R)$ for a finite quantum group (H, R).

In Section 1, we recall the definition of the Brauer groups $\mathrm{BQ}(k, H)$ and $\mathrm{BC}(k, H, R)$ when (H, R) is a coquasitriangular Hopf algebra. In Section 2, we consider the braided Hopf algebra $\mathcal{H}_R$ of a finite quantum group (H, R) constructed by S. Majid in [20]. The algebra $\mathcal{H}_R$ is a left coideal subalgebra of the quantum double $D(H)$ though it is not a Hopf subalgebra in the usual sense. It turns out that any YD H-module can be treated as an $\mathcal{H}_R$-bimodule. In other words, there exists a covariant functor from the category of Yetter–Drinfel'd H-modules to the category of $\mathcal{H}_R$-bimodules (see Proposition 2.7). This fact enables us to define a generalized cotensor product in the YD H-module category.

In Section 3, we consider YD H-module algebras that are $\mathcal{H}_R^*$-bigalois objects in the sense of [26]. These bigalois objects form a monoidal category under the generalized cotensor product. We construct a group $\mathrm{Gal}(\mathcal{H}_R)$ consisting of $\mathcal{H}_R^*$-bigalois objects which are quantum commutative. The group $\mathrm{Gal}(\mathcal{H}_R)$ plays the main role in the computation of the Brauer group $\mathrm{BC}(k, H, R)$. When (H, R) is triangular the group $\mathrm{Gal}(\mathcal{H}_R)$ is an abelian group.

In Section 4, we establish a group homomorphism $\widetilde{\pi}$ from the Brauer group $\mathrm{BC}(k, H, R)$ to the group $\mathrm{Gal}(\mathcal{H}_R)$. In order to define the homomorphism $\widetilde{\pi}$, we have to show that any element of $\mathrm{BC}(k, H, R)$ is represented by an H-Azumaya algebra which is an H^{op}-Galois extension of its coinvariants. Such an H-Azumaya algebra is called a Galois R-Azumaya algebra. The centralizer of the coinvariants of a Galois R-Azumaya algebra turns out to be an $\mathcal{H}_R^*$-bigalois object. The kernel of the homomorphism $\widetilde{\pi}$ is isomorphic to the usual Brauer group $\mathrm{Br}(k)$ of k. Thus the quotient group $\mathrm{BC}(k, H, R)/\mathrm{Br}(k)$ is determined by the group $\mathrm{Gal}(\mathcal{H}_R)$ of bigalois objects.

In Section 5, we calculate the group $\mathrm{Gal}(\mathcal{H}_R)$, where (H, R) is the Sweedler CQT Hopf algebra. In this case the exact sequence (23) is split and the Brauer group $\mathrm{BC}(k, H_4, R)$ is determined.

The main result of this paper has been included in the author's expository paper [37] without proof. The readers would get a better overview of the Brauer group theory of Hopf algebras from [37].

1. Preliminaries

Let k be a fixed commutative ring with unit. Throughout all algebras, unadorned $\otimes$, Hom are over k. A module (or an algebra) is said to be *finite* if it is faithfully projective (i.e., faithful, finitely generated and projective) as a k-module. A *finite quantum group* is a finite Hopf algebra over k with a coquasitriangular (CQT) structure. That is, there is an invertible element $R \in (H \otimes H)^*$, the convolution algebra of $H \otimes H$, subject to the following conditions:

(CQT1) $R(h \otimes 1) = R(1 \otimes h) = \varepsilon(h)$,
(CQT2) $R(x \otimes yz) = \sum R(x_{(1)} \otimes z)R(x_{(2)} \otimes y)$,
(CQT3) $R(yz \otimes x) = \sum R(y \otimes x_{(1)})R(z \otimes x_{(2)})$,
(CQT4) $\sum R(x_{(1)} \otimes y_{(1)})x_{(2)}y_{(2)} = \sum R(x_{(2)} \otimes y_{(2)})y_{(1)}x_{(1)}$

for all x, y and $z \in H$. Since H is faithfully projective, we may identify R with an invertible element $\sum R^1 \otimes R^2$ in $H^* \otimes H^*$. In this case, (H^*, R) is a quasitriangular Hopf algebra, namely, R as an element in the dual Hopf algebra $(H^*, \underline{\Delta}, \underline{\varepsilon})$ satisfies the conditions:

(QT1) $\sum \underline{\varepsilon}(R^1)R^2 = \sum R^1\underline{\varepsilon}(R^2) = 1$,
(QT2) $\sum \underline{\Delta}(R^1) \otimes R^2 = \sum R^1 \otimes r^1 \otimes R^2r^2$,
(QT3) $\sum R^1 \otimes \underline{\Delta}(R^2) = \sum R^1r^1 \otimes r^2 \otimes R^2$,
(QT4) $R\underline{\Delta}(p) = \underline{\Delta}^{\mathrm{op}}(p)R$

for all $p \in H^*$, where $r = R$. To a CQT structure R, we associate two Hopf algebra maps:

$$\Theta_l : H^{\mathrm{cop}} \to H^*, \qquad \Theta_l(h)(l) = R(h \otimes l),$$
$$\Theta_r : H^{\mathrm{op}} \to H^*, \qquad \Theta_r(h)(l) = R(l \otimes h).$$

Since Θ_l is a Hopf algebra map, we deduce that $R(S \otimes S) = R$. Recall the definition of the Brauer group of a Hopf algebra with a bijective antipode and some related notions. Let H be a Hopf algebra with a bijective antipode (not necessarily finite). A Yetter–Drinfel'd H-module (simply, YD H-module) M is a crossed H-bimodule [39]. That is, M is a k-module which is at once a left H-module and a right H-comodule satisfying the following equivalent compatibility conditions [14, 5.1.1]:

(i) $\sum h_{(1)} \cdot m_{(0)} \otimes h_{(2)} m_{(1)} = \sum (h_{(2)} \cdot m)_{(0)} \otimes (h_{(2)} \cdot m)_{(1)} h_{(1)}$,
(ii) $\sum (h \cdot m)_{(0)} \otimes (h \cdot m)_{(1)} = \sum h_{(2)} \cdot m_{(0)} \otimes h_{(3)} m_{(1)} S^{-1}(h_{(1)})$,

where the sigma notations for a comodule and for a comultiplication can be found in the reference books [28]. Denote by $\mathcal{Q}^H$ the category of YD H-modules and YD H-module morphisms. A YD H-module algebra is a YD H-module A such that A is a left H-module algebra and a right H^{op}-comodule algebra. For the details of H-(co)module algebras we refer to [1,23,28]. Let A and B be two YD H-module algebras. The braided product algebra $A \# B$ defined below is again a YD H-module algebra:

$$(a \# b)(c \# d) = \sum a c_{(0)} \# (c_{(1)} \cdot b) d$$

for all $a, c \in A$ and $b, d \in B$. The H-module and H^{op}-comodule structures of $A \# B$ are the diagonal H-module and co-diagonal H^{op}-comodule structures of $A \otimes B$ respectively. More details on braided product # can be found in [6].

In [6] we defined the Brauer group of a Hopf algebra H by considering isomorphism classes of H-Azumaya algebras. A YD H-module algebra A is said to be *H-Azumaya* if it is finite as a k-module and if the following two YD H-module algebra maps are isomorphisms:

$$F : A \# \overline{A} \to \mathrm{End}(A), \qquad F\big(a \# \bar{b}\big)(x) = \sum a x_{(0)} (x_{(1)} \cdot b),$$
$$G : \overline{A} \# A \to \mathrm{End}(A)^{\mathrm{op}}, \qquad G\big(\bar{a} \# b\big)(x) = \sum a_{(0)} (a_{(1)} \cdot x) b,$$

where $\overline{A}$ is the H-opposite YD H-module algebra of A, that is, $\overline{A} = A$ as a YD H-module, but with the multiplication given by

$$\bar{a} \cdot \bar{b} = \sum \overline{b_{(0)} (b_{(1)} \cdot a)}$$

for $\bar{a}, \bar{b} \in \overline{A}$ (see [6] for the details). For a finite YD H-module M, the endomorphism algebra $\mathrm{End}_k(M)$ is a YD H-module algebra with H-structures given by

$$(h \cdot f)(m) = \sum h_{(1)} \cdot f\big(S(h_{(2)}) \cdot m\big),$$
$$\sum f_{(0)}(m) \otimes f_{(1)} = \sum f(m_{(0)})_{(0)} \otimes S^{-1}(m_{(1)}) f(m_{(0)})_{(1)}$$

for $f \in \mathrm{End}(M)$ and $m \in M$. The elementary H-Azumaya algebra $\mathrm{End}(M)^{\mathrm{op}}$ has the different H-structures from those of $\mathrm{End}(M)$ (see [6] for the details).

Two H-Azumaya algebras A and B are Brauer equivalent (denoted $A \sim B$) if there exist two finite YD H-modules M and N such that $A \# \mathrm{End}(M) \cong B \# \mathrm{End}(N)$. Note that $A \sim B$ if and only if A is H-Morita equivalent to B (see [6, Theorem 2.10]). The relation $\sim$ is an equivalence relation on the set $B(k, H)$ of isomorphism classes of H-Azumaya algebras and the quotient set of $B(k, H)$ modulo $\sim$ is a group, called the Brauer group of the Hopf algebra H, denoted by $\mathrm{BQ}(k, H)$. An element in $\mathrm{BQ}(k, H)$ represented by an H-Azumaya algebra A is indicated by $[A]$. The unit in $\mathrm{BQ}(k, H)$ is represented by $\mathrm{End}(M)$ for any finite YD H-module M.

Now let H be a CQT Hopf algebra with a CQT structure R. If M is a right H-(or H^{op}-)comodule, the Hopf algebra map Θ_l induces a left H-module structure on M as follows:

$$h \triangleright_1 m = \Theta_l(h) \cdot m = \sum m_{(0)} R(h \otimes m_{(1)}) \tag{1}$$

for $h \in H$ and $m \in M$. The H-action (1) together with the original H-coaction makes M into a YD H-module, cf. [7,14]. Denote by $\mathbf{M}_R^H$ the category of YD H-modules with the left H-module structure (1) coming from the right H-comodule structure. It is obvious that $\mathbf{M}_R^H$ is a full braided monoidal subcategory of $\mathcal{Q}^H$.

When $A = M$ is a right H^{op}-comodule algebra, (1) makes A into a left H-module algebra and hence a YD H-module algebra. In the sequel, a YD H-module algebra A is called an *R-module algebra* if the H-action on A comes from the H^{op}-coaction on A through R. An R-module algebra is said to be *R-Azumaya* if it is H-Azumaya. The subset of $\mathrm{BQ}(k, H)$ consisting of the elements represented by the R-Azumaya algebras turns out to be a subgroup of $\mathrm{BQ}(k, H)$, denoted by $\mathrm{BC}(k, H, R)$. It is obvious that $\mathrm{BC}(k, H, R)$ contains the Brauer group $\mathrm{Br}(k)$.

Dually, if H is a QT Hopf algebra with a QT structure R, then a left H-module algebra A is simultaneously a YD H-module algebra with the right H^{op}-comodule structure given by

$$A \to A \otimes H^{\mathrm{op}}, \quad a \mapsto \sum R^2 \cdot a \otimes R^1$$

for all $a \in A$. The subset of $\mathrm{BQ}(k, H)$ consisting of the elements represented by the H-Azumaya algebras with right H^{op}-comodule structures stemming from left H-module structures in the above way, turns out to be a subgroup of $\mathrm{BQ}(k, H)$, denoted by $\mathrm{BM}(k, H, R)$. It is obvious that $\mathrm{BM}(k, H, R)$ contains the Brauer group $\mathrm{Br}(k)$.

The Brauer group $\mathrm{BQ}(k, H)$ is a special case of the Brauer group $\mathrm{Br}(\mathcal{C})$ of a braided monoidal category $\mathcal{C}$ as introduced in [34]. The fact that $\mathrm{BC}(k, H, R)$ is a subgroup of $\mathrm{BQ}(k, H)$ when (H, R) is a CQT Hopf algebra, can be explained in a categorical way. If $\mathcal{D}$ is a full braided monoidal subcategory of a braided monoidal category $\mathcal{C}$, then the Brauer group $\mathrm{Br}(\mathcal{D})$ is a subgroup of $\mathrm{Br}(\mathcal{C})$. This fact allows us to consider various subgroups of the Brauer group $\mathrm{Br}(\mathcal{C})$ of a braided monoidal category $\mathcal{C}$ whenever $\mathcal{C}$ contains certain closed braided subcategories. For example, if (H, R) is a CQT Hopf algebra, then the category $\mathbf{M}_R^H$ of right H-comodules is a full braided monoidal subcategory of the braided category $\mathcal{Q}^H$ of YD H-modules with the braiding φ given by:

$$M \otimes N \to N \otimes M, \quad m \otimes n \mapsto \sum n_{(0)} \otimes m_{(0)} R(n_{(1)} \otimes m_{(1)}),$$

where $m \in M$ and $n \in N$. The Brauer group $\mathrm{Br}(\mathbf{M}_R^H)$ of $\mathbf{M}_R^H$ is indeed $\mathrm{BC}(k, H, R)$. When H is a finite Hopf algebra, it is well-known that the category of YD H-modules is equivalent to the category of left $D(H)$-modules (see [18]), where $D(H)$ is the Drinfel'd double of H. So we have that $\mathrm{BQ}(k,H)=\mathrm{BM}(k,D(H),R)=\mathrm{BC}(k,D(H)^*,R)$, where R is the canonical quasitriangular structure on $D(H)$.

To end this section let us recall the notion of a Hopf Galois extension by a Hopf algebra H. A right H-comodule algebra A is said to be H-Galois if the canonical k-module map

$$\beta : A\otimes_{A_0} A \to A\otimes H, \quad a\otimes b\mapsto \sum ab_{(0)}\otimes b_{(1)}$$

is an isomorphism, where

$$A_0=\Big\{x\in A \;\Big|\; \sum x_{(0)}\otimes x_{(1)} = x\otimes 1\Big\}$$

is the coinvariant subalgebra of A. For a general Hopf Galois theory one may refer to [23, 26,27].

2. The braided Hopf algebra $\mathcal{H}_R$

Every CQT Hopf algebra (H,R) gives rise to a braided Hopf algebra $\mathcal{H}_R$ in the braided monoidal category $\mathbf{M}_R^H$. This process is called transmutation. In this section, we study the braided Hopf algebra (or the braided group) $\mathcal{H}_R$ of a CQT Hopf algebra (H,R) constructed by Majid [20] and establish a relationship between the category $\mathcal{Q}^H$ of YD H-modules and the category of $\mathcal{H}_R$-bimodules. At the end of the section, we will define a generalized cotensor product in the category $\mathcal{Q}^H$. We start with Majid's construction of the braided group $\mathcal{H}_R$ from (H,R).

Lemma 2.1 [20, Theorem 4.1]. *Let (H,R) be a CQT Hopf algebra. Then there is a braided Hopf algebra $\mathcal{H}_R$ in the category $\mathbf{M}_R^H$ described as follows in terms of H. As a k-module and coalgebra, $\mathcal{H}_R$ coincides with H. The multiplication $\star$ and the antipode S_R are given by*

$$\begin{aligned} h\star l &= \sum l_{(2)}h_{(2)}R\big(S^{-1}(l_{(3)})l_{(1)}\otimes h_{(1)}\big),\\ S_R(h) &= \sum S(h_{(2)})R\big(S^2(h_{(3)})S(h_{(1)})\otimes h_{(4)}\big), \end{aligned} \tag{2}$$

where $h,l\in H$. As an object in $\mathbf{M}_R^H$, $\mathcal{H}_R$ has the adjoint right coaction:

$$\rho(h)=\sum h_{(2)}\otimes S(h_{(1)})h_{(3)} \quad \textit{for all } h\in\mathcal{H}_R.$$

For further details on transmutation and braided groups, we refer to [20]. In [12] Doi and Takeuchi constructed a double Hopf algebra for a CQT Hopf algebra (H,R) (not necessarily finite). This double Hopf algebra, denoted by $D[H]$, is equal to $H\otimes H$ as a coalgebra with the multiplication given by

$$(h\otimes l)\big(h'\otimes l'\big)=\sum hh'_{(2)}\otimes l_{(2)}l'R\big(h'_{(1)}\otimes l_{(1)}\big)R\big(S\big(h'_{(3)}\big)\otimes l_{(3)}\big)$$

for h, l, h' and $l' \in H$. The antipode of $D[H]$ is given by

$$S(h\otimes l)=\big(1\otimes S(l)\big)\big(S(h)\otimes 1\big)$$

for all $h, l \in H$.

Write $h \bowtie l$ for an element in $D[H]$ and $H \bowtie H$ for $D[H]$. Since H is finite, the canonical Hopf algebra map $\Theta_l : H \to H^{*\mathrm{op}}$ given by $\Theta_l(h)(l) = R(h\otimes l)$ induces an Hopf algebra map from $D[H]$ to $D(H)$, the Drinfel'd quantum double $H^{*\mathrm{op}} \bowtie H$ (see [13]).

$$\Phi : D[H]\to D(H),\quad \Phi(h\bowtie l)=\Theta_l(h)\bowtie l.$$

When Θ_l is an isomorphism, we can identify $D[H]$ with $D(H)$. Any YD H-module is automatically a left $D[H]$-module. Moreover, the following lemma claims that $\mathcal{H}_R$ can be embedded into $D[H]$.

Lemma 2.2. *The following k-module map is an injective algebra map*:

$$\phi:\mathcal{H}_R\to D[H],\quad \phi(h)=\sum S^{-1}(h_{(2)})\bowtie h_{(1)}.$$

Proof. Given $h, l \in \mathcal{H}_R$, we have

$$\begin{aligned}
\phi(h\star l)&=\sum \phi(l_{(2)}h_{(2)})R\big(S^{-1}(l_{(3)})l_{(1)}\otimes h_{(1)}\big)\\
&=\sum S^{-1}(l_{(3)}h_{(3)})\bowtie l_{(2)}h_{(2)}R\big(S^{-1}(l_{(4)})l_{(1)}\otimes h_{(1)}\big)\\
&=\sum S^{-1}(h_{(4)})S^{-1}(l_{(3)})\bowtie l_{(2)}h_{(3)}R\big(S^{-1}(l_{(4)})\otimes h_{(1)}\big)R(l_{(1)}\otimes h_{(2)})\\
&=\sum S^{-1}(h_{(4)})S^{-1}(l_{(3)})\bowtie h_{(2)}l_{(1)}R\big(S^{-1}(l_{(4)})\otimes h_{(1)}\big)R(l_{(2)}\otimes h_{(3)})\\
&=\sum\big(S^{-1}(h_{(2)})\bowtie h_{(1)}\big)\big(S^{-1}(l_{(2)})\bowtie l_{(1)}\big)\\
&=\phi(h)\phi(l).
\end{aligned}$$

It is obvious that ϕ is injective. □

Proposition 2.3. *$\mathcal{H}_R$ is a left $D[H]$-comodule algebra.*

Proof. Define a k-module map from $\mathcal{H}_R$ to $D[H]\otimes\mathcal{H}_R$ as follows:

$$\chi:\mathcal{H}_R\to D[H]\otimes\mathcal{H}_R,\quad \chi(h)=\sum\big(S^{-1}(h_{(3)})\bowtie h_{(1)}\big)\otimes h_{(2)}.$$

It is easy to check that χ is a left $D[H]$-comodule map. We have to show that χ is an algebra map. Indeed, if $h, l \in \mathcal{H}_R$, then

$$
\begin{aligned}
\chi(h \star l) &= \sum \chi(h_{(2)} l_{(1)}) R\big(l_{(2)} \otimes S(h_{(1)}) h_{(3)}\big) \\
&= \sum S^{-1}(h_{(4)} l_{(3)}) \bowtie h_{(2)} l_{(1)} \otimes h_{(3)} l_{(2)} R\big(l_{(4)} \otimes S(h_{(1)}) h_{(5)}\big) \\
&= \sum S^{-1}(h_{(4)} l_{(3)}) \bowtie h_{(2)} l_{(1)} \otimes h_{(3)} l_{(2)} R(l_{(4)} \otimes h_{(5)}) \\
&\quad \times R\big(S^{-1}(l_{(5)}) \otimes h_{(1)}\big) \\
&= \sum S^{-1}(l_{(4)} h_{(5)}) \bowtie h_{(2)} l_{(1)} \otimes h_{(3)} l_{(2)} R(l_{(3)} \otimes h_{(4)}) \\
&\quad \times R\big(S^{-1}(l_{(5)}) \otimes h_{(1)}\big) \\
&= \sum S^{-1}(l_{(4)} h_{(5)}) \bowtie h_{(2)} l_{(1)} \otimes l_{(3)} h_{(4)} R(l_{(2)} \otimes h_{(3)}) \\
&\quad \times R\big(S^{-1}(l_{(5)}) \otimes h_{(1)}\big) \\
&= \sum S^{-1}(l_{(6)} h_{(6)}) \bowtie h_{(2)} l_{(1)} \otimes l_{(3)} h_{(5)} R\big(S^{-1}(l_{(4)}) l_{(2)} \otimes h_{(4)}\big) \\
&\quad \times R(l_{(5)} \otimes h_{(3)}) R\big(S^{-1}(l_{(7)}) \otimes h_{(1)}\big) \\
&= \sum \big(S^{-1}(h_{(4)}) \bowtie h_{(1)}\big)\big(S^{-1}(l_{(5)}) \bowtie l_{(1)}\big) \otimes l_{(3)} h_{(3)} \\
&\quad \times R\big(S^{-1}(l_{(4)}) l_{(2)} \otimes h_{(2)}\big) \\
&= \sum \big(S^{-1}(h_{(3)}) \bowtie h_{(1)}\big)\big(S^{-1}(l_{(3)}) \bowtie l_{(1)}\big) \otimes h_{(2)} \star l_{(2)} \\
&= \chi(h)\chi(l). \qquad \square
\end{aligned}
$$

Lemma 2.2 and Proposition 2.3 show that $\mathcal{H}_R$ can be embedded into $D[H]$ as a left coideal subalgebra. In fact, $\mathcal{H}_R$ can be further embedded into $D(H)$ as a left coideal subalgebra.

Corollary 2.4. *The composite algebra map*:

$$\mathcal{H}_R \xrightarrow{\phi} D[H] \xrightarrow{\Phi} D(H)$$

is injective.

Proof. Since H is faithfully flat over k, we have that the kernel of Φ is $\mathrm{Ker}(\Theta_l) \bowtie H$. If $\Phi\phi(h) = 0$ for some $h \in \mathcal{H}_R$, then

$$\phi(h) = \sum S^{-1}(h_{(2)}) \bowtie h_{(1)} \in \mathrm{Ker}(\Theta_l) \bowtie H.$$

It follows that

$$
\begin{aligned}
1 \bowtie h &= \sum \Theta_l(1) \bowtie \varepsilon\big(S^{-1}(h_{(2)})\big) h_{(1)} \\
&= \sum (\varepsilon_{H^*} \otimes \iota)\big(\Theta_l\big(S^{-1}(h_{(2)})\big) \bowtie h_{(1)}\big) \\
&= (\varepsilon_{H^*} \otimes \iota)\Phi\phi(h) = 0
\end{aligned}
$$

where ι is the identity map. This implies that $h = 0$ and hence $\Phi\phi$ is injective. Moreover, since Φ is a Hopf algebra map, $\Phi\phi(\mathcal{H}_R)$ is a left coideal subalgebra of $D(H)$. □

Let us now consider Yetter–Drinfel'd H-modules and $\mathcal{H}_R$-bimodules. Let M be a Yetter–Drinfel'd module over H, or a left $D(H)$-module. The following composite map:

$$\mathcal{H}_R \otimes M \xrightarrow{\phi\otimes\iota} D[H] \otimes M \xrightarrow{\Phi\otimes\iota} D(H) \otimes M$$

makes M into a left $\mathcal{H}_R$-module. If we write $-\triangleright$ for the above left action, then we have the explicit formula:

$$\begin{aligned} h -\triangleright m &= \sum S^{-1}(h_{(2)}) \triangleright_1 (h_{(1)} \cdot m) \\ &= \sum (h_{(2)} \cdot m_{(0)}) R\big(S^{-1}(h_{(4)}) \otimes h_{(3)} m_{(1)} S^{-1}(h_{(1)})\big) \end{aligned} \tag{3}$$

for $h \in \mathcal{H}_R$, $m \in M$ and $\triangleright_1$ as in (1).

Since there is an augmentation map ε on $\mathcal{H}_R$, we may define the $\mathcal{H}_R$-invariant set of a left $\mathcal{H}_R$-module M which is

$$M^{\mathcal{H}_R} = \big\{m \in M \mid h -\triangleright m = \varepsilon(h)m,\ \forall h \in \mathcal{H}_R\big\}.$$

When a left $\mathcal{H}_R$-module comes from a YD H-module, the invariant k-module can be characterized as follows:

Lemma 2.5. *Let M be a YD H-module. Then*

$$M^{\mathcal{H}_R} = \Big\{m \in M \;\Big|\; h \cdot m = h \triangleright_1 m = \sum m_{(0)} R(h \otimes m_{(1)}),\ \forall h \in H\Big\}.$$

Proof. By definition of the action of $\mathcal{H}_R$ on M, we have

$$h -\triangleright m = \sum S^{-1}(h_{(2)}) \triangleright_1 (h_{(1)} \cdot m)$$

for any $h \in \mathcal{H}_R$ and $m \in M$. It follows that the latter set is contained in $M^{\mathcal{H}_R}$.

Conversely, if $m \in M^{\mathcal{H}_R}$, then we have

$$\begin{aligned} h \cdot m &= \sum h_{(3)} \triangleright_1 \big(S^{-1}(h_{(2)}) \triangleright_1 (h_{(1)} \cdot m)\big) \\ &= \sum h_{(2)} \triangleright_1 (h_{(1)} -\triangleright m) \\ &= \sum h_{(2)} \triangleright_1 \big(\varepsilon(h_{(1)})m\big) \\ &= h \triangleright_1 m \end{aligned}$$

for any $h \in \mathcal{H}_R$ and $m \in M$. □

Following Lemma 2.5, we obtain that the invariant submodule $M^{\mathcal{H}_R}$ of a YD H-module M is the maximal submodule sitting in the subcategory $\mathbf{M}_R^H$, where the left H-action is the induced one $\triangleright_1$. Thus we get a covariant functor κ:

$$\kappa : \mathcal{Q}^H \to \mathbf{M}_R^H, \quad \kappa(M) = M^{\mathcal{H}_R}.$$

It follows that a *YD H-module is an object in $\mathbf{M}_R^H$ if and only if $\mathcal{H}_R$ acts trivially on M, i.e., $M = M^{\mathcal{H}_R}$*. Observe that the functor κ has a left adjoint functor, the embedding functor from $\mathbf{M}_R^H$ to $\mathcal{Q}^H$.

Now we define a right $\mathcal{H}_R$-module structure on a YD H-module M. Observe that the right H-comodule structure of M induces two left H-module structures. The first one is (1), and the second one is given by

$$h \triangleright_2 m = \sum m_{(0)} R\big(S(m_{(1)}) \otimes h\big) \tag{4}$$

for $h \in H$ and $m \in M$. With this second left H-action on M, M becomes a right $D[H]$-module.

Lemma 2.6. *Let M be a YD H-module. Then M is a right $D[H]$-module defined by*

$$m \leftharpoondown (h \bowtie l) = S(l) \triangleright_2 \big(S(h) \cdot m\big) \tag{5}$$

for $h, l \in H$ and $m \in M$. Moreover, if A is a YD H-module algebra, then (5) *makes A into a right $D[H]^{\mathrm{cop}}$-module algebra.*

Proof. Since M is a left H-module under both actions $\cdot$ and $\triangleright_2$, it is sufficient to show that

$$m \leftharpoondown \big[(1 \bowtie l)(h \bowtie 1)\big] = \big[m \leftharpoondown (1 \bowtie l)\big] \leftharpoondown (h \bowtie 1) = S(h) \cdot \big(S(l) \triangleright_2 m\big)$$

for $h, l \in H$ and $m \in M$. Indeed, we have

$$\begin{aligned}
m &\leftharpoondown \big[(1 \bowtie l)(h \bowtie 1)\big] \\
&= \sum m \leftharpoondown (h_{(2)} \bowtie l_{(2)}) R(h_{(1)} \otimes l_{(1)}) R\big(S(h_{(3)}) \otimes l_{(3)}\big) \\
&= \sum S(l_{(2)}) \triangleright_2 \big(S(h_{(2)}) \cdot m\big) R(h_{(1)} \otimes l_{(1)}) R\big(S(h_{(3)}) \otimes l_{(3)}\big) \\
&= \sum S(h_{(3)}) \cdot \big(S(l_{(3)}) \triangleright_2 m\big) R(h_{(4)} \otimes l_{(4)}) R\big(S(h_{(2)}) \otimes l_{(2)}\big) \\
&\qquad \times R(h_{(1)} \otimes l_{(1)}) R\big(S(h_{(5)}) \otimes l_{(5)}\big) \\
&= S(h) \cdot \big(S(l) \triangleright_2 m\big)
\end{aligned}$$

for $h, l \in H$ and $m \in M$. The second statement is obvious. $\square$

Unfortunately, the right $D[H]$-module structure (5) does not commute with the canonical left $D[H]$-module structure induced by Hopf algebra map Φ to make M into a $D[H]$-bimodule. However the right $\mathcal{H}_R$-module structure on M given by

$$M \otimes \mathcal{H}_R \xrightarrow{\iota\otimes\phi} M \otimes D[H] \to M,$$

together with the left $\mathcal{H}_R$-module structure (3) makes M into an $\mathcal{H}_R$-bimodule. Write $\triangleleft\!-$ for the above right action of $\mathcal{H}_R$, then we have the explicit formula:

$$\begin{aligned} m \triangleleft\!- h &= \sum S(h_{(1)}) \triangleright_2 (h_{(2)} \cdot m) \\ &= \sum (h_{(3)} \cdot m_{(0)}) R\big(h_{(4)} m_{(1)} S^{-1}(h_{(2)}) \otimes h_{(1)}\big) \end{aligned} \tag{6}$$

for $m \in M$ and $h \in \mathcal{H}_R$.

Proposition 2.7. *Let M be a YD H-module. Then M is an $\mathcal{H}_R$-bimodule via* (3) *and* (6).

Proof. Given $h, l \in \mathcal{H}_R$ and $m \in M$, we have to prove

$$(l \multimap m) \triangleleft\!- h = l \multimap (m \triangleleft\!- h),$$

i.e.,

$$\begin{aligned} &\sum S(h_{(1)}) \triangleright_2 \big(h_{(2)} \cdot \big(S^{-1}(l_{(2)}) \triangleright_1 (l_{(1)} \cdot m)\big)\big) \\ &\quad = \sum S^{-1}(l_{(2)}) \triangleright_1 \big(l_{(1)} \cdot \big(S(h_{(1)}) \triangleright_2 (h_{(2)} \cdot m)\big)\big). \end{aligned}$$

Indeed, we have

$$\begin{aligned} &\sum S^{-1}(l_{(2)}) \triangleright_1 \big(l_{(1)} \cdot \big(S(h_{(1)}) \triangleright_2 (h_{(2)} \cdot m)\big)\big) \\ &\quad = \sum S^{-1}(l_{(4)}) \triangleright_1 \big(S(h_{(2)}) \triangleright_2 (l_{(2)} h_{(4)} \cdot m)\big) R(l_{(1)} \otimes h_{(3)}) R\big(l_{(3)} \otimes S(h_{(1)})\big) \\ &\quad = \sum S(h_{(3)}) \triangleright_2 \big(S^{-1}(l_{(5)}) \triangleright_1 (l_{(2)} h_{(6)} \cdot m)\big) R(l_{(1)} \otimes h_{(5)}) \\ &\qquad \times R\big(l_{(3)} \otimes S(h_{(1)})\big) R\big(l_{(6)} \otimes S(h_{(4)})\big) R\big(S^{-1}(l_{(4)}) \otimes S(h_{(2)})\big) \\ &\quad = \sum S(h_{(1)}) \triangleright_2 \big(S^{-1}(l_{(3)}) \triangleright_1 (l_{(2)} h_{(4)} \cdot m)\big) R(l_{(1)} \otimes h_{(3)}) R\big(l_{(4)} \otimes S(h_{(2)})\big) \\ &\quad = \sum S(h_{(1)}) \triangleright_2 \big(S^{-1}(l_{(3)}) \triangleright_1 (h_{(3)} l_{(1)} \cdot m)\big) R(l_{(2)} \otimes h_{(4)}) R\big(l_{(4)} \otimes S(h_{(2)})\big) \\ &\quad = \sum S(h_{(1)}) \triangleright_2 \big(h_{(2)} \cdot \big(S^{-1}(l_{(2)}) \triangleright_1 (l_{(1)} \cdot m)\big)\big), \end{aligned}$$

where the first, the second and the forth equation follow from the following identities respectively:

$$h \rhd_2 (l \cdot m) = \sum l_{(2)} \cdot (h_{(2)} \rhd_2 m) R(l_{(1)} \otimes h_{(1)}) R\big(S(l_{(3)} \otimes h_{(3)})\big),$$
$$l \rhd_1 (h \rhd_2 m) = \sum h_{(2)} \rhd_2 (l_{(2)} \rhd_1 m) R\big(S(l_{(1)}) \otimes h_{(1)}\big) R(l_{(3)} \otimes h_{(3)}),$$
$$h \rhd_1 (l \cdot m) = \sum l_{(2)} \cdot (h_{(2)} \rhd_1 m) R\big(h_{(1)} \otimes S^{-1}(l_{(1)})\big) R\big(S(h_{(3)} \otimes l_{(3)})\big)$$

for $h, l \in H$ and $m \in M$. So M is an $\mathcal{H}_R$-bimodule. □

Remark that the right $\mathcal{H}_R$-invariant of a YD H-module is different from the left one described in Lemma 2.5. One may apply the same argument in Lemma 2.5 to obtain the right invariant set of a YD H-module M:

$$\big\{m \in M \mid h \cdot m = h \rhd_2 m,\ \forall h \in H\big\}.$$

Like the set of left invariants, the set of right invariants is the maximal right–right YD H-submodule of M. Combining Lemmas 2.2, 2.6 and Propositions 2.3, 2.7, we obtain the following:

Corollary 2.8. *If A is a YD H-module algebra, then A is an $\mathcal{H}_R$-bimodule algebra in the sense that*

$$\begin{aligned} h \multimap (ab) &= \sum (h_{(-1)} \rightharpoondown a)(h_{(0)} \multimap b), \\ (ab) \mathrel{\triangleleft\!\!-} h &= \sum (a \mathrel{\triangleleft\!\!-} h_{(0)})(b \leftharpoondown h_{(-1)}) \end{aligned} \tag{7}$$

for $a, b \in A$ and $h \in \mathcal{H}_R$, where $\chi(h) = \sum h_{(-1)} \otimes h_{(0)} \in D[H] \otimes \mathcal{H}_R$, $\rightharpoondown$ and $\leftharpoondown$ stand for the left and right actions of $D[H]$ on a YD H-module.

Proof. We show the second equation and leave the first one to the readers. Given $a, b \in A$ and $h \in \mathcal{H}_R$ we have

$$\begin{aligned} (ab) \mathrel{\triangleleft\!\!-} h &= \sum S(h_{(1)}) \rhd_2 \big(h_{(2)} \cdot (ab)\big) \\ &= \sum \big(S(h_{(2)}) \rhd_2 (h_{(3)} \cdot a)\big)\big(S(h_{(1)}) \rhd_2 (h_{(4)} \cdot b)\big) \\ &= \sum \big(a \leftharpoondown \big(S^{-1}(h_{(3)}) \bowtie h_{(2)}\big)\big)\big(b \leftharpoondown \big(S^{-1}(h_{(4)}) \bowtie h_{(1)}\big)\big) \\ &= \sum (a \mathrel{\triangleleft\!\!-} h_{(0)})(b \leftharpoondown h_{(-1)}). \qquad \square \end{aligned}$$

In the sequel, we define a generalized cotensor product in the category of Yetter–Drinfel'd modules of a CQT Hopf algebra (H, R).

Given two YD H-modules X and Y, let $X \wedge Y$ be the cotensor product

$$\Big\{\sum x_i \otimes y_i \mid \sum (x_i \mathrel{\triangleleft\!\!-} h) \otimes y_i = \sum x_i \otimes (h \multimap y_i),\ \forall h \in \mathcal{H}_R\Big\}.$$

Observe that $X \wedge Y$ is still an $\mathcal{H}_R$-bimodule with the left and right $\mathcal{H}_R$-module structures stemming from the left $\mathcal{H}_R$-module structure of X and the right $\mathcal{H}_R$-module structure of Y respectively. Does this $\mathcal{H}_R$-bimodule structure of $X \wedge Y$ come from a YD H-module structure on $X \wedge Y$? To answer this question, we need to characterize the cotensor product $X \wedge Y$.

Lemma 2.9. *Let X, Y be two YD H-modules. Then*

$$X \wedge Y = \Big\{ \sum x_i \otimes y_i \in X \otimes Y \mid \sum h_{(1)} \cdot x_i \otimes h_{(2)} \rhd_1 y_i = \sum h_{(1)} \rhd_2 x_i \otimes h_{(2)} \cdot y_i, \ \forall h \in H \Big\}.$$

Proof. Let T be the following set:

$$\Big\{ \sum x_i \otimes y_i \in X \otimes Y \mid \sum h_{(1)} \cdot x_i \otimes h_{(2)} \rhd_1 y_i = \sum h_{(1)} \rhd_2 x_i \otimes h_{(2)} \cdot y_i, \forall h \in H \Big\}.$$

In order to simplify the computation, we will write $x \otimes y$ for an element $\sum x_i \otimes y_i$ in $X \otimes Y$ in the sequel. Given $x \otimes y \in X \wedge Y$ and $h \in H$, we have

$$\begin{aligned}
&\sum h_{(1)} \cdot x \otimes h_{(2)} \rhd_1 y \\
&\quad = \sum h_{(1)} \rhd_2 \big(S(h_{(2)}) \rhd_2 (h_{(3)} \cdot x) \big) \otimes h_{(4)} \rhd_1 y \\
&\quad = \sum h_{(1)} \rhd_2 (x \lhd\!\!- h_{(2)}) \otimes h_{(3)} \rhd_1 y \\
&\quad = \sum h_{(1)} \rhd_2 x \otimes h_{(3)} \rhd_1 (h_{(2)} -\!\!\rhd y) \\
&\quad = \sum h_{(1)} \rhd_2 x \otimes h_{(4)} \rhd_1 \big(S^{-1}(h_{(3)}) \rhd_1 (h_{(2)} \cdot y) \big) \\
&\quad = \sum h_{(1)} \rhd_2 x \otimes h_{(2)} \cdot y.
\end{aligned}$$

So $x \otimes y$ belongs to T and we have that $X \wedge Y \subseteq T$.

Conversely, if $x \otimes y \in T$, and $h \in \mathcal{H}_R$, we have

$$\begin{aligned}
(x \lhd\!\!- h) \otimes y &= \sum S(h_{(1)}) \rhd_2 (h_{(2)} \cdot x) \otimes y \\
&= \sum S(h_{(1)}) \rhd_2 (h_{(2)} \cdot x) \otimes S^{-1}(h_{(4)}) h_{(3)} \rhd_1 y \\
&= \sum S(h_{(1)}) \rhd_2 (h_{(2)} \rhd_2 x) \otimes S^{-1}(h_{(4)}) \rhd_1 (h_{(3)} \cdot y) \\
&= \sum x \otimes S^{-1}(h_{(2)}) \rhd_1 (h_{(1)} \cdot y) \\
&= x \otimes (h -\!\!\rhd y).
\end{aligned}$$

It follows that $T \subseteq X \wedge Y$. □

Lemma 2.9 results in an alternative definition of $X \wedge Y$ which is more applicable when we test whether an element is in $X \wedge Y$. Moreover, it leads to the following left H-action on $X \wedge Y$ given by

$$h \cdot \sum (x_i \otimes y_i) = \sum h_{(1)} \cdot x_i \otimes h_{(2)} \triangleright_1 y_i = \sum h_{(1)} \triangleright_2 x_i \otimes h_{(2)} \cdot y_i \tag{8}$$

whenever $\sum x_i \otimes y_i \in X \wedge Y$ and $h \in H$. To show that (8) is a left H-module structure on $X \wedge Y$, one simply applies Lemma 2.9. Nevertheless, this left H-module structure fits in a YD H-module structure with the right H-comodule structure inheriting from $X \otimes Y$.

Proposition 2.10. *$X \wedge Y$ with the H-action* (8) *and the right H-coaction inheriting from $X \otimes Y$ is a YD H-module.*

Proof. We show that $X \wedge Y$ is an H-subcomodule of $X \otimes Y$. Again we write $x \otimes y \in X \wedge Y$ for an element in $X \wedge Y$. It is sufficient to verify that

$$\sum x_{(0)} \otimes y_{(0)} \langle p, y_{(1)} x_{(1)} \rangle \in X \wedge Y$$

for all $p \in H^*$ and $x \otimes y \in X \wedge Y$. Indeed, we have for all $h \in H$,

$$\begin{aligned}
&\sum h_{(1)} \cdot x_{(0)} \otimes h_{(2)} \triangleright_1 y_{(0)} \langle p, y_{(1)} x_{(1)} \rangle \\
&\quad = \sum h_{(1)} \cdot (p_{(2)} \cdot x) \otimes h_{(2)} \triangleright_1 (p_{(1)} \cdot y) \\
&\quad = \sum p_{(3)} \cdot (h_{(2)} \cdot x) \otimes p_{(2)} \cdot (h_{(3)} \triangleright_1 y) \langle p_{(1)}, h_{(4)} \rangle \big\langle p_{(4)}, S^{-1}(h_{(1)}) \big\rangle \\
&\quad = \sum p_{(3)} \cdot (h_{(2)} \triangleright_2 x) \otimes p_{(2)} \cdot (h_{(3)} \cdot y) \langle p_{(1)}, h_{(4)} \rangle \big\langle p_{(4)}, S^{-1}(h_{(1)}) \big\rangle \\
&\quad = \sum h_{(1)} \triangleright_2 (p_{(2)} \cdot x) \otimes h_{(2)} \cdot (p_{(1)} \cdot y) \\
&\quad = \sum h_{(1)} \triangleright_2 x_{(0)} \otimes h_{(2)} \cdot y_{(0)} \langle p, y_{(1)} x_{(1)} \rangle,
\end{aligned}$$

where we abuse the use of the $\cdot$ for both the action of H on M and the dual action of H^* on M in order to reduce new symbols. It follows that $X \wedge Y$ is a right H-subcomodule of $X \otimes Y$.

Next we show that the right H-comodule $X \wedge Y$ with the left H-module structure (8) is a YD H-module. For $h \in H$ and $x \otimes y \in X \wedge Y$, we have

$$\begin{aligned}
\rho\big(h \cdot (x \otimes y)\big) &= \sum \rho(h_{(1)} \cdot x \otimes h_{(2)} \triangleright_1 y) \\
&= \sum h_{(2)} \cdot x_{(0)} \otimes h_{(5)} \triangleright_1 y_{(0)} \otimes h_{(6)} y_{(1)} S^{-1}(h_{(4)}) h_{(3)} x_{(1)} S^{-1}(h_{(1)}) \\
&= \sum h_{(2)} \cdot (x_{(0)} \otimes y_{(0)}) \otimes h_{(3)} y_{(1)} x_{(1)} S^{-1}(h_{(1)})
\end{aligned}$$

for any $h \in H$ and $x \otimes y \in X \wedge Y$. $\square$

In order to show that the canonical $\mathcal{H}_R$-bimodule structure on $X \wedge Y$ stems from the YD H-module structure defined above, we have to show that

$$\begin{aligned} h \rightharpoonup \Big(\sum x_i \otimes y_i\Big) &= \sum (h \rightharpoonup x_i) \otimes y_i, \\ \Big(\sum x_i \otimes y_i\Big) \leftharpoonup h &= \sum x_i \otimes (y_i \leftharpoonup h) \end{aligned} \tag{9}$$

whenever $h \in \mathcal{H}_R$ and $\sum x_i \otimes y_i \in X \wedge Y$. We verify the first formula and leave the second one to the readers. Indeed,

$$\begin{aligned} h \rightharpoonup (x \otimes y) &= \sum S^{-1}(h_{(2)}) \triangleright_1 \big(h_{(1)} \cdot (x \otimes y)\big) \\ &= \sum S^{-1}(h_{(3)}) \triangleright_1 (h_{(1)} \cdot x \otimes h_{(2)} \triangleright_1 y) \\ &= \sum S^{-1}(h_{(2)}) \triangleright_1 (h_{(1)} \cdot x) \otimes y \\ &= (h \rightharpoonup x) \otimes y, \end{aligned}$$

for any $h \in \mathcal{H}_R$ and $x \otimes y \in X \wedge Y$. Thus the right (or left) $\mathcal{H}_R^*$-comodule structure of $X \wedge Y$ only comes from the right one of X (or the left one of Y). The $\mathcal{H}_R$-bimodule structure (9) of $X \wedge Y$ shows that the generalized cotensor product $\wedge$ is associative.

Now we consider the cotensor product of two YD H-module algebras. Let $\#_R$ be the braided product in the category $\mathbf{M}_R^H$ to differ from the braided product in $\mathcal{Q}^H$. This makes sense when a YD H-module algebra A can be treated as an algebra in $\mathbf{M}_R^H$ by forgetting the H-module structure of A and endowing with the induced H-module structure (1). Let X and Y be two YD H-module algebras. If there is no confusion we will write $\sum x_i \# y_i$ (or simply $x \# y$) for an element in $X \wedge Y$ as we can multiply them in $X \#_R Y$.

Proposition 2.11. *If X and Y are two YD H-module algebras, then $X \wedge Y$ is a YD H-module algebra and $X \wedge Y$ is a subalgebra of $X \#_R Y$.*

Proof. By Proposition 2.10, $X \wedge Y$ is a YD H-module. It remains to be shown that $X \wedge Y$ is a left H-module algebra and a right H^{op}-comodule algebra.

First we have to show that $X \wedge Y$ is a subalgebra of $X \#_R Y$. Write $x \otimes y$ and $x' \otimes y'$ for two arbitrary elements of $X \wedge Y$. We show that

$$(x \# y)\big(x' \# y'\big) = \sum x x'_{(0)} \otimes y_{(0)} y' R\big(x'_{(1)} \otimes y_{(1)}\big)$$

is in $X \wedge Y$. For $h \in H$, we have

$$\begin{aligned} &\sum h_{(1)} \cdot \big(x x'_{(0)}\big) \otimes h_{(2)} \triangleright_1 \big(y_{(0)} y'\big) R\big(x'_{(1)} \otimes y_{(1)}\big) \\ &\quad= \sum (h_{(1)} \cdot x b_{(1)})\big(h_{(2)} \cdot x'_{(0)}\big) \otimes (h_{(3)} \triangleright_1 y_{(0)})\big(h_{(4)} \triangleright_1 y'\big) R\big(x'_{(1)} \otimes y_{(1)}\big) \\ &\quad= \sum (h_{(1)} \cdot x)\big(h_{(2)} \cdot x'_{(0)}\big) \otimes y_{(0)} y'_{(0)} R(h_{(3)} \otimes y_{(1)}) R\big(h_{(4)} \otimes y'_{(1)}\big) R\big(x'_{(1)} \otimes y_{(2)}\big) \end{aligned}$$

$$
\begin{aligned}
&= \sum (h_{(1)} \cdot x)\big(h_{(4)} \cdot x'_{(0)}\big) \otimes y_{(0)} y'_{(0)} R\big(h_{(5)} x'_{(1)} S^{-1}(h_{(3)}) \otimes y_{(1)}\big) \\
&\qquad \times R(h_{(2)} \otimes y_{(2)}) R\big(h_{(6)} \otimes y'_{(1)}\big) \\
&= \sum (h_{(1)} \cdot x \otimes h_{(2)} \triangleright_1 y)\big(h_{(3)} \cdot x' \otimes h_{(4)} \triangleright_1 y'\big).
\end{aligned}
$$

Similarly, one may obtain

$$
\begin{aligned}
&\sum h_{(1)} \triangleright_2 \big(x x'_{(0)}\big) \otimes h_{(2)} \cdot \big(y_{(0)} y'\big) R\big(x'_{(1)} \otimes y_{(1)}\big) \\
&\quad = \sum (h_{(1)} \triangleright_2 x \otimes h_{(2)} \cdot y)\big(h_{(3)} \triangleright_2 x' \otimes h_{(4)} \cdot y'\big).
\end{aligned}
$$

Thus we obtain that

$$
\begin{aligned}
&\sum h_{(1)} \cdot \big(x x'_{(0)}\big) \otimes h_{(2)} \triangleright_1 \big(y_{(0)} y'\big) R\big(x'_{(1)} \otimes y_{(1)}\big) \\
&\quad = \sum (h_{(1)} \cdot x \otimes h_{(2)} \triangleright_1 y)\big(h_{(3)} \cdot x' \otimes h_{(4)} \triangleright_1 y'\big) \\
&\quad = \sum (h_{(1)} \triangleright_2 x \otimes h_{(2)} \cdot y)\big(h_{(3)} \triangleright_2 x' \otimes h_{(4)} \cdot y'\big) \\
&\quad = \sum h_{(1)} \triangleright_2 \big(x x'_{(0)}\big) \otimes h_{(2)} \cdot \big(y_{(0)} y'\big) R\big(x'_{(1)} \otimes y_{(1)}\big).
\end{aligned}
$$

By Lemma 2.9, $X \wedge Y$ is a subalgebra of $X \#_R Y$ and hence a H^{op}-comodule subalgebra of $X \#_R Y$. Moreover, the previous computations actually showed that $X \wedge Y$ is a left H-module algebra with the H-action (8). It follows from Proposition 2.10 that $X \wedge Y$ is a YD H-module algebra. □

To end this section we present the dual comodule version of (7) which is needed in the next section. Observe that the dual coalgebra $\mathcal{H}_R^*$ is a left $D[H]^*$-module quotient coalgebra of the dual Hopf algebra $D[H]^*$ in the sense that the following coalgebra map is a surjective $D[H]^*$-module map:

$$
\phi^* : D[H]^* \to \mathcal{H}_R^*, \quad p \bowtie q \mapsto q S^{*-1}(p),
$$

where $D[H]^* = H^* \bowtie H^*$ is equal to $H^* \otimes H^*$ as an algebra but has the dual comultiplication of the multiplication of $D[H]$. Thus a left (or right) $D[H]^*$-comodule M is a left (or right) $\mathcal{H}_R^*$-comodule in the natural way through ϕ^*. In order to distinguish $D[H]^*$ or $\mathcal{H}_R^*$-comodule structures from the H-comodule structures (e.g., a YD H-module has all three comodule structures) we use different uppercase Sweedler sigma notations:

(i) $\sum x^{[-1]} \otimes x^{[0]}$, $\sum x^{[0]} \otimes x^{[1]}$ stand for left and right $D[H]^*$-comodule structures, respectively.
(ii) $\sum x^{(-1)} \otimes x^{(0)}$, $\sum x^{(0)} \otimes x^{(1)}$ stand for left and right $\mathcal{H}_R^*$-comodule structures, respectively.

Now let X be a YD H-module algebra. Then X is both a left and right $D[H]$-module algebra, and therefore an $\mathcal{H}_R$-bimodule algebra in the sense of (7). Thus the dual comodule versions of the formulas in (7) read as follows:

$$\begin{aligned}\sum (ab)^{(0)} \otimes (ab)^{(1)} &= \sum a^{[0]} b^{(0)} \otimes a^{[1]} \rightharpoondown b^{(1)},\\ \sum (ab)^{(-1)} \otimes (ab)^{(0)} &= \sum b^{[-1]} \rightharpoondown a^{(-1)} \otimes a^{(0)} b^{[0]}\end{aligned} \tag{10}$$

for $a, b \in X$, where $\rightharpoondown$ is the left action of $D[H]^*$ on $\mathcal{H}_R^*$. We will call X a *right* (*or left*) $\mathcal{H}_R^*$*-comodule algebra* in the sense of (10).

Finally, for a YD H-module M, we will write $M_\diamond$ (or ${}_\diamond M$) for the right (or left) $\mathcal{H}_R^*$-coinvariants. For instance,

$$M_\diamond = \Big\{ m \in M \;\Big|\; \sum m^{(0)} \otimes m^{(1)} = m \otimes \varepsilon \Big\}.$$

It is obvious that $M_\diamond = M^{\mathcal{H}_R}$. If we let k be the trivial YD H-module, then

$$M_\diamond = k \wedge M, \qquad {}_\diamond M = M \wedge k.$$

Moreover, if A is a YD H-module algebra, then $A_\diamond$ and ${}_\diamond A$ are subalgebras of A.

3. The group Gal($\mathcal{H}_R$)

In this section, we construct a group Gal($\mathcal{H}_R$) of 'bigalois' objects for $\mathcal{H}_R$. The group Gal($\mathcal{H}_R$) plays the vital part in this paper. Let A be a right $D[H]^*$-comodule algebra. Then A is a right $\mathcal{H}_R^*$-comodule algebra in the sense of (10).

Definition 3.1. Let A be a right $D[H]^*$-comodule algebra. The extension $A/A_\diamond$ is said to be a right $\mathcal{H}_R^*$-Galois extension if the k-module map

$$\beta^r : A \otimes_{A_\diamond} A \to A \otimes \mathcal{H}_R^*, \quad \beta^r(a \otimes b) = \sum a^{(0)} b \otimes a^{(1)}$$

is an isomorphism. Similarly, if A is a left $D[H]^*$-comodule algebra, then $A/{}_\diamond A$ is said to be left Galois if the k-module map

$$\beta^l : A \otimes_{{}_\diamond A} A \to \mathcal{H}_R^* \otimes A, \quad \beta^l(a \otimes b) = \sum b^{(-1)} \otimes a b^{(0)}$$

is an isomorphism. If in addition the subalgebra ${}_\diamond A$ (or $A_\diamond$) is trivial and A is faithfully flat over k, then A is called a *left* (or *right*) $\mathcal{H}_R^*$*-Galois object*. A right $D[H]^*$-comodule algebra is a right $\mathcal{H}_R^*$-Galois object if and only if the functor $A \otimes -$ defines an category equivalence from category of left k-modules to the category of $(A, \mathcal{H}_R^*)$-Hopf modules (see [26]). For more details on Hopf quotient Galois theory, the readers may refer to [21,26,27].

The objects we are interested in are those $\mathcal{H}_R^*$-bigalois objects which are both left and right $\mathcal{H}_R^*$-Galois such that the left and right $\mathcal{H}_R^*$-coactions commute. Denote by $\mathcal{E}(\mathcal{H}_R)$ the category of YD H-module algebras which are $\mathcal{H}_R^*$-bigalois objects. The morphisms in $\mathcal{E}(\mathcal{H}_R)$ are YD H-module algebra isomorphisms. This is because any YD H-module algebra map between two Galois objects A and B yields that B is an $(A, \mathcal{H}_R^*)$-Hopf module and hence $B \cong A \otimes k = A$ by the equivalence mentioned in the previous paragraph. We show that the category $\mathcal{E}(\mathcal{H}_R)$ is closed under the cotensor product $\wedge$.

Proposition 3.2. *If X, Y are two objects of $\mathcal{E}(\mathcal{H}_R)$, then $X \wedge Y$ is an object of $\mathcal{E}(\mathcal{H}_R)$.*

Proof. From Proposition 2.11 we know that $X \wedge Y$ is a YD H-module algebra. It remains to show that $X \wedge Y$ is an $\mathcal{H}_R^*$-bigalois object. Note that the $\mathcal{H}_R$-bimodule structure on $X \wedge Y$ induced by its YD H-module structure is given by the left $\mathcal{H}_R$-module structure of X and the right $\mathcal{H}_R$-module structure of Y (see (9)). Thus the right (or left) $\mathcal{H}_R^*$-comodule structure of $X \wedge Y$ comes from the right one of X (or the left one of Y).

From the remark at the end of Section 2, we have

$$(X \wedge Y)_\diamond = k \wedge (X \wedge Y) = (k \wedge X) \wedge Y = k \wedge Y = k.$$

Similarly the left coinvariant subalgebra of $X \wedge Y$ is trivial as well. To show that $X \wedge Y$ is a right $\mathcal{H}_R^*$-Galois object, we need to consider the YD H-module $(X \otimes X) \wedge Y$, where $X \otimes X$ is the YD H-module of the YD H-module algebra $X \# X$ in the category $\mathcal{Q}^H$.

Now one may take a while to check that the following diagram is commutative:

$$\begin{array}{ccc} (X \wedge Y) \otimes (X \wedge Y) & \xrightarrow{\beta^r_{X \wedge Y}} & (X \wedge Y) \otimes \mathcal{H}_R^* \\ & {\scriptstyle\tau} \searrow \qquad \nearrow {\scriptstyle\delta} & \\ & (X \otimes X) \wedge Y & \end{array}$$

where τ is the k-module map given by

$$\tau\big((x \# y) \otimes (x' \# y')\big) = \sum \big(x \otimes x'^{(0)}\big) \# \big(x'^{(1)} \triangleright_1 y\big) y'$$

and the k-module map δ is defined by

$$\delta\big(\big(x \otimes x'\big) \# y\big) = \sum x^{(0)} x' \# y \otimes x^{(1)}.$$

However, we have to show that τ and δ are well-defined. We leave the easier verification of τ to the readers, and show that δ is well-defined.

Observe that the multiplication map of X: $X \otimes X \to X$ is an $\mathcal{H}_R$-bimodule map because X is an $\mathcal{H}_R$-bimodule algebra in the sense of Corollary 2.8. The multiplication map then induces an $\mathcal{H}_R$-bimodule map:

$$\mu : (X \otimes X) \wedge Y \to X \wedge Y.$$

If we can prove that the map η:

$$(X \otimes X) \wedge Y \to (X \otimes X) \wedge Y \otimes \mathcal{H}_R^*, \quad (x \otimes x') \# y \mapsto \sum (x^{(0)} \otimes x') \# y \otimes x^{(1)}$$

is well-defined, then δ is well-defined because δ is actually the composite map of η with μ:

$$\delta : (X \otimes X) \wedge Y \xrightarrow{\eta} (X \otimes X) \wedge Y \otimes \mathcal{H}_R^* \xrightarrow{\mu} X \wedge Y \otimes \mathcal{H}_R^*.$$

To show that η is well-defined, it is equivalent to show that for any $l \in \mathcal{H}_R$, and $(x \otimes x') \# y \in (X \otimes X) \wedge Y$, the element

$$(l \rightharpoonup x \otimes x') \# y$$

is still in $(X \otimes X) \wedge Y$. This is the case since

$$\begin{aligned}((l \rightharpoonup x \otimes x') \leftharpoonup h) \# y &= \sum \big[((l \rightharpoonup x) \leftharpoonup h_{(0)}) \otimes (x' \leftharpoonup h_{(-1)}) \big] \# y \\ &= \sum \big[l \rightharpoonup (x \leftharpoonup h_{(0)}) \otimes (x' \leftharpoonup h_{(-1)}) \big] \# y \\ &= \sum (l \rightharpoonup x \otimes x') \# (h \rightharpoonup y)\end{aligned}$$

for any $h, l \in \mathcal{H}_R$ and $(x \otimes x') \# y \in (X \otimes X) \wedge Y$.

Since both τ and δ are obviously isomorphisms, we obtain that $\beta_{X \wedge Y}$ is an isomorphism. It follows that $X \wedge Y$ is a right $\mathcal{H}_R^*$-Galois extension of k. Similarly, one may show that $X \wedge Y$ is a left $\mathcal{H}_R^*$-Galois extension of k.

Finally we have to show that $X \wedge Y$ is faithfully flat over k. Observe that

$$\begin{aligned}X \otimes (X \wedge Y) &\cong (X \otimes X) \wedge Y \\ &\cong (X \otimes \mathcal{H}_R^*) \wedge Y \\ &\cong X \otimes (\mathcal{H}_R^* \wedge Y) \\ &\cong X \otimes Y,\end{aligned}$$

where $\mathcal{H}_R^* = H^*$ as an object in $\mathcal{E}(\mathcal{H}_R)$ is defined below and the last isomorphism will be proved in Proposition 3.4. Since $X \otimes Y$ and X are faithfully flat, it follows that $X \wedge Y$ is faithfully flat. Thus $X \wedge Y$ is an object in $\mathcal{E}(\mathcal{H}_R)$. □

Now let H^* be the convolution algebra of H. There is a canonical YD H-module structure on H^* such that H^* is a YD H-module algebra. For $h^*, p \in H^*$ and $h \in H$, we define

$$\begin{aligned}h \cdot p &= \sum p_{(1)} \langle p_{(2)}, h \rangle, \quad H\text{-action}, \\ h^* \cdot p &= \sum h^*_{(2)} p S^{-1}(h^*_{(1)}), \quad H\text{-coaction}\end{aligned} \tag{11}$$

where we use S for the antipodes of both H and H^* in order to simplify the notations and we will do the same in the sequel. Before we show that H^* is an object in $\mathcal{E}(\mathcal{H}_R)$ we need to work out the comultiplication of $\mathcal{H}_R^*$. Since H is finite, we may think of the CQT structure R of H as an element $\sum R^1 \otimes R^2$ in $H^* \otimes H^*$. Then we have

$$\begin{aligned}\Delta_R(p) &= \sum R^2 r^2 p_{(2)} \otimes r^1 p_{(1)} S^{(-1)}\big(R^1\big)\\ &= \sum R^2 p_{(1)} r^2 \otimes p_{(2)} r^1 S^{(-1)}\big(R^1\big)\end{aligned}$$

where $r = R$, $p \in H^*$.

Lemma 3.3. *H^* is an object in $\mathcal{E}(\mathcal{H}_R)$.*

Proof. It is sufficient to show that the induced $\mathcal{H}_R$-bimodule structure (3) and (6) on H^* is the same as the dual $\mathcal{H}_R$-bimodule structure stemming from the comultiplication Δ_R of $\mathcal{H}_R^*$. Indeed, given $p \in H^*$, $h \in \mathcal{H}_R$, we have

$$\begin{aligned}h \multimap p &= \sum S^{-1}(h_{(2)}) \triangleright_1 (h_{(1)} \cdot p)\\ &= \sum \Theta_l\big(S^{-1}(h_{(3)})\big)(h_{(1)} \cdot p) S^{(-1)}\big(\Theta_l\big(S^{-1}(h_{(2)})\big)\big)\\ &= \sum \Theta_l\big(S^{-1}(h_{(3)})\big) p_{(1)} \Theta_l(h_{(2)}) \langle p_{(2)}, h_{(1)}\rangle\\ &= \sum R^2 p_{(1)} r^2 \langle p_{(2)}, h_{(1)}\rangle \big\langle r^1, h_{(2)}\big\rangle \big\langle S^{(-1)}\big(R^1\big), h_{(3)}\big\rangle\\ &= \sum R^2 p_{(1)} r^2 \big\langle p_{(2)} r^1 S^{(-1)}\big(R^1\big), h\big\rangle\\ &= \sum p^{(1)} \big\langle p^{(2)}, h\big\rangle.\end{aligned}$$

Similarly, one have $p \multimapinv h = \sum p^{(2)} \langle p^{(1)}, h\rangle$ for any $h \in \mathcal{H}_R$ and $p \in H^*$. Since $\mathcal{H}_R^*$ is a quotient coalgebra of $D[H]^*$, we have that H^* with the $\mathcal{H}_R^*$-bicomodule structure Δ_R is an $\mathcal{H}_R^*$-bigalois object. $\square$

Denote by I the object H^* described in Lemma 3.3. In fact, I is the unit of the category $\mathcal{E}(\mathcal{H}_R)$. Before we prove this, we need to figure out the relation between $\mathcal{H}_R^*$-comodule structure and the H^*-comodule structures of a YD H-module. Let M be a YD H-module. We use the following summation notation for the dual H^*-comodule structure of the left H-module structure of M:

$$M \to M \otimes H^*, \quad m \mapsto \sum m_{[0]} \otimes m_{[1]},$$

and the usual Sweedler notation $\sum m_{(0)} \otimes m_{(1)}$ for the H-comodule structure of M. The right and left $\mathcal{H}_R^*$-comodule structure will be indicated by the Sweedler 'uppercase' sigma notations.

It is not difficult to check that the right $D[H]^*$-comodule structure of M reads as follows:

$$M \to M \otimes D[H]^*, \quad m \mapsto \sum m_{[0](0)} \otimes \Theta_r(m_{[0](1)}) \bowtie m_{[1]}.$$

Similarly, one may get the dual left $D[H]^*$-comodule structure of (5). It follows from (7) that we obtain the corresponding dual right $\mathcal{H}_R^*$-comodule structure of (3) and the dual left $\mathcal{H}_R^*$-comodule structure of (6) respectively:

$$\begin{aligned} &M \to M \otimes \mathcal{H}_R^*, \quad \sum m^{(0)} \otimes m^{(1)} = \sum m_{[0](0)} \otimes m_{[1]} S^{-1}\big(\Theta_r(m_{[0](1)})\big), \\ &M \to \mathcal{H}_R^* \otimes M, \quad \sum m^{(-1)} \otimes m^{(0)} = \sum \Theta_l(m_{[0](1)}) m_{[1]} \otimes m_{[0](0)}. \end{aligned} \tag{12}$$

Proposition 3.4. *The category $\mathcal{E}(\mathcal{H}_R)$ is a monoidal category with product $\wedge$ and the unit I.*

Proof. It is sufficient to show that $I \wedge X \cong X \cong X \wedge I$ for any $X \in \mathcal{E}(\mathcal{H}_R)$. We show that $I \wedge X \cong X$. The proof of $X \wedge I \cong X$ is similar. Let ρ^+ be the composite map of the flip map with the right $\mathcal{H}_R^*$-comodule structure of X. We show that

$$\rho^+ : X \to I \wedge X, \quad \rho^+(x) = \sum x^{(1)} \# x^{(0)}$$

is the desired isomorphism in $\mathcal{E}(\mathcal{H}_R)$. By Lemma 3.3, we have

$$\begin{aligned} \sum \big(x^{(1)} \triangleleft\!\!- h\big) \# x^{(0)} &= \sum x^{(2)} \langle x^{(1)}, h\rangle \# x^{(0)} \\ &= \sum x^{(1)} \# \big(h -\!\!\triangleright x^{(0)}\big) \end{aligned}$$

for $h \in \mathcal{H}_R$ and $x \in X$. So ρ^+ is a well-defined isomorphism with the inverse given by

$$I \wedge X \to X, \quad \sum p_i \# x_i \mapsto \sum p_i(1) x_i.$$

Secondly, we verify that ρ^+ is an algebra map. For $x, y \in X$ and $h \in \mathcal{H}_R$, we compute $\rho^+(x)\rho^+(y)$.

$$\begin{aligned} \rho^+(x)\rho^+(y) &= \sum \big(x^{(1)} \# x^{(0)}\big)\big(y^{(1)} \# y^{(0)}\big) \\ &= \sum x^{(1)} y^{(1)}_{(0)} \# x^{(0)}_{(1)} y^{(0)} R\big(y^{(1)}_{(1)} \otimes x^{(1)}_{(1)}\big) \\ &= \sum x^{(1)} \big(\Theta_r\big(x^{(0)}_{(1)}\big) \cdot y^{(1)}\big) \# x^{(0)}_{(0)} y^{(0)} \\ &= \sum x_{[1]} \Theta_r\big(S(x_{[0](3)})\big) \Theta_r(x_{[0](2)}) y_{[1]} \Theta_r\big(S(y_{[0](1)})\big) \\ &\quad \times \Theta_r\big(S(x_{[0](1)})\big) \# x_{[0](0)} y_{[0](0)} \end{aligned}$$

$$= \sum x_{[1]} y_{[1]} \Theta_r \big(S(y_{[0](1)} x_{[0](1)})\big) \# x_{[0](0)} y_{[0](0)}$$
$$= \sum (xy)^{(1)} \otimes (xy)^{(0)}$$
$$= \sum \rho^+(xy).$$

Finally it is not hard to check that ρ^+ is a YD H-module morphism, hence a morphism in $\mathcal{E}(\mathcal{H}_R)$. $\square$

Note that the proof of Proposition 3.4 deduced that the coalgebra $\mathcal{H}_R^*$ with the convolution algebra structure is a *braided Hopf algebra* in $\mathbf{M}_R^H$ (see the definition from [19,29]). A more categorical study of the braided Hopf algebra $\mathcal{H}_R^*$ will be included in the forthcoming paper [40].

Denote by $E(\mathcal{H}_R)$ the set of the isomorphism classes of objects in $\mathcal{E}(\mathcal{H}_R)$. Propositions 3.2, 3.4 say that $E(\mathcal{H}_R)$ is a semigroup. The rest of this section is devoted to show that $E(\mathcal{H}_R)$ contains a group.

Let X be an object in $\mathcal{E}(\mathcal{H}_R)$. Let $\overline{X}$ be the opposite algebra in $\mathbf{M}_R^H$. That is, $\overline{X} = X$ as a right H^{op}-comodule, but with the multiplication given by

$$\bar{x} \circ \bar{y} = \sum \overline{y_{(0)} x_{(0)}} R(y_{(1)} \otimes x_{(1)})$$

when $\bar{x}, \bar{y} \in \overline{X}$. Since the H-action on X does not give an H-module algebra structure on $\overline{X}$, we have to define a new H-action on $\overline{X}$ such that $\overline{X}$ together with the inherited H^{op}-comodule structure from X is a YD H-module algebra. Let H act on $\overline{X}$ as follows

$$\begin{aligned} h \rightharpoonup \bar{x} &= \sum \overline{S(h_{(4)}) \cdot x_{(0)}} R(h_{(6)} \otimes x_{(2)}) R\big(x_{(1)} \otimes S(h_{(3)})\big) \\ &\quad \times R\big(h_{(5)} \otimes S(h_{(2)})\big) u^{-1}(h_{(1)}) \\ &= \sum \overline{h^u_{(3)} \cdot \big(h_{(2)} \triangleright_2 (h_{(5)} \triangleright_1 x)\big)} R\big(S(h_{(4)}) \otimes h_{(1)}\big) \end{aligned} \tag{13}$$

where $h \in \mathcal{H}_R$, $\bar{x} \in \overline{X}$, $h^u = \sum S(h_{(2)}) u^{-1}(h_{(1)})$ and $u = \sum S(R^2) R^1 \in H^*$ is the Casimir element of H^*. Since the square of the antipode of H^* is an inner automorphism induced by the Casimir element u, we have the formulae (see [17]):

$$\begin{aligned} &\sum u(h_{(1)}) h_{(2)} = \sum S^2(h_{(1)}) u(h_{(2)}), \\ &\sum u^{-1}(h_{(1)}) S^{-1}(h_{(2)}) = \sum S(h_{(1)}) u^{-1}(h_{(2)}) \end{aligned} \tag{14}$$

for any $h \in H$. We will use the formulas (14) quite often in the sequel. For instance, one may change the order of the actions $\triangleright_1, \triangleright_2$ and $\cdot$ in the formula (13) in order to have an alternative formula:

$$h \rightharpoonup \bar{x} = \sum \overline{h_{(1)} \triangleright_2 \left(h_{(4)} \triangleright_1 \left(S^{-1}(h_{(3)}) \cdot x\right)\right)} R\left(S^2(h_{(5)}) \otimes h_{(2)}\right) \tag{15}$$

for $h \in \mathcal{H}_R$ and $x \in \overline{X}$.

Lemma 3.5. *Let X be an object in $\mathcal{E}(\mathcal{H}_R)$. Then the right H^{op}-comodule algebra $\overline{X}$ with the H-action* (13) *is a YD H-module algebra.*

Proof. We show that $\overline{X}$ together with (13) is a YD H-module, and leave to the readers the tedious check that $\overline{X}$ is left H-module algebra.

First we show that (13) is a left H-module structure on $\overline{X}$. Given $h, l \in H$ and $\bar{x} \in \overline{X}$, we have

$$\begin{aligned}
&l \rightharpoonup \left(h \rightharpoonup \bar{x}\right) \\
&\quad = \sum l \rightharpoonup \overline{S(h_{(4)}) \cdot x_{(0)}} R(h_{(6)} \otimes x_{(2)}) R\left(x_{(1)} \otimes S(h_{(3)})\right) R\left(h_{(5)} \otimes S(h_{(2)})\right) u^{-1}(h_{(1)}) \\
&\quad = \sum \overline{S(l_{(4)}) \cdot \left(S(h_{(4)}) \cdot x_{(0)}\right)} R\left(l_{(6)} \otimes S(h_{(4)}) x_{(2)} h_{(8)}\right) R\left(S(h_{(5)}) x_{(1)} h_{(7)} \otimes S(l_{(3)})\right) \\
&\qquad \times R\left(l_{(5)} \otimes S(l_{(2)})\right) u^{-1}(l_{(1)}) R(h_{(10)} \otimes x_{(4)}) R\left(x_{(3)} \otimes S(h_{(3)})\right) \\
&\qquad \times R\left(h_{(9)} \otimes S(h_{(2)})\right) u^{-1}(h_{(1)}) \\
&\quad = \sum \overline{S(h_{(6)} l_{(6)}) \cdot x_{(0)}} R(l_{(8)} \otimes h_{(8)}) R\left(l_{(9)} \otimes S(h_{(3)})\right) R(l_{(10)} \otimes x_{(3)}) \\
&\qquad \times R(h_{(5)} \otimes l_{(5)}) R\left(x_{(1)} \otimes S(h_{(4)})\right) R\left(h_{(7)} \otimes S(l_{(3)})\right) R\left(l_{(7)} \otimes S(l_{(2)})\right) \\
&\qquad \times R(h_{(10)} \otimes x_{(4)}) R\left(x_{(2)} \otimes S(h_{(4)})\right) R\left(h_{(9)} \otimes S(h_{(2)})\right) u^{-1}(h_{(1)}) u^{-1}(l_{(1)}) \\
&\quad = \sum \overline{S(l_{(4)} h_{(4)}) \cdot x_{(0)}} R(h_{(5)} \otimes l_{(5)}) R(l_{(8)} \otimes h_{(8)}) R\left(l_{(9)} h_{(9)} \otimes S(h_{(2)})\right) \\
&\qquad \times R\left(h_{(7)} l_{(7)} \otimes S(l_{(2)})\right) R\left(x_{(1)} \otimes S(l_{(4)} h_{(4)})\right) \\
&\qquad \times R(l_{(10)} h_{(10)} \otimes x_{(2)}) u^{-1}(h_{(1)}) u^{-1}(l_{(1)}) \\
&\quad = \sum \overline{S^{-1}(l_{(3)} h_{(3)}) \cdot x_{(0)}} \left(\left(u^{-1} \otimes u^{-1}\right) R_{21} R\right)(l_{(5)} \otimes h_{(5)}) \\
&\qquad \times R\left(S^{-1}(l_{(4)} h_{(4)}) \otimes l_{(1)} h_{(1)}\right) R(l_{(6)} h_{(6)} \otimes x_{(2)}) R\left(x_{(1)} \otimes S^{-1}(l_{(2)} h_{(2)})\right) \\
&\quad = \sum \overline{S^{-1}(l_{(3)} h_{(3)}) \cdot x_{(0)}} u^{-1}(l_{(5)} h_{(5)}) R\left(S^{-1}(l_{(4)} h_{(4)}) \otimes l_{(1)} h_{(1)}\right) \\
&\qquad \times R(l_{(6)} h_{(6)} \otimes x_{(2)}) R\left(x_{(1)} \otimes S^{-1}(l_{(2)} h_{(2)})\right) \\
&\quad = \sum \overline{S(l_{(4)} h_{(4)}) \cdot x_{(0)}} R\left(l_{(5)} h_{(5)} \otimes S(l_{(2)} h_{(2)})\right) R(l_{(6)} h_{(6)} \otimes x_{(2)}) \\
&\qquad \times R\left(x_{(1)} \otimes S(l_{(3)} h_{(3)})\right) u^{-1}(l_{(1)} h_{(1)}) \\
&\quad = lh \rightharpoonup \bar{x},
\end{aligned}$$

where we used the identity $(u^{-1} \otimes u^{-1}) R_{21} R = \Delta(u^{-1})$ (see [17] for the proof), and $R_{21} = \sum R^2 \otimes R^1$.

Next we show that $\overline{X}$ is a YD H-module. Given $h \in H$ and $\bar{x} \in \overline{X}$, we compute $\rho(h \rightharpoonup \bar{x})$.

$$\begin{aligned}\rho\big(h \rightharpoonup \bar{x}\big) &= \sum \overline{\big(S(h_{(4)}) \cdot x_{(0)}\big)_{(0)}} \otimes \big(S(h_{(4)}) \cdot x_{(0)}\big)_{(1)} R(h_{(6)} \otimes x_{(2)}) \\ &\quad \times R\big(x_{(1)} \otimes S(h_{(3)})\big) R\big(h_{(5)} \otimes S(h_{(2)})\big) u^{-1}(h_{(1)}) \\ &= \sum \overline{S(h_{(5)}) \cdot x_{(0)}} \otimes S(h_{(4)}) x_{(1)} h_{(6)} R\big(x_{(2)} \otimes S(h_{(3)})\big) \\ &\quad \times R(h_{(8)} \otimes x_{(3)}) R\big(h_{(7)} \otimes S(h_{(2)})\big) u^{-1}(h_{(1)}) \\ &= \sum \overline{S(h_{(5)}) \cdot x_{(0)}} \otimes x_{(2)} S(h_{(3)}) h_{(6)} u^{-1}(h_{(1)}) \\ &\quad \times R(h_{(8)} \otimes x_{(3)}) R\big(x_{(1)} \otimes S(h_{(4)}) R(h_{(6)} \otimes S(h_{(3)})\big) \\ &= \sum \overline{S(h_{(5)}) \cdot x_{(0)}} \otimes x_{(2)} h_{(7)} S^{-1}(h_{(1)}) u^{-1}(h_{(2)}) \\ &\quad \times R(h_{(8)} \otimes x_{(3)}) R\big(x_{(1)} \otimes S(h_{(4)})\big) R\big(h_{(7)} \otimes S(h_{(2)})\big) \\ &= \sum \overline{S(h_{(5)}) \cdot x_{(0)}} \otimes h_{(8)} x_{(3)} S^{-1}(h_{(1)}) u^{-1}(h_{(2)}) \\ &\quad \times R(h_{(7)} \otimes x_{(2)}) R\big(x_{(1)} \otimes S(h_{(4)})\big) R\big(h_{(6)} \otimes S(h_{(3)})\big) \\ &= \sum h_{(2)} \rightharpoonup \overline{x_{(0)}} \otimes h_{(3)} x_{(1)} S^{-1}(h_{(1)}).\end{aligned}$$

So $\overline{X}$ is a YD H-module algebra. □

Now let us look at the $\mathcal{H}_R$-bimodule structure of $\overline{X}$ stemming from the new YD H-module structure of $\overline{X}$.

Lemma 3.6. *The $\mathcal{H}_R$-bimodule structure on $\overline{X}$ is given by*

$$\begin{aligned} h \,{-}\!\!\triangleright\, \bar{x} &= \sum \overline{S^{-1}(h_{(2)}) \cdot (h_{(1)} \triangleright_2 x)} \equiv \overline{h \twoheadrightarrow x}, \\ \bar{x} \,\triangleleft\!\!{-}\, h &= \sum \overline{S(h_{(1)}) \cdot (h_{(2)} \triangleright_1 x)} \equiv \overline{x \twoheadleftarrow h}. \end{aligned} \tag{16}$$

Proof. We verify the right $\mathcal{H}_R$-action, and leave the left $\mathcal{H}_R$-action to be checked by the readers. Indeed, for $h \in \mathcal{H}_R$ and $\bar{x} \in \overline{X}$, we have

$$\begin{aligned}\bar{x} \,\triangleleft\!\!{-}\, h &= \sum S(h_{(1)}) \triangleright_2 \big(h_{(2)} \rightharpoonup \bar{x}\big) \\ &= \sum \overline{h_{(4)} \triangleright_1 \big(S^{-1}(h_{(3)}) \cdot x\big)} u(h_{(1)}) R\big(S(h_{(5)}) \otimes S^{-1}(h_{(3)})\big) \quad \text{(using (15))} \\ &= \sum \overline{S^{-1}(h_{(4)}) \cdot (h_{(7)} \triangleright_1 x)} R^{-1}\big(h_{(6)} \otimes S^{-1}(h_{(5)})\big) \\ &\quad \times R\big(h_{(8)} \otimes S^{-1}(h_{(3)})\big) R\big(S(h_{(9)}) \otimes S^{-1}(h_{(2)})\big) u(h_{(1)}) \\ &= \sum \overline{S^{-1}(h_{(2)}) \cdot (h_{(4)} \triangleright_1 x)} u^{-1}(h_{(3)}) u(h_{(1)})\end{aligned}$$

$$= \sum \overline{S(h_{(3)}) \cdot (h_{(4)} \triangleright_1 x)} u^{-1}(h_{(2)}) u(h_{(1)})$$
$$= \sum \overline{S(h_{(1)}) \cdot (h_{(2)} \triangleright_1 x)}. \qquad \square$$

From (16), one obtains that the $\mathcal{H}_R$-bimodule structure of $\overline{X}$ is given by a new $\mathcal{H}_R$-bimodule structure of X defined by $\twoheadleftarrow$ and $\twoheadrightarrow$. We will use a different Sweedler uppercase sigma notation to denote the dual $\mathcal{H}_R^*$-bicomodule structure of the new $\mathcal{H}_R$-bimodule structure of X:

$$\sum x^{\langle -1\rangle} \otimes x^{\langle 0\rangle}, \qquad \sum x^{\langle 0\rangle} \otimes x^{\langle 1\rangle}.$$

Thus the $\mathcal{H}_R$-bimodule structure (16) on $\overline{X}$ can be translated into the $\mathcal{H}_R^*$-bicomodule structure on $\overline{X}$ in the following way:

$$\begin{aligned} \sum \bar{x}^{(0)} \otimes \bar{x}^{(1)} &= \sum \overline{x^{\langle 0\rangle}} \otimes x^{\langle 1\rangle} = \sum \overline{x_{(0)[0]}} \otimes \Theta_l\big(S(x_{(1)})\big) S^{-1}(x_{(0)[1]}), \\ \sum \bar{x}^{(-1)} \otimes \bar{x}^{(0)} &= \sum x^{\langle -1\rangle} \otimes \overline{x^{\langle 0\rangle}} = \sum S(x_{(0)[1]}) \Theta_r(x_{(1)}) \otimes \overline{x_{(0)[0]}}. \end{aligned} \tag{17}$$

Now let us recall that a YD H-module algebra A is said to be *quantum commutative* if

$$ab = \sum b_{(0)}(b_{(1)} \cdot a) \tag{18}$$

for any $a, b \in A$. That is, A is a commutative algebra in $\mathcal{Q}^H$.

Lemma 3.7. *Let A be a quantum commutative YD H-module algebra. Let $\overline{A}$ be the opposite algebra in $\mathbf{M}_R^H$. Then the multiplication of $\overline{A}$ reads as follows*:

$$\bar{a} \circ \bar{b} = \sum \overline{a_{(0)}\big(S(a_{(1)}) \twoheadrightarrow b\big)} = \sum \overline{(a \twoheadleftarrow b_{(1)}) b_{(0)}} \tag{19}$$

for $a, b \in A$, where the actions $\twoheadleftarrow$ and $\twoheadrightarrow$ are defined in (16).

Proof. Let $\bar{a}$ and $\bar{b}$ be two elements in $\overline{A}$. By definition, we have

$$\bar{a} \circ \bar{b} = \sum \overline{b_{(0)}(b_{(1)} \triangleright_1 a)}.$$

Since A is also quantum commutative in $\mathcal{Q}^H$, we have

$$\bar{a} \circ \bar{b} = \sum \overline{b_{(0)}(b_{(1)} \triangleright_1 a)} = \sum \overline{S(b_{(1)}) \cdot (b_{(2)} \triangleright_1 a) b_{(0)}} = \sum \overline{(a \twoheadleftarrow b_{(1)}) b_{(0)}}.$$

Similarly, we have

$$\bar{a} \circ \bar{b} = \sum \overline{\big(S(a_{(1)}) \triangleright_2 b\big) a_{(0)}} = \sum \overline{a_{(0)}\big(a_{(1)} \cdot \big(S(a_{(2)}) \triangleright_2 b\big)\big)} = \sum \overline{a_{(0)}\big(S(a_{(1)}) \twoheadrightarrow b\big)}$$

for any $\bar{a}, \bar{b} \in \overline{A}$. $\square$

Proposition 3.8. *Let X be an object in $\mathcal{E}(\mathcal{H}_R)$ such that X is quantum commutative in $\mathcal{Q}^H$. Then the opposite algebra $\overline{X}$ in $\mathbf{M}_R^H$ is an object in $\mathcal{E}(\mathcal{H}_R)$.*

Proof. We have to show that $\overline{X}$ is an $\mathcal{H}_R^*$-bigalois objects. Note that the sets of left and right $\mathcal{H}_R$-invariants are equal to the sets of the right and left $\mathcal{H}_R$-invariants of X respectively by Lemmas 3.6 and 2.5 and the remark preceding to Corollary 2.8. Thus $\overline{X}$ has trivial left and right $\mathcal{H}_R^*$-coinvariants. Let $f : X \otimes X \to X \otimes X$ be the k-module map defined by $f(x \otimes y) = \sum x_{(0)} \otimes S(x_{(1)}) \twoheadrightarrow y$. It is easy to see that f is a k-module isomorphism. Let us compute the canonical Galois k-module map $\beta^l_{\overline{X}}$ from $\overline{X} \otimes \overline{X}$ to $\mathcal{H}_R^* \otimes \overline{X}$. By applying the formulae (17) and (19), we obtain

$$\begin{aligned}
\beta^l_{\overline{X}}(\bar{y} \otimes \bar{x}) &= \sum \bar{x}^{(-1)} \otimes \bar{y} \circ \bar{x}^{(0)} \\
&= \sum x^{\langle -1 \rangle} \otimes \bar{y}\overline{x^{\langle 0 \rangle}} \\
&= \sum x^{\langle -1 \rangle} \otimes \overline{y_{(0)}\big(S(y_{(1)}) \twoheadrightarrow x^{\langle 0 \rangle}\big)} \\
&= \sum \big(S(y_{(1)}) \twoheadrightarrow x\big)^{\langle -1 \rangle} \otimes \overline{y_{(0)}\big(S(y_{(1)}) \twoheadrightarrow x\big)^{\langle 0 \rangle}}
\end{aligned}$$

for $\bar{x}, \bar{y} \in \overline{X}$. Since X is right $\mathcal{H}_R^*$-Galois, we have the canonical isomorphism β^r_X:

$$\beta^r_X(x \otimes y) = \sum x^{(0)} y \otimes x^{(1)} = \sum x_{[0](0)} y \otimes x_{[1]} S^{-1}\big(\Theta_r(x_{[0](1)})\big)$$

for any $x, y \in X$. The map β^r_X induces an isomorphism

$$\gamma^r : X \otimes X \to X \otimes \mathcal{H}_R^*$$

given by $\gamma^r(y \otimes x) = \sum y x_{(0)[0]} \otimes S(x_{(0)[1]}) \Theta_r(x_{(1)}) = \sum y x^{\langle 0 \rangle} \otimes x^{\langle -1 \rangle}$. Identifying the k-module X with $\overline{X}$ the map $\beta^l_{\overline{X}}$ is the following composite k-module isomorphism:

$$X \otimes X \xrightarrow{f} X \otimes X \xrightarrow{\gamma^r} X \otimes \mathcal{H}_R^* \xrightarrow{\tau} \mathcal{H}_R^* \otimes X$$

where τ is the flip map.

Similarly, one may verify that $\beta^r_{\overline{X}}$ is the composite isomorphism:

$$X \otimes X \xrightarrow{g} X \otimes X \xrightarrow{\gamma^l} \mathcal{H}_R^* \otimes X \xrightarrow{\tau} X \otimes \mathcal{H}_R^*,$$

where g and γ^l are given by

$$g(x \otimes y) = \sum x \twoheadleftarrow y_{(1)} \otimes y_{(0)}, \qquad \gamma^l(x \otimes y) = \sum x^{\langle 1 \rangle} \otimes x^{\langle 0 \rangle} y$$

for $x, y \in X$. So $\overline{X}$ is indeed an $\mathcal{H}_R^*$-bigalois object, and hence an object in $\mathcal{E}(\mathcal{H}_R)$. $\square$

Now we are able to prove our main theorem in this section. Denote by $\mathrm{Gal}(\mathcal{H}_R)$ the subset of $E(\mathcal{H}_R)$ consisting of the isomorphism classes of objects in $\mathcal{E}(\mathcal{H}_R)$ that are quantum commutative in $\mathcal{Q}^H$.

Theorem 3.9. *The set* $\mathrm{Gal}(\mathcal{H}_R)$ *is a group.*

Proof. First of all we show that $\mathrm{Gal}(\mathcal{H}_R)$ is a sub-semigroup of $E(\mathcal{H}_R)$. It is obvious that I is a quantum commutative algebra and $[I] \in \mathrm{Gal}(\mathcal{H}_R)$. Suppose that $[X]$ and $[Y]$ are two elements of $\mathrm{Gal}(\mathcal{H}_R)$. We have to verify that $X \wedge Y$ is quantum commutative. For simplicity, we will write $x \# y$ for an element $\sum x_i \# y_i$ of $X \wedge Y$. Given $x \# y, a \# b \in X \wedge Y$, we compute $\sum (x_{(0)} \# y_{(0)})(y_{(1)}x_{(1)} \rightharpoonup (a \# b))$.

$$\begin{aligned}
&\sum (x_{(0)} \# y_{(0)})\big(y_{(1)}x_{(1)} \rightharpoonup (a \# b)\big)\\
&\quad= \sum (x_{(0)} \# y_{(0)})(y_{(1)}x_{(1)} \triangleright_2 a \# y_{(2)}x_{(2)} \cdot b)\\
&\quad= \sum x_{(0)}\big(S(y_{(1)})y_{(2)}x_{(1)} \triangleright_2 a\big) \# y_{(0)}(y_{(3)}x_{(2)} \cdot b)\\
&\quad= \sum x_{(0)}(x_{(1)} \triangleright_2 a) \# y_{(0)}\big(y_{(1)} \cdot (x_{(2)} \cdot b)\big)\\
&\quad= \sum x_{(0)}(x_{(1)} \cdot a) \# y_{(0)}\big(y_{(1)} \cdot (x_{(2)} \triangleright_1 b)\big)\\
&\quad= \sum a x_{(0)} \# (x_{(1)} \triangleright_1 b) y\\
&\quad= (a \# b)(x \# y).
\end{aligned}$$

This means that $X \wedge Y$ is quantum commutative as well.

Secondly, we show that $\overline{X}$ is quantum commutative if X is quantum commutative. Indeed, given $\bar{x}, \bar{y} \in \overline{X}$, we have

$$\begin{aligned}
\sum \overline{y_{(0)}} \circ \big(y_{(1)} \rightharpoonup \bar{x}\big) &= \sum \overline{y_{(0)}} \circ \overline{S(y_{(4)}) \cdot x_{(0)}} R(y_{(6)} \otimes x_{(2)}) R\big(x_{(1)} \otimes S(y_{(3)})\big)\\
&\quad \times R\big(y_{(5)} \otimes S(y_{(2)})\big) u^{-1}(y_{(1)})\\
&= \sum \overline{\big(S(y_{(6)}) \cdot x_{(0)}\big) y_{(0)}} R\big(S(y_{(5)}) x_{(1)} y_{(7)} \otimes y_{(1)}\big)\\
&\quad \times R(y_{(9)} \otimes x_{(3)}) R\big(x_{(2)} \otimes S(y_{(4)})\big) R\big(y_{(8)} \otimes S(y_{(3)})\big) u^{-1}(y_{(2)})\\
&= \sum \overline{\big(S(y_{(5)}) \cdot x_{(0)}\big) y_{(0)}} R\big(S(y_{(4)}) x_{(1)} y_{(7)} \otimes y_{(1)}\big) R(y_{(8)} \otimes x_{(3)})\\
&\quad \times R\big(x_{(2)} y_{(7)} \otimes S(y_{(3)})\big) u^{-1}(y_{(2)})\\
&= \sum \overline{\big(S(y_{(5)}) \cdot x_{(0)}\big) y_{(0)}} R\big(S(y_{(4)}) x_{(1)} y_{(7)} \otimes y_{(1)}\big) R(y_{(8)} \otimes x_{(3)})\\
&\quad \times R\big(S(x_{(2)} y_{(7)}) \otimes y_{(2)}\big) u^{-1}(y_{(3)})\\
&= \sum \overline{\big(S(y_{(4)}) \cdot x_{(0)}\big) y_{(0)}} R\big(S(y_{(3)}) \otimes y_{(1)}\big) R(y_{(5)} \otimes x_{(1)}) u^{-1}(y_{(2)})\\
&= \sum \overline{\big(S(y_{(4)}) \cdot x_{(0)}\big) y_{(0)}} u^{-1}(y_{(3)}) R\big(S^{-1}(y_{(2)}) \otimes y_{(1)}\big) R(y_{(5)} \otimes x_{(1)})
\end{aligned}$$

$$= \sum \overline{\big(S(y_{(1)}) \cdot x_{(0)}\big) y_{(0)}} R(y_{(2)} \otimes x_{(1)})$$
$$= \sum \overline{y_{(0)} x_{(0)}} R(y_{(1)} \otimes x_{(1)})$$
$$= \bar{x} \circ \bar{y}.$$

Thus we have proved that $[\overline{X}]$ is an element of $\mathrm{Gal}(\mathcal{H}_R)$ if $[X] \in \mathrm{Gal}(\mathcal{H}_R)$. Finally we show that $[\overline{X}]$ is the inverse of $[X]$ in $\mathrm{Gal}(\mathcal{H}_R)$.

In order to show that $X \wedge \overline{X} \cong I$ in $\mathcal{E}(\mathcal{H}_R)$, it is sufficient to construct a non-zero YD H-module algebra map from I to $X \wedge \overline{X}$. Since X is a right $\mathcal{H}_R^*$-Galois object we have the isomorphism:

$$\beta^r : X \otimes X \to X \otimes \mathcal{H}_R^*.$$

Write $\sum U_i(p) \otimes V_i(p)$ for the inverse image $(\beta^r)^{-1}(1 \otimes p)$ of an element $p \in \mathcal{H}_R^*$. We claim that $\sum U_i(p) \otimes \overline{V_i(p)}$ is in $X \wedge \overline{X}$ for any $p \in \mathcal{H}_R^*$. To show this we need to verify that

$$\sum \big(U_i(p) \mathrel{\triangleleft\!\!-} h\big) \# \overline{V_i(p)} = \sum U_i(p) \# \big(h \mathrel{-\!\!\triangleright} \overline{V_i(p)}\big),$$

or

$$\sum \big(U_i(p) \mathrel{\triangleleft\!\!-} h\big) \# V_i(p) = \sum U_i(p) \# \big(h \twoheadrightarrow V_i(p)\big),$$

for any $h \in \mathcal{H}_R$.

Indeed, if $p \in \mathcal{H}_R^*$, we have the formulae

$$\beta^r\Big(\sum U_i(p) \otimes V_i(p)\Big) = \sum U_i(p)^{(0)} V_i(p) \otimes U_i(p)^{(1)} = 1 \otimes p,$$
$$\gamma^r\Big(\sum U_i(p) \otimes V_i(p)\Big) = \sum U_i(p) V_i(p)^{\langle 0\rangle} \otimes V_i(p)^{\langle -1\rangle} = 1 \otimes p.$$

Similarly, writing $\sum X_j(p) \otimes Y_j(p)$ for the element $(\beta^l)^{-1}(p \otimes 1)$ if $p \in \mathcal{H}_R^*$, then we have

$$\sum X_j(p)^{\langle 1\rangle} \otimes X_j(p)^{\langle 0\rangle} Y_j(p) = p \otimes 1$$

and

$$\sum X_j\big(x^{\langle 1\rangle}\big) \otimes Y_j\big(x^{\langle 1\rangle}\big) x^{\langle 0\rangle} = x \otimes 1 \tag{20}$$

for any $x \in X$. Applying formula (20), we obtain

$$\begin{aligned}\sum\big(\iota\otimes\gamma^r\big)\big(1\otimes U_i(p)\otimes V_i(p)\big) &= 1\otimes 1\otimes p\\ &= \sum U_i(p)V_i(p)^{\langle 0\rangle}\otimes 1\otimes V_i(p)^{\langle -1\rangle}\\ &= \sum U_i(p)X_j(q)\otimes Y_j(q)V_i(p)^{\langle 0\rangle}\otimes V_i(p)^{\langle -1\rangle}\\ &= \sum\big(\iota\otimes\gamma^r\big)U_i(p)X_j(q)\otimes Y_j(q)\otimes V_i(p),\end{aligned}$$

where $q=V_i(p)^{\langle 1\rangle}$. Since $\iota\otimes\gamma^r$ is an isomorphism, we obtain

$$\sum 1\otimes U_i(p)\otimes V_i(p)=\sum U_i(p)X_j(q)\otimes Y_j(q)\otimes V_i(p)^{\langle 0\rangle}, \tag{21}$$

where again $q=V_i(p)^{\langle 1\rangle}$. Now let $\beta^l\otimes\iota$ act on both sides of (21), we get

$$\sum U_i(p)^{(-1)}\otimes U_i(p)^{(0)}\otimes V_i(p)=\sum V_i(p)^{\langle 1\rangle}\otimes U_i(p)\otimes V_i(p)^{\langle 0\rangle}.$$

It follows that

$$\sum\big(U_i(p)\triangleleft\!\!- h\big)\,\#\,V_i(p)=\sum U_i(p)\,\#\,\big(h\twoheadrightarrow V_i(p)\big),$$

for any $h\in\mathcal{H}_R$, and hence $\sum U_i(p)\,\#\,\overline{V_i(p)}$ is in $X\wedge\overline{X}$.

Next we show that the well-defined map

$$\omega: I\to X\wedge\overline{X},\quad \omega(p)=\big(\beta^r\big)^{-1}(1\otimes p)$$

is an algebra map. In order to simplify the notations, we write $a\,\#\,\bar b$ and $c\,\#\,\bar d$ for $\omega(p)$ and $\omega(q)$ respectively, where $p,q\in I$. Since

$$\sum a^{(0)}b\otimes a^{(1)}=\sum a_{[0](0)}b\otimes a_{[1]}S^{-1}\big(\Theta_r(a_{[0](1)})\big)=1\otimes p$$

is equivalent to

$$\sum a_{[0]}b_{(0)}\otimes a_{[1]}\Theta_r(b_{(1)})=1\otimes p,$$

It is sufficient to show that

$$\sum x_{[0]}y_{(0)}\otimes x_{[1]}\Theta_r(y_{(1)})=1\otimes pq,$$

where

$$x\,\#\,\bar y=\big(a\,\#\,\bar b\big)\big(c\,\#\,\bar d\big)=\sum\big(ac_{(0)}\,\#\,\overline{d_{(0)}b_{(0)}}\big)R(d_{(1)}c_{(1)}\otimes b_{(1)}).$$

Indeed, we have

$$\sum x_{[0]}y_{(0)} \otimes x_{[1]}\Theta_r(y_{(1)}) \\ = \sum a_{[0]}c_{(0)[0]}d_{(0)}b_{(0)} \otimes a_{[1]}c_{(0)[1]}\Theta_r(d_{(1)})\Theta_r(b_{(1)})R(d_{(2)}c_{(1)} \otimes b_{(2)}).$$

Applying the equations:

$$\sum c_{(0)[0]}d_{(0)} \otimes c_{(0)[1]}\Theta_r(d_{(1)}) \otimes S^{-1}\big(\Theta_l(d_{(2)}c_{(1)})\big) \\ = \sum c_{[0](0)}d_{(0)} \otimes R^{-1}\big(c_{[1]}\Theta_r(d_{(2)}) \otimes S^{-1}\big(\Theta_l(c_{[0](1)}d_{(1)})\big)R\big) \\ = 1 \otimes (\iota \otimes S)\big(R^{-1}(q \otimes 1)R\big) \\ = \sum 1 \otimes S\big(R^1\big)qr^1 \otimes S\big(R^2r^2\big)$$

where $R = r$, we obtain

$$\sum x_{[0]}y_{(0)} \otimes x_{[1]}\Theta_r(y_{(1)}) \\ = \sum a_{[0]}b_{(0)} \otimes a_{[1]}S\big(R^1\big)qr^1\Theta_r(b_{(1)})\big\langle S\big(R^2r^2\big), b_{(2)}\big\rangle \\ = \sum a_{[0]}b_{(0)} \otimes a_{[1]}\big(S\big(R^1\big) \cdot q\big)\Theta_r(b_{(1)})\big\langle S\big(R^2, b_{(2)}\big)\big\rangle \\ = \sum a_{[0]}b_{(0)} \otimes a_{[1]}q_{(0)}\Theta_r(b_{(1)})R\big(S(q_{(1)}) \otimes S(b_{(2)})\big) \\ = \sum a_{[0]}b_{(0)} \otimes a_{[1]}q_{(0)}\big(q_{(1)} \cdot \Theta_r(b_{(1)})\big) \\ = \sum a_{[0]}b_{(0)} \otimes a_{[1]}\Theta_r(b_{(1)})q \\ = 1 \otimes pq.$$

So ω is indeed an algebra map.

Finally, we show that ω is a YD H-module map so that it is an $\mathcal{H}_R$-bimodule map (or a $\mathcal{H}_R^*$-bicomodule map) as well. Given $p \in I$ and $h \in \mathcal{H}_R$, we have

$$\sum U_i(p)_{[0]}V_i(p)_{(0)} \otimes U_i(p)_{[1]}\Theta_r\big(V_i(p)_{(1)}\big) = 1 \otimes p.$$

It implies that

$$h \cdot \omega(p) = \sum h_{(1)} \cdot U_i(p) \,\#\, h_{(2)} \triangleright_1 \overline{V_i(p)} \\ = \sum U_i(p)_{[0]} \,\#\, \overline{V_i(p)_{(0)}}\big\langle h, U_i(p)_{[1]}\Theta_r\big(V_i(p)_{(1)}\big)\big\rangle \\ = \sum U_i(p_{(1)}) \,\#\, \overline{V_i(p_{(1)})}\langle h, p_{(2)}\rangle \\ = \omega(h \cdot p).$$

To show that ω is H^{op}-colinear, we verify that ω is left H^*-linear. Indeed, if $p \in H^*$ and $q \in I$, we have

$$\begin{aligned}\omega(p\cdot q) &= \sum \omega\big(p_{(2)}qS^{-1}(p_{(1)})\big)\\ &= \sum \omega(p_{(2)})\omega(q)\omega\big(S^{-1}(p_{(1)})\big)\\ &= \sum \omega(q)_{(0)}\big(\omega(q)_{(1)}\cdot\omega(p_{(2)})\big)\omega\big(S^{-1}(p_{(1)})\big)\\ &= \sum \omega(q)_{(0)}\omega(p_{(2)})\omega\big(S^{-1}(p_{(1)})\big)\big\langle\omega(q)_{(1)}, p_{(3)}\big\rangle\\ &= \sum \omega(q)_{(0)}\big\langle\omega(q)_{(1)}, p\big\rangle,\end{aligned}$$

where we used the facts that $X \wedge \overline{X}$ is quantum commutative, that ω is an algebra map and that ω is H-linear. So ω is indeed a YD H-module algebra map, and $I \cong X \wedge \overline{X}$ in $\mathcal{E}(\mathcal{H}_R)$. This proved that $[\overline{X}]$ is a right inverse of $[X]$ in $\mathrm{Gal}(\mathcal{H}_R)$. Since any element of $\mathrm{Gal}(\mathcal{H}_R)$ has a right inverse element in $\mathrm{Gal}(\mathcal{H}_R)$, $\mathrm{Gal}(\mathcal{H}_R)$ is a group. □

When (H, R) is a cotriangular Hopf algebra, the braided monoidal category $\mathbf{M}_R^H$ is a symmetric monoidal category. Since the multiplication of a generalized cotensor product $A \wedge B$ is defined in $\mathbf{M}_R^H$, we expect $\mathrm{Gal}(\mathcal{H}_R)$ to be an abelian group, and this is the case.

Proposition 3.10. *If* (H, R) *is a cotriangular Hopf algebra, then* $\mathrm{Gal}(\mathcal{H}_R)$ *is an abelian group.*

Proof. Let $[A]$ and $[B]$ be two elements of $\mathrm{Gal}(\mathcal{H}_R)$. We prove that $A \wedge B \cong B \wedge A$ in $\mathcal{E}(\mathcal{H}_R)$. Let Ψ be the braiding from $A \#_R B$ to $B \#_R A$. We show that Ψ restricts to an isomorphism from $A \wedge B$ to $B \wedge A$:

$$\Psi : A \wedge B \to B \wedge A, \quad \sum a_i \# b_i \mapsto \sum b_{i(0)} \# a_{i(0)} R(b_{i(1)} \otimes a_{i(1)}).$$

First we show that $\Psi(A \wedge B) \subseteq B \wedge A$. For simplicity, we write $a \# b$ for an element in $A \wedge B$. Given $h \in H$, we have

$$h\cdot(a \# b) = \sum h_{(1)}\cdot a \# h_{(2)} \triangleright_1 b = \sum h_{(1)} \triangleright_2 a \# h_{(2)}\cdot b.$$

Applying Ψ on both sides of the two equations, we obtain

$$\begin{aligned}\Psi\big(h\cdot(a \# b)\big) &= \sum \Psi(h_{(1)}\cdot a \# h_{(2)} \triangleright_1 b)\\ &= \sum b_{(0)} \# h_{(2)}\cdot a_{(0)} R\big(b_{(1)} \otimes h_{(3)}a_{(1)}S^{-1}(h_{(1)})\big)R(h_{(4)} \otimes b_{(2)})\\ &= \sum b_{(0)} \# h_{(2)}\cdot a_{(0)} R\big(b_{(1)} \otimes h_{(3)}a_{(1)}S^{-1}(h_{(1)})\big)R\big(b_{(2)} \otimes S^{-1}(h_{(4)})\big)\\ &= \sum b_{(0)} \# h_{(2)}\cdot a_{(0)} R\big(b_{(1)} \otimes S^{-1}(h_{(4)})h_{(3)}a_{(1)}S^{-1}(h_{(1)})\big)\\ &= \sum b_{(0)} \# h_{(2)}\cdot a_{(0)} R\big(b_{(1)} \otimes a_{(1)}S^{-1}(h_{(1)})\big)\\ &= \sum h_{(1)} \triangleright_2 b_{(0)} \# h_{(2)}\cdot a_{(0)} R(b_{(1)} \otimes a_{(1)}),\end{aligned}$$

where the third equation holds because R is cotriangular. On the other hand,

$$\begin{aligned}
\Psi\big(h\cdot(a\,\#\,b)\big) &= \sum \Psi(h_{(1)} \triangleright_2 a\,\#\,h_{(2)}\cdot b)\\
&= \sum h_{(3)}\cdot b_{(0)}\,\#\,a_{(0)} R\big(h_{(4)} b_{(1)} S^{-1}(b_{(2)})\otimes a_{(1)}\big) R\big(S(a_{(2)})\otimes h_{(1)}\big)\\
&= \sum h_{(3)}\cdot b_{(0)}\,\#\,a_{(0)} R\big(h_{(4)} b_{(1)} S^{-1}(b_{(2)})\otimes a_{(1)}\big) R(h_{(1)}\otimes a_{(2)})\\
&= \sum h_{(3)}\cdot b_{(0)}\,\#\,a_{(0)} R\big(h_{(4)} b_{(1)} S^{-1}(b_{(2)}) h_{(1)}\otimes a_{(1)}\big)\\
&= \sum h_{(1)}\cdot b_{(0)}\,\#\,a_{(0)} R(h_{(2)}\otimes a_{(1)}) R(b_{(1)}\otimes a_{(2)})\\
&= \sum h_{(1)}\cdot b_{(0)}\,\#\,h_{(2)} \triangleright_1 a_{(0)} R(b_{(1)}\otimes a_{(1)}).
\end{aligned}$$

It follows from Lemma 2.9 that the element $\Psi(a\,\#\,b)$ is in $B\wedge A$. Moreover, we have proved that $\Psi(h\cdot(a\,\#\,b)) = h\cdot\Psi(a\,\#\,b)$. That is, Ψ is an H-module map and hence a YD H-module map from $A\wedge B$ to $B\wedge A$. Since Ψ is the restriction of the braiding on $A\wedge B$, we have that $\Psi_{B\wedge A}\circ\Psi_{A\wedge B} = \mathrm{Id}_{A\wedge B}$. So Ψ is an isomorphism.

Now it remains to show that Ψ is an algebra map. To simplify the notations we let $a\,\#\,b$ and $c\,\#\,d$ be two elements in $A\wedge B$. Then

$$\begin{aligned}
\Psi\big((a\,\#\,b)(c\,\#\,d)\big) &= \sum b_{(0)}\,\#\,a_{(0)}c_{(0)} R(d_{(1)}b_{(1)}\otimes c_{(1)}a_{(1)}) R(c_{(2)}\otimes b_{(2)})\\
&= \sum b_{(0)}\,\#\,a_{(0)}c_{(0)} R(d_{(1)}\otimes c_{(1)}a_{(1)}) R(b_{(1)}\otimes c_{(2)}a_{(2)}) R(c_{(3)}\otimes b_{(2)})\\
&= \sum b_{(0)}\,\#\,a_{(0)}c_{(0)} R(d_{(1)}\otimes c_{(1)}a_{(1)}) R(b_{(1)}\otimes c_{(2)}a_{(2)})\\
&\qquad\times R\big(b_{(2)}\otimes S^{-1}(c_{(3)})\big)\\
&= \sum b_{(0)}\,\#\,a_{(0)}c_{(0)} R(d_{(1)}\otimes c_{(1)}a_{(1)}) R(b_{(1)}\otimes a_{(2)})\\
&= \sum b_{(0)}\,\#\,a_{(0)}c_{(0)} R(d_{(1)}\otimes a_{(1)}) R(d_{(2)}\otimes c_{(1)}) R(b_{(1)}\otimes a_{(2)})\\
&= \sum (b_{(0)}\,\#\,a_{(0)})(\#c_{(0)}) R(d_{(1)}\otimes c_{(1)}) R(b_{(1)}\otimes a_{(1)})\\
&= \Psi(a\,\#\,b)\Psi(c\,\#\,d).
\end{aligned}$$

Thus we have proved that Ψ is an algebra isomorphism and that $A\wedge B\cong B\wedge A$ in $\mathcal{E}(\mathcal{H}_R)$. So $\mathrm{Gal}(\mathcal{H}_R)$ is an abelian group. □

4. The exact sequence

In this section, we investigate the R-Azumaya algebras which are Galois extensions of its coinvariants, and establish a group homomorphism from $\mathrm{BC}(k,H,R)$ to the group $\mathrm{Gal}(\mathcal{H}_R)$ constructed in the previous section. The main result will be the exact sequence

(23). In the sequel, for simplifying notations we will write M_0 for the coinvariant set $M^{\operatorname{co} H}$ of a right H-comodule:

$$\Big\{m \in M \mid \sum m_{(0)} \otimes m_{(1)} = m \otimes 1\Big\}.$$

We start with a special elementary R-Azumaya algebra.

Lemma 4.1. *Let $M = H^{\mathrm{op}}$ be the right regular H^{op}-comodule, and let A be the induced R-Azumaya algebra* $\operatorname{End}(M)$*. Then $A \cong H^{*\mathrm{op}} \# H^{\mathrm{op}}$, where the left H^{op}-action on $H^{*\mathrm{op}}$ is given by $h \cdot p = \sum p_{(1)} \langle p_{(2)}, S^{-1}(h)\rangle = S^{-1}(h) \rightharpoonup p$, whenever $h \in H^{\mathrm{op}}$ and $p \in H^{*\mathrm{op}}$.*

Proof. Let $\pi : H^{\mathrm{op}} \to \operatorname{End}(M)$ be the representation of the regular left H^{op}-module. We claim that π is a right H^{op}-colinear algebra map. Indeed, by definition, the right H^{op}-comodule structure of $\pi(h)$, for $h \in H^{\mathrm{op}}$, is given by

$$\begin{aligned}
\rho\big(\pi(h)\big)(x \otimes 1) &= \sum \big(\pi(h)(x_{(0)})\big)_{(0)} \otimes S^{-1}(x_{(1)})\big(\pi(h)(x_{(0)})\big)_{(1)} \\
&= \sum (x_{(1)}h)_{(1)} \otimes S^{-1}(x_{(2)})(x_{(1)}h)_{(2)} \\
&= \sum x_{(1)}h_{(1)} \otimes S^{-1}(x_{(3)})x_{(2)}h_{(2)} \\
&= \sum \pi(h_{(1)})(x) \otimes h_{(2)}
\end{aligned}$$

for any $x \in M$. It follows that $\rho(\pi(h)) = \sum \pi(h_{(1)}) \otimes h_{(2)}$. So π is a right H^{op}-colinear algebra map. This fact implies that the right H^{op}-comodule algebra A is a smash product $A_0 \# H^{\mathrm{op}}$ [10, 1.4], where A_0 is the coinvariant subalgebra of A.

Now we show that A_0 is isomorphic to $H^{*\mathrm{op}}$. It is obvious that $A_0 = \operatorname{End}^H(M)$, the subalgebra of all H^{op}-colinear endomorphisms of M. We know that the k-module map

$$\lambda : H^{*\mathrm{op}} \to \operatorname{End}^H\big(H^{\mathrm{op}}\big), \quad \lambda(p)(x) = \sum p(x_{(1)})x_{(2)}$$

for $p \in H^{*\mathrm{op}}$ and $x \in H^{\mathrm{op}}$, is an algebra isomorphism.

Finally, for $p \in H^{*\mathrm{op}}$, $h \in H^{\mathrm{op}}$ and $x \in M$, we have

$$\begin{aligned}
\pi(h)\big(\lambda(p)(x)\big) &= \sum x_{(2)}h\langle p, x_{(2)}\rangle \\
&= \sum \big(S^{-1}(h_{(1)}) \rightharpoonup p\big)\big(\pi(h_{(2)})(x)\big).
\end{aligned}$$

It follows that the action of H^{op} on $H^{*\mathrm{op}}$ is

$$h \cdot p = S^{-1}(h) \rightharpoonup p = \sum p_{(1)}\big\langle p_{(2)}, S^{-1}(h)\big\rangle$$

for $h \in H^{\mathrm{op}}$ and $p \in H^{*\mathrm{op}}$. $\square$

Corollary 4.2. *Any element of* $\mathrm{BC}(k, H, R)$ *can be represented by an* R*-Azumaya algebra that is a smash product.*

Proof. Let $[A]$ be an element of $\mathrm{BC}(k, H, R)$. Since $\mathrm{End}(H^{\mathrm{op}})$ represents the unit of $\mathrm{BC}(k, H, R)$, we have $[A \# \mathrm{End}(H^{\mathrm{op}})] = [A]$. Now the composite algebra map

$$H^{\mathrm{op}} \xrightarrow{\lambda} \mathrm{End}\big(H^{\mathrm{op}}\big) \hookrightarrow A \# \mathrm{End}\big(H^{\mathrm{op}}\big)$$

is still H^{op}-colinear. It follows that $A \# \mathrm{End}(H^{\mathrm{op}})$ is a smash product algebra $B \# H^{\mathrm{op}}$ where $B = (A \# \mathrm{End}(H^{\mathrm{op}}))_0$. □

Since any smash product algebra is a Galois extension of its coinvariants, we have that *any element of* $\mathrm{BC}(k, H, R)$ *can be represented by an* R*-Azumaya algebra which is an* H^{op}*-Galois extension of its coinvariants.*

Lemma 4.3. *Let* A *be an* R*-Azumaya algebra. If* A *is an* H^{op}*-Galois extension of* A_0*, then* $\overline{A}$ *is a* H^{op}*-Galois extension of* A_0^{op}.

Proof. Since A/A_0 is H^{op}-Galois, we have the canonical isomorphism:

$$\beta'_A : A \otimes_{A_0} A \to A \otimes H^{\mathrm{op}}, \quad a \otimes b \mapsto \sum a_{(0)} b \otimes b_{(1)}.$$

Since the flip map τ is a k-module isomorphism from $A \otimes_{A_0} A$ to $\overline{A} \otimes_{A_0^{\mathrm{op}}} \overline{A}$, β'_A gives an isomorphism

$$\eta : \overline{A} \otimes_{A_0^{\mathrm{op}}} \overline{A} \to \overline{A} \otimes H^{\mathrm{op}}, \quad \eta\big(\bar{a} \otimes \bar{b}\big) = \sum \overline{a_{(0)} b} \otimes a_{(1)}.$$

Define a k-module map:

$$\xi : \overline{A} \otimes H^{\mathrm{op}} \to \overline{A} \otimes H^{\mathrm{op}}, \quad \bar{a} \otimes h \mapsto \sum \overline{a_{(0)}} \otimes h_{(3)} R\big(h_{(2)} \otimes a_{(1)} S(h_{(1)})\big).$$

We show that ξ is a k-module isomorphism. As remarked in the previous section, we may view R as an element $\sum R^1 \otimes R^2$ in $H^* \otimes H^*$ which is a QT structure of H^*. Then the element $u = \sum R^2 S^{-1}(R^1)$ is the Casimir element of H^* that is invertible. Thus we may rewrite the k-module map ξ as the following composite map:

$$\overline{A} \otimes H^{\mathrm{op}} \xrightarrow{\iota \otimes \tilde{u}} \overline{A} \otimes H^{\mathrm{op}} \xrightarrow{\sigma} \overline{A} \otimes H^{\mathrm{op}}$$

where $\tilde{u}$ is defined by $\tilde{u}(h) = h \leftharpoonup u = \sum h_{(2)} u(h_{(1)})$, and σ is defined by

$$\sigma\big(\bar{a} \otimes h\big) = \sum \overline{h_{(1)} \triangleright_1 a} \otimes h_{(2)} = \sum \overline{a_{(0)}} \otimes h_{(2)} R(h_{(1)} \otimes a_{(1)})$$

for all $a \in A$ and $h \in H^{\mathrm{op}}$. Since $\widetilde{u}$ and σ are k-module isomorphisms, we have that ξ is an isomorphism. It is easy to check that $\beta_{\overline{A}} = \xi \beta'_A \tau$. So $\beta_{\overline{A}}$ is an isomorphism, and $\overline{A}/A_0^{\mathrm{op}}$ is an H^{op}-Galois extension. □

In the sequel, an R-Azumaya algebra A is said to be *Galois* if it is a right H^{op}-Galois extension of its coinvariant subalgebra A_0. Let A be a Galois R-Azumaya algebra. Denote by $\pi(A)$ the centralizer subalgebra $C_A(A_0)$ of A_0 in A. It is clear that $\pi(A)$ is an H^{op}-comodule subalgebra of A. The Miyashita–Ulbrich–Van Oystaeyen (MUVO) action [22, 30,32] of H on $\pi(A)$ is given by

$$h \rightharpoonup a = \sum X_i(h) a Y_i(h), \tag{22}$$

where $\sum X_i(h) \otimes Y_i(h) = \beta^{-1}(1 \otimes h)$, for $h \in H$. It is well-known (e.g., see [6,30]) that $\pi(A)$ together with the action (22) is a new YD H-module algebra. Moreover, $\pi(A)$ is quantum commutative in the sense of (18). By Corollary 2.8, $\pi(A)$ is an $\mathcal{H}_R$-bimodule algebra, or A is an $\mathcal{H}_R^*$-bicomodule algebra.

Lemma 4.4. *Let A be a Galois R-Azumaya algebra. Then $\pi(A)/k$ is an $\mathcal{H}_R^*$-biextension.*

Proof. Given $a \in \pi(A)_\diamond$, then by Lemma 2.5, we have $h \rightharpoonup a = h \triangleright_1 a$ for any $h \in H$. Then for any element $b \in A$, we have

$$ab = \sum b_{(0)}(b_{(1)} \rightharpoonup a) = \sum b_{(0)}(b_{(1)} \triangleright_1 a).$$

This means that a is an element in the left H-center of A that is trivial [7]. So $\pi(A)_\diamond = k$. Similarly, for $a \in {}_\diamond\pi(A)$, we have

$$ab = \sum b_{(0)}(b_{(1)} \rightharpoonup a) = \sum b_{(0)}(b_{(1)} \triangleright_2 a) = \sum b_{(0)} a_{(0)} R^{-1}(a_{(1)} \otimes b_{(1)}),$$

for any $b \in A$. This implies that $\sum a_{(0)} b_{(0)} R(a_{(1)} \otimes b_{(1)}) = ba$ for any $b \in A$. So a is in the right H-center of A that is trivial as well. It follows that $\pi(A)/k$ is an $\mathcal{H}_R^*$-biextension. □

Next we show that $\pi(A)$ is faithfully flat over k. To this end we consider the algebra $A \#_R \mathcal{H}_R^*$. There is a left $A \# \overline{A} = A^e$ module structure on $A \#_R \mathcal{H}_R^*$ as follows:

$$\big(a \# \bar{b}\big) \cdot (c \otimes p) = \sum a\big(S(b_{(2)}) \triangleright_2 c\big) b_{(0)} \otimes \big(S(b_{(1)}) \multimap p\big)$$

for $a \# \bar{b} \in A^e$ and $c \otimes p \in A \#_R \mathcal{H}_R^*$. It is not hard to verify that $A \#_R \mathcal{H}_R^*$ is an object in the category ${}_{A^e}\mathcal{Q}^H$ which is equivalent to $\mathcal{Q}^H$ through the pair of functors $((-)^A, A \otimes -)$ (see [7, Proposition 2.6] for further details).

Let Γ be the H^{op}-comodule subalgebra of $A \#_R \mathcal{H}_R^*$:

$$\big(A \#_R \mathcal{H}_R^*\big)^A = \big\{x \in A \#_R \mathcal{H}_R^* \;\big|\; (b \# 1)x = \big(1 \# \bar{b}\big)x,\ \forall b \in A\big\}.$$

Then $A \#_R \mathcal{H}_R^* \cong A \otimes \Gamma$ by [7, Proposition 2.6]. Thus Γ is a faithfully flat algebra over k since A and $\mathcal{H}_R^*$ are faithfully flat.

Lemma 4.5. *Let A be a Galois R-Azumaya algebra. Then $\pi(A) \cong \Gamma$ and hence $\pi(A)$ is faithfully flat over k.*

Proof. It is sufficient to prove that $\Gamma = \pi(A) \wedge \mathcal{H}_R^*$. Let $x = a \otimes p$ be an element in $\pi(A) \wedge \mathcal{H}_R^*$. We verify that $(b \# 1)x = (1 \# \bar{b})x$ for any $b \in A$. Indeed, we have

$$\begin{aligned}
(1 \# \bar{b})(a \otimes p) &= \sum \big(S(b_{(2)}) \rhd_2 a\big) b_{(0)} \otimes \big(S(b_{(1)}) \rightharpoonup p\big) \\
&= \sum S(b_{(2)}) \rhd_2 \big(a \leftharpoonup S(b_{(1)})\big) b_{(0)} \otimes p \\
&= \sum \big(S(b_{(3)}) \rhd_2 \big(S^2(b_{(2)}) \rhd_2 \big(S(b_{(1)}) \cdot a\big)\big)\big) b_{(0)} \otimes p \\
&= \sum \big(S(b_{(1)}) \cdot a\big) b_{(0)} \otimes p \\
&= ba \otimes p = (b \# 1)(a \otimes p),
\end{aligned}$$

whenever $a, b \in A$ and $p \in \mathcal{H}_R^*$. Thus we have proved that $\pi(A) \wedge \mathcal{H}_R^*$ is contained in Γ.

Conversely, let $A_0 \# 1$ be the subalgebra of $A \#_R \mathcal{H}_R^*$. It is easy to see that $\pi(A) \#_R \mathcal{H}_R^*$ is the centralizer of $A_0 \# 1$ in $A \#_R \mathcal{H}_R^*$. Thus $\Gamma \subseteq \pi(A) \#_R \mathcal{H}_R^*$. Let $x = a \otimes p$ be an element in Γ. For any element $h \in H^{\mathrm{op}}$, there exists a unique element $\sum X_i(h) \otimes Y_i(h) \in A \otimes_{A_0} A$ such that $\sum X_i(h) Y_i(h)_{(0)} \otimes Y_i(h)_{(1)} = 1 \otimes h$, or equivalently

$$\sum X_i(h)_{(0)} Y_i(h) \otimes X_i(h)_{(1)} = 1 \otimes S^{-1}(h).$$

Thus we have

$$\begin{aligned}
(a \leftharpoonup h) \otimes p &= \sum S(h_{(1)}) \rhd_2 (h_{(2)} \cdot a) \otimes p \\
&= \sum S(h_{(1)}) \rhd_2 \big(X_i(h_{(2)}) a Y_i(h_{(2)})\big) \otimes p \\
&= \sum S(h_{(1)}) \rhd_2 \big(S\big(X_i(h_{(2)})_{(2)}\big) \rhd_2 a\big) X_i(h_{(2)})_{(0)} Y_i(h_{(2)}) \\
&\qquad \otimes \big(S\big(X_i(h_{(2)})_{(1)}\big) \rightharpoonup p\big) \\
&= \sum S(h_{(1)}) \rhd_2 \big(S\big(S^{-1}(h_{(2)})\big) \rhd_2 a\big) \otimes \big(S\big(S^{-1}(h_{(3)})\big) \rightharpoonup p\big) \\
&= a \otimes (h \rightharpoonup p),
\end{aligned}$$

which proves that Γ is contained in $\pi(A) \wedge \mathcal{H}_R^*$. □

Recall from [30, Lemma 1.3] that when a Galois H^{op}-comodule algebra A is an Azumaya algebra, the centralizer $\pi(A)$ is a right H^*-Galois extension of k with respect to the MUVO action (22). This is no longer the case when A is an R-Azumaya algebra.

However, we will see that $\pi(A)$ would be an $\mathcal{H}_R^*$-Galois object, instead of an H^*-Galois object.

Proposition 4.6. *Let A be a Galois R-Azumaya algebra, and let $\pi(A)$ be as above. Then $\pi(A)$ is an object in* $\mathrm{Gal}(\mathcal{H}_R)$.

Proof. Let $F : A \mathbin{\#} \overline{A} \to \mathrm{End}(A)$ be the canonical H-linear algebra isomorphism. It is easy to see that F induces an algebra isomorphism $\pi(A) \mathbin{\#} \overline{\pi(A)} \to \mathrm{End}_{A_0-A_0}(A)$, where the latter is the subalgebra of all A_0-biendomorphisms of A. Since A/A_0 is H^{op}-Galois, we have the Doi–Takeuchi k-module isomorphism:

$$\delta : \mathrm{Hom}\big(H, \pi(A)\big) \to \mathrm{End}_{A_0-A_0}(A), \quad \delta(f)(a) = \sum a_{(0)} f(a_{(1)})$$

[11, 3.2]. Define a k-module map α as follows:

$$\alpha : \pi(A) \otimes \pi(A) \to \pi(A) \otimes H^*, \quad a \otimes b \mapsto \sum a_{[0]} b_{(0)} \otimes a_{[1]} \Theta_r(b_{(1)}).$$

One may take a while to check that α and δ fit in the following commutative diagram:

$$\begin{array}{ccc} \pi(A) \otimes \pi(A) & \xrightarrow{F} & \mathrm{End}_{A_0-A_0}(A) \\ {\scriptstyle \alpha}\downarrow & & \uparrow{\scriptstyle \delta} \\ \pi(A) \otimes H^* & \xrightarrow{\cong} & \mathrm{Hom}\big(H, \pi(A)\big). \end{array}$$

Note that here we view F as a k-module isomorphism from $\pi(A) \otimes \pi(A)$ to $\mathrm{End}_{A_0-A_0}(A)$. It follows from the above commutative diagram that α is a k-module isomorphism. It is evident that the canonical Galois k-module map β^r is now the composite isomorphism:

$$\pi(A) \otimes \pi(A) \xrightarrow{\alpha} \pi(A) \otimes H^* \xrightarrow{\eta} \pi(A) \otimes H^*$$

where η is given by $\eta(a \otimes p) = \sum a_{(0)} \otimes p S^{-1}(\Theta_r(a_{(1)}))$ whenever $a \in \pi(A)$ and $p \in H^*$. So we obtain that $\pi(A)$ is a right $\mathcal{H}_R^*$-Galois object.

Similarly, let $G : \overline{A} \mathbin{\#} A \to \mathrm{End}(A)^{\mathrm{op}}$ be the canonical H-linear algebra isomorphism. Then one has the commutative diagram

$$\begin{array}{ccc} \pi(A) \otimes \pi(A) & \xrightarrow{G} & \mathrm{End}_{A_0-A_0}(A)^{\mathrm{op}} \\ {\scriptstyle \alpha'}\downarrow & & \uparrow{\scriptstyle \delta} \\ \pi(A) \otimes H^* & \xrightarrow{\cong} & \mathrm{Hom}\big(H, \pi(A)\big) \end{array}$$

where α' is given by $\alpha'(a \otimes b) = \sum a_{(0)[0]}b \otimes a_{(0)[1]}\Theta_l(a_{(1)})$ for $a, b \in \pi(A)$. Let ζ and η' be the k-linear automorphisms of $\pi(A) \otimes H^*$ given by

$$\zeta(a \otimes p) = \sum a_{[0]} \otimes S(a_{[1]})p, \qquad \eta'(a \otimes p) = \sum a_{(0)} \otimes pS\big(\Theta_l(a_{(1)})\big)$$

for any $a \in A$ and $p \in H^*$. We have

$$\eta'\zeta\alpha'(a \otimes b) = \sum ab_{[0](0)} \otimes S\big(\Theta_l(b_{[0](1)})b_{[1]}\big)$$

for any $a, b \in \pi(A)$. It follows that the Galois k-module map β^l is the composite isomorphism $(S^{-1} \otimes \iota)\tau\eta'\zeta\alpha'$, where τ is the flip map. So $\pi(A)$ is a left $\mathcal{H}_R^*$-Galois object. This completes the proof. □

Now we are ready to show that π induces a group homomorphism from the Brauer group $\mathrm{BC}(k, H, R)$ to the group $\mathrm{Gal}(\mathcal{H}_R)$.

Proposition 4.7. *Let A and B be two Galois R-Azumaya algebras. Then we have $\pi(A \# B) = \pi(A) \wedge \pi(B)$.*

Proof. It is obvious that $\pi(A \# B) \subseteq \pi(A) \# \pi(B)$ because $A_0 \otimes B_0 = A_0 \# B_0 \subseteq (A \# B)_0$. For an element $h \in H^{\mathrm{op}}$, we let

$$\beta_A^{-1}(1 \otimes h) = \sum X_j(h) \otimes Y_j(h) \in A \otimes_{A_0} A,$$
$$\beta_B^{-1}(1 \otimes h) = \sum U_i(h) \otimes V_i(h) \in B \otimes_{B_0} B.$$

Then we have

$$\sum \big(X_j(h) \# 1\big) \otimes \big(Y_j(h) \# 1\big) = \beta_{A\#B}^{-1}(1 \otimes h) = \sum \big(1 \# U_i(h)\big) \otimes \big(1 \# V_i(h)\big).$$

This implies that the MUVO action of H on $\pi(A \# B)$ can be written in two ways:

$$\begin{aligned} h \rightharpoonup (a \# b) &= \sum \big(X_j(h) \# 1\big)(a \# b)\big(Y_j(h) \# 1\big) \\ &= \sum \big(1 \# U_i(h)\big)(a \# b)\big(1 \# V_i(h)\big) \end{aligned}$$

where $a \# b$ should be read as a sum of elements in $A \# B$. Precisely, we have for $a \# b \in \pi(A \# B)$,

$$\begin{aligned} h \rightharpoonup (a \# b) &= \sum \big(X_j(h) \# 1\big)(a \# b)\big(Y_j(h) \# 1\big) \\ &= \sum X_j(h)aY_j(h)_{(0)} \# Y_j(h)_{(1)} \triangleright_1 b \\ &= \sum h_{(1)} \rightharpoonup a \# h_{(2)} \triangleright_1 b. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} h \rightharpoonup (a \# b) &= \sum \big(1 \# U_i(h)\big)(a \# b)\big(1 \# V_i(h)\big) \\ &= \sum a_{(0)} \# \big(a_{(1)} \triangleright_1 U_i(h)\big) b V_i(h) \\ &= \sum a_{(0)} \# U_i(h)_{(0)} b V_i(h) R\big(a_{(1)} \otimes U_i(h)_{(1)}\big) \\ &= \sum a_{(0)} R\big(a_{(1)} \otimes S^{-1}(h_{(1)})\big) \# h_{(2)} \rightharpoonup b \\ &= \sum h_{(1)} \triangleright_2 a \# h_{(2)} \rightharpoonup b. \end{aligned}$$

This means that $a \# b$ is in $\pi(A) \wedge \pi(B)$ by Lemma 3.3. It follows that $\pi(A \# B) \subseteq \pi(A) \wedge \pi(B)$.

Conversely, if $a \# b$ is an element of $\pi(A) \wedge \pi(B)$, we show that $a \# b \in \pi(A \# B)$. Indeed, given $x \# y \in (A \# B)_0$, we have

$$\sum x_{(0)} \# y_0 \otimes y_{(1)} x_{(1)} = x \otimes y \otimes 1,$$

or

$$\sum x_{(0)} \# y \otimes x_{(1)} = \sum x \# y_{(0)} \otimes S(y_{(1)}).$$

These two formulae lead to the equations:

$$\begin{aligned} (a \# b)(u \# v) &= \sum a u_{(0)} \# (x_{(1)} \triangleright_1 b) v \\ &= \sum x_{(0)} (x_{(1)} \rightharpoonup a) \# (x_{(2)} \triangleright_1 b) v \\ &= \sum x_{(0)} (x_{(1)} \triangleright_2 a) \# (x_{(2)} \rightharpoonup b) y \\ &= \sum x_{(0)} (x_{(1)} \triangleright_2 a) \# y_{(0)} (y_{(1)} x_{(2)} \rightharpoonup b) \\ &= \sum x_{(0)} (x_{(1)} \triangleright_2 a) \# y b \\ &= \sum x \big(S(y_{(1)}) \triangleright_2 a\big) \# y_{(0)} b \\ &= \sum x a_{(0)} \# (a_{(1)} \triangleright_1 y) b \\ &= (x \# y)(a \# b), \end{aligned}$$

where we used the quantum commutativity (18). This implies that $a \# b \in \pi(A \# B)$. So $\pi(A) \wedge \pi(B) \subseteq \pi(A \# B)$, and hence they are equal. □

Lemma 4.8. *Let M be a finite right H^{op}-comodule, and $A = \mathrm{End}(M)$ be the elementary R-Azumaya algebra. If A is a Galois R-Azumaya algebra, then $\pi(A) \cong I$.*

Proof. Since M is a right H^{op}-comodule, we may view M as a left H^*-module. The representation map

$$\lambda: H^* \to A, \quad \lambda(p)(m) = p \cdot m = \sum m_{(0)} \langle p, m_{(1)} \rangle$$

sends H^* into the subalgebra $\pi(A)$ because $A_0 = \mathrm{End}_{H^*}(A)$. Thus λ is an algebra map from I to $\pi(A)$. If we can prove that λ is a YD H-module map, then λ becomes an $\mathcal{H}_R$-bimodule map, and hence an isomorphism between the two Galois objects in $\mathrm{Gal}(\mathcal{H}_R)$. By definition (11), λ is right H^{op}-colinear. We show that λ is left H-linear as well.

To show that $\lambda(h \cdot p) = h \rightharpoonup \lambda(p)$ for $h \in H$ and $p \in I$, it is sufficient (or equivalent) to show that

$$\lambda(p) f = \sum f_{(0)} \lambda(f_{(1)} \cdot p)$$

for any $f \in A$. Given $m \in M$ and $f \in A$, we have

$$\sum f_{(0)} \lambda(f_{(1)} \cdot p)(m) = \sum f_{(0)}(m_{(0)}) \langle p, m_{(1)} f_{(1)} \rangle.$$

Since

$$\sum f_{(0)}(m) \otimes f_{(1)} = \sum f(m_{(0)})_{(0)} \otimes S(m_{(1)}) f(m_{(0)})_{(1)},$$

we have

$$\begin{aligned} \sum f_{(0)} \lambda(f_{(1)} \cdot p)(m) &= \sum f_{(0)}(m_{(0)}) \langle p, m_{(1)} f_{(1)} \rangle \\ &= \sum f(m)_{(0)} \big\langle p, f(m)_{(1)} \big\rangle \\ &= p \cdot f(m) \\ &= \lambda(p) f(m). \end{aligned}$$

This proves that λ is a YD H-module algebra map, and hence an isomorphism because I and $\pi(A)$ are $\mathcal{H}_R^*$-bigalois objects. □

Lemma 4.9. *π induces a group homomorphism $\widetilde{\pi}$ from $\mathrm{BC}(k, H, R)$ to $\mathrm{Gal}(\mathcal{H}_R)$, where $\widetilde{\pi}([A]) = [\pi(A)]$ and A is a Galois R-Azumaya algebra representing the class $[A]$ in $\mathrm{BC}(k, H, R)$.*

Proof. First we show that $\widetilde{\pi}$ is well-defined. Suppose that A and B are two Galois R-Azumaya algebras representing the same class in $\mathrm{BC}(k, H, R)$. Then there are two finite right H^{op}-modules M, N such that

$$A \# \mathrm{End}(M) \cong B \# \mathrm{End}(N).$$

Let H^{op} be the regular right H^{op}-comodule. By Lemma 4.1, $\mathrm{End}(H^{\mathrm{op}})$ is a Galois R-Azumaya algebra. Since $\mathrm{End}(M)\,\#\,\mathrm{End}(H^{\mathrm{op}}) \cong \mathrm{End}(M \otimes H^{\mathrm{op}})$ is still a Galois R-Azumaya algebra, by Lemma 4.8 we have

$$\pi\big(\mathrm{End}(M)\,\#\,\mathrm{End}\big(H^{\mathrm{op}}\big)\big) \cong \pi\big(\mathrm{End}\big(M \otimes H^{\mathrm{op}}\big)\big) \cong I.$$

This implies that

$$\begin{aligned}\pi(A) &\cong \pi(A) \wedge I\\ &\cong \pi\big(A\,\#\,\mathrm{End}(M)\,\#\,\mathrm{End}\big(H^{\mathrm{op}}\big)\big)\\ &\cong \pi\big(B\,\#\,\mathrm{End}(N)\,\#\,\mathrm{End}\big(H^{\mathrm{op}}\big)\big)\\ &\cong \pi(B) \wedge I\\ &\cong \pi(B).\end{aligned}$$

So we obtain that $\widetilde{\pi}([A]) = \widetilde{\pi}([B])$, and $\widetilde{\pi}$ is well-defined. □

In order to figure out the kernel of $\widetilde{\pi}$, we need two more preparations. Recall from [3] that an action of a Hopf algebra H on an algebra A is called an *inner action* if there is an invertible element u in the convolution algebra $\mathrm{Hom}(H, A)$ such that

$$h \cdot a = \sum u(h_{(1)})au^{-1}(h_{(2)})$$

for any $a \in A$ and $h \in H$. If in addition, u is an algebra map, then the action of H is called a *strongly inner action*.

Lemma 4.10. *Let A be a Galois R-Azumaya algebra such that $\pi(A) \cong I$. Then the action of $H^{*\mathrm{cop}}$ (or the coaction of H^{op}) on A is strongly inner.*

Proof. By assumption, there is a YD H-module algebra isomorphism $\psi : I \to \pi(A)$. Thus the action and the coaction of H on $\pi(A)$ are determined by the corresponding action and the coaction of H on I through ψ. Namely, we have:

$$\begin{aligned}h \rightharpoonup \psi(p) &= \sum \psi(p_{(1)})\langle p_{(2)}, h\rangle,\\ h^* \cdot \psi(p) &= \sum \psi\big(h^*_{(2)}\big)\psi(p)\psi\big(S^{-1}\big(h^*_{(1)}\big)\big)\end{aligned}$$

where $p \in I, h^* \in H^{*\mathrm{op}}$ and $h \in H$. In particular, the H^{op}-coaction on $\pi(A)$ is strongly inner. We show that this inner action extends to the inner action on A. Indeed, given $a \in A$ and $h^* \in H^{*\mathrm{op}}$, we have

$$\begin{aligned} h^* \cdot a &= \sum a_{(0)} \langle h^*, a_{(1)} \rangle \\ &= \sum \psi\big(h^*_{(3)} S^{-1}\big(h^*_{(2)}\big)\big) a_{(0)} \langle h^*_{(1)}, a_{(1)} \rangle \\ &= \sum \psi\big(h^*_{(3)}\big) a_{(0)} \big(a_{(1)} \rightharpoonup \psi\big(S^{-1}\big(h^*_{(2)}\big)\big)\big) \langle h^*_{(1)}, a_{(2)} \rangle \\ &= \sum \psi\big(h^*_{(4)}\big) a_{(0)} \psi\big(S^{-1}\big(h^*_{(3)}\big)\big) \langle S^{-1}\big(h^*_{(2)}\big), a_{(1)} \rangle \langle h^*_{(1)}, a_{(2)} \rangle \\ &= \sum \psi\big(h^*_{(2)}\big) a \psi\big(S^{-1}\big(h^*_{(1)}\big)\big). \end{aligned}$$

This means that the algebra map $\psi : H^{*\mathrm{op}} \to \pi(A) \hookrightarrow A$ induces a strongly inner action of $H^{*\mathrm{op}}$ on A. □

Lemma 4.11. *Let A be a Galois R-Azumaya algebra such that $\pi(A) \cong I$, and B any R-Azumaya algebra. Then as algebras*

(a) $A \otimes B \cong A \# B$,
(b) $\overline{A} \cong A^{\mathrm{op}}$,
(c) *A is an Azumaya algebra.*

Proof. (a) Let $\psi : I \to \pi(A)$ be an isomorphism of the two Galois objects in $\mathcal{E}(\mathcal{H}_R)$. We define a k-module map ξ as follows:

$$\xi : A \otimes B \to A \# B, \quad \xi(a \otimes b) = \sum a\psi\big(\Theta_r\big(S(b_{(1)})\big)\big) \# b_{(0)}.$$

It is easy to see that ξ is an isomorphism. We verify that ξ is an algebra map as well. Indeed, for $a, c \in A$ and $b, d \in B$, then

$$\begin{aligned} \xi\big((a \otimes b)(c \otimes d)\big) &= \sum ac\psi\big(\Theta_r\big(S(d_{(1)} b_{(1)})\big)\big) \# b_{(0)} d_{(0)} \\ &= \sum a\psi\big(\Theta_r\big(S(b_{(3)})\big)\big)\psi\big(\Theta_r(b_{(2)})\big) c\psi\big(\Theta_r\big(S(d_{(1)})\big)\big) \\ &\quad \times \psi\big(\Theta_r\big(S(b_{(1)})\big)\big) \# b_{(0)} d_{(0)} \\ &= \sum a\psi\big(\Theta_r\big(S(b_{(2)})\big)\big)\big[\Theta_r(b_{(1)}) \cdot \big(c\psi\big(\Theta_r(d_{(1)})\big)\big)\big] \# b_{(0)} d_{(0)} \\ &= \sum \big(a\psi\big(\Theta_r\big(S(b_{(1)})\big) \# b_{(0)}\big)\big(c\psi\big(\Theta_r(d_{(1)})\big)\big) \# d_{(0)}\big) \\ &= \xi(a \otimes b)\xi(c \otimes d). \end{aligned}$$

For (b) and (c), the proof of part (a) shows that there is a k-module map $\nu : \overline{A} \to A^{\mathrm{op}}$ given by

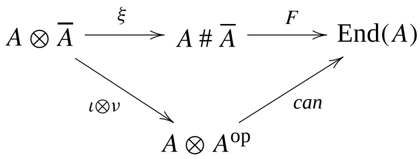
$$\nu(\bar{a}) = \sum \psi\big(\Theta_r\big(S(a_{(1)})\big)\big) a_{(0)}.$$

We show that ν is the desired algebra isomorphism.

First ν is an algebra map. Given $\bar{a}, \bar{b} \in \overline{A}$, we have

$$\begin{aligned}
\nu\big(\bar{a}\bar{b}\big) &= \sum \nu\big(\overline{b_{(0)}a_{(0)}}\big)R(b_{(1)}\otimes a_{(1)}) \\
&= \sum \psi\big(\Theta_r\big(S(a_{(1)}b_{(1)})\big)\big)b_{(0)}a_{(0)}R(b_{(2)}\otimes a_{(2)}) \\
&= \sum \psi\big(\Theta_r\big(S(b_{(2)}a_{(2)})\big)\big)b_{(0)}a_{(0)}R(b_{(1)}\otimes a_{(1)}) \\
&= \sum \psi\big(\Theta_r\big(S(b_{(1)}a_{(2)})\big)\big)\big(\Theta_r(a_{(1)})\cdot b\big)a_{(0)} \\
&= \sum \psi\big(\Theta_r\big(S(b_{(1)}a_{(3)})\big)\big)\psi\big(\Theta_r(a_{(2)})\big)b\psi\big(\Theta_r\big(S(a_{(1)})\big)\big)a_{(0)} \\
&= \sum \psi\big(\Theta_r\big(S(b_{(1)})\big)\big)b_{(0)}\psi\big(\Theta_r\big(S(a_{(1)})\big)\big)a_{(0)} \\
&= \nu\big(\bar{a}\big)\circ\nu\big(\bar{b}\big).
\end{aligned}$$

Now one may easily check that the following diagram commutes:



Since $F\circ\xi$ is an isomorphism, we obtain that $\iota\otimes\nu$ is injective and *can* is surjective. Since all the algebras involved are finite, ν and *can* are isomorphisms. So A is an Azumaya algebra and $\overline{A}\cong A^{\mathrm{op}}$ as algebras. □

Theorem 4.12. *We have an exact sequence of group homomorphisms*:

$$1\to \mathrm{Br}(k)\xrightarrow{\iota}\mathrm{BC}(k,H,R)\xrightarrow{\widetilde{\pi}}\mathrm{Gal}(\mathcal{H}_R). \tag{23}$$

Proof. Suppose that A, B are two Galois R-Azumaya algebras such that $[A],[B]\in\mathrm{Ker}(\widetilde{\pi})$. By Lemma 4.11, A and B are Azumaya algebras, and $A\,\#\,B\cong A\otimes B$. This implies that there is a group homomorphism

$$\zeta:\mathrm{Ker}(\widetilde{\pi})\to\mathrm{Br}(k),\qquad \zeta\big([A]\big)=[A]$$

by forgetting the H-structures on A, where $[A]\in\mathrm{Ker}(\widetilde{\pi})$ is represented by a Galois R-Azumaya algebra A. It is evident that $\zeta\circ\iota=\mathrm{id}$, the identity map on $\mathrm{Br}(k)$. If we can show that ζ is also injective, then $\mathrm{Ker}(\widetilde{\pi})\cong\mathrm{Br}(k)$. Indeed, if $\zeta([A])=1\in\mathrm{Br}(k)$, then there is a finite k-module M such that $A\cong\mathrm{End}(M)$ as an algebra. By Lemma 4.10, the coaction of $H^{*\mathrm{op}}$ on A is strongly inner. So there is an $H^{*\mathrm{op}}$-coaction on M such that M is a right $H^{*\mathrm{op}}$-comodule and $A\cong\mathrm{End}(M)$ as $H^{*\mathrm{op}}$-comodule algebra. This implies that $[A]=[\mathrm{End}(M)]=1$ in $\mathrm{BC}(k,H,R)$. It follows that the sequence (23) is exact. □

Note that the exact sequence (23) indicates that the factor group $\mathrm{BC}(k,H,R)/\mathrm{Br}(k)$ is completely determined by the $\mathcal{H}_R^*$-bigalois objects. In particular, when k is an algebraic closed field, $\mathrm{BC}(k,H,R)$ is a subgroup of $\mathrm{Gal}(\mathcal{H}_R)$.

Now let us look at some special cases. First let H be a commutative Hopf algebra. Then H has a trivial coquasitriangular structure $R = \varepsilon \otimes \varepsilon$. In this case, $\mathcal{H}_R$ is equal to H as an algebra and $D[H] = H \otimes H$ is the tensor product algebra. An R-Azumaya algebra is an Azumaya algebra which is a right H-comodule algebra with the trivial left H-action. On the other hand, the $\mathcal{H}_R$-bimodule structures (3) and (6) of a YD H-module M coincide and are exactly the left H-module structure of M. So in this case an object in the category $\mathcal{E}(\mathcal{H}_R)$ is nothing but an H^*-Galois object which is automatically an H^*-bigalois object since H^* is cocommutative. So the group $\mathrm{Gal}(\mathcal{H}_R)$ is the group $E(H^*)$ of H^*-Galois objects with the cotensor product over H^*. So we obtain the following exact sequence due to Beattie.

Corollary 4.13 [2]. *Let H be a finite commutative Hopf algebra. Then the following group sequence is exact and split*:

$$1 \to \mathrm{Br}(k) \xrightarrow{\iota} \mathrm{BC}(k, H) \xrightarrow{\widetilde{\pi}} E(H^*) \to 1$$

where the group map $\widetilde{\pi}$ is surjective and split because any H^*-Galois object B is equal to $\pi(B \# H)$ and the smash product $B \# H$ is a right H-comodule Azumaya algebra which represents an element in $\mathrm{BC}(k, H)$.

Secondly we let R be a non-trivial coquasitriangular structure of H, but let H be a commutative and cocommutative finite Hopf algebra over k. In this case, $\mathcal{H}_R$ is isomorphic to H as an algebra and becomes a Hopf algebra. An object in $\mathrm{Gal}(\mathcal{H}_R)$ is an H^*-bigalois object. It is not difficult to check that YD H-module (or H-bimodule) structures commute with both H^*-Galois structures.

Let θ be the Hopf algebra map corresponding to the coquasitriangular structure R, that is,

$$\theta : H \to H^*, \quad \theta(h)(l) = R(l \otimes h)$$

for $h, l \in H$. Let $\rightharpoonup$ be the induced H-action on a right H-comodule M:

$$h \rightharpoonup m = \sum m_{(0)}\theta(h)(m_{(1)}) = \sum m_{(0)}R(m_{(1)} \otimes h)$$

for $h \in H$ and $m \in M$. In [31], Ulbrich constructed a group $D(\theta, H^*)$ consisting of isomorphism classes of H^*-bigalois objects which are also H-bimodule algebras such that all H and H^* structures commute, and satisfy the following additional conditions interpreted by means of R, cf. [31, (14), (16)]:

$$\begin{aligned} h \rightharpoondown\!\triangleright a &= \sum a_{(0)} \triangleleft\!- h_{(1)} R\big(a_{(1)} \otimes S(h_{(2)})\big) R\big(S(h_{(3)}) \otimes a_{(2)}\big), \\ \sum x_{(0)}(a \triangleleft\!- x_{(1)}) &= \sum (x_{(1)} \rightharpoonup a) x_{(0)}. \end{aligned} \tag{24}$$

Let us check that any object A in the category $\mathcal{E}(\mathcal{H}_R)$ satisfies the conditions (24) so that A represents an element of $D(\theta, H^*)$. Indeed, since H is commutative and cocommutative, we have

$$\begin{aligned} h \multimap a &= \sum (h_{(2)} \cdot a_{(0)}) R\big(S^{-1}(h_{(4)}) \otimes h_{(3)} a_{(1)} S^{-1}(h_{(1)})\big) \\ &= \sum (h_{(1)} \cdot a_{(0)}) R\big(S(h_{(2)}) \otimes a_{(1)}\big) \\ &= \sum (h_{(2)} \cdot a_{(0)}) R(a_{(1)} \otimes h_{(1)}) R\big(a_{(2)} \otimes S(h_{(3)})\big) R\big(S(h_{(4)}) \otimes a_{(3)}\big) \\ &= \sum (a_{(0)} \triangleleft\!- h_{(1)}) R\big(a_{(1)} \otimes S(h_{(2)})\big) R\big(S(h_{(3)}) \otimes a_{(2)}\big), \end{aligned}$$

and

$$\begin{aligned} \sum x_{(0)} (a \triangleleft\!- x_{(1)}) &= \sum x_{(0)} (x_{(1)} \cdot a_{(0)}) R(a_{(1)} \otimes x_{(1)}) \\ &= \sum a_{(0)} x_{(0)} R(a_{(1)} \otimes x_{(1)}) \quad \text{(by q.c.)} \\ &= \sum (x_{(1)} \rightharpoonup a) x_{(0)} \end{aligned}$$

for any $a, x \in A$ and $h \in H$. It follows that the group $\mathrm{Gal}(\mathcal{H}_R)$ is contained in $D(\theta, H^*)$. As a consequence, we obtain Ulbrich's exact sequence [31, 1.10]:

$$1 \to \mathrm{Br}(k) \to \mathrm{BD}(\theta, H^*) \xrightarrow{\pi_\theta} D(\theta, H^*)$$

for a commutative and cocommutative finite Hopf algebra with a Hopf algebra map θ from H to H^*. In particular, when $H = kG$, a group Hopf algebra of an abelian group, we get the exact sequence [9, 1.2]:

$$1 \to \mathrm{Br}(k) \to \mathrm{B}_\phi(k, G) \xrightarrow{\pi} \mathrm{Galz}(k, G),$$

where $\phi : G \times G \to U(k)$ is a bicharacter map.

5. An example

In this section, we let k be a field with $\mathrm{ch}(k) \neq 2$. Let H_4 be the Sweedler 4-dimensional Hopf algebra over k. That is, H_4 is generated by two elements g and h satisfying

$$g^2 = 1, \qquad h^2 = 0, \qquad gh + hg = 0.$$

The comultiplication, the counit and the antipode are given as follows:

$$\begin{aligned} &\Delta(g) = g \otimes g, && \Delta(h) = 1 \otimes h + h \otimes g, \\ &\varepsilon(g) = 1, && \varepsilon(h) = 0, \\ &S(g) = g, && S(h) = gh. \end{aligned}$$

There is a family of CQT structures R_t on H_4 parameterized by $t \in k$ as follows:

R_t	1	g	h	gh
1	1	1	0	0
g	1	-1	0	0
h	0	0	t	$-t$
gh	0	0	t	t

It is not hard to check that the Hopf algebra maps Θ_l and Θ_r induced by R_t are as follows:

$$\Theta_l:\ H_4^{\mathrm{cop}} \to H_4^*, \quad \Theta_l(g) = \bar{1} - \bar{g} = x, \quad \Theta_l(h) = t\big(\bar{h} - \overline{gh}\big) = txy,$$
$$\Theta_r:\ H_4^{\mathrm{op}} \to H_4^*, \quad \Theta_r(g) = \bar{1} - \bar{g} = x, \quad \Theta_r(h) = t\big(\bar{h} + \overline{gh}\big) = ty$$

where $\{\bar{1}, \bar{g}, \bar{h}, \overline{gh}\}$ is the dual basis of H_4^*. When t is non-zero, Θ_l and Θ_r are isomorphisms, so that H_4 is a self-dual Hopf algebra.

The deformation algebra $\mathcal{H}_{R_t}$ is a four-dimensional commutative algebra generated by two elements x and y satisfying the relations:

$$x^2 = 1, \qquad xy - yx = 0, \qquad y^2 = t(1 - x).$$

The double algebra $D[H_4]$ with respect to R_t is generated by four elements, g_1, g_2, h_1 and h_2 subject to the following relations:

$$g_i^2 = 1, \qquad h_i^2 = 0, \qquad g_i h_j + h_j g_i = 0,$$
$$g_1 g_2 = g_2 g_1, \qquad h_1 h_2 + h_2 h_1 = t(1 - g_1 g_2).$$

The comultiplication of $D[H_4]$ is easy because the Hopf subalgebras generated by g_i, h_i, $i = 1, 2$, are isomorphic to H_4. Thus the algebra embedding ϕ reads as follows:

$$\Phi : \mathcal{H}_{R_t} \to D[H_4], \quad \phi(x) = g_1 g_2, \quad \phi(y) = g_1(h_2 - h_1).$$

Let us consider the triangular case where $R = R_0$ and write $\mathcal{H}_R$ for $\mathcal{H}_{R_0}$. The dual coalgebra $C = \mathcal{H}_R^*$ has a linear basis $\{e, a, b, c\}$ with comultiplication and counit given by

$$\Delta(e) = e \otimes e, \quad \Delta(a) = a \otimes a, \quad \Delta(b) = b \otimes e + e \otimes b, \quad \Delta(c) = c \otimes a + a \otimes c,$$
$$\varepsilon(e) = 1, \quad \varepsilon(a) = 1, \quad \varepsilon(b) = 0, \quad \varepsilon(c) = 0.$$

It is easy to see that $C = C_e \oplus C_a$, where $C_e = ke + kb$ and $C_a = ka + kc$.

Lemma 5.1. *If A is an object in $\mathcal{E}(\mathcal{H}_R)$, then there is a linear basis $\{1, u, v, w\}$ of A such that*

$$\rho(1) = 1 \otimes e, \qquad \rho(u) = u \otimes a,$$
$$\rho(v) = v \otimes e + 1 \otimes b, \qquad \rho(w) = w \otimes a + u \otimes c. \tag{25}$$

Proof. Since A is a C-Galois object, it is a four-dimensional algebra. The right C-comodule of A decomposes into

$$A = A\square_C(C_e \oplus C_a) = (A\square_C C_e) \oplus (A\square_C C_a) = A_e \oplus A_a.$$

The spaces A_e and A_a are two-dimensional spaces and A_e contains the unit. Let $A_e = k + kv'$ and $A_a = ku' + kw'$. Then $\rho(v') = v' \otimes e + \mu \otimes b$ for some $\mu \in k$ because $(\iota \otimes \varepsilon)\rho(v') = v'$ and $(\iota \otimes \Delta)\rho(v') = (\rho \otimes \iota)\rho(v')$. Since A is C-Galois, μ is non-zero. Set $v = \mu^{-1}v'$. We have $\rho(v) = v \otimes e + 1 \otimes b$.

Similarly, one may find an element $u \in A_a$ such that $\rho(u) = u \otimes a$ because a is a group-like element, and an element $w \in A_a$ such that $\rho(w) = w \otimes a + u \otimes c$. The set $\{1, v, u, w\}$ forms a basis of A. □

Corollary 5.2. *Let A be a Galois object in $\mathcal{E}(\mathcal{H}_R)$. Then there exist a basis $\{1, u, v, w\}$ of A such that the action of $\mathcal{H}_R$ on the basis is as follows*:

$$\begin{aligned} &x \cdot 1 = 1, \quad x \cdot u = -u, \quad x \cdot v = v, \quad x \cdot w = -w, \\ &y \cdot 1 = 0, \quad y \cdot u = 0, \quad y \cdot v = 1, \quad y \cdot w = u. \end{aligned} \tag{26}$$

Let A be an object in $\mathcal{E}(\mathcal{H}_R)$. We choose a basis $\{1, v, u, w\}$ satisfying the properties of Lemma 5.1 and Corollary 5.2. We consider the possible YD H_4-module structures on A such that the induced $\mathcal{H}_R^*$-comodule structure and $\mathcal{H}_R$-module structure on the basis $\{1, u, v, w\}$ are (25) and (26) respectively.

Let X and Y be the matrix representations in $\mathbf{M}_{4\times 4}$ of $x, y \in \mathcal{H}_R$. Then X and Y have the forms with respect to the basis $\{1, u, v, w\}$:

$$X = \left(\begin{array}{cc|cc} 1 & & & \\ & -1 & & \\ \hline & & 1 & \\ & & & -1 \end{array}\right), \qquad Y = \left(\begin{array}{cc|cc} & & 1 & \\ & & & 1 \\ \hline & & & \\ & & & \end{array}\right),$$

where the blank entries are zeros. Since $R_0(h, l) = R_0(l, h) = 0$ for any element $l \in H_4$, we have $h \triangleright_2 m = 0$ for $m \in M$, where M is a right H_4-comodule. Thus the matrix representation of $h_1 \in D[H_4]$ in $\mathbf{M}_{4\times 4}$ is the zero matrix. Let G_i and H_i be the representation matrices of g_i and h_i in $\mathbf{M}_{4\times 4}$, $i = 1, 2$. Since $x = g_1 g_2$ and $y = g_1(h_2 - h_1)$, we have $X = G_1 G_2$ and $Y = G_1 H_2$ because $H_1 = 0$.

Since G_1 anti-commutes with Y and $G_1^2 = I_4$, we obtain that G_1 (and consequently G_2) are of the forms:

$$G_1 = \left(\begin{array}{cc|cc} 1 & & c & \\ & a & & b \\ \hline & & -1 & \\ & & & -a \end{array}\right), \qquad G_2 = \left(\begin{array}{cc|cc} 1 & & c & \\ & -a & & b \\ \hline & & -1 & \\ & & & a \end{array}\right),$$

where $a^2 = 1$ and $b, c \in k$. It is easy to see that G_1 and G_2 have two different eigenvalues 1 and -1. If we choose a different basis of A, say, $\{1, \underline{u}, \underline{v}, \underline{w}\}$, then G_1 and G_2 can be of the following forms:

$$G_1 = \left(\begin{array}{cc|cc} 1 & & & \\ & 1 & & \\ \hline & & -1 & \\ & & & -1 \end{array}\right), \qquad G_2 = \left(\begin{array}{cc|cc} 1 & & & \\ & -1 & & \\ \hline & & -1 & \\ & & & 1 \end{array}\right).$$

However, the matrix H_2 depends on the choice of $a = \pm 1$. So it has the following two types of forms:

$$\text{(i) } H_2 = \left(\begin{array}{cc|cc} & & 1 & \\ & & & 1 \\ \hline & & & \\ & & & \end{array}\right), \qquad \text{(ii) } H_2 = \left(\begin{array}{cc|cc} & & 1 & \\ & & & 0 \\ \hline 0 & & & \\ & -1 & & \end{array}\right),$$

Thus we obtain the following:

Proposition 5.3. *Let A be an object in $\mathcal{E}(\mathcal{H}_R)$. There is a basis $\{1, u, v, w\}$ of A such that the H_4-module structure and the $\mathcal{H}_R$-module structure are either*

$$\text{Type I:} \quad \begin{array}{llll} g \cdot 1 = 1, & g \cdot u = -u, & g \cdot v = -v, & g \cdot w = w, \\ h \cdot 1 = 0, & h \cdot u = 0, & h \cdot v = 1, & h \cdot w = u, \\ x \cdot 1 = 1, & x \cdot u = -u, & x \cdot v = v, & x \cdot w = -w, \\ y \cdot 1 = 0, & y \cdot u = 0, & y \cdot v = 1, & y \cdot w = u \end{array} \tag{27}$$

or

$$\text{Type II:} \quad \begin{array}{llll} g \cdot 1 = 1, & g \cdot u = -u, & g \cdot v = -v, & g \cdot w = w, \\ h \cdot 1 = 0, & h \cdot u = -w, & h \cdot v = 1, & h \cdot w = 0 \\ x \cdot 1 = 1, & x \cdot u = -u, & x \cdot v = v, & x \cdot w = -w, \\ y \cdot 1 = 0, & y \cdot u = w, & y \cdot v = 1, & y \cdot w = 0. \end{array} \tag{28}$$

An object A in $\mathcal{E}(\mathcal{H}_R)$ is said to be of *type* I if A has the structures (27), and it is said to of *type* II if it satisfies (28). Since the H_4-comodule structure of A is partially killed by the coquasitriangular structure R_0, we can not obtain the comodule structure of A in the same way as we obtained the module structure of A. However, we have not analyzed the multiplication of A and the quantum commutativity of A.

Let $\{1, u, v, w\}$ be the basis we chose in Proposition 5.3 so that the H_4-action on A are of the forms (27) or (28). Let U, V and W be the matrix representation of the regular multiplication of u, v and w in A.

Proposition 5.4. *Let A be an object in $\mathcal{E}(\mathcal{H}_R)$ with the H_4-module structure* (27) *on a basis $\{1, u, v, w\}$. Then A is a generalized quaternion algebra $(\frac{\alpha, \beta}{k})$ with $\alpha \neq 0$.*

Proof. Since A is an H_4-module algebra, the matrices U, V, W and G_2, H_2 must satisfy the commutation rules stemming from the smash product $A \# H_4$. Thus we have the following relations:

$$\begin{array}{lll} G_2U = -UG_2, & G_2V = -VG_2, & G_2W = WG_2, \\ H_2U = UH_2, & H_2V = VH_2 + G_2, & H_2W = WH_2 + UG_2. \end{array}$$

A further computation shows that U, V and W are of the forms:

$$U = \left(\begin{array}{cc|cc} & \alpha & & \\ 1 & & & \\ \hline & & & \alpha \\ & & 1 & \end{array}\right), \qquad V = \left(\begin{array}{cc|cc} & & \beta & \\ & & & -\beta \\ \hline 1 & & & \\ & -1 & & \end{array}\right),$$

and $W = UV$, for some $\alpha, \beta \in k$. This implies that A is a generalized quaternion algebra with generators u and v satisfying the relations: $u^2 = \alpha$, $v^2 = \beta$ and $uv + vu = 0$.

Next we show that $\alpha \neq 0$. Since A is an $\mathcal{H}_R^*$-Galois object with the right $\mathcal{H}_R^*$-coaction given by (25), we have $\beta_r(u \otimes u) = u^2 \otimes a = \alpha \otimes a$. The bijectivity of β_r implies that α is non-zero. □

If an object A in $\mathcal{E}(\mathcal{H}_R)$ is of type I, then A is necessary a generalized quaternion algebra and is a right H_4^*-Galois object. Using a similar argument to the one made above, we obtain the following:

Proposition 5.5. *Let A be an object in $\mathcal{E}(\mathcal{H}_R)$ with the H_4-module structure given by* (28). *Then A is a commutative algebra $k\langle\sqrt{\alpha}\,\rangle \otimes k\langle\sqrt{\beta}\,\rangle$ for some $\alpha \neq 0, \beta \in k$, where the two generators are v and w and $u = -vw$.*

Note that if an object in $\mathcal{E}(\mathcal{H}_R)$ is of type II, then the H_4-module algebra is not an H_4^*-Galois object. Once we know the H_4-module algebra structure of an object in $\mathcal{E}(\mathcal{H}_R)$, we are able to work out the H_4-comodule structure of A by utilizing the quantum commutativity. Let us first translate the q.c. formula into its dual version. Suppose that A is a q.c. YD H-module algebra. Denote by $\sum a_{[0]} \otimes a_{[1]} \in A \otimes H^*$ the dual coaction of H^* on element a. Then the quantum commutativity of A can be stated in terms of the dual action and dual coaction of H^*:

$$ab = \sum (a_{[1]} \rightharpoonup b)a_{[0]} \tag{29}$$

for any elements $a, b \in A$, where $h^* \rightharpoonup a = \sum a_{(0)}\langle h^*, a_{(1)}\rangle$ for $h^* \in H^*$ and $a \in A$.

Proposition 5.6. *Let A be an object in $\mathcal{E}(\mathcal{H}_R)$ with a basis $\{1, u, v, w\}$ satisfying Proposition* 5.3.

(i) *If A is of type* I, *then the H_4-comodule structure is given by*

$$\rho(u) = u \otimes 1 - 2w \otimes gh, \qquad \rho(v) = v \otimes g + 2\beta \otimes h,$$
$$\rho(w) = w \otimes g. \tag{30}$$

(ii) *If A is of type* II*, then the H_4-comodule structure of A is given by*

$$\rho(u) = u \otimes 1, \qquad \rho(v) = v \otimes g + 2\beta \otimes h,$$
$$\rho(w) = w \otimes g - 2u \otimes h. \tag{31}$$

Proof. In case A is of type I the formulae given in (30) are uniquely determined by the H_4-module structure of A and are given by the MUVO action of H_4^* since A is an H_4^*-Galois object (see [6,30]). Suppose that A is of type II. In this case A is not an H_4^*-Galois object. So the H_4-comodule structure of A is not from a MUVO action on A. However, we may still recover the H_4-comodule structure from the quantum commutativity and the H_4-module structure of A. It is sufficient (and necessary) to obtain the dual action of H_4^* of the coaction of H_4. Since H_4^* is isomorphic to H_4 we simply need to work out the action of g and h on the generators v, w of $A = k\langle\sqrt{\alpha}\,\rangle \otimes k\langle\sqrt{\beta}\,\rangle$. Recall that the Hopf algebra map $\Phi : D[H_4] \to D(H_4)$ induced by R_0 restricts to an isomorphism on sub-Hopf algebra generated by group-like elements g_1, g_2. Thus the dual action of g is the same as the action of g_1 given by matrix representation G_1. It remains now to recover the dual action of h. By assumption we have $v^2 = \beta$, $w^2 = \alpha$ and $u = -vw = -wv$. The dual coaction of the H_4-action is as follows:

$$\sum u_{[0]} \otimes u_{[1]} = u \otimes g + w \otimes h, \qquad \sum v_{[0]} \otimes v_{[1]} = v \otimes g + w \otimes h,$$

$$\sum w^0 \otimes w^1 = w \otimes 1.$$

Now the quantum commutativity of A implies that

$$uv = \sum (u_{[1]} \rightharpoonup v)u_{[0]} = (g \rightharpoonup v)u + (h \rightharpoonup v)w = -vu + (h \rightharpoonup v)w,$$
$$vu = \sum (v_{[1]} \rightharpoonup u)v_{[0]} = (g \rightharpoonup u)v + h \rightharpoonup u = uv + h \rightharpoonup u,$$
$$vw = \sum (v_{[1]} \rightharpoonup w)v_{[0]} = (g \rightharpoonup w)v + h \rightharpoonup w = -wv + h \rightharpoonup w.$$

It follows that $h \rightharpoonup v = 2uvw^{-1} = 2v^2 = 2\beta$, $h \rightharpoonup u = 0$ and $h \rightharpoonup w = -2u$. Thus the corresponding H_4-comodule structure of A is then given by

$$\rho(u) = u \otimes 1, \qquad \rho(v) = v \otimes g + 2\beta \otimes h, \qquad \rho(w) = w \otimes g - 2u \otimes h. \qquad \square$$

Now we are able to classify the YD H_4-module structures of all the objects in $\mathcal{E}(\mathcal{H}_R)$.

Theorem 5.7. *Let A be an object in $\mathcal{E}(\mathcal{H}_R)$. Then A is either of type* I *or is of type* II.

(i) *If A is of type* I, *then $A = (\frac{\alpha,\beta}{k})$ is a generalized quaternion algebra for some $\alpha \neq 0$, $\beta \in k$ with generators u, v satisfying $u^2 = \alpha$, $v^2 = \beta$. The YD H_4-module structures of A are given by*

$$\begin{aligned} &g \cdot u = -u, \qquad g \cdot v = -v, \qquad h \cdot u = 0, h \cdot v = 1, \\ &\rho(u) = u \otimes 1 - 2uv \otimes gh, \qquad \rho(v) = v \otimes g + 2\beta \otimes h. \end{aligned} \tag{32}$$

In this case, the induced $\mathcal{H}_R^*$-bicomodule structures are as follows:

$$\begin{aligned} \rho_l(1) &= e \otimes 1, & \rho_r(1) &= 1 \otimes e, \\ \rho_l(u) &= a \otimes u, & \rho_r(u) &= u \otimes a, \\ \rho_l(v) &= e \otimes v + b \otimes 1, & \rho_r(v) &= v \otimes e + 1 \otimes b, \\ \rho_l(uv) &= a \otimes uv + c \otimes u, & \rho_r(uv) &= uv \otimes a + u \otimes c. \end{aligned} \tag{33}$$

(ii) *If A is of type* II, *then $A = k\langle\sqrt{\alpha}\,\rangle \otimes k\langle\sqrt{\beta}\,\rangle$ for some $\alpha \neq 0$, $\beta \in k$ with generators u, v satisfying $u^2 = \alpha$, $v^2 = \beta$ and $uv = vu$. The YD H_4-module structures are given by*

$$\begin{aligned} &g \cdot u = u, \qquad g \cdot v = -v, \qquad h \cdot u = 0, \qquad h \cdot v = 1, \\ &\rho(u) = u \otimes g + 2uv \otimes h, \qquad \rho(v) = v \otimes g + 2\beta \otimes h. \end{aligned} \tag{34}$$

In this case, the induced $\mathcal{H}_R^*$-bicomodule structures are as follows:

$$\begin{aligned} \rho_l(1) &= e \otimes 1, & \rho_r(1) &= 1 \otimes e, \\ \rho_l(u) &= a \otimes u, & \rho_r(u) &= u \otimes a, \\ \rho_l(v) &= e \otimes v + b \otimes 1, & \rho_r(v) &= v \otimes e + 1 \otimes b, \\ \rho_l(uv) &= a \otimes uv + c \otimes u, & \rho_r(uv) &= uv \otimes a - u \otimes c. \end{aligned} \tag{35}$$

Proof. The only ones left to be shown are the $\mathcal{H}_R^*$-bicomodule structures of A in each case. Since we know the YD H_4-module structures of A in each case, the actions of $\mathcal{H}_R$ on A follow from the definitions (3) and (6). □

Let A be an object in $\mathcal{E}(\mathcal{H}_R)$. We denote A by $\langle\frac{\alpha,\beta}{k}\rangle$ if A is of type I, and by $k\langle\sqrt{\alpha}, \sqrt{\beta}\,\rangle$ if A is of type II. Let

$$A = \left\langle \frac{\alpha, \beta}{k} \right\rangle \quad \text{and} \quad B = \left\langle \frac{\alpha', \beta'}{k} \right\rangle$$

be two objects in $\mathcal{E}(\mathcal{H}_R)$ of type I. We compute the product $A \wedge B$. Observing the standard $\mathcal{H}_R^*$-bicomodule structures of $\langle\frac{\alpha,\beta}{k}\rangle$ from Theorem 5.7, we may easily find that $A \wedge B$ is generated by two elements $\underline{u} = u \# u'$ and $\underline{v} = v \# 1 + 1 \# v$. A routine computation shows that $\underline{u}$ and $\underline{v}$ generate a generalized quaternion algebra $(\frac{\alpha\alpha',\beta+\beta'}{k})$. This fact suggests that the subset Γ of isomorphism classes represented by objects of type I in $\mathcal{E}(\mathcal{H}_R)$ form a subgroup of $\mathrm{Gal}(\mathcal{H}_R)$.

Proposition 5.8. *Γ is a subgroup of* $\mathrm{Gal}(\mathcal{H}_R)$ *and is isomorphic to* $k^+ \times k^\bullet / k^{\bullet 2}$.

Proof. Suppose that $[\langle\frac{\alpha,\beta}{k}\rangle]$ and $[\langle\frac{\alpha',\beta'}{k}\rangle]$ are two elements of Γ. In the preceding argument, we showed that $\langle\frac{\alpha,\beta}{k}\rangle \wedge \langle\frac{\alpha',\beta'}{k}\rangle$ as an algebra is isomorphic to $(\frac{\alpha\alpha',\beta+\beta'}{k})$. If $\langle\frac{\alpha,\beta}{k}\rangle \wedge \langle\frac{\alpha',\beta'}{k}\rangle$ has the YD H-module structure of type I, i.e.,

$$\left\langle\frac{\alpha,\beta}{k}\right\rangle \wedge \left\langle\frac{\alpha',\beta'}{k}\right\rangle = \left\langle\frac{\alpha\alpha',\beta+\beta'}{k}\right\rangle,$$

then Γ is a group. Since $\underline{u} = u \,\#\, u'$ and $\underline{v} = v \,\#\, 1 + 1 \,\#\, v$ are the two generators of $\langle\frac{\alpha,\beta}{k}\rangle \wedge \langle\frac{\alpha',\beta'}{k}\rangle$, it is enough to check that the action and coaction of H_4 on $\underline{u}$ and $\underline{v}$ satisfy (32). Indeed, we have

$$\begin{aligned}
\rho(\underline{u}) &= (u \# 1 \otimes 1 - 2w \# 1 \otimes gh)\big(1 \# u' \otimes 1 - 1 \# 2w' \otimes gh\big)\\
&= u \# u' \otimes 1 - 2\big(w \# u' + u \# w'\big) \otimes gh\\
&= \underline{u} \otimes 1 - 2\underline{u}\underline{v} \otimes gh\\
&= \underline{u} \otimes 1 - 2\underline{w} \otimes gh, \quad \text{and}\\
\rho(\underline{v}) &= v \# 1 \otimes g + 2\beta \# 1 \otimes h + 1 \# v' \otimes g + 1 \# 2\beta' \otimes h\\
&= \big(v \# 1 + 1 \# v'\big) \otimes g + 2\big(\beta + \beta'\big)\big(1 \# 1'\big) \otimes h\\
&= \underline{v} \otimes g + 2\big(\beta + \beta'\big) \otimes h,
\end{aligned}$$

where $w = uv$ and $\underline{w} = \underline{u}\underline{v}$. Similarly one may check that

$$g \cdot \underline{u} = -\underline{u}, \qquad g \cdot \underline{v} = -\underline{v}, \qquad h \cdot \underline{u} = 0, \qquad h \cdot \underline{v} = 1.$$

So we have proved that $\langle\frac{\alpha,\beta}{k}\rangle \wedge \langle\frac{\alpha',\beta'}{k}\rangle = \langle\frac{\alpha\alpha',\beta+\beta'}{k}\rangle$.

Next we show that the subgroup Γ fits in the following split and exact sequence of group homomorphisms:

$$1 \to k^+ \to \Gamma \to k^\bullet / k^{\bullet 2} \to 1,$$

where k^+ is the additive group of k and $k^\bullet$ is the multiplicative group of k. Let

$$A = \left\langle\frac{\alpha,\beta}{k}\right\rangle.$$

Assign to $\langle\frac{\alpha,\beta}{k}\rangle$ the quadratic extension $k\langle\sqrt{\alpha}\,\rangle$. Then we get a group homomorphism

$$\lambda : \Gamma :\to Q(k)$$

from Γ into the group $Q(k)$ of quadratic extensions. It is obvious that λ is surjective. We show that the kernel of λ is isomorphic to k^+. Recall that the group $Q(k)$ is isomorphic to the group $k^\bullet/k^{\bullet 2}$ (see [38]). Moreover,

$$\lambda\left[\left\langle\frac{\alpha,\beta}{k}\right\rangle\right]=1 \quad \text{if and only if} \quad \alpha\in k^{\bullet 2}.$$

It follows that

$$\operatorname{Ker}(\lambda)=\left\{\left[\left\langle\frac{\alpha,\beta}{k}\right\rangle\right]\ \middle|\ \alpha\in k^{\bullet 2}\right\},$$

which is easily seen to be isomorphic to the additive group k^+. Finally the exact sequence is split because the map $\iota: Q(k)\to\Gamma$ given by

$$\iota(\alpha)=\left[\left\langle\frac{\alpha,0}{k}\right\rangle\right]$$

is a well-defined group homomorphism and $\iota\cdot\lambda=\mathrm{Id}_{Q(k)}$. □

Theorem 5.9. *The group* $\mathrm{Gal}(\mathcal{H}_R)$ *is isomorphic to* $\Gamma\rtimes\mathbb{Z}_2$, *where the multiplication rule is given by*

$$\big((\alpha,\beta)\rtimes i\big)\big((\alpha',\beta')\rtimes j\big)=\big((-1)^{ij}\alpha\alpha',\beta+\beta'\big)\rtimes(i+j).$$

Proof. Let D be the object $k\langle\sqrt{1},\sqrt{0}\rangle$ of type II in $\mathcal{E}(\mathcal{H}_R)$. Consider the object $D^2=D\wedge D$. It is easy to see from (35) that the two elements $\underline{u}=u\,\#\,u$ and $\underline{v}=1\,\#\,v+v\,\#\,1$ generate the algebra D^2 and satisfy the relations:

$$\underline{u}^2=-1,\qquad \underline{v}^2=0,\qquad \underline{u}\,\underline{v}+\underline{v}\,\underline{u}=0.$$

Thus D^2 is the generalized quaternion algebra $(\frac{-1,0}{k})$. Now it is straightforward to check that $\underline{u}$ and $\underline{v}$ satisfy (32), and it follows that

$$D^2=\left\langle\frac{-1,0}{k}\right\rangle.$$

By Proposition 5.8, the object D is of order 2 if $-1\in k^{\bullet 2}$, and is of order 4 if $-1\notin k^{\bullet 2}$.

Next we show that any object A of type II in $\mathcal{E}(\mathcal{H}_R)$ is a product of D with an object of type I. Suppose that $A=k\langle\sqrt{\alpha},\sqrt{\beta}\rangle$ is an object of type II for some $\alpha\in k^\bullet$ and $\beta\in k$. We show that the product $\langle\frac{\alpha,\beta}{k}\rangle\wedge D$ is equal to $k\langle\sqrt{\alpha},\sqrt{\beta}\rangle$. It is easy to see that $\langle\frac{\alpha,\beta}{k}\rangle\wedge D$ is generated by two elements $\underline{u}=u\,\#\,u'$ and $\underline{v}=v\,\#\,1+1\,\#\,v'$, where u,v and u',v' are generators of $\langle\frac{\alpha,\beta}{k}\rangle$ and D respectively. We have $\underline{u}^2=\alpha$, $\underline{v}^2=\beta$ and $\underline{u}\,\underline{v}=\underline{v}\,\underline{u}$. Thus

$$\left\langle\frac{\alpha,\beta}{k}\right\rangle\wedge D=k\langle\sqrt{\alpha}\rangle\otimes k\langle\sqrt{\beta}\rangle$$

as algebras. Now we check that $\underline{u}$ and $\underline{v}$ satisfy (34). Indeed, we have

$$\begin{aligned}
g\cdot\underline{u} &= g\cdot u\,\#\,(g\cdot u') = -u\,\#\,(-u') = \underline{u},\\
g\cdot\underline{v} &= g\cdot v\,\#\,1 + 1\,\#\,(g\cdot v') = -v\,\#\,1 - 1\,\#\,v' = -\underline{v},\\
h\cdot\underline{u} &= u\,\#\,(h\cdot u') + h\cdot u\,\#\,g\cdot u' = 0,\\
h\cdot\underline{v} &= h\cdot v\,\#\,1 + 1\,\#\,(h\cdot v') = 1\,\#\,1 + 0 = 1
\end{aligned}$$

and

$$\begin{aligned}
\rho(\underline{u}) &= (u\,\#\,1\otimes 1 - 2uv\,\#\,1\otimes gh)\big(1\,\#\,u'\otimes g + 1\,\#\,2u'v'\otimes h\big)\\
&= u\,\#\,u'\otimes g + 2\big(uv\,\#\,u' + u\,\#\,u'v'\big)\otimes gh\\
&= \underline{u}\otimes 1 + 2\underline{uv}\otimes gh, \quad \text{and}\\
\rho(\underline{v}) &= v\,\#\,1\otimes g + 2\beta\,\#\,1\otimes h + 1\,\#\,v'\otimes g + 1\,\#\,2\beta'\otimes h\\
&= \big(v\,\#\,1 + 1\,\#\,v'\big)\otimes g + 2\big(\beta+\beta'\big)\big(1\,\#\,1'\big)\otimes h\\
&= \underline{v}\otimes g + 2\big(\beta+\beta'\big)\otimes h.
\end{aligned}$$

Similarly, one can show that

$$D\wedge\left\langle\frac{\alpha,\beta}{k}\right\rangle = k\big\langle\sqrt{\alpha},\sqrt{\beta}\,\big\rangle$$

for any $\alpha\in k^{\bullet}$ and $\beta\in k$. Thus we have proved that any object in $\mathcal{E}(\mathcal{H}_R)$ is either a generalized quaternion algebra $\langle\frac{\alpha,\beta}{k}\rangle$ or a product $\langle\frac{\alpha,\beta}{k}\rangle\wedge D$, where $\alpha\in k^{\bullet}$, $\beta\in k$ and $D = k\langle\sqrt{1},\sqrt{0}\rangle$. This fact implies that the group $\mathrm{Gal}(\mathcal{H}_R)$ is an abelian group generated by the subgroup Γ and the element $[D]$.

Define a map ϑ from $\mathrm{Gal}(\mathcal{H}_R)$ into $\Gamma\rtimes\mathbb{Z}_2$ as follows:

$$\vartheta\left(\left[\left\langle\frac{\alpha,\beta}{k}\right\rangle\right]\right) = \left[\left\langle\frac{\alpha,\beta}{k}\right\rangle\right]\rtimes 0,\quad\text{and}\quad \vartheta\left(\left[k\big\langle\sqrt{\alpha},\sqrt{\beta}\,\big\rangle\right]\right) = \left[\left\langle\frac{\alpha,\beta}{k}\right\rangle\right]\rtimes 1.$$

It is clear from the definition that $\vartheta(D) = (1,0)\rtimes 1$. Since

$$k\big\langle\sqrt{\alpha},\sqrt{\beta}\,\big\rangle = \left\langle\frac{\alpha,\beta}{k}\right\rangle\wedge D,\quad \left\langle\frac{\alpha,\beta}{k}\right\rangle\wedge D = D\wedge\left\langle\frac{\alpha,\beta}{k}\right\rangle\quad\text{and}\quad D\wedge D = \left\langle\frac{-1,0}{k}\right\rangle,$$

ϑ is an isomorphism. □

Theorem 5.10. *The homomorphism $\widetilde{\pi}$ is surjective and we have an exact sequence*:

$$1\to \mathrm{Br}(k)\to \mathrm{BC}(k,H_4,R)\xrightarrow{\widetilde{\pi}} \mathrm{Gal}(\mathcal{H}_R)\to 1. \tag{36}$$

Proof. If A is an object of type I in $\mathrm{Gal}(\mathcal{H}_R)$, then A is some generalized quaternion algebra $\langle\frac{\alpha,\beta}{k}\rangle$, $\alpha \neq 0$ and $\beta \in k$. When $\beta \neq 0$, $\langle\frac{\alpha,\beta}{k}\rangle$ is an R-Azumaya algebra if we forget the left H_4-module structure. Since the coinvariant subalgebra of $\langle\frac{\alpha,\beta}{k}\rangle$ is trivial, we have

$$\pi\left(\left\langle\frac{\alpha,\beta}{k}\right\rangle\right) = \left(\frac{\alpha,\beta}{k}\right) \quad \text{if } \beta \neq 0.$$

To get the preimage of $\langle\frac{\alpha,0}{k}\rangle$ for $\alpha \in k^\bullet$, we choose the R-Azumaya algebra $\langle\frac{\alpha,1}{k}\rangle \# \langle\frac{1,-1}{k}\rangle$. Since π is monoidal we have that

$$\pi\left(\left\langle\frac{\alpha,1}{k}\right\rangle \# \left\langle\frac{1,-1}{k}\right\rangle\right) = \pi\left(\left\langle\frac{\alpha,1}{k}\right\rangle\right) \wedge \left\langle\frac{1,-1}{k}\right\rangle\Big) = \left\langle\frac{\alpha,0}{k}\right\rangle.$$

For an object $k\langle\sqrt{\alpha},\sqrt{\beta}\,\rangle$ of type II in $\mathrm{Gal}(\mathcal{H}_R)$, we choose a Galois R-Azumaya algebra A such that

$$\pi(A) = \left\langle\frac{\alpha,\beta}{k}\right\rangle$$

(assured by the foregoing arguments). Then it is easy to check that

$$\pi\left(A \# k\langle\sqrt{1}\rangle\right) = k\langle\sqrt{\alpha},\sqrt{\beta}\,\rangle$$

for $\alpha \in k^\bullet$ and $\beta \in k$. Thus by Theorem 5.7, $\widetilde{\pi}$ is an epimorphism, and hence the sequence (36) is exact. □

Recall that the Brauer–Wall group $\mathrm{BW}(k)$ is $\mathrm{BC}(k, k\mathbb{Z}_2, R')$, where $k\mathbb{Z}_2$ is the sub-Hopf algebra of H_4 generated by the group-like element $g \in H_4$, and R' is the restriction of R to $k\mathbb{Z}_2$. The following well-known exact sequence is a special case of (23):

$$1 \to \mathrm{Br}(k) \to \mathrm{BW}(k) \xrightarrow{\widetilde{\pi}} Q_2(k) \to 1, \tag{37}$$

where $Q_2(k) = Q(k) \rtimes \mathbb{Z}_2$ is nothing but $\mathrm{Gal}(\mathcal{H}_{R'})$ and $\mathcal{H}_{R'} \cong k\mathbb{Z}_2$, here $H = k\mathbb{Z}_2$.

The sequence (37) can be also obtained if we restrict the homomorphism $\widetilde{\pi}$ in (36) to the subgroup $\mathrm{BW}(k)$ of $\mathrm{BC}(k, H_4, R)$. The group $\widetilde{\pi}(\mathrm{BW}(k))$ consist of all objects of form: $(\frac{\alpha,0}{k})$ of type I and $k(\sqrt{\alpha}\,) \otimes k(\sqrt{0}\,)$ of type II, which is isomorphic to $Q_2(k)$.

Recall from [35] that the CQT Hopf algebra map $H_4 \to \mathbb{Z}_2$ sending g to g and h to zero induces a group homomorphism γ from $\mathrm{BC}(H_4, R)$ onto $\mathrm{BW}(k)$, where $\gamma([A]) = [A]$, and the later $[A]$ has only grading.

In order to distinguish the group homomorphism $\widetilde{\pi}$, we use π_2 and π_4 (consequently $\widetilde{\pi}_2, \widetilde{\pi}_4$) to denote the canonical monoidal functors for CQT Hopf algebras $(k\mathbb{Z}_2, R')$ and (H_4, R) respectively. Let A be $\langle\frac{\alpha,\beta}{k}\rangle$, $\alpha, \beta \in k^\bullet$ with H_4-coaction given by (32). Then

$$\pi_4\left(\left\langle\frac{\alpha,\beta}{k}\right\rangle\right) = \left\langle\frac{\alpha,\beta}{k}\right\rangle.$$

If we forget the coaction of h, then $\langle\frac{\alpha,\beta}{k}\rangle$ represents an element in $\mathrm{BW}(k)$. Let u, v be the canonical generators of $\langle\frac{\alpha,\beta}{k}\rangle$. Then $A_0 = k + ku$. It is easy to see that

$$\pi_2\left(\left\langle\frac{\alpha,\beta}{k}\right\rangle\right) = C_{A_0}(A) = A_0 = k\big\langle\sqrt{\alpha}\,\big\rangle.$$

Now let $A = \mathrm{End}(H^{\mathrm{op}})\,\#\,k\langle\sqrt{1}\,\rangle$. Then $\pi_4(A) = D$ and

$$\pi_2(A) = \pi\big(\mathrm{End}\big(H^{\mathrm{op}}\big)\big) \wedge \pi\big(k\big\langle\sqrt{1}\big\rangle\big) = k\big\langle\sqrt{1}\big\rangle.$$

since $k\langle\sqrt{1}\,\rangle$ is now a Galois graded Azumaya algebra. Thus we have proved that γ fits in the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Br}(k)\cap K & \longrightarrow & K & \xrightarrow{\widetilde{\pi}_4} & k^+ & \longrightarrow & 1\\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\iota} & & \\
1 & \longrightarrow & \mathrm{Br}(k) & \longrightarrow & \mathrm{BC}(k, H_4, R) & \xrightarrow{\widetilde{\pi}_4} & \mathrm{Gal}(\mathcal{H}_R) & \longrightarrow & 1\\
 & & \downarrow & & {\scriptstyle\gamma}\downarrow & & {\scriptstyle p}\downarrow & & \\
1 & \longrightarrow & \mathrm{Br}(k) & \longrightarrow & \mathrm{BW}(k) & \xrightarrow{\widetilde{\pi}_2} & Q_2(k) & \longrightarrow & 1,
\end{array}$$

where K is the kernel of γ, ι is the inclusion map and p is the projection from $k^+ \times Q_2(k)$ onto $Q_2(k)$. Here $\widetilde{\pi}_4(K) = k^+$ because $\widetilde{\pi}_2 \circ \gamma = p \circ \widetilde{\pi}_4$. By definition of γ we have $\mathrm{Br}(k) \cap K = 1$. It follows that $K \cong k^+$. Since γ is split, we obtain that the Brauer group $\mathrm{BC}(k, H_4, R)$ is isomorphic to the direct product group $k^+ \times \mathrm{BW}(k)$, which coincides with Theorem 8 in [35].

In this case, we have an exact and split sequence, cf. [35]:

$$1 \to k^+ \to \mathrm{BC}(k, H_4, R) \to \mathrm{BW}(k) \to 1 \tag{38}$$

where k^+ is the additive group that is isomorphic to the group of H_4-bigalois objects [25].

Recently, G. Carnovale proved in [8] that the Brauer group $\mathrm{BC}(k, H_4, R_t)$ is isomorphic to $\mathrm{BC}(k, H_4, R_0)$ for any $t \neq 0$ although (H_4, R_t) is not coquasitriangularly isomorphic to (H_4, R_0) when $t \neq 0$ (see [24]).

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Further reading

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