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# Casimir effect in (2 + 1)-dimensional noncommutative theories

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**Abstract**

We study the Dirichlet Casimir effect for a complex scalar field on two noncommutative spatial coordinates plus a commutative time. To that end, we introduce Dirichlet-like boundary conditions on a curve contained in the spatial plane, in such a way that the correct commutative limit can be reached. We evaluate the resulting Casimir energy for two different curves: (a) Two parallel lines separated by a distance $L$, and (b) a circle of radius $R$. In the first case, the resulting Casimir energy agrees exactly with the one corresponding to the commutative case, regardless of the values of $L$ and of the noncommutativity scale $\theta$, while for the latter the commutative behaviour is only recovered when $R \gg \sqrt{\theta}$. Outside of that regime, the dependence of the energy with $R$ is substantially changed due to noncommutative corrections, becoming regular for $R \to 0$.


In the Casimir effect [1], a nice interplay between the geometry of a spatial region and the vacuum fluctuations of a field conspire to produce an observable effect: the Casimir force. The properties of such a force do depend on the kind of field theory considered, on the nature of the boundary conditions imposed, and on the number of spatial dimensions. The physical reason is that the properties above will determine the kind of vacuum fluctuations that are allowed in each spatial region, and whose competing effects produce the Casimir force.

On the other hand, Noncommutative Quantum Field Theories (NCQFT's) [2], are endowed with an intrinsic scale, due to the fundamental commutation relation:

$$[x_\mu, x_\nu] = i\theta_{\mu\nu}, \quad \mu, \nu = 0, 1, \dots, d, \tag{1}$$

where $\theta_{\mu\nu}$ is a constant antisymmetric tensor. The resulting existence of a 'granularity' for the coordinates resolution, with its corresponding scale $\theta$ playing the role of a minimal area, suggests the possibility that noncommutativity might affect the properties of the Casimir force introducing corrections depending on $\sqrt{\theta}/L$ (where $L$ is a length related to the 'size' of the system).

Besides this immediate, merely dimensional argument, we should expect also interesting results to emerge when a NC-QFT is subject to boundary conditions on a nontrivial region: firstly, the boundary conditions are certainly problematic by themselves, since they are imposed on elements in a noncommutative algebra. In particular, the act of imposing a boundary condition on a codimension-1 manifold will set the spatial resolution along one spatial coordinate to zero. Secondly, NCQFT's have been associated to *incompressible* quantum fluids [3,4], whose fluctuations are (because of that property) expected to be more sensitive to the existence of boundaries than in the commutative case.

In this Letter, we consider the Casimir effect for the NCQFT of a complex scalar field in $2 + 1$ dimensions. In this case only two spacetime coordinates may be noncommutative; we shall assume them to be the two spatial ones (which form a Moyal plane), while the time is a commutative object. Our main motivation for considering precisely this situation is that concrete physical systems do exist where noncommutativity is naturally realized in exactly that way: indeed, when a strong constant magnetic field is applied to a two-dimensional system, a projection to the lowest Landau level justifies a noncommutative description [5,6]. On the other hand, since the time coordinate remains commutative, the Hamiltonian still plays the role of the generator of time translations in the usual way, hence many standard Quantum Field Theory tools have the same interpreta-

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tion that in the commutative case. In particular, a path integral formula for the vacuum persistence amplitude can be applied to obtain the vacuum energy.

In this way we shall be able to disentangle new effects that result from the interference of noncommutativity and boundary conditions, from the ones that, even in the absence of boundaries, could still modify the vacuum energy.

Some works have already dealt with the issue of imposing boundary conditions within the context of NCQFT [7–9]. However, both the kind of system considered and the approach followed are different; therefore the ensuing conclusions are incommensurable. For example, in [7], the time coordinate is regarded as noncommutative, while in [8] and [9] noncommutativity is introduced for manifolds without boundaries.

The complex scalar field $\varphi$, on which boundary conditions are to be imposed on the curve $\mathcal{C}$, shall be equipped with a standard free Euclidean action $S_0$:

$$S_0(\varphi^*,\varphi)=\int d^3x\left(\partial_\mu\varphi^*\star\partial_\mu\varphi+m^2\varphi^*\star\varphi\right), \tag{2}$$

where the Moyal product involves just the two spatial coordinates $x_j$, $j=1,2$:

$$f(x_0,x_1,x_2)\star g(x_0,x_1,x_2) \equiv \lim_{y\to x} e^{\frac{i}{2}\theta_{jk}\frac{\partial}{\partial x_j}\frac{\partial}{\partial y_k}} f(x_0,x_1,x_2)g(x_0,y_1,y_2). \tag{3}$$

To impose the boundary conditions for the field on $\mathcal{C}$, we use the procedure of adding to the Lagrangian a term that introduces an interaction with $\mathcal{C}$, in such a way that the boundary conditions emerge when the interaction is strong. This procedure, already used in the Commutative Quantum Field Theory (CQFT) case [10], is here much simpler than attempting to impose the boundary conditions on the field.

To briefly review this approach, let us apply it to the commutative version of our first example, namely, a region $\mathcal{C}$ that corresponds to two straight lines at $x_2=0$ and $x_2=L$. In this case, the total Euclidean action $S=S_0+S_I$ includes an interaction with $\mathcal{C}$:

$$S_I(\varphi,\varphi^*)=\lambda\int_{x_0,x_1}\left[\varphi^*(x_0,x_1,0)\varphi(x_0,x_1,0)+\varphi^*(x_0,x_1,L)\varphi(x_0,x_1,L)\right]. \tag{4}$$

The vacuum energy $E_0$, may be obtained from the path integral expression

$$e^{-TE_0}=\frac{\int\mathcal{D}\varphi^*\mathcal{D}\varphi e^{-S}}{\int\mathcal{D}\varphi^*\mathcal{D}\varphi e^{-S_0}}, \tag{5}$$

where the denominator subtracts the $L\to\infty$ contribution, and $T$ is assumed to tend to infinity. Then

$$E_0=\lim_{T\to\infty,\lambda\to\infty}\frac{1}{T}\operatorname{Tr}\log(1+\Delta D), \tag{6}$$

where $\Delta$ is the free propagator and $D$ is an operator whose kernel is defined by

$$S_I=\int_{x,y}\varphi^*(x)D(x,y,L,\lambda)\varphi(y). \tag{7}$$

Of course, in this case $E$ is expected to be proportional to the length of the lines (in the $x_1$ direction). Since that length is regarded as infinite, in practice one deals with the linear density of energy. The $\lambda\to\infty$ limit is, on the other hand, taken in order to enforce Dirichlet boundary conditions.

Let us now generalize this example to the noncommutative case, considering an action $S^\star=S_0+S_I^\star$ where $S_I^\star\equiv S_I^{\star(L)}+S_I^{\star(0)}$ with:

$$S_I^{\star(L)}\equiv\lambda\int_x\varphi^*\star\delta_2^L\star\varphi, \tag{8}$$

and $\delta_2^L\equiv\delta(x_2-L)$. $S_I^{\star(0)}$ corresponds to setting $L\equiv 0$ above. The Casimir energy will then be obtained by applying (5) to the action $S^\star$.

Introducing the Fourier transform of the field with respect to the $x_0$ and $x_1$ variables,

$$\varphi(x_0,x_1,x_2)=\int\frac{d\omega}{2\pi}\int\frac{dp}{2\pi}e^{i(\omega x_0+px_1)}\tilde\varphi(\omega,p,x_2), \tag{9}$$

and using the properties of the $\star$-product, we may write the interaction term at $x_2=L$ as follows:

$$S_I^{\star(L)}=\lambda\int\frac{d\omega}{2\pi}\int\frac{dp}{2\pi}\int dx_2\,\delta(x_2-L)\times\tilde\varphi^*\left(\omega,p,x_2+\frac{\theta p}{2}\right)\tilde\varphi\left(\omega,p,x_2+\frac{\theta p}{2}\right), \tag{10}$$

and a similar expression for $S_I^{\star(0)}$. Performing now the change of variables

$$\tilde\varphi(\omega,p,x_2)=\psi\left(\omega,p,x_2-\frac{\theta p}{2}\right) \tag{11}$$

(which yields no Jacobian in the path integral), and taking into account the invariance of the free kernel under translations in $x_2$, one sees that the action becomes:

$$S=\int\frac{d\omega}{2\pi}\int\frac{dp}{2\pi}\int dx_2\,\psi^*(\omega,p,x_2)\times\left\{\left(\omega^2+p^2-\partial_2^2+m^2\right)+\lambda\left[\delta(x_2-L)+\delta(x_2)\right]\right\}\times\psi(\omega,p,x_2). \tag{12}$$

Note that $\theta$ has disappeared from the action, and indeed, this expression coincides with the one we would have obtained in the commutative case. This means that the vacuum energy $E_0$ for the noncommutative model is identical to the commutative one, regardless of the value of $\lambda$. In particular, for the Dirichlet case ($\lambda\to\infty$), we conclude that the Casimir force in the NCQFT agrees, for this geometry, with the CQFT one.

This property could seem to be surprising at first, but then one should realize that it is a consequence of the fact that this boundary divides space into two noncompact subsets. And the noncommutative effects seem to be controlled by the ratio between the area enclosed by the boundary and the minimal area $\theta$. It should be noted that the agreement with the CQFT result is realized after performing a field redefinition that depends on $\theta$. This means that the $\langle\varphi\varphi^*\rangle$ propagator in the presence of the boundaries will not be equal to its commutative counterpart, in spite of the fact that they will produce the same result for the Casimir energy.

Let us now consider the qualitatively different case of a circular defect; to be more precise, assuming a free action as before, we now consider the NCQFT analog of a commutative interaction term:

$$S_I = \lambda \int d^3x\, \delta(r-R)\varphi^*\varphi, \tag{13}$$

in the $\lambda \to \infty$ limit. A difficulty one immediately faces is to find a natural way to introduce the noncommutative version of the $\delta(r-R)$-function. However, that is not strictly necessary: we only need to assign a meaning to the integral of $\delta(r-R)$ times a function (as it appears in the interaction term). From the defining properties of the $\delta$ distribution in the commutative case, we recall that it only depends on the values of the function on the $x_1^2 + x_2^2 = R^2$ circle. And there is a basis for the space of fields where this problem looks somewhat simpler, since it is compatible with rotation symmetry in the noncommutative plane: the so-called 'matrix basis' [11]. Here, functions that depend only on $R$ are diagonal, and one can then attribute a clear meaning to the interaction term, as one that only depends on the value of the field on an eigenspace of $x_1^2 + x_2^2$.

Using the same conventions as in [12], we shall assume the interaction term to have the form:

$$S_I^\star = \lambda \int d^3x\, f_{NN} \star \varphi^* \star f_{NN} \star \varphi, \tag{14}$$

where no sum over $N$ is meant. As it has been shown in [12], $f_{NN}$ is a radial function. And certainly it yields for the interaction term a result that only depends on the function at a radius which is determined by $N$: recalling the relation $x^2 + y^2 \leftrightarrow 2(N+\frac{1}{2})\theta$, $R \approx \sqrt{2N\theta}$, which becomes a continuous variable in the commutative limit (large $N$). For small $N$, only discrete values of $R$ are possible: as expected, there is an 'area quantization' effect and one cannot confine the field to a region that whose area is not a multiple of the minimal one. The commutative Casimir energy for this case behaves like $R^{-1}$, which in our case would correspond to $N^{-\frac{1}{2}}$.

Decomposing the field variables in the matrix base, $S_I^\star = \lambda \int_{x_0} \varphi_{NN}^*(t)\varphi_{NN}(t)$.[1] Then, the vacuum energy becomes:

$$E_0(N) = \int \frac{d\omega}{2\pi} \log\bigl(1 + \lambda \Delta_{N,N;N,N}(\omega)\bigr), \tag{15}$$

where $\Delta_{n_1,n_2,n_3,n_4}$ is the free propagator written in the matrix basis. We may obtain it by a simple redefinition from the $(1+1)$-dimensional one presented explicitly in [13], the result being:

$$\Delta_{n_1 n_2, n_3 n_4}(\omega) = \delta_{n_1+n_3, n_2+n_4} \int_0^\infty dx\, x^{n_2-n_1} e^{-x} \times \sqrt{\frac{n_1!\, n_4!}{n_2!\, n_3!}} \frac{\mathbb{L}_{n_1}^{n_2-n_1}(x)\mathbb{L}_{n_4}^{n_3-n_4}(x)}{\omega^2 + m^2 + \frac{2}{\theta}x}, \tag{16}$$

[1] Global factors are absorbed in a redefinition of the field variable because the action is exactly quadratic.

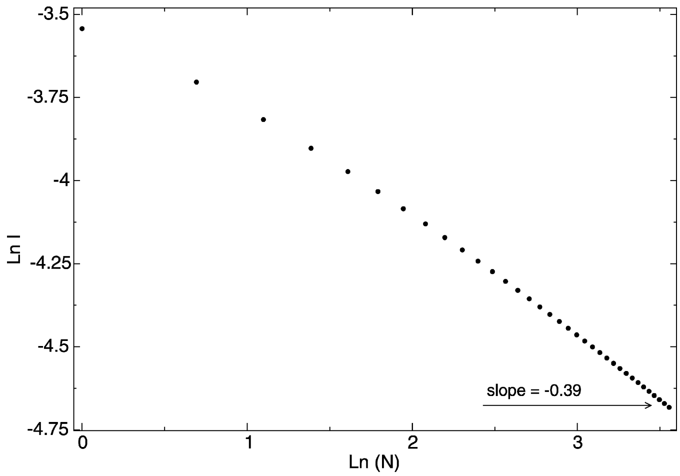


Fig. 1. Numerical evaluation of the integral in (19).

where the $\mathbb{L}_b^a$ denote associated Laguerre polynomials. In our case only part of the diagonal elements of this object appear, so that the expression for the vacuum energy becomes:

$$E_0(N) = \frac{1}{2\pi} \int_{\omega \in \mathbb{R}} \log\left(1 + \lambda \int_0^\infty dx\, e^{-x} \frac{[\mathbb{L}^N(x)]^2}{\omega^2 + m^2 + \frac{2}{\theta}x}\right), \tag{17}$$

where $\mathbb{L}^N$ is the Laguerre polynomial of order $N$.

The previous result for the vacuum energy is the starting point for our derivation of more explicit expressions, in different limits and for particular cases.

We first assume $m = 0$; thus, changing variables: $\omega \to \theta^{-1/2}\omega$, we have:

$$E_0(N) = \frac{1}{2\pi\sqrt{\theta}} \int_{\omega \in \mathbb{R}} \log\left(1 + \lambda\theta \int_0^\infty dx\, e^{-x} \frac{[\mathbb{L}^N(x)]^2}{\omega^2 + 2x}\right). \tag{18}$$

If the condition $\lambda\theta \ll 1$ is met, we have:

$$\begin{aligned} E_0(N) &\approx \frac{1}{2\pi\sqrt{\theta}} \lambda\theta \int_{\omega \in \mathbb{R}} \int_0^\infty dx\, e^{-x} \frac{[\mathbb{L}^N(x)]^2}{\omega^2 + 2x} \\ &= \frac{1}{2\pi\sqrt{\theta}} \lambda\theta\pi \int_0^\infty dx\, e^{-x} \frac{[\mathbb{L}^N(x)]^2}{\sqrt{2x}}. \end{aligned} \tag{19}$$

Of course this is a convergent integral. We performed a numerical evaluation of (19) for different values of $N$, the results of which are shown in Fig. 1. Note that close to the origin $E_0$ is well behaved; for large $N$ we should have instead an asymptotic behaviour $\sim 1/\sqrt{N}$.

We see that, up to our maximum $N$, (19) does not yet reach its asymptotic regime, which corresponds to a $-\frac{1}{2}$ slope.[2] In Fig. 2 we plot the slope ($\alpha$) of the previous graph versus $\log N$.

[2] We have defined the coefficient of the power law as the one an experimentalist would use, namely, $\alpha = \frac{\partial \log(\Delta E)}{\partial \log(N)}$.

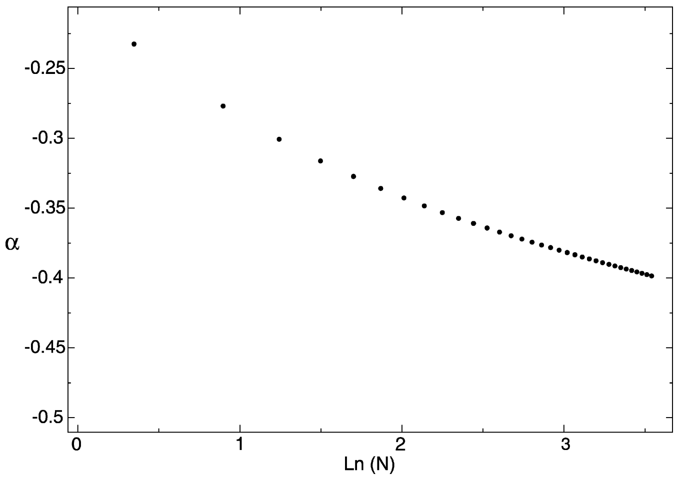


Fig. 2. Slope ($\alpha$) of Fig. 1.

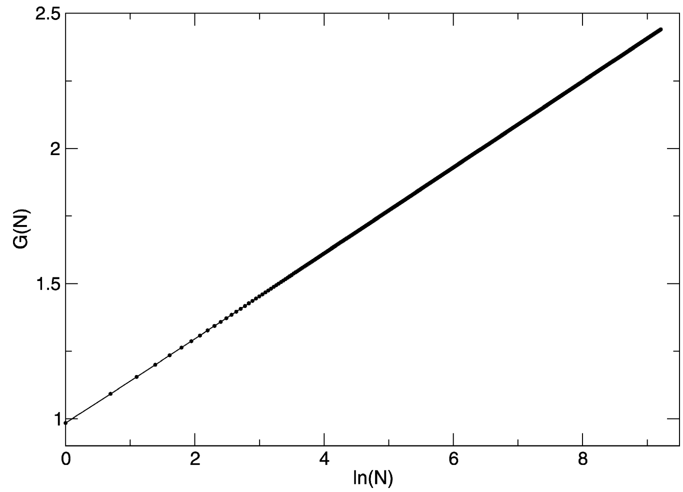


Fig. 3. Numerical evaluation of the function $G(N)$.

To see that the asymptotic power law will be such that $\alpha \to -\frac{1}{2}$, we step back to:

$$E_0(N) = \frac{1}{2\pi\sqrt{\theta}} \int_{\omega\in\mathbb{R}} \log\left(1 + \lambda\theta \int_0^\infty e^{-x} \frac{[\mathbb{L}^N(x)]^2}{\omega^2 + 2x}\, dx\right). \quad (20)$$

In the large-$N$ limit, we may use the property:

$$\lim_{n\to\infty} \mathbb{L}^n\left(\frac{z^2}{4n}\right) = J_0(z), \quad (21)$$

where $J_0$ is the Bessel $J$ function of order zero. This approximation can be used inside the integral in (20), because the pre-factor reduces the effective domain of integration. Thus, if $N \gg 1$:

$$\int_0^\infty dx\, \frac{e^{-x}[\mathbb{L}^N(x)]^2}{\omega^2 + 2x} \approx \frac{1}{2N} \int_0^\infty dz\, z \frac{e^{-\frac{z^2}{4N}}(J_0(z))^2}{\omega^2 + \frac{2z^2}{4N}}. \quad (22)$$

To proceed, we only need $\lambda\theta$ to be bounded, so that for a large enough $N$ ($\lambda\theta/N \ll 1$), we shall have:

$$E_0(N) \approx \frac{1}{2\pi\sqrt{\theta}} \int_{\omega\in\mathbb{R}} \frac{\lambda\theta}{2N} \int_0^\infty \frac{e^{-\frac{z^2}{4N}}(J_0(z))^2}{\omega^2 + \frac{z^2}{2N}} z\, dz, \quad (23)$$

or:

$$\Delta E \approx \frac{\lambda\theta}{2\sqrt{2\theta N}} \int_0^\infty \left(J_0(z)\right)^2 e^{\frac{-z^2}{4N}}\, dz. \quad (24)$$

In this manner we have managed to extract a $\frac{1}{\sqrt{2\theta N}}$ dependence, but we still have to deal with the function:

$$G(N) = \int_0^\infty \left(J_0(z)\right)^2 e^{-\frac{z^2}{4N}}\, dz. \quad (25)$$

A numerical study of this function shows that it diverges logarithmically (a plot is shown in Fig. 3), so the asymptotic power law holds, as we have claimed before.[3] We thus see the result converges to the asymptotic regime slowly, since the limit is approached logarithmically. Writing $1/N^{\frac{1}{2}+\epsilon}$, we have, for example, $\epsilon(N = 8000) \simeq -0.07$. In spite of the slow convergence, the $1/N^{\frac{1}{2}}$ power law is indeed asymptotically reached in the large $N$ regime.

We now show more explicitly that the interaction term (14) is delta-like in the commutative limit. In order to do that, consider the Fourier transform of $f_{NN}$ [12].

$$\frac{\hat{f}_{NN}(k)}{\sqrt{\theta}} = 2\pi\sqrt{\theta} e^{-\frac{\theta k^2}{4}} \mathbb{L}^{(N)}\left(\frac{2N\theta k^2}{4N}\right)$$
$$\approx 2\pi\sqrt{\theta} e^{-\frac{\theta k^2}{4}} J_0(\sqrt{2N\theta}k). \quad (26)$$

So that the inverse reads:

$$\frac{f_{NN}(r)}{\sqrt{\theta}} = \int_0^{2\pi}\int_0^\infty \frac{\sqrt{\theta}}{2\pi} e^{-\frac{\theta k^2}{4}} J_0(Rk) e^{ikr\cos\beta} k\, dk\, d\beta, \quad (27)$$

which using the integral representation of $J_0$ gives

$$\frac{f_{NN}(r)}{\sqrt{\theta}} = \int_0^\infty \sqrt{\theta} e^{-\frac{\theta k^2}{4}} J_0(Rk) J_0(kr) k\, dk. \quad (28)$$

So, in the limit $\theta \to 0$, orthogonality relation

$$\int_0^\infty J_\alpha(xv) J_\alpha(xu) x\, dx = \frac{1}{u}\delta(u - v),$$

yield to

$$\frac{f_{NN}(r)}{\sqrt{\theta}} \approx \frac{\sqrt{\theta}}{R}\delta(r - R). \quad (29)$$

On the other hand, because of the previous relation, for the second $\delta$-like factor we have the correspondence

[3] Using our definition of $\alpha$ we find: $\alpha = -\frac{1}{2} + \underbrace{\frac{B}{A + B\log(N)}}_{G(N)}$.

$$\delta(0) \leftrightarrow \frac{1}{R} A\left(\frac{R^2}{\theta}\right), \tag{30}$$

where $A(q)$ is given by

$$A(q) = \int_0^\infty \left(J_0(x)\right)^2 e^{-\frac{x^2}{4q}} x\,dx. \tag{31}$$

We have seen numerically that $A(q) \approx 1.77\sqrt{q}$, thus $\delta(0) \approx \frac{1}{\sqrt{\theta}}$. We the see that the interaction term is

$$\frac{\sqrt{\theta}}{R^2}\lambda\theta \int \delta_R \varphi^* \varphi, \tag{32}$$

that, using the assumptions

$$\begin{cases} \lambda\theta \ll N, \\ R^2 \approx 2N\theta, \end{cases}$$

the asymptotic form of the noncommutative interaction term could be rewritten as

$$S_I^\star \sim g(\theta)\xi \int d^3x\, \delta_R \varphi^* \varphi,$$

where $g(\theta) \equiv \frac{1}{2\sqrt{\theta}}$ is a large constant with dimension of mass, while $\xi \equiv \frac{\lambda\theta}{N}$ is a small (and can be assumed to be fixed) dimensionless constant. This produces then the 'hard' $\delta$-like form in the asymptotic regime, as claimed at the beginning.

We have seen that, since the defect encloses a bounded region, the vacuum energy shift is seriously modified with respect to the commutative case. In particular, close to zero size the energy is finite, what can be shown without resorting to any approximation.

Noncommutativity effects on the energy extend to large distances, as it was shown the correction to commutative exponent for the power law $\epsilon$ goes to zero as $1/\log(N)$.

We have also studied the finite-mass case, where we found that the commutative power law is reached at shorter distances. The asymptotic behavior was studied numerically from expression:

$$\Delta E = \frac{\lambda\theta}{4N\sqrt{\theta}} \int_0^\infty \frac{z\,dz}{\sqrt{\mu^2 + \frac{2z^2}{4N}}} e^{-\frac{z^2}{4N}} \left(J_0(z)\right)^2, \tag{33}$$

where $\mu^2 = \theta m^2$, which was deduced from (17).

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