Hexavalent half-arc-transitive graphs of order $4p$

Xiuyun Wang, Yan-Quan Feng
Department of Mathematics, Beijing Jiaotong University, Beijing 100044, PR China

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A B S T R A C T
A graph is half-arc-transitive if its automorphism group acts transitively on its vertex set and edge set, but not arc set. It was shown by [Y.-Q. Feng, K.S. Wang, C.X. Zhou, Tetravalent half-arc-transitive graphs of order $4p$, European J. Combin. 28 (2007) 726–733] that all tetravalent half-arc-transitive graphs of order $4p$ for a prime $p$ are non-Cayley and such graphs exist if and only if $p - 1$ is divisible by 8. In this paper, it is proved that each hexavalent half-arc-transitive graph of order $4p$ is a Cayley graph and such a graph exists if and only if $p - 1$ is divisible by 12, which is unique for a given order. This result contributes to the classification of half-arc-transitive graphs of order $4p$ of general valencies.

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1. Introduction

Throughout this paper graphs are assumed to be finite, simple and undirected, but with an implicit orientation of the edges when appropriate. For a graph $X$, let $V(X)$, $E(X)$, $A(X)$ and $\text{Aut}(X)$ be the vertex set, the edge set, the arc set and the automorphism group of $X$, respectively. Let $D_{2n}$ be the dihedral group of order $2n$, and $\mathbb{Z}_n$ the cyclic group of order $n$ as well as the ring of integers modulo $n$. Denote by $\mathbb{Z}_n^*$ the multiplicative group of $\mathbb{Z}_n$ consisting of numbers coprime to $n$, and for a prime $p$, denote by $\mathbb{Z}_p^m$ the elementary abelian group $\mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ ($m$ times). For a finite group $G$ and a subset $S$ of $G$ such that $1 \not\in S$ and $S = S^{-1}$, the Cayley graph $\text{Cay}(G, S)$ on $G$ with respect to $S$ is defined to have vertex set $G$ and edge set $\{(g, sg) \mid g \in G, s \in S\}$. A graph $X$ is isomorphic to a Cayley graph on $G$ if and only if its automorphism group $\text{Aut}(X)$ has a subgroup isomorphic to $G$, acting regularly on vertices (see [1, Lemma 16.3]).

A graph $X$ is said to be vertex-transitive, edge-transitive or arc-transitive if $\text{Aut}(X)$ acts transitively on $V(X)$, $E(X)$, or $A(X)$, respectively. A graph is said to be half-arc-transitive provided that it is vertex-transitive and edge-transitive, but not arc-transitive. More generally, by a half-arc-transitive action of
a subgroup $G$ of $\text{Aut}(X)$ on a graph $X$ we shall mean a vertex-transitive and edge-transitive, but not arc-transitive action of $G$ on $X$. In this case, we shall say that the graph $X$ is $G$-half-arc-transitive.

The investigation of half-arc-transitive graphs was initiated by Tutte [2] and he proved that a vertex- and edge-transitive graph with odd valency must be arc-transitive. In 1970 Bouwer [3] constructed a $2k$-valent half-arc-transitive graph for every $k \geq 2$ and later more such graphs were constructed (see [4–10]). Let $p$ be a prime. It is well known that there are no half-arc-transitive graphs of order $p$ or $p^2$ [11], and by Cheng and Oxley [12], there are no half-arc-transitive graphs of order $2p$. Alspach and Xu [4] classified half-arc-transitive graphs of order $3p$ and Wang [10] classified half-arc-transitive graphs of order $4p$ and $p^2$ [11].

### Proposition 2.3

Let $X$ be a graph. Then $X$ is half-arc-transitive if and only if $X$ is both vertex-transitive and edge-transitive, but not arc-transitive.

### Proposition 2.4

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### Proposition 2.5

Let $X$ be a graph. Then $X$ is half-arc-transitive if and only if $X$ is both vertex-transitive and edge-transitive, but not arc-transitive.

### Proposition 2.6

Let $X$ be a graph. Then $X$ is half-arc-transitive if and only if $X$ is both vertex-transitive and edge-transitive, but not arc-transitive.

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Let $X$ be a graph. Then $X$ is half-arc-transitive if and only if $X$ is both vertex-transitive and edge-transitive, but not arc-transitive.

### Proposition 2.9

Let $X$ be a graph. Then $X$ is half-arc-transitive if and only if $X$ is both vertex-transitive and edge-transitive, but not arc-transitive.

### Proposition 2.10

Let $X$ be a graph. Then $X$ is half-arc-transitive if and only if $X$ is both vertex-transitive and edge-transitive, but not arc-transitive.

### Proposition 2.11

Let $X$ be a graph. Then $X$ is half-arc-transitive if and only if $X$ is both vertex-transitive and edge-transitive, but not arc-transitive.
Let $D$ be a symmetric $(v, k, \lambda)$-design and denote by $V$ and $V'$ the sets of points and blocks of $D$, respectively. The incidence graph $B(D)$ of $D$ has vertex set $V \cup V'$ and edge set $\{xy \mid x \in V, y \in V', x \neq y\}$, and the graph $B'(D)$ is the incidence graph of the complementary design of $D$. Thus $B'(D)$ has vertex set $V \cup V'$ and edge set $\{xy \mid x \in V, y \in V', x \neq y\}$. Denote by $H(11)$ the unique symmetric $(11, 5, 2)$-design. Let $n > 2$ be an integer. The symmetric design $PG(n - 1, q)$ has as its point and blocks the points and hyperplanes, respectively, of the $(n - 1)$-dimensional projective space over $GF(q)$ with the incidence relation being determined by inclusion. The following proposition can be extracted from Theorem 2.4 and Table 1 in [12].

**Proposition 2.5.** Let $X$ be a connected edge-transitive graph of order $2p$, where $p$ is a prime. Then $X$ is symmetric. Assume $p \geq 7$. If $X$ has valency 3 then one of the following holds:

1. $X \cong G(2 \cdot 7, 3)$, the Heawood graph of order 14, and $\text{Aut}(G(2 \cdot 7, 3)) \cong \text{PGL}(2, 7)$;
2. $X \cong G(2p, 3)$, $p \geq 13$ and 3 \mid (p - 1)$, with $\text{Aut}(G(2p, 3)) \cong (\mathbb{Z}_p \times \mathbb{Z}_3) \times \mathbb{Z}_2$.

If $X$ has valency 6 then one of the following holds:

3. $X \cong B(\text{PG}(2, 5))$, $p = 31$ and $\text{Aut}(B(\text{PG}(2, 5))) \cong P^1GL(3, 5) \rtimes \mathbb{Z}_2$;
4. $X \cong B'(H(11))$, $p = 11$ and $\text{Aut}(B'(H(11))) = \text{PSL}(2, 11) \rtimes \mathbb{Z}_2$;
5. $X \cong G(2p, 6)$ and 6 \mid (p - 1)$, with $\text{Aut}(G(2 \cdot 7, 6)) \cong S_7 \rtimes \mathbb{Z}_2$ and $\text{Aut}(G(2p, 6)) \cong (\mathbb{Z}_p \times \mathbb{Z}_6) \times \mathbb{Z}_2$ for $p \geq 13$.

Let $G$ act transitively on a set $\Omega$. Then $G$ induces an action on $\Omega \times \Omega$ defined by $(x, y)^g = (x^g, y^g)$ for $(x, y) \in \Omega \times \Omega$ and $g \in G$. The orbits of $G$ on $\Omega \times \Omega$ are called **orbits** of $G$. The orbital $\Delta = \{(x, x) \mid x \in \Omega\}$ of $G$ is trivial and all other orbits of $G$ in $(\Omega \times \Omega) \setminus \Delta$ are nontrivial. Let $\Theta$ be a nontrivial orbital of $G$. The pair $(\Theta, \Theta)$ is a directed graph with vertex set $\Omega$ and directed edge set $\Theta$, called the **orbital digraph** of $G$ relative to $\Theta$. For any orbital $\Theta$ of $G$, it is easy to show that $\Theta^* = \{ (\alpha, \beta) \mid (\beta, \alpha) \in \Theta \}$ is also an orbital of $G$, called the **paired orbital** of $\Theta$, and $\Theta$ is said to be self-paired if $\Theta^* = \Theta$. Clearly, if $\Theta$ is a non-self-paired orbital then the underlying graph of $(\Theta, \Theta)$ is $G$-arc-transitive. Conversely, if $X$ is a half-arc-transitive graph then $X$ is an underlying graph of an orbital digraph $(V(X), \Theta)$ of $\text{Aut}(X)$ for some non-self-paired orbital $\Theta$. In this case, $\text{Aut}(X)$ is a subgroup of the automorphism group of the digraph $(V(X), \Theta)$. Thus, we have the following proposition.

**Proposition 2.6.** Let $X$ be a connected half-arc-transitive graphs of valency $2n$. Let $A = \text{Aut}(X)$ and let $A_u$ be the stabilizer of $u \in V(X)$ in $A$. Then each prime divisor of $|A_u|$ is a divisor of $n!$. In particular, if $X$ has valency 6 then $A_u$ is a $[2, 3]$-group.

The following proposition is due to Burnside.

**Proposition 2.7 ([25, Theorem 8.5.3]).** Let $p$ and $q$ be primes and let $m$ and $n$ be non-negative integers. Then, any group of order $p^m q^n$ is solvable.

### 3. Main result

In this section we classify hexavalent half-arc-transitive graphs of order $4p$. First we introduce some examples of half-arc-transitive graphs. Let $p$ be a prime such that $p - 1$ is divisible by 12, and let $G = \langle a, b \mid a^p = b^4 = 1, b^{-1}ab = a^r \rangle$ with $r^2 = -1 \mod p$. Note that $r$ is an element of order 4 in $\mathbb{Z}_p^*$ and the group $G$ is independent of the choice of $r$. Since $\mathbb{Z}_p^*$ is cyclic, there are exactly two elements of order 6 in $\mathbb{Z}_p^*$, say $\varepsilon$ and $\varepsilon^{-1}$. Define

$${\mathcal{C}}(4p) := \text{Cay}(G, \{b, b^{-1}, ab, (ab)^{-1}, a^\varepsilon b, (a^\varepsilon b)^{-1}\}).$$

Let $\alpha$ be the automorphism of $G$ induced by $a \mapsto a^\varepsilon$ and $b \mapsto b$. It is easy to show that $\alpha$ is an isomorphism from the Cayley graph $\text{Cay}(G, \{b, b^{-1}, ab, (ab)^{-1}, a^\varepsilon b, (a^\varepsilon b)^{-1}\})$ to the Cayley graph $\text{Cay}(G, \{b, b^{-1}, ab, (ab)^{-1}, a^{-\varepsilon} b, (a^{-\varepsilon} b)^{-1}\})$. Thus, $\mathcal{C}(4p)$ is independent of the choice of $\varepsilon$. The following is the main result of this paper.

**Theorem 3.1.** Let $p$ be a prime. Then $X$ is a hexavalent half-arc-transitive graph of order $4p$ if and only if $12 \mid (p - 1)$ and $X \cong \mathcal{C}(4p)$. 

Lemma 3.2. Let $p$ be a prime. If there is a hexavalent half-arc-transitive Cayley graph on a group $G$ of order $4p$ then $p \geq 7$ and $G \cong \langle a, b | a^p = b^4 = 1, b^{-1}ab = a' \rangle$ for some square root $r$ of $-1$ modulo $p$. Moreover, for each prime $p$ satisfying $12 | (p - 1)$, the Cayley graph $\mathcal{C}(4p)$ as defined above is half-arc-transitive.

Proof. Let $X = \text{Cay}(G, S)$ be a hexavalent half-arc-transitive Cayley graph on the group $G$ of order $4p$ with respect to $S$. Then $X$ is connected because there are no half-arc-transitive graphs of order $p$ or $2p$ (see [12]). It follows that $|S|=6, S^{-1} = S$ and $\langle S \rangle = G$. By Proposition 2.3, $G$ is non-abelian and by Proposition 2.2, $p \geq 7$. From the elementary group theory we know that up to isomorphism there are three non-abelian groups of order $4p$ for an odd prime $p \geq 7$:

- $G_1(p) = \langle a, b | a^{2p} = b^2 = 1, b^{-1}ab = a^{-1} \rangle$,
- $G_2(p) = \langle a, b | a^{2p} = 1, b^2 = a^r, b^{-1}ab = a^{-1} \rangle$,
- $G_3(p) = \langle a, b | a^{2p} = b^4 = 1, b^{-1}ab = a' \rangle$, $r^2 = -1 \pmod{p}$.

By Proposition 2.3, there is no involution in $S$, and then since $G = \langle S \rangle$, one has $G \neq G_1(p)$. Suppose $G = G_2(p)$. Then $S$ contains at least one element of order 4 and its inverse. Each element of order 4 is of the form $a^i b^j$ or $a^{-i} b^{-j}$ for an integer $i$. The automorphism of $G_2(p)$ induced by $b \mapsto b^{-1}, a \mapsto a$, maps $a^i b^j$ to $(a^i b^j)^{-1}$ for any integer $j$ and fixes $(a)$ pointwise. This is impossible by Proposition 2.3. Thus, $G = G_3(p)$.

Now let $12 | (p - 1)$ and $\mathcal{C}(4p) = \text{Cay}(G, S)$ with $G = \langle a, b | a^p = b^4 = 1, b^{-1}ab = a' \rangle$ ($r^2 = -1 \pmod{p}$) and $S = \{b, b^{-1}, ab, (ab)^{-1}, a^r b, (a^r b)^{-1}\}$, where $\varepsilon$ is an element of order 6 in $\mathbb{Z}_p^*$. To finish the proof of the lemma, we only need to show that $\mathcal{C}(4p)$ is half-arc-transitive. Note that the fact $\varepsilon$ is an element of order 6 in $\mathbb{Z}_p^*$ implies that $\varepsilon^2 = -1 = 0$ in $\mathbb{Z}_p^*$. Hence $\varepsilon \neq 1$ is invertible, so the map $a \mapsto a^r, b \mapsto ab$ induces an automorphism of $G$, say $\alpha$. Thus, $b^a = ab, (ab)^a = a'd = a^r b$ and $(a' b)^a = b$. It follows that $\alpha \in \text{Aut}(G, S)$ and hence $\mathcal{C}(4p)$ is edge-transitive. Furthermore, one may easily show that Aut($G, S$) = $\langle \alpha \rangle \cong \mathbb{Z}_3$. Let $A = \text{Aut}(\mathcal{C}(4p))$. Denote by $A_1$ the stabilizer of the vertex 1 in $A$ and by $A_1^*$ the subgroup of $A_1$ fixing every neighborhood of 1 in $\mathcal{C}(4p)$. We first claim that $A_1^* = 1$.

Depict the induced subgraph by the set of vertices having distance less than 3 and some vertices having distance 3 from 1 in $\mathcal{C}(4p)$ as Fig. 1. One may see that for any $u \in \{b, ab, a' b\}, v \in \{b^{-1}, (ab)^{-1}, (a' b)^{-1}\}$, there is exactly one 4-cycle passing through 1, $u$ and $v$. Thus $A_1^*$ fixes all 4-cycles passing through 1 pointwise. In particular, $A_1^*$ fixes $\{a^{-r} b^2, a^{-r} b^2, a^r b^2, ab^2, b^2, a^{-r} b^2, a^{-r} b^2, a^{-r} b^2, a^{-r} b^2, a^{-r} b^2\}$ pointwise. Furthermore, $A_1^*$ fixes the

![Fig. 1. An induced subgraph in the Cayley graph $\mathcal{C}(4p)$.](image)
set \{a^\epsilon, a^{(e-1)r}\} and the set \{a^e, a^{(1-e)r}\} which are neighbors of \(a^e b\) and \(ab\), respectively. One may compute that
\[
N(a^\epsilon) = \{a^{1+\epsilon}b, a^{2\epsilon}b, a^{-\epsilon}b^{-1}, a^{-\epsilon(1+\epsilon)}b^{-1}\},
\]
\[
N(a^{(e-1)r}) = \{a^{-1}b, a^{2\epsilon-1}b, a^{1-\epsilon}b^{-1}, a^{-1(1-\epsilon)}b^{-1}\},
\]
\[
N(a^e) = \{a^{2\epsilon}b, a^{1+\epsilon}b, a^{-1}b^{-1}, a^{-1-\epsilon}b^{-1}\},
\]
\[
N(a^{(1-e)r}) = \{a^{-\epsilon}b, a^{2-\epsilon}b, a^{-1-\epsilon}b^{-1}, a^{-1(1+\epsilon)}b^{-1}\}.
\]
Note that \(N(a^\epsilon) \cup N(a^{(e-1)r}) \cap \{N(a^e) \cup N(a^{(1-e)r})\} = \{a^{e+1}b\}.\) Thus, \(A^e\) fixes \(a^{e+1}b\) and the 6-cycle \((1, a^e b, a^{e+1} b, a^{e+1} b, a^e b, ab)\) pointwise. This means that \(A^e\) fixes \(a^\epsilon, a^{(e-1)r}, a^e\) and \(a^{(1-e)r}\), and hence \(A^e\) fixes every neighbor of \(a^e b\) and \(ab\). Similarly, one may show that \(A^e\) fixes every neighbor of \(b^{-1}\) and \((ab)^{-1}\). It follows that \(A^e\) fixes every vertex at distance 2 from 1. By the connectivity and the vertex-transitivity of \(C_4(4p), A^e\) fixes every vertex in \(C_4(4p),\) that is, \(A^e = 1.\)

Thus \(A^e\) is isomorphic to a subgroup of \(S_6.\) If \(5 \mid |A^e|\) then the constituent \(A^e_{N_1(1)}\) of \(A^e\) on the neighborhood \(N(1)\) of 1 in \(C_4(4p)\) is 2-transitive, implying that \(C_4(4p)\) is 2-arc-transitive, which is impossible by Fig. 1, because some 2-arcs in \(C_4(4p)\) are contained in 4-cycles and some are not. Thus, \(|A^e|\) is a divisor of 144. Since \(A^e = A^e_{N_1(1)}\), \(|A : R(G)|\) is a divisor of 144. It follows that \(|A : R(G) \times (\alpha)|\) is a divisor of 48. Let \(P = \langle R(\alpha) \rangle.\) Then \(P \trianglelefteq R(G)\) and \(P\) is a Sylow \(p\)-subgroup of \(A^e\). Since \(a^e = a^{-1}\), one has \(P^m = P,\) implying \(R(G) \times (\alpha) \leq N_4(P).\) Thus, \(|A : N_4(P)|\) divides 48. Note that \(|A : N_4(P)|\) is the number of Sylow \(p\)-subgroups of \(A^e.\) Then, \(|A : N_4(P)| = mp + 1\) for some integer \(m\) and \(mp + 1\) divides 48. Since \(p \equiv 1 \pmod{12},\) one has \(mp + 1 = 1,\) forcing \(P\) to be normal in \(A^e.\)

Consider the quotient graph \(C_4(4p) / P\) of \(C_4(4p)\) corresponding to the orbits of \(P,\) that is, the graph with the orbits of \(P\) as vertices and with two orbits being adjacent if there are edges of \(C_4(4p)\) between these two orbits. Since \(a^e \in A,\) the normality of \(P\) in \(A^e\) implies that \(C_4(4p) / P\) is a 4-cycle, say \(C_4(4p) / P = \langle B_0, B_1, B_2, B_3 \rangle.\) The induced subgraph \((B_i, B_{i+1})\) of \(B_i \cup B_{i+1}\) in \(C_4(4p) / P\) for each \(i \in \mathbb{Z}_4\) is a cubic edge-transitive graph of order 2p, which is arc-transitive by Proposition 2.5. Let \(K\) be the kernel of \(A^e\) acting on \(V(C_4(4p) / P).\) Since \(|B_i| = p,\) for each \(i \in \mathbb{Z}_4,\) \(K\) is primitive on \(B_i.\) Suppose that \(K\) is unfaithful on \(B_i.\) Then the kernel of \(K\) on \(B_i\) is transitive on \(B_{i+1}\) because \(K\) is primitive on \(B_{i+1},\) which implies that the induced subgraph \((B_i, B_{i+1})\) is isomorphic to \(K_{p, p}.\) If it follows that \(p = 3,\) contrary to the fact that \(12 \mid (p - 1).\) Thus, \(K\) acts faithfully on \(B_i,\) and by Proposition 2.5, \(|K_1| = 3\) and \(|K| = 3p.\) Since \(C_4(4p) / P\) is a 4-cycle, \(A^e / K\) is isomorphic to a subgroup of \(D_8.\) Thus, \(|A^e / K| = 4\) or 8, that is, \(|A| = 12p\) or \(24p.\) Thus, \(|A : R(G) \times (\alpha)| = 1,\) or 2, implying \(R(G) \times (\alpha) \leq A.\) Note that \(R(G)\) is characteristic in \(R(G) \times (\alpha),\) forcing \(R(G) \leq A,\) that is, \(C_4(4p)\) is a normal Cayley graph. By Proposition 2.1, \(A_1 = \Aut(G, S)\). Since \(\Aut(G, S) = (\alpha) \cong \mathbb{Z}_3, C_4(4p)\) is half-arc-transitive. □

The lexicographic product \(X[Y]\) of graphs \(X\) and \(Y\) is the graph with vertex set \(V(X[Y]) = V(X) \times V(Y)\) and with two vertices \(u = (x_1, y_1)\) and \(v = (x_2, y_2)\) adjacent whenever \(x_1\) is adjacent to \(x_2,\) or \(x_1 = x_2\) and \(y_1\) is adjacent to \(y_2.\) To finish the proof of Theorem 3.1, it suffices to prove the following lemma.

**Lemma 3.3.** Let \(p\) be a prime and \(X\) a hexavalent half-arc-transitive graph of order \(4p.\) Then \(12 \mid (p - 1)\) and \(X \cong C_4(4p).\)

**Proof.** Since there are no half-arc-transitive graphs of order \(p\) or \(2p\) (see [12]), \(X\) is connected. Let \(A = \Aut(X).\) Recall that \(X\) is an underlying graph of an orbital digraph \(D := (V(X), \Theta)\) of \(A\) for some non-self-paired orbital \(\Theta.\) Thus, \(A \leq \Aut(D)\) and \(D\) is a directed graph with out- and in-valency equal to 3. Let \(u \in V(X)\) and denote by \(A_u\) the stabilizer of \(u\) in \(A.\) By Proposition 2.6, \(A_u\) is a \(\{2, 3, p\}\)-group and hence \(A\) is a \(\{2, 3, p\}\)-group with \(|A|\) not divisible by \(p^2.\) The edge-transitivity of \(X\) implies that \(12p \mid |A|\). By Proposition 2.2, \(p \geq 7.\) Let \(N\) be a minimal normal subgroup and \(P\) a Sylow \(p\)-subgroup of \(A.\) Then \(|P| = p.\) We first prove the following claim.

**Claim:** \(P \leq A.\)

Suppose that all minimal normal subgroups of \(A\) are nonsolvable. Then \(N \cong T^m\) where \(T\) is a non-abelian simple \(\{2, 3, p\}\)-group. Since \(p^2 \mid |A|\) and \(p \geq 7,\) we have, by [26, pp. 12–14], that \(m = 1\) and \(N = T\) is one of five groups given in Table 1. By the simplicity of \(N,\) \(N\) has orbits of length \(p, 2p\) or \(4p,\)
and hence $|N : N_u|=p, 2p, 4p$, where $N_u$ is the stabilizer of $u$ in $N$. Since $N_u$ is contained in a maximal subgroup of $N$, Table 1 and Proposition 2.4 combined together imply that $N \cong L_2(7)$ or $L_3(3)$. Suppose first that $N \cong L_2(7)$. Then $|V(X)|=28$, and since, by assumption, $A$ has no solvable minimal normal subgroup, we have that $C_A(N)=1$. Consequently $A \leq \text{Aut}(N)$, which implies that $|A| \leq |N||\text{Out}(N)|$.

By the ATLAS [27], $|\text{Out}(N)| = 2$ and thus either $|A| = |N|$ or $|A| = 2|N|$. It follows that $N$ has at most two orbits on $V(X)$. If $N$ is transitive on $V(X)$ then $|N_u|=6$, and $X$ is the underlying graph of an orbital digraph of $N$ for some non-self-paired orbital. However, with the use of program software MAGMA [28] one can see that this is impossible. If $N$ has two orbits on $X$ then the fact that $A \leq \text{Aut}(N)$ implies that $A = \text{PGL}(2,7)$ and thus $|A_u|=12$. However, by the ATLAS [27], $A_u$ is a maximal subgroup of $A$, contradicting Proposition 2.4. The case $N \cong L_3(3)$ is excluded in a similar manner; details are left to the reader.

We have proved that $A$ has at least one solvable minimal normal subgroup, say $N$. This $N$ is elementary abelian. Since $|V(X)| = 4p$, $N$ cannot be a 3-group. If $N$ is a $p$-group then $N$ is a normal Sylow $p$-subgroup of $A$ and the claim is true. In what follows we assume that $N$ is an elementary abelian 2-group, say $N \cong \mathbb{Z}_2^r$ for some integer $r$. Let $K$ be the kernel of $A$ acting on the quotient graph $X_u$ of $X$ corresponding to the orbits of $N$. Clearly, $N \leq K$ and since $|V(X)| = 4p$, orbits of $N$ on $V(X)$ are of length 2 or 4.

Suppose the orbits of $N$ are of length 4. Now, furthermore, $|X_u|=p$ and $X_u$ has valency 2 or 6. If $X_u$ has valency 2 then $X_u=C_p$, say $X_u = (B_0, B_1, \ldots, B_{p-1})$ with $B_i$ and $B_{i+1}$ adjacent for each $i \in \mathbb{Z}_p$. The induced subgraph $(B_i, B_{i+1})$ of $B_i \cup B_{i+1}$ in $X$ is an edge-transitive cubic graph of order 8, and therefore it is isomorphic to the three-dimensional hypercube $Q_3$. Note that $\text{Out}(Q_3)=S_4 \times \mathbb{Z}_2$. Then $K$ is faithful on each $B_i$ and $K \leq S_4$, forcing $N \cong \mathbb{Z}_2^2$. If $X_u$ has valency 6, the stabilizer $K_u$ of $u$ in $K$ fixes each neighborhood of $u$ in $X$ because $K$ fixes every orbit of $N$. It follows that $K_u = 1$ and $N \cong \mathbb{Z}_2^2$. Thus, $NP$ is a regular subgroup of $A$, that is $X$ is a Cayley graph on $NP$. By Lemma 3.2, $NP$ has a cyclic Sylow 2-subgroup, a contradiction.

Hence the orbits of $N$ are of length 2. Then, $|X_u|=2p$ and $X_u$ has valency 3 or 6. If $X_u$ has valency 3 then $X \cong X_u(2K_3)$. Note that $X_u$ is edge-transitive because $X$ is edge-transitive. By Proposition 2.5, $X_u$ is arc-transitive and hence $X \cong X_u(2K_3)$ is arc-transitive, a contradiction. Thus, $X_u$ has valency 6. In this case, $K_u = 1$ and $K \cong \mathbb{Z}_2$. And one may view $A/N$ as a group of automorphisms of $X_u$, that is, $A/N \cong \text{Aut}(X_u)$. Set $H = A/N$. Then $X_u$ is $H$-half-arc-transitive. Note that $6p \upharpoonright |H|$. By Proposition 2.5, $X_u$ is isomorphic to $B(PG(2,5))$, B(H(11)) or $G(2p,6)$ with $6 \mid (p-1)$. Suppose $X_u \cong B(PG(2,5))$. Then $H$ is a $\{2,3,31\}$-subgroup of $\text{Aut}(B(PG(2,5))) = PGL(3,5) \ltimes \mathbb{Z}_2$ and $186 \mid |H|$, implying that $H \cap PGL(3,5)$ is a proper subgroup of $PGL(3,5)$ with order divisible by 93. By MAGMA [28], $|H \cap PGL(3,5)| \geq 93$. Thus, $|H|=186$ and $|A|=3 \cdot 4 \cdot 31$. Note that $A$ has normal Sylow 31-subgroups and so $A$ is solvable. Then, $X$ is a Cayley graph on a Hall $\{2,31\}$-subgroup of $A$. By Lemma 3.2, $4 \upharpoonright (31-1)$, a contradiction. Suppose $X_u \cong B(H(11))$. Then $H$ is a $\{2,3,11\}$-subgroup of $\text{Aut}(B(H(11))) \cong \text{PSL}(2,11) \rtimes \mathbb{Z}_2$ and $33 \mid |\text{PSL}(2,11)|$, which is impossible because $\text{PSL}(2,11)$ has no proper subgroup with order divisible by 33. Thus, $X_u \cong G(2p,6)$ with $6 \mid (p-1)$. First assume $p \geq 11$. By Proposition 2.5, $\text{Aut}(G(2p,6))$ has a normal Sylow $p$-subgroup, implying that $PN/N \leq A/N$. It follows that $PN \not\leq A$. Clearly, $P$ is characteristic in $PN$ and hence $P \not\leq A$. The claim holds. Now assume $p < 11$. Then $X_u \cong G(2,7,6)$. In this case, $X_u$ has vertex set $V \cup V$ with $V = \{i \mid i \in \mathbb{Z}_7\}$ and $V = \{i \mid i \in \mathbb{Z}_7\}$, and edge set $\{ij \mid i, j \in \mathbb{Z}_7, i \neq j\}$. Recall that $X_u$ is $H$-half-arc-transitive. Let $H^*$ be the subgroup of $H$ fixing $V$ and $V^*$ setwise. Then $|H \cap H^*| = 2$. Clearly, for each $i \in \mathbb{Z}_7$, one has $H_i \leq H^*$, $H_i = H_{i'}$ and $H^*$ acts faithfully on $V$, where $H_i$ and $H_{i'}$ are the stabilizers of $i$ and $i'$ in $H$, respectively. By half-arc-transitivity of $H$, $H_{0}$ has two orbits of length 3 on $V \setminus \{0\}$, say $O_1$ and $O_2$. If $H_0(=H_0^*)$ is faithful on $O_1$, then $H_0(=H_0^*)$ is faithful on $O_1$. 

### Table 1
Non-abelian simple $\{2,3,p\}$-groups ($p \geq 7$), extracted from [27,26].

<table>
<thead>
<tr>
<th>Group</th>
<th>$p$</th>
<th>Order</th>
<th>Indices of maximal subgroups</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_2(7)$</td>
<td>7</td>
<td>$2^3 \cdot 3 \cdot p$</td>
<td>8, $p$</td>
</tr>
<tr>
<td>$L_2(8)$</td>
<td>7</td>
<td>$2^3 \cdot 3^2 \cdot p$</td>
<td>9, 36, 4p</td>
</tr>
<tr>
<td>$L_2(17)$</td>
<td>17</td>
<td>$2^3 \cdot 3^2 \cdot p$</td>
<td>18, 6p, 8p, 9p</td>
</tr>
<tr>
<td>$L_3(3)$</td>
<td>13</td>
<td>$2^3 \cdot 3^3 \cdot p$</td>
<td>144, p, 18p</td>
</tr>
<tr>
<td>$U_3(3)$</td>
<td>7</td>
<td>$2^3 \cdot 3^3 \cdot p$</td>
<td>36, 4p, 9p</td>
</tr>
</tbody>
</table>
then as a permutation group of degree 7 on \( V \), \( H^* \) contains a transposition or a 3-cycle. Note that \( H^* \) is primitive on \( V \) and \( V' \). By [29, Theorem 13.3], \( H^* \) is 2-transitive on \( V \) and \( V' \). This implies that \( H_0 = H_{0'} \) is transitive on \( V \setminus \{0\} \), implying that \( H \) is arc-transitive on \( X_0 \), a contradiction. Thus, \( H_0 \) is faithful on \( O_1 \), implying that \( H_0 \leq S_3 \). It follows that \( |H^*| \leq 42 \) and by Sylow theorem, \( H^* \) has a characteristic Sylow 7-subgroup, implying that \( H \) has a normal Sylow 7-subgroup, that is, \( PN/N \leq A/N \). Thus, \( PN \leq A \) and since \( P \) is characteristic in \( PN \), one has \( P \leq A \). This completes the proof of the claim.

Now consider the quotient graph \( X\rho \) of \( X \) corresponding to the orbits of \( P \). Then \( |V(X\rho)| = 4 \) and \( X\rho \) has valency 2 or 3. Let \( K \) be the kernel of \( A \) acting on \( V(X\rho) \).

If \( X\rho \) has valency 3, then \( X\rho \) is the complete graph \( K_3 \). In this case, \( K_3 \) fixes every out-neighbor of \( u \) in the directed graph \( D \), which implies \( K_3 \) is isomorphic to \( Z_3 \) and \( A/P \leq S_4 \). Since \( 12p \mid |A| \), one has \( A/P \cong A_4 \) or \( S_4 \). Let \( R \) be a Sylow 2-subgroup of \( A_4 \). Then \( R \cong Z_2 \times Z_2 \) and \( RP \) acts regularly on \( V(X) \). This means that \( X \) is a Cayley graph on \( RP \). By Lemma 3.2, \( RP \) has a cyclic Sylow 2-subgroup, a contradiction.

Thus, \( X\rho \) has valency 2, that is, \( X\rho \) is a 4-cycle, say \( X\rho = (B_0, B_1, B_2, B_3) \) with \( B_i \) and \( B_{i+1} \) adjacent for each \( i \in \mathbb{Z}_4 \). The induced subgraph \( T = (B_i, B_{i+1}) \) of \( B_i \cup B_{i+1} \) in \( X \) is an edge-transitive cubic graph of order \( 2p \), and by half-arc-transitivity of \( X \), all edges in \( T \) have the same direction either from \( B_i \) to \( B_{i+1} \) or from \( B_{i+1} \) to \( B_i \) in the directed graph \( D \). Thus, \( 3p \mid |K| \) and \( A/K \cong \mathbb{Z}_4 \). By Proposition 2.5, \( T \) is the Heawood graph of order 14 for \( p = 7 \) or the Cayley graph \( G(2p, 3) \) for a prime \( p \geq 13 \) such that \( p - 1 \) is a multiple of 3. If \( T \) is the Heawood graph then \( \text{Aut}(T) \cong \text{PGL}(2, 7) \) and hence \( K \leq \text{PSL}(2, 7) \). Note that \( P \leq A \) implies that \( A \) is solvable. Thus, \( K \) is a proper subgroup of \( \text{PSL}(2, 7) \) and since \( 21 \mid |K| \), \( K \) must be a maximal subgroup of \( \text{PSL}(2, 7) \) isomorphic to \( Z_2 \times Z_2 \), implying \( |K| = 21 \) and \( |A| = 84 \). In this case, \( T \) is the Cayley graph \( G(2p, 3) \) for a prime \( p \geq 13 \) such that \( p - 1 \) is a multiple of 3, then by Proposition 2.5, \( |\text{Aut}(T)| = 6p \) and hence \( |K| \leq 3p \). It follows that \( |A| = 12p \) for \( p \geq 7 \). Thus, \( A_3 \) is a Sylow 3-subgroup of \( A \). Let \( R \) be a Sylow 2-subgroup of \( A \). Then \( RP \) is a regular subgroup of \( A \). By Lemma 3.2, one may let \( X = \text{Cay}(G, S) \), where \( G = \langle a, b \mid a^6 = b^3 = 1, b^{-1}ab = a' \rangle \) (\( r^2 = -1 \) (mod \( p \)) and \( |S| = 6 \). Note that \( r^2 = -1 \) (mod \( p \)). By the normality of \( P \) in \( A, P \leq R(G) \). Set \( C = C_4(P) \). Since there is no involution in \( G \) which commutes with \( a, C \) is a \( \{3, p\} \)-group. Suppose \( 3 \mid |C| \). Then \( A_3 \leq C \). And since \( |A| = 12p, A_3 \) is normal in \( C \) and hence characteristic in \( C \). Since \( C \leq A \), one has \( A_3 \leq C \), a contradiction. Thus, \( P \). Since \( A/C = A/P \leq \text{Aut}(P) \cong \mathbb{Z}_p \) for \( p \mid |A/P| \), one has \( R(G)/P \leq A/P \), implying \( R(G) \leq A \). Thus \( X = \text{Cay}(G, S) \) is a normal Cayley graph. By Proposition 2.1, \( A = R(G) \times \text{Aut}(G, S) \) and since \( |A| = 12p \), there is an element \( \beta \) of order \( 3 \) in \( \text{Aut}(G, S) \).

Each element of order 4 in \( G \) is of the form \( a'b \) or \( a'b^{-1} \) for some integer \( i \). Note that the connectivity of \( X \) implies \( \text{Aut}(S) = G \). This means that \( S \) contains at least two elements of order 4 and since \( \text{Aut}(G) \) is transitive on the set \( \{a'b \mid 0 \leq i \leq p - 1\} \), one may assume \( b \in S \). Suppose that \( S \) contains exactly two elements of order 4. By Proposition 2.3, \( S \) contains no involutions and hence \( S = \{a, a^{-1}, a', a'^{-1}, b, b^{-1}\} \) for some integers \( i,j \). However, the automorphism of \( G \) induced by \( a \mapsto a^{-1}, b \mapsto b \) fixes \( S \) and maps \( a' \) to \( a'^{-1} \), contrary to Proposition 2.3. Suppose that \( S \) contains exactly four elements of order 4. Since for each \( k \in \mathbb{Z}_p \) the map \( a \mapsto a^k, b \mapsto b \) induces an automorphism of \( G \), one may assume \( S = \{a', a^{-1}, b, b^{-1}, ab, (ab)^{-1} \} \). In this case, the automorphism of \( G \) induced by \( a \mapsto a^{-1}, b \mapsto ab \) fixes \( S \) and maps \( a' \) to \( a'^{-1} \), contrary to Proposition 2.3. It follows that \( S \) consists of elements of order 4. Since one may assume \( S = \{b, b^{-1}, ab, (ab)^{-1}, a'b, (a'b)^{-1}\} \) for some \( i \in \mathbb{Z}_p \), clearly, \( i \neq 0, 1 \). Note that there is no automorphism of \( G \) mapping \( b \) to \( b^{-1} \). Hence, \( \langle \beta \rangle \) has two orbits on \( S \), that is, \( \{b, ab, a'b\} \) and \( \{b^{-1}, (ab)^{-1}, (a'b)^{-1}\} \). Furthermore, one may assume that \( \beta \) permutes \( b, ab \) and \( a'b \) cyclically. Thus, \( \eta b = ab \) and \( \beta a = (ab)^{-1} \beta = a'b(ab)^{-1} = a'^{-1} \). And \( b = (ab)^{-1} = a'^{-1}ab \), implying \( p^2 = i + 1 = 0 \) (mod \( p \)) in \( \mathbb{Z}_p \). In this case, \( p^2 = 1 \) (mod \( p \)) and \( l \neq -1 \), forcing that \( i \) is an element of order 6 in \( \mathbb{Z}_p^* \). Thus, \( 6 \mid (p - 1) \). Since \( 4 \mid (p - 1) \), one has \( 12 \mid (p - 1) \) and \( X \cong C_4(4p) \).

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References