Hexavalent half-arc-transitive graphs of order $4p$

Xiuyun Wang, Yan-Quan Feng

Department of Mathematics, Beijing Jiaotong University, Beijing 100044, PR China

**ARTICLE INFO**

Article history:
Received 5 July 2008
Accepted 20 November 2008
Available online 13 January 2009

**ABSTRACT**

A graph is half-arc-transitive if its automorphism group acts transitively on its vertex set and edge set, but not arc set. It was shown by [Y.-Q. Feng, K.S. Wang, C.X. Zhou, Tetravalent half-arc-transitive graphs of order $4p$, European J. Combin. 28 (2007) 726–733] that all tetravalent half-arc-transitive graphs of order $4p$ for a prime $p$ are non-Cayley and such graphs exist if and only if $p−1$ is divisible by 8. In this paper, it is proved that each hexavalent half-arc-transitive graph of order $4p$ is a Cayley graph and such a graph exists if and only if $p−1$ is divisible by 12, which is unique for a given order. This result contributes to the classification of half-arc-transitive graphs of order $4p$ of general valencies.

© 2008 Elsevier Ltd. All rights reserved.

1. Introduction

Throughout this paper graphs are assumed to be finite, simple and undirected, but with an implicit orientation of the edges when appropriate. For a graph $X$, let $V(X)$, $E(X)$, $A(X)$ and $\text{Aut}(X)$ be the vertex set, the edge set, the arc set and the automorphism group of $X$, respectively. Let $D_{2n}$ be the dihedral group of order $2n$, and $\mathbb{Z}_n$ the cyclic group of order $n$ as well as the ring of integers modulo $n$. Denote by $\mathbb{Z}_n^*$ the multiplicative group of $\mathbb{Z}_n$ consisting of numbers coprime to $n$, and for a prime $p$, denote by $\mathbb{Z}_p^m$ the elementary abelian group $\mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ ($m$ times). For a finite group $G$ and a subset $S$ of $G$ such that $1 \not\in S$ and $S = S^{-1}$, the Cayley graph $\text{Cay}(G, S)$ on $G$ with respect to $S$ is defined to have vertex set $G$ and edge set $\{\{g, sg\} | g \in G, s \in S\}$. A graph $X$ is isomorphic to a Cayley graph on $G$ if and only if its automorphism group $\text{Aut}(X)$ has a subgroup isomorphic to $G$, acting regularly on vertices (see [1, Lemma 16.3]).

A graph $X$ is said to be vertex-transitive, edge-transitive or arc-transitive if $\text{Aut}(X)$ acts transitively on $V(X)$, $E(X)$, or $A(X)$, respectively. A graph is said to be half-arc-transitive provided that it is vertex-transitive and edge-transitive, but not arc-transitive. More generally, by a half-arc-transitive action of

**E-mail address:** yqfeng@bjtu.edu.cn (Y.-Q. Feng).

0195–6698/$ – see front matter © 2008 Elsevier Ltd. All rights reserved.
There are no half-arc-transitive graphs with fewer than 24 vertices.

Proposition 2.3.

Proposition 2.1

2. Preliminary results

Let $\text{Cay}(G, S)$ be a Cayley graph. Given $g \in G$, define the permutation $R(g)$ on $G$ by $x \mapsto xg$, $x \in G$. Then $R(G) = \{R(g) \mid g \in G\}$ is a permutation group isomorphic to $G$, called the right regular representation of $G$. The Cayley graph $\text{Cay}(G, S)$ is vertex-transitive because it admits $R(G)$ as a regular subgroup of the automorphism group $\text{Aut}($\text{Cay}(G, S)$)$. Furthermore, the group $\text{Aut}(G, S) = \{\alpha \in \text{Aut}(G) \mid S^\alpha = S\}$ is also a subgroup of $\text{Aut}($\text{Cay}(G, S)$)$. Actually, $\text{Aut}(G, S)$ is a subgroup of $\text{Aut}($\text{Cay}(G, S)$)_1$, the stabilizer of the vertex 1 in $\text{Aut}($\text{Cay}(G, S)$)$. A Cayley graph $\text{Cay}(G, S)$ is said to be normal if $\text{Aut}($\text{Cay}(G, S)$)$ contains $R(G)$ as a normal subgroup. The following proposition is fundamental for normal Cayley graphs.

**Proposition 2.1 ([22, Proposition 1.5]).** Let $X = \text{Cay}(G, S)$ be a Cayley graph on a finite group $G$ with respect to $S$. Let $A = \text{Aut}(X)$ and let $A_1$ be the stabilizer of 1 in $A$. Then $X$ is normal if and only if $A_1 = \text{Aut}(G, S)$.

Now we state a simple observation about half-arc-transitive graphs (see [13]).

**Proposition 2.2.** There are no half-arc-transitive graphs with fewer than 27 vertices.

The following proposition is straightforward (see [24, Propositions 2.1 and 2.2]).

**Proposition 2.3.** Let $X = \text{Cay}(G, S)$ be a half-arc-transitive graph. Then, there is no involution in $S$, and no $\alpha \in \text{Aut}(G, S)$ such that $s^\alpha = s^{-1}$ for some $s \in S$. In particular, there are no half-arc-transitive Cayley graphs on abelian groups.

Li et al. [7] considered primitive half-arc-transitive graphs.

**Proposition 2.4 ([7, Theorem 1.4]).** There are no vertex-primitive half-arc-transitive graphs of valency less than 10.

To state the classification of connected cubic and hexavalent symmetric graphs of order $2p$, $p$ a prime, due to Cheng and Oxley [12], we need to define the following graphs. Let $V$ and $V'$ be two disjoint copies of $\mathbb{Z}_p$, say $V = \{i \mid i \in \mathbb{Z}_p\}$ and $V' = \{i' \mid i \in \mathbb{Z}_p\}$. Let $r$ be a positive integer dividing $p - 1$ and $H(p, r)$ the unique subgroup of $\mathbb{Z}_p$ of order $r$. Define the graph $G(2p, r)$ to have vertex set $V \cup V'$ and edge set $\{xy \mid x, y \in \mathbb{Z}_p, y - x \in H(p, r)\}$. Note that $G(2p, p - 1) \cong K_{p, p}$. Let $\tau$ and $\rho$ be the maps defined as following: $i' = i + 1$ and $i^\tau = i' + 1$, and $i^\rho = -(i')$ and $i^\rho = -i$. Then, the graph $G(2p, r)$ is a Cayley graph since $(\tau, \rho)$ is regular on $V(G(2p, r))$. 
Let $D$ be a symmetric $(v, k, \lambda)$-design and denote by $V$ and $V'$ the sets of points and blocks of $D$, respectively. The incidence graph $B(D)$ of $D$ has vertex set $V \cup V'$ and edge set $\{xy \mid x \in V, y \in V', x \in y\}$, and the graph $B'(D)$ is the incidence graph of the complementary design of $D$. Thus $B'(D)$ has vertex set $V \cup V'$ and edge set $\{xy \mid x \in V, y \in V', x \notin y\}$. Denote by $H(11)$ the unique symmetric $(11, 5, 2)$-design. Let $n > 2$ be an integer. The symmetric design $PG(n - 1, q)$ has as its point and blocks the points and hyperplanes, respectively, of the $(n - 1)$-dimensional projective space over $GF(q)$ with the incidence relation being determined by inclusion. The following proposition can be extracted from Theorem 2.4 and Table 1 in [12].

**Proposition 2.5.** Let $X$ be a connected edge-transitive graph of order $2p$, where $p$ is a prime. Then $X$ is symmetric. Assume $p \geq 7$. If $X$ has valency 3 then one of the following holds:

1. $X \cong G(2 \cdot 7, 3)$, the Heawood graph of order 14, and $Aut(G(2 \cdot 7, 3)) = PGL(2, 7)$;
2. $X \cong G(2p, 3)$, $p \geq 13$ and 3 | $(p - 1)$, with $Aut(G(2p, 3)) \cong (\mathbb{Z}_p \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$.

If $X$ has valency 6 then one of the following holds:

3. $X \cong B(PG(2, 5))$, $p = 31$ and $Aut(B(PG(2, 5))) = PGL(3, 5) \rtimes \mathbb{Z}_2$;
4. $X \cong B'(H(11))$, $p = 11$ and $Aut(B'(H(11))) = PSL(2, 11) \rtimes \mathbb{Z}_2$;
5. $X \cong G(2p, 6)$ and 6 | $(p - 1)$, with $Aut(G(2p, 6)) \cong (\mathbb{Z}_2 \rtimes (\mathbb{Z}_3 \rtimes \mathbb{Z}_2)$ and $Aut(G(2p, 6)) \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_6) \rtimes \mathbb{Z}_2$ for $p \geq 3$.

Let $G$ act transitively on a set $\Omega$. Then $G$ induces an action on $\Omega \times \Omega$ defined by $(x, y)^g = (x^g, y^g)$ for $(x, y) \in \Omega \times \Omega$ and $g \in G$. The orbits of $G$ on $\Omega \times \Omega$ are called orbitals of $G$. The orbital $\Delta = \{(x, x) \mid x \in \Omega\}$ of $G$ is trivial and all other orbitals of $G$ in $(\Omega \times \Omega) \setminus \Delta$ are nontrivial. Let $\Theta$ be a nontrivial orbital of $G$. The pair $(\Theta, \Theta)$ is a directed graph with vertex set $\Theta$ and directed edge set $\Theta_e$, called the orbital digraph of $G$ relative to $\Theta$. For any orbital $\Theta$ of $G$, it is easy to show that $\Theta^* = \{(\alpha, \beta) \mid (\beta, \alpha) \in \Theta\}$ is also an orbital of $G$, called the paired orbital of $\Theta$, and $\Theta$ is said to be self-paired if $\Theta^* = \Theta$. Clearly, if $\Theta$ is a non-self-paired orbital then the underlying graph of $(\Theta, \Theta)$ is $G$-half-arc-transitive. Conversely, if $X$ is a half-arc-transitive graph then $X$ is an underlying graph of an orbital digraph $(V(X), \Theta)$ of $Aut(X)$ for some non-self-paired orbital $\Theta$. In this case, $Aut(X)$ is a subgroup of the automorphism group of the digraph $(V(X), \Theta)$. Thus, we have the following proposition.

**Proposition 2.6.** Let $X$ be a connected half-arc-transitive graphs of valency $2n$. Let $A = Aut(X)$ and let $A_u$ be the stabilizer of $u \in V(X)$ in $A$. Then each prime divisor of $|A_u|$ is a divisor of $n!$. In particular, if $X$ has valency 6 then $A_u$ is a $[2, 3]$-group.

The following proposition is due to Burnside.

**Proposition 2.7** ([25, Theorem 8.5.3]). Let $p$ and $q$ be primes and let $m$ and $n$ be non-negative integers. Then, any group of order $p^mq^n$ is solvable.

3. Main result

In this section we classify hexavalent half-arc-transitive graphs of order $4p$. First we introduce some examples of half-arc-transitive graphs. Let $p$ be a prime such that $p - 1$ is divisible by 12, and let $G = \langle a, b \mid a^p = b^4 = 1, b^{-1}ab = a^r \rangle$ with $r^2 = -1 \pmod p$. Note that $r$ is an element of order 4 in $\mathbb{Z}_p^*$ and the group $G$ is independent of the choice of $r$. Since $\mathbb{Z}_p^*$ is cyclic, there are exactly two elements of order 6 in $\mathbb{Z}_p^*$, say $\varepsilon$ and $\varepsilon^{-1}$. Define

$C(4p) := Cay(G, \{b, b^{-1}, ab, (ab)^{-1}, a^\varepsilon b, (a^\varepsilon b)^{-1}\}).$

Let $\alpha$ be the automorphism of $G$ induced by $a \mapsto a^\varepsilon^{-1}$ and $b \mapsto b$. It is easy to show that $\alpha$ is an isomorphism from the Cayley graph $Cay(G, \{b, b^{-1}, ab, (ab)^{-1}, a^\varepsilon b, (a^\varepsilon b)^{-1}\})$ to the Cayley graph $Cay(G, \{b, b^{-1}, ab, (ab)^{-1}, a^{-\varepsilon} b, (a^{-\varepsilon} b)^{-1}\})$. Thus, $C(4p)$ is independent of the choice of $\varepsilon$. The following is the main result of this paper.

**Theorem 3.1.** Let $p$ be a prime. Then $X$ is a hexavalent half-arc-transitive graph of order $4p$ if and only if $12 \mid (p - 1)$ and $X \cong C(4p)$.
The sufficiency is proved as a part of Lemma 3.2 by showing that $C(4p)$ is half-arc-transitive, and the necessity is proved in Lemma 3.3.

**Lemma 3.2.** Let $p$ be a prime. If there is a hexavalent half-arc-transitive Cayley graph on a group $G$ of order $4p$ then $p \geq 7$ and $G \cong \langle a, b | a^p = b^4 = 1, b^{-1}ab = a' \rangle$ for some square root $r$ of $-1$ modulo $p$. Moreover, for each prime $p$ satisfying $12 \mid (p - 1)$, the Cayley graph $C(4p)$ as defined above is half-arc-transitive.

**Proof.** Let $X = \text{Cay}(G, S)$ be a hexavalent half-arc-transitive Cayley graph on the group $G$ of order $4p$ with respect to $S$. Then $X$ is connected because there are no half-arc-transitive graphs of order $p$ or $2p$ (see [12]). It follows that $|S| = 6$, $S^{-1} = S$ and $(S) = G$. By Proposition 2.3, $G$ is non-abelian and by Proposition 2.2, $p \geq 7$. From the elementary group theory we know that up to isomorphism there are three non-abelian groups of order $4p$ for an odd prime $p \geq 7$:

- $G_1(p) = \langle a, b | a^{2p} = b^2 = 1, b^{-1}ab = a^{-1} \rangle$,
- $G_2(p) = \langle a, b | a^{2p} = b = a', b^{-1}ab = a^{-1} \rangle$,
- $G_3(p) = \langle a, b | a^p = b^4 = 1, b^{-1}ab = a' \rangle$,  \hspace{1cm} $r^2 = -1 \pmod{p}$.

By Proposition 2.3, there is no involution in $S$, and then since $G = (S)$, one has $G \neq G_1(p)$. Suppose $G = G_2(p)$. Then $S$ contains at least one element of order 4 and its inverse. Each element of order 4 is of the form $ab^i$ or $a^{-1}b^i$ for an integer $i$. The automorphism of $G_2(p)$ induced by $b \mapsto b^{-1}$, $a \mapsto a$, maps $ab^i$ to $(ab)^{-1}$ for any integer $j$ and fixes $(a)$ pointwise. This is impossible by Proposition 2.3. Thus, $G = G_3(p)$.

Now let $12 \mid (p - 1)$ and $C(4p) = \text{Cay}(G, S)$ with $G = \langle a, b | a^p = b^4 = 1, b^{-1}ab = a' \rangle$ ($r^2 = -1 \pmod{p}$) and $S = \{b, b^{-1}, ab, (ab)^{-1}, a'b, (a'b)^{-1}\}$, where $e$ is an element of order 6 in $\mathbb{Z}_p^*$. To finish the proof of the lemma, we only need to show that $C(4p)$ is half-arc-transitive. Note that the fact $e$ is an element of order 6 in $\mathbb{Z}_p^*$ implies that $e^2 - e + 1 = 0$ in $\mathbb{Z}_p^*$. Hence $e \neq 1$ is invertible, so the map $a \mapsto a^{-1}, b \mapsto ab$ induces an automorphism of $G$, say $\alpha$. Thus, $b^a = ab, (ab)^a = a'b$ and $(a'b)^a = b$. It follows that $\alpha \in \text{Aut}(G, S)$ and hence $C(4p)$ is edge-transitive. Furthermore, one may easily show that $\text{Aut}(G, S) = \langle e \rangle \cong \mathbb{Z}_3$. Let $A = \text{Aut}(C(4p))$. Denote by $A_1$ the stabilizer of the vertex 1 in $A$ and by $A_1^*$ the subgroup of $A_1$ fixing every neighborhood of 1 in $C(4p)$. We first claim that $A_1^* = 1$.

Depict the induced subgraph by the set of vertices having distance less than 3 and some vertices having distance 3 from 1 in $C(4p)$ as Fig. 1. One may see that for any $u \in \{b, ab, a'b\}, v \in \{b^{-1}, (ab)^{-1}, (a'b)^{-1}\}$, there is exactly one 4-cycle passing through $u$ and $v$. Thus $A_1^*$ fixes all 4-cycles passing through 1 pointwise. In particular, $A_1^*$ fixes $\{a^{-e}b^2, a^{-e}b^2, a^e b^2, a^-b^2, a^2 b^2, a^{-e}b^2, a^2 b^2, a^-b^2, a^2 b^2\}$ pointwise. Furthermore, $A_1^*$ fixes the

![Fig. 1. An induced subgraph in the Cayley graph $C(4p)$.](image-url)
Let $p$ be a prime and $X$ a hexavalent half-arc-transitive graph of order $4p$. Then $12 \mid (p-1)$ and $X \cong C(4p)$.

**Proof.** Since there are no half-arc-transitive graphs of order $p$ or $2p$ (see [12]), $X$ is connected. Let $A = \text{Aut}(X)$. Recall that $X$ is an underlying graph of an orbital digraph $D := (V(X), \mathcal{O})$ of $A$ for some non-self-paired orbital $\mathcal{O}$. Thus, $A \leq \text{Aut}(D)$ and $D$ is a directed graph with out- and in-valency equal to 3. Let $u \in V(X)$ and denote by $A_u$ the stabilizer of $u$ in $A$. By Proposition 2.6, $A_u$ is a $\{2, 3\}$-group and hence $A$ is a $\{2, 3, p\}$-group with $|A|$ not divisible by $p^2$. The edge-transitivity of $X$ implies that $12p \mid |A|$. By Proposition 2.2, $p \geq 7$. Let $N$ be a minimal normal subgroup and $P$ a Sylow $p$-subgroup of $A$. Then $|P| = p$. We first prove the following claim.

**Claim:** $P \trianglelefteq A$.

Suppose that all minimal normal subgroups of $A$ are nonsolvable. Then $N \cong T^m$ where $T$ is a non-abelian simple $\{2, 3, p\}$-group. Since $p^2 \nmid |A|$ and $p \geq 7$, we have, by [26, pp. 12–14], that $m = 1$ and $N = T$ is one of five groups given in Table 1. By the simplicity of $N$, $N$ has orbits of length $p$, $2p$ or $4p$,
and hence $|N : N_u| = p$, 2p, 4p, where $N_u$ is the stabilizer of $u$ in $N$. Since $N_u$ is contained in a maximal subgroup of $N$, Table 1 and Proposition 2.4 combined together imply that $N \cong L_2(7)$ or $L_2(3)$. Suppose first that $N \cong L_2(7)$. Then $|V(X)| = 28$, and since, by assumption, $A$ has no solvable minimal normal subgroup, we have that $C_A(N) = 1$. Consequently $A \leq \text{Aut}(N)$, which implies that $|A| \leq |N||\text{Out}(N)|$. By the ATLAS [27], $|\text{Out}(N)| = 2$ and thus either $|A| = |N|$ or $|A| = 2|N|$. It follows that $N$ has at most two orbits on $V(X)$. If $N$ is transitive on $V(X)$ then $|N_u| = 6$, and $X$ is the underlying graph of an orbital digraph of $N$ for some non-self-paired orbital. However, with the use of program software MAGMA [28] one can see that this is impossible. If $N$ has two orbits on $X$ then the fact that $A \leq \text{Aut}(N)$ implies that $A = \text{PGL}(2, 7)$ and thus $|A_u| = 12$. However, by the ATLAS [27], $A_u$ is a maximal subgroup of $A$, contradicting Proposition 2.4. The case $N \cong L_2(3)$ is excluded in a similar manner; details are left to the reader.

We have proved that $A$ has at least one solvable minimal normal subgroup, say $N$. This $N$ is elementary abelian. Since $|V(X)| = 4p$, $N$ cannot be a 3-group. If $N$ is a $p$-group then $N$ is a normal Sylow $p$-subgroup of $A$ and the claim is true. In what follows we assume that $N$ is an elementary abelian 2-group, say $N \cong \mathbb{Z}_2^r$ for some integer $r$. Let $K$ be the kernel of $A$ acting on the quotient graph $X_0$ of $X$ corresponding to the orbits of $N$. Clearly, $N \leq K$ and since $|V(X)| = 4p$, orbits of $N$ on $V(X)$ are of length 2 or 4.

Suppose the orbits of $N$ are of length 4. Now, furthermore, $|X_0| = p$ and $X_0$ has valency 2 or 6. If $X_N$ has valency 2 then $X_N = C_p$, say $X_N = (B_0, B_1, \ldots, B_{p-1})$ with $B_i$ and $B_{i+1}$ adjacent for each $i \in \mathbb{Z}_p$. The induced subgraph $(B_i, B_{i+1})$ of $B_0 \cup B_{i+1}$ in $X$ is an edge-transitive cubic graph of order 8, and therefore it is isomorphic to the three-dimensional hypercube $Q_3$. Note that $\text{Aut}(Q_3) = S_4 \times \mathbb{Z}_2$. Then $K$ is faithful on each $B_i$ and $K \leq S_4$, forcing $N = \mathbb{Z}_2^2$. If $X_N$ has valency 6, the stabilizer $K_u$ of $u$ in $K$ fixes each neighborhood of $u$ in $X$ because $K$ fixes each orbit of $N$. It follows that $K_u = 1$ and $N = \mathbb{Z}_2^2$. Thus, $NP$ is a regular subgroup of $A$, that is $X$ is a Cayley graph on $NP$. By Lemma 3.2, $NP$ has a cyclic Sylow 2-subgroup, a contradiction.

Hence the orbits of $N$ are of length 2. Then, $|X_0| = 2p$ and $X_N$ has valency 3 or 6. If $X_N$ has valency 3 then $X \cong X_0[2K_3]$. Note that $X_0$ is edge-transitive because $X$ is edge-transitive. By Proposition 2.5, $X_0$ is arc-transitive and hence $X \cong X_0[2K_3]$ is arc-transitive, a contradiction. Thus, $X_N$ has valency 6. In this case, $K_u = 1$ and $K = N \cong \mathbb{Z}_2$. And one may view $A/N$ as a group of automorphisms of $X_N$, that is, $A/N \leq \text{Aut}(X_N)$. Set $H = A/N$. Then $X_N$ is $H$-half-arc-transitive. Note that $6p \mid |H|$. By Proposition 2.5, $X_N$ is isomorphic to $B(\text{PG}(2, 5)), B'(H(11))$ or $G(2p, 6)$ with $6 \mid (p - 1)$. Suppose $X_N \cong B(\text{PG}(2, 5))$. Then $H$ is a $(2, 3, 31)$-subgroup of $\text{Aut}(B(\text{PG}(2, 5))) = P\Gamma L(3, 5) \rtimes \mathbb{Z}_2$ and $186 \mid |H|$, implying that $H \cap P\Gamma L(3, 5)$ is a proper subgroup of $P\Gamma L(3, 5)$ with order divisible by 93. By MAGMA [28], $|H \cap P\Gamma L(3, 5)| = 93$. Thus, $|H| = 186$ and $|A| = 3 \cdot 4 \cdot 31$. Note that $A$ has normal Sylow 31-subgroups and so $A$ is solvable. Then, $X$ is a Cayley graph on a Hall $(2, 31)$-subgroup of $A$. By Lemma 3.2, 4 $(31 - 1)$, a contradiction. Suppose $X_N \cong B'(H(11))$. Then $H$ is a $(2, 3, 11)$-subgroup of $\text{Aut}(B'(H(11))) \cong \text{PSL}(2, 11) \rtimes \mathbb{Z}_2$ and $33 \mid |\text{PSL}(2, 11)|$, which is impossible because $\text{PSL}(2, 11)$ has no proper subgroup with order divisible by 33. Thus, $X_N \cong G(2p, 6)$ with $6 \mid (p - 1)$. First assume $p > 11$. By Proposition 2.5, $\text{Aut}(G(2p, 6))$ has a normal Sylow $p$-subgroup, implying that $PN/N \leq A/N$. It follows that $PN \not\leq A$. Clearly, $P$ is characteristic in $PN$ and hence $P \not\leq A$. The claim holds. Now assume $p < 11$. Then $X_N \cong G(2, 7, 6)$. In this case, $X_N$ has vertex set $V' \cup V$ with $V = \{i \mid i \in \mathbb{Z}_7\}$ and $V' = \{i' \mid i \in \mathbb{Z}_7\}$, and edge set $\{ij' \mid i, j \in \mathbb{Z}_7, i \neq j\}$. Recall that $X_N$ is $H$-half-arc-transitive. Let $H^*$ be the subgroup of $H$ fixing $V$ and $V'$ setwise. Then $|H : H^*| = 2$. Clearly, for each $i \in \mathbb{Z}_7$, one has $H_i \leq H^*$, $H_i = H_0$ and $H^*$ acts faithfully on $V$, where $H_i$ and $H_0$ are the stabilizers of $i$ and $i'$ in $H$, respectively. By half-arc-transitivity of $H$, $H_0$ has two orbits of length 3 on $V \setminus \{0\}$, say $O_1$ and $O_2$. If $H_0 (= H_{o'})$ is unfaithful on $O_1$

<table>
<thead>
<tr>
<th>Group</th>
<th>$p$</th>
<th>Order</th>
<th>Indices of maximal subgroups</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_2(7)$</td>
<td>7</td>
<td>$2^3 \cdot 3 \cdot p$</td>
<td>8, p</td>
</tr>
<tr>
<td>$L_2(8)$</td>
<td>7</td>
<td>$2^3 \cdot 3^2 \cdot p$</td>
<td>9, 36, 4p</td>
</tr>
<tr>
<td>$L_2(17)$</td>
<td>17</td>
<td>$2^3 \cdot 3^2 \cdot p$</td>
<td>18, 6p, 8p, 9p</td>
</tr>
<tr>
<td>$L_2(3)$</td>
<td>13</td>
<td>$2^3 \cdot 3^3 \cdot p$</td>
<td>144, p, 18p</td>
</tr>
<tr>
<td>$U_2(3)$</td>
<td>7</td>
<td>$2^3 \cdot 3^3 \cdot p$</td>
<td>36, 4p, 9p</td>
</tr>
</tbody>
</table>

Table 1

Non-abelian simple $\{2, 3, p\}$-groups ($p \geq 7$), extracted from [27,26].
then as a permutation group of degree 7 on \( V \), \( H^* \) contains a transposition or a 3-cycle. Note that \( H^* \) is primitive on \( V \) and \( V' \). By [29, Theorem 13.3], \( H^* \) is 2-transitive on \( V \) and \( V' \). This implies that \( H_0 = H_{0'} \) is transitive on \( V \setminus \{0\} \), implying that \( H \) is arc-transitive on \( X_0 \), a contradiction. Thus, \( H_0 \) is faithful on \( O_1 \), implying that \( H_0 \leq S_2 \). It follows that \( |H^*| \mid 42 \) and by Sylow theorem, \( H^* \) has a characteristic Sylow 7-subgroup, implying that \( H \) has a normal Sylow 7-subgroup, that is, \( PN/N \leq A/N \). Thus, \( PN \leq A \) and since \( P \) is characteristic in \( PN \), one has \( P \leq A \). This completes the proof of the claim.

Now consider the quotient graph \( X_P \) of \( X \) corresponding to the orbits of \( P \). Then \( |V(X_P)| = 4 \) and \( X_P \) has valency 2 or 3. Let \( K \) be the kernel of \( A \) acting on \( V(X_P) \).

If \( X_P \) has valency 3, then \( X_P \) is the complete graph \( K_4 \). In this case, \( K_4 \) fixes every out-neighbor of \( u \) in the directed graph \( D \), which implies \( K_u = 1 \). Thus, \( P = K = \mathbb{Z}_p \) and \( A/P \leq S_4 \). Since \( 12p \mid |A| \), one has \( A/P = A_4 \) or \( S_4 \). Let \( R \) be a Sylow 2-subgroup of \( A_4 \). Then \( R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \) and \( RP \) acts regularly on \( V(X) \). This means that \( X \) is a Cayley graph on \( RP \). By Lemma 3.2, \( RP \) has a cyclic Sylow 2-subgroup, a contradiction.

Thus, \( X_P \) has valency 2, that is, \( X_P \) is a 4-cycle, say \( X_P = (B_0, B_1, B_2, B_3) \) with \( B_i \) and \( B_{i+1} \) adjacent for each \( i \in \mathbb{Z}_4 \). The induced subgraph \( T = (B_i, B_{i+1}) \) of \( B_0 \cup B_{i+1} \) in \( X \) is an edge-transitive cubic graph of order 2p, and by half-arc-transitivity of \( X \), all edges in \( T \) have the same direction either from \( B_i \) to \( B_{i+1} \) or from \( B_{i+1} \) to \( B_i \) in the directed graph \( D \). Thus, \( 3p \mid |K| \) and \( A/K \cong \mathbb{Z}_4 \). By Proposition 2.5, \( T \) is the Heawood graph of order 14 for \( p \neq 7 \) or the Cayley graph \( G(2p, 3) \) for a prime \( p \geq 13 \) such that \( p - 1 \) is a multiple of 3. If \( T \) is the Heawood graph then \( \text{Aut}(T) \cong \text{PGL}(2, 7) \) and hence \( K \leq \text{PSL}(2, 7) \). Note that \( P \leq A \) implies that \( A \) is solvable. Thus, \( K \) is a proper subgroup of \( \text{PSL}(2, 7) \) and since \( 21 \mid |K| \), \( K \) must be a maximal subgroup of \( \text{PSL}(2, 7) \) isomorphic to \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), implying \( |K| = 21 \) and \( |A| = 84 \). If \( T \) is the Cayley graph \( G(2p, 3) \) for a prime \( p \geq 13 \) such that \( p - 1 \) is a multiple of 3 then, by Proposition 2.5, \( |\text{Aut}(T)| = 6p \) and hence \( |K| = 3p \). It follows that \( |A| = 12p \) for \( p \geq 7 \). Thus, \( A_u \) is a Sylow 3-subgroup of \( A \). Let \( R \) be a Sylow 2-subgroup of \( A \). Then \( RP \) is a regular subgroup of \( A \). By Lemma 3.2, one may let \( X = \text{Cay}(G, S) \), where \( G = (a, b \mid a^p = b^p = 1, b^{-1}ab = a^i) \) (\( i^2 = -1 \) (mod \( p \))) and \(|S| = 6 \). Note that \( i^2 = -1 \) (mod \( p \)) implies \( 4 \mid (p - 1) \). By the normality of \( P \) in \( A, P \leq R(G) \). Set \( C = C_n(P) \). Since there is no involution in \( G \) which commutes with \( a, C \) is a \( \{3, p\} \)-group. Suppose \( 3 \mid |C| \). Then \( A_u \leq C \).

And since \( |A| = 12p \), \( A_u \) is normal in \( C \) and hence characteristic in \( C \). Since \( C \leq A \), one has \( A_u \leq A \), a contradiction. Thus, \( P \in C \). Since \( A/C = A/P \leq \text{Aut}(P) \cong \mathbb{Z}_{p-1} \), one has \( R(G)/P \leq A/P \), implying \( R(G) \leq A \). Thus \( X = \text{Cay}(G, S) \) is a normal Cayley graph. By Proposition 2.1, \( A = R(G) \times \text{Aut}(G, S) \) and since \( |A| = 12p \), there is an element \( \beta \) of order 3 in \( \text{Aut}(G, S) \).

Each element of order 4 in \( G \) is of the form \( ab^i \) or \( ab^{-i} \) for some integer \( i \). Note that the connectivity of \( X \) implies \( |S| = G \). This means that \( S \) contains at least two elements of order 4 and since \( \text{Aut}(G) \) is transitive on the set \( \{a^i \mid 0 \leq i \leq p - 1\} \), one may assume \( b \in S \). Suppose that \( S \) contains exactly two elements of order 4. By Proposition 2.3, \( S \) contains no involutions and hence \( S = \{a^i, a^j, a^i, a^j, b, b^{-1}\} \) for some integers \( i, j \). However, the automorphism of \( G \) induced by \( a \mapsto a^{-1}, b \mapsto b \) fixes \( S \) and maps \( a^i \) to \( a^{-j} \), contrary to Proposition 2.3. Suppose that \( S \) contains exactly four elements of order 4. Since for each \( z \in \mathbb{Z}_p^* \), the map \( a \mapsto a^z \) induces an automorphism of \( G \), one may assume \( S = \{a^i, a^{-i}, b, b^{-1}, ab, (ab)^{-1}\} \). In this case, the automorphism of \( G \) induced by \( a \mapsto a^{-1}, b \mapsto ab \) fixes \( S \) and maps \( a^i \) to \( a^{-i} \), contrary to Proposition 2.3. It follows that \( S \) consists of elements of order 4. Since for each \( z \in \mathbb{Z}_p^* \) the map \( a \mapsto a^z \) induces an automorphism of \( G \), one may assume \( S = \{a^i, a^{-i}, b, b^{-1}, ab, (ab)^{-1}\} \). In this case, the automorphism of \( G \) mapping \( b \) to \( b^{-1} \). Hence, \( \langle \beta \rangle \) has two orbits on \( S \), that is, \( \{ab, a^i \} \) and \( \{b^{-1}, (ab)^{-1}, (ab)^{-1}\} \). Furthermore, one may assume that \( \beta \) permutes \( b, ab \) and \( a^i \) cyclically. Thus, \( b^{-1} = a^i \) and \( ab^{-1} = (ab)^{-1} = ab \). And \( (ab)^{-1} = a^{i+3} \), implying \( i^2 = -1 \) (mod \( p \)) and \( i \neq -1 \), forcing that \( i \) is an element of order 6 in \( \mathbb{Z}_p^* \). Thus, \( 6 \mid (p - 1) \). Since \( 4 \mid (p - 1) \), one has \( 12 \mid (p - 1) \) and \( X \cong C(E) \).

\section*{Acknowledgements}

This work was supported by the National Natural Science Foundation of China (10871021) and the Specialized Research Fund for the Doctoral Program of Higher Education in China (20060004026).

\section*{References}