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Nonbounding and Slaman triples

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Abstract

We consider the relationship of the lattice-theoretic properties and the jump-theoretic properties satisfied by a recursively enumerable Turing degree. The existence is shown of a high$_2$ r.e. degree which does not bound what we call the *base* of any Slaman triple.

1. Introduction

Much work has been done in the study of the recursively enumerable (r.e.) sets and degrees, and in the study of the Turing degrees in general, establishing connections between the order-theoretic properties and the (Turing) jump-theoretic properties of the elements of these structures. A general pattern has been the discovery of instances of sets and degrees with "low" jump resembling recursive sets and the recursive degree, and those with "high" jump resembling complete r.e. sets and the complete r.e. degree $\mathbf{0}'$, respectively. We mention a few well-known examples; others may be found in, to cite a few sources, [15], [11], [2], or [3].

In the lattice $\mathscr{E}^*$ of r.e. sets modulo finite sets, Soare [14] has shown that if A is low (that is, if $A' \equiv_T \emptyset'$), then $\mathscr{E}^*$ is isomorphic to $\mathscr{L}^*(A)$, the lattice of r.e. supersets of A. In the r.e. degrees $\mathbf{R}$, Robinson [9] has extended the Sacks Splitting Theorem [10] (which states that any nonrecursive, r.e. degree can be "split," that is, expressed as the join of two incomparable r.e. degrees) to show that the property of splitting holds over any low r.e. degree. (That is, if $\mathbf{a} > \mathbf{d}$ and $\mathbf{d}$ is low then there are incomparable degrees $\mathbf{b}$ and $\mathbf{c}$ above $\mathbf{d}$ such that $\mathbf{b} \cup \mathbf{c} = \mathbf{a}$.) As for high r.e. sets and degrees (i.e., those with jump as high as possible, namely $\emptyset''$ or $\mathbf{0}''$, respectively), Martin [8] has

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shown that every high r.e. degree contains a maximal set, and Cooper [1] has shown that every high r.e. degree bounds a minimal pair.

We can further refine our analysis of these degree structures by considering the classes determined by evaluating equivalence under iterated jumps. The high–low hierarchy thus obtained provides a natural and nontrivial partitioning of both the Turing degrees and the recursively enumerable Turing degrees. We denote the jump classes as usual as $\mathbf{H}_n = \{\mathbf{d} \in \mathbf{S} \mid \mathbf{d}^{(n)} = \mathbf{0}^{(n+1)}\}$ and $\mathbf{L}_n = \{\mathbf{d} \in \mathbf{S} \mid \mathbf{d}^{(n)} = \mathbf{0}^{(n)}\}$, where $\mathbf{S}$ is the particular degree structure under consideration. It is natural to ask whether some or all of these classes may be defined in a particular structure according to some order-theoretic formula.

In the general Turing degrees $\mathbf{D}$, much progress has been made. Shore [11] has produced results on jump classes which show that all classes of the form $\mathbf{H}_n$ and $\mathbf{L}_n$, $n \geqslant 3$, are definable within $\mathbf{D}(\leqslant \mathbf{0}')$, the Turing degrees below $\mathbf{0}'$. Cooper [2] has shown that the Turing jump itself is definable in $\mathbf{D}$, implying that all classes $\mathbf{H}_n$ and $\mathbf{L}_n$, $n \geqslant 1$, are definable within $\mathbf{D}$.

In the r.e. Turing degrees $\mathbf{R}$, however, the situation is less resolved. Very recent work of Nies, Shore, and Slaman would prove the definability within $\mathbf{R}$ of all jump classes of the form $\mathbf{H}_n$ and $\mathbf{L}_n$ for $n \geqslant 3$ as well as the class $\mathbf{H}_2$ (personal communication). However, the problem of determining the definability or nondefinability within $\mathbf{R}$ of the classes $\mathbf{H}_1$, $\mathbf{L}_1$, and $\mathbf{L}_2$ is still open. [1] The class $\mathbf{L}_2$ is of special interest because general techniques have been developed both for permitting beneath a non-low$_2$ r.e. degree [12] and for permitting beneath a low$_2$ r.e. degree [4], suggesting some hope of proving the definability of this jump class within $\mathbf{R}$.

Shore and Slaman have found two properties which separate the high r.e. degrees from the low$_2$ r.e. degrees. To state these properties, we need the following definitions.

Definition 1.1. Two r.e. degrees $\mathbf{a} > \mathbf{d}$ form a *nonsplitting pair* iff $\mathbf{a}$ does not split over $\mathbf{d}$, i.e.,

$$\neg\exists \mathbf{b}\exists \mathbf{c}[\mathbf{d} < \mathbf{b}, \mathbf{c} < \mathbf{a} \ \&\ \mathbf{b} \cup \mathbf{c} = \mathbf{a}].$$

The existence of such a pair was shown by Lachlan in [6].

Definition 1.2. A *Slaman triple* is a triple of r.e. degrees $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ such that
(i) $\mathbf{u}$ is not recursive;
(ii) $\mathbf{w} \not\leqslant \mathbf{v}$; and
(iii) $\mathbf{0} < \mathbf{b} \leqslant \mathbf{u} \Rightarrow \mathbf{w} \leqslant \mathbf{b} \cup \mathbf{v}$.
We call the degree $\mathbf{u}$ the *base* of the triple.

[1] Cooper has recently announced the existence of an automorphism of $\mathbf{R}$ which moves a low degree to a nonlow degree, implying the nondefinability of $\mathbf{L}_1$ within $\mathbf{R}$.

The existence of such a triple was first shown by T. Slaman, which is why such a triple has come to be frequently referred to by this name. The separating theorems of Shore and Slaman may now be stated.

Theorem 1.3 (Shore and Slaman [12]). *No low$_2$ r.e. degree bounds any nonsplitting pair, or bounds any Slaman triple.*

Theorem 1.4 (Shore and Slaman [13]). *Any high r.e. degree bounds some nonsplitting pair, and bounds some Slaman triple.*

Furthermore, Shore and Slaman are able to prove the existence of both a nonsplitting pair $(\mathbf{a}, \mathbf{d})$ and a Slaman triple $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ with the "top" degree (i.e., $\mathbf{a}$ or $\mathbf{u} \cup \mathbf{v} \cup \mathbf{w}$, respectively) low$_3$. Based on these results, the property of not bounding either a nonsplitting pair or a Slaman triple might define $\mathbf{L}_2$, but could not define any other jump class, within $\mathbf{R}$.

Downey et al. have shown

Theorem 1.5 (Downey et al. [3]). *There exists a high$_2$ r.e. degree which does not bound any minimal pair.*

(This result has been proven independently by both Lerman and Kučera, as cited in [3].) Note that if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ form a Slaman triple then $\mathbf{u}$ and $\mathbf{v}$ form a minimal pair. It follows, as observed by Shore, that there exists a high$_2$ r.e. degree which does not bound any Slaman triple. This result shows that the property of not bounding any Slaman triple does not define $\mathbf{L}_2$ in $\mathbf{R}$.

In this paper we prove a related algebraic fact about the r.e. degrees which rules out a heretofore possible defining formula for $\mathbf{L}_2$ within $\mathbf{R}$:

Theorem 1.6. *There exists a high$_2$ r.e. degree which does not bound the base of any Slaman triple.*

Thus the property of not bounding the base of a Slaman triple cannot define any jump class within $\mathbf{R}$.

While the *statement* of this theorem is very similar to the theorem of Downey et al., and while the proof uses some of the same techniques, it should be noted that there is *no* high$_2$ r.e. degree which does not bound the "base" (i.e., one half) of a minimal pair; in fact, it can be easily shown that any *nonrecursive* r.e. degree bounds at least half of a minimal pair.

It is not known whether every non-low$_2$ r.e. degree bounds a nonsplitting pair. [2]

The remainder of this paper is devoted to proving Theorem 1.6. We give the formal requirements and an intuitive sketch of the proof in the next section, before giving

[2] Cooper and Yi have very recently answered this question negatively by proving the existence of a high$_2$ r.e. degree which does not bound the base of any nonsplitting pair.

the formal proof in the subsequent sections. The notation used is generally standard and follows that of [15], with the following exceptions: we write $\phi(x)$ to denote the largest number actually used by the computation $\Phi^Y(x)$; and when referring to nodes or sets of nodes in the tree of strategies, we denote the true path as *TP*, the apparent true path as *ATP*, the correct part of the true path as *CTP*, and the genuine true path as *GTP*.

2. The requirements and the intuition

In this section we state the requirements that our construction will satisfy in order to prove Theorem 1.6, which we hereafter refer to as the Main Theorem. We first state the theorem in a form which refers directly to the desired conditions rather than to the defined triples.

Main Theorem. *There exists a high$_2$ r.e. degree* $\mathbf{a}$ *such that for any triple of r.e. degrees* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ *with* $\mathbf{0} < \mathbf{u} \leqslant \mathbf{a}$ *and* $\mathbf{w} \nleqslant \mathbf{v}$, *there exists* $\mathbf{b}$ *with* $\mathbf{0} < \mathbf{b} \leqslant \mathbf{u}$ *and* $\mathbf{w} \nleqslant \mathbf{b} \cup \mathbf{v}$.

We will build r.e. set A, p.r. functional Δ, various r.e. sets $B = B_{\Phi,U,V,W}$, p.r. functionals $\Gamma = \Gamma_{\Phi,U,V,W,\Theta}$, and $\Lambda = \Lambda_{\Phi,U,V,W,Z}$ to meet the requirements described below. We first consider the high$_2$ requirements and the non-bounding requirements separately.

2.1. The high$_2$ requirements.

Our strategy to make A high$_2$, and in fact our technique for combining the high$_2$ requirements with the nonbounding requirements, is very similar to that found in the proof of Downey et al. that there is a high$_2$ r.e. degree which does not bound any minimal pair [3]. We will build a p.r. functional Δ such that for every $x \in \omega$,

$$\mathscr{P}_x : \quad \lim_t \lim_s \Delta^A(x,s,t) = \mathrm{Cof}(x),$$

where $\mathrm{Cof} = \{x \mid W_x \text{ cofinite}\}$ is the canonical Σ_3-complete set.

When assigning requirements of this form to nodes on the tree of strategies, we break these requirements up into subrequirements of the form

$$\mathscr{P}_{x,m} : \quad \lim_s \Delta^A(x,s,t_m) = 1 \Leftrightarrow [m,\infty) \subseteq W_x,$$

where t_m will be defined by the (unique) strategy on the correct (part of the) true path working on this requirement in such a way that $m \leqslant t_m$ and $t_m < t_{m'}$ whenever $m < m'$. Note that the $\mathscr{P}_{x,m}$ requirements are really psuedo-requirements, in the sense that for fixed x, it suffices that we meet requirement $\mathscr{P}_{x,m}$ for almost every m, rather than for all $m \in \omega$. In fact, the known result that we cannot make the set A high implies that we cannot meet $\mathscr{P}_{x,m}$ for all $x \in \omega$ and for all $m \in \omega$.

Our strategy to satisfy a requirement of the form $\mathscr{P}_x$ will be the following. We define values of $\Delta^A(x,s,t)$ to 0 in such a way that uses are strictly increasing in any argument; although we define all such uses at the beginning of the construction, one could alternatively define only finitely many values at any given stage. For each $m \in \omega$, we will have a $\mathscr{P}_{x,m}$-strategy β test whether $[m,\infty) \subseteq W_x$. During each stage at which we observe additional evidence toward a positive answer, β will change the definition of $\Delta^A(x,s,t)$ from 0 to 1 for finitely many values of s and for finitely many values of t, by putting the corresponding δ-uses into the set A. The values of t will be those t such that $t_l < t \leqslant t_m$, where t_m is the value of the parameter associated with the particular $\mathscr{P}_{x,m}$-strategy β currently acting, and t_l is the parameter associated ith the longest $\mathscr{P}_{x,l}$-strategy $\alpha < \beta$. We may roughly think of t_l as t_{m-1}, although for technical reasons this will not always be true. In a slight variant on the terminology of the Lempp–Lerman framework [7], we say that these are the values of t over which strategy β has *control*. The maximum such value of s will be the current stage. The minimum value of s for which we change the value of $\Delta^A(x,s,t)$ will be the least s such that we have not already changed this value, and such that $\delta(x,s,t)$ is above the A-restraint desired by stronger priority strategies. Determining this minimum value is what prevents strategy β from injuring stronger priority strategies.

Note that the question as to whether $[m,\infty) \subseteq W_x$ is a Π_2 question. Therefore the outcomes of the $\mathscr{P}_{x,m}$-strategy β will correspond to either finite action or infinite action. In the case of finite measurement and action, strategies of weaker priority than β are initialized finitely often but are eventually no longer affected by β. In the case of infinite action by β, a strategy η of weaker priority needs merely to wait until certain δ-uses have been put into A by β, before believing the computation that η is attempting to measure. We will arrange that the A-restraint at any node along the true path reaches a finite limit along the stages at which that node is eligible to act, and that furthermore, strategies which lie along the true path and are working beneath the infinite outcome of β will have correct information about this finite limit. Therefore at any given stage, such a strategy will know exactly which δ-uses strategy β will put into set A at some future stage.

What we have described above is the operation of a $\mathscr{P}_{x,m}$-strategy working in isolation. We will need to modify this procedure in certain situations where the $\mathscr{P}_{x,m}$-strategy must respect a stronger priority $\mathscr{Q}$-strategy; however, we postpone description of this modification until after we have described our strategy to meet the nonbounding requirements in isolation.

2.2. *The nonbounding requirements.*

To motivate the overall coordination of substrategies that we use to build a set of nonbounding degree, we first restate the theorem in a form which is logically equivalent to the first statement, but which (it is hoped) makes our overall strategy more transparent.

Main Theorem (Restated). *There exists a high*$_2$ *degree* $\mathbf{a}$ *such that for any triple of r.e. degrees* $\mathbf{u},\mathbf{v},\mathbf{w}$ *with* $\mathbf{u} \leqslant \mathbf{a}$ *there exists r.e. degree* $\mathbf{b} \leqslant \mathbf{u}$ *such that*

$$\mathbf{b} \leqslant \mathbf{0} \text{ or } \mathbf{w} \leqslant \mathbf{b} \cup \mathbf{v} \Rightarrow \mathbf{u} \leqslant \mathbf{0} \text{ or } \mathbf{w} \leqslant \mathbf{v}.$$

To show that $\mathbf{a} = \deg(A)$ does not bound the base of any Slaman triple, we meet

$$\mathscr{Q}_{\Phi,U,V,W}\colon \quad U = \Phi^A \Rightarrow (W \leqslant_T V) \text{ or } (U \leqslant_T \emptyset) \text{ or } (\exists B \leqslant_T U)[\forall\Theta(\mathscr{N}_\theta) \ \& \ \forall Z(\mathscr{S}_Z)]$$

where the subrequirements are given by

$$\mathscr{N}_\theta = \mathscr{N}_{\Phi,U,V,W,\Theta}\colon \quad W = \Theta^{B\oplus V} \Rightarrow \exists\Gamma(W = \Gamma^V),$$

$$\mathscr{S}_Z = \mathscr{S}_{\Phi,U,V,W,Z}\colon \quad \bar{B} = Z \Rightarrow \exists\Lambda(U = \Lambda).$$

The requirement that $B \leq_T U$ is never assigned to the tree of strategies, but will be satisfied by the nature of the construction. Note also that by making $\bar{B}$ infinite, we only need to consider those r.e. sets Z which are infinite.

The general strategy is as follows: faced with a triple of degrees as represented by r.e. sets U, V, W such that U appears reducible to A via p.r. functional Φ, we respond by building an r.e. set B which will serve, if necessary, as a counter-example to the claim that $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ forms a Slaman triple. We will ensure that $B \leq_T U$ by permitting along the stages which are genuine for the $\mathscr{Q}$-strategy responsible for building the set B.

The requirement $\mathscr{Q}$ may be satisfied in any one of five different manners:

(I) We measure $\limsup_s \ell(\Phi^A,\mathrm{U})$ finite.

(II) Some $\mathscr{N}_\theta$ has Π_3 outcome $W = \Gamma^V$, showing that $W \leq_T V$.

(III) Some $\mathscr{S}_z$ has Π_2 outcome $U = \Lambda$, showing that $U \leq_T \emptyset$.

(IV) We measure $\liminf_s \ell(\Phi^A,\mathrm{U})$ finite.

(V) All $\mathscr{N}_\theta$ and $\mathscr{S}_z$ have Σ_3 outcome $\Theta^{B\oplus V} \neq W$ and Σ_2 outcome $\bar{B} \neq Z$ (respectively), so that $\mathbf{0} < \mathbf{b} \leqslant \mathbf{u}$ and $\mathbf{w} \nleqslant \mathbf{b} \cup \mathbf{v}$.

These different potential manners of satisfaction will be reflected in our assignment of requirements to nodes on the tree of strategies, and in the Assignment Lemma 3.12, which asserts that our assignment of requirements achieves our purposes.

As usual in tree arguments, we delay assignment of a substrategy on the tree of strategies until a node at which the current assumed manner of satisfaction of a given $\mathscr{Q}$ requirement has been refuted. We follow the notational convention of using calligraphic type for requirements and italic type for a strategy working on the corresponding requirement; thus, R is always an $\mathscr{R}$-strategy.

Substrategies of the form N_Θ will build Γ so that $W = \Gamma^V$ in response to increasing length of agreement between W and $\Theta^{B\oplus V}$, keeping Γ correct via B-restraint. Since N_Θ may have $\limsup_s \ell(\Phi^A, U)$ infinite but $\liminf_s \ell(\Phi^A, U)$ finite, we must check this by measuring. As we descend any branch of the tree and make progressively larger nodes eligible to act, we require that $\liminf_s \ell(\Phi^A, U \oplus W \oplus V)$ becomes progressively larger. We link down to the appropriate node at stages during which we measure that a θ-use has increased, thus making N_Θ's functional Γ undefined on some fixed argument and therefore providing the lowered B-restraint that weaker priority $\mathscr{S}_Z$-strategies will need. We may consider such a node β to be working on the subrequirement

$$\mathscr{M}_{\theta,y}: \quad W \upharpoonright (y+1) = \Theta^{B \oplus V} \upharpoonright (y+1) \Rightarrow W \upharpoonright (y+1) = \Gamma_\beta^V \upharpoonright (y+1)$$

obtained by giving the outermost universally quantified variable in $\mathscr{N}_\theta$ the finite bound $(y+1)$. Since the functional $\Gamma_\beta = \Gamma_\alpha$ is actually being built by the $\mathscr{N}_\theta$-strategy α which β believes to be on the correct true path, definitions of the form $\Gamma_\alpha^V(y)$ made by one $\mathscr{M}_{\theta,y}$-substrategy of α must also be valid for any and every other $\mathscr{M}_{\theta,y}$-substrategy of α.

Substrategies of the form S_Z will attempt to show that $\bar{B} \neq Z$, but will simultaneously build Λ so that $U \upharpoonright (\text{dom } \Lambda) = \Lambda \upharpoonright (\text{dom } \Lambda)$ in response to increasing evidence that $\bar{B} = Z$. Such a strategy would like to find a witness z such that either z never appears in set Z and the strategy withholds the number from B, whereby $B \cup Z \neq \omega$, or else z appears in Z and the $\mathscr{S}_Z$-strategy is allowed to put z into B, whereby $B \cap Z \neq \emptyset$. To be allowed to put a number z into set B, strategy S_Z must have both (i) z greater than the B-restraint applied against S_Z, and (ii) permission from the set U which is "recent" in terms of stages which are genuine for the $\mathscr{Q}$-strategy above S_Z. (Say that a stage is *genuine* for a strategy iff that strategy is made eligible to act at that stage.) We will arrange that the B-restraint applied against a strategy along the true path reaches a finite limit along those stages at which the strategy is eligible to act, so that an $\mathscr{S}_Z$-strategy with set Z infinite will not be affected by the B-restraint after a certain stage. We create a link from an $\mathscr{S}_Z$-strategy wanting to put an element z into B up to the $\mathscr{Q}$-strategy which is responsible for building set B, for the sake of communicating the desired U-permission from Q down to S_Z, if this U-permission in fact occurs.

If the desired U-permission occurs by the next stage at which this link is travelled, then S_Z puts number z into set B and is thereby finitarily satisfied. If the desired U-permission does not occur, then S_Z increases the domain of Λ to reflect this strategy's belief that $U \upharpoonright z$ has "settled down," i.e., $U_s \upharpoonright z = U \upharpoonright z$, where s is the current stage.

Once S_Z has defined $\Lambda \upharpoonright z = U \upharpoonright z$ for some domain z, the strategy does not want $U \upharpoonright z$ to change at a later stage when S_Z is no longer linked up to Q, since U-permission at such a stage will generally not be synchronized with the lowered B-restraint that S_Z also needs. We cannot restrain the opponent's set U directly, but we can restrain $U \upharpoonright z$ indirectly, by restraining our set $A \upharpoonright (\phi(z-1)+1)$. This we do, associating this restraint with S_Z, until the next stage at which S_Z has found a new, larger number $z' \in Z$ which it would like to put into set B.

This general technique is known as the "gap–cogap" method, and occurs in many $\mathbf{0}'''$ arguments. The idea is that at any given stage, either the $\mathscr{S}_Z$-strategy has its gap "open," at which time the strategy requests no A-restraint but is poised to take advantage of any U-permission, or else the $\mathscr{S}_Z$-strategy has its gap "closed," at which time the strategy protects $U \upharpoonright (\text{dom } \Lambda)$ by restraining $A \upharpoonright (\phi(\text{dom } \Lambda - 1)+1)$. Either the $\mathscr{S}_Z$-strategy is finitarily satisfied, or else the strategy opens and closes infinitely many gaps with no U-permission during the "open" intervals. In either case, the $\liminf_s$

of the A-restraint applied by S_Z against a weaker priority strategy which is correct about S_Z's outcome will be only finite.

This concludes our intuitive description of the general strategy to meet the nonbounding requirements in isolation. We next discuss the interaction of a nonbounding strategy $Q = Q_{\Phi,U,V,W}$ and its substrategies of form N_Θ and S_Z with a high$_2$ strategy $P_{x,0}$ and its substrategies of the form $P_{x,m}$.

Consider first the case in which Q has stronger priority than $P_{x,0}$. In this case, if and when a different manner of satisfaction for requirement $\mathscr{Q}$ than previously assumed is discovered along a given branch of the tree of strategies, we do not have the luxury of reassigning requirement $P_{x,0}$ and its subrequirements. The reason for this is that $\Delta(-,-,-)$ is a *global* functional; definitions made by one $\mathscr{P}_{x,m}$-strategy on the tree must be valid also for any and every other $\mathscr{P}_{x,m}$-strategy on the tree. Therefore the substrategies of Q must anticipate the actions of such $P_{x,m}$.

If a Q-substrategy S_Z assumes that a strategy $P_{x,m}$ with priority locally stronger but globally weaker than S_Z has finite outcome, this is no problem, since the strategy can live with finitely many initializations. However, if a Q-substrategy S_Z assumes the infinite outcome of $P_{x,m}$, the following complication may occur.

It may happen that S_Z finds some new, sufficiently large $z \in Z$, say at stage s_0, with $\phi(z)[s_0]$ smaller than any δ-use which S_Z expects $P_{x,m}$ to put into A in the future. Strategy S_Z will then open a gap, dropping its A-restraint, and wait until the next genuine Q-expansionary stage s_1 to check whether the desired U-permission occurred. If not, then S_Z would like to increase the domain of its function Λ up to level z and then restrain $A \upharpoonright \phi(z)[s_1]$. If $\phi(z)[s_0] = \phi(z)[s_1]$ then this is no problem. However, since we dropped S_Z's A-restraint during the opening of the gap, the use $\phi(z)$ may have increased so that $\phi(z)[s_1]$ is now larger than some δ-use which S_Z expects $P_{x,m}$ to eventually put into A. If S_Z is correct about $P_{x,m}$'s outcome, then the injury to S_Z's desired A-restraint may result in a $U \upharpoonright (\mathrm{dom}\ \Lambda)$ change occurring at a stage when S_Z is not able to take advantage of it. If this sequence of events occurs infinitely often, as it may, then we will have $\Lambda \neq^* U$.

The solution we use here is that described in [3]. Namely, we "force" the infinite outcome of $P_{x,m}$ by allowing S_Z to put into A those δ-uses which it expects to enter A, and then changing the appropriate Δ-definitions to value 1, even though $P_{x,m}$ has not yet observed the evidence that we would ordinarily require before doing this. Strategy S_Z delays the close of this gap until use $\phi(z)$ is "cleared" of smaller δ-uses of the form $\delta(x,s,t)$ with $\Delta^A(x,s,t) = 0$ and $\delta(x,s,t)$ greater than the A-restraint which S_Z assumes is applied against $P_{x,m}$. If $\phi(z)$ is eventually cleared, then we close this gap as before. If $\phi(z)$ is never cleared (as may happen), then strategy S_Z has succeeded in showing that Φ is really partial, thus requirement $\mathscr{Q}$ is satisfied. Therefore an $\mathscr{S}_Z$-strategy will have two different infinitary outcomes, the leftmost (which we label as g) corresponding to the case in which S_Z opens and closes infinitely many new gaps (and hence $\Lambda = U$), and the other infinitary outcome (which we label as u) corresponding to the case in which S_Z keeps a gap open cofinitely often because $\phi(z)$ is never cleared of δ-uses (and hence Φ has unbounded use on fixed argument z, so is partial). Note that in outcome g, a link from S_Z up to Q is created and removed infinitely often, but in outcome u, a single link from S_Z up to Q will be in place for cofinitely many stages.

The price paid for using this technique is that we will now have defined $\lim_s \Delta^A(x,s,t_m) = 1$, even though we may actually have $[m,\infty) \not\subseteq W_x$. Thus we have failed to satisfy pseudo-requirement $\mathscr{P}_{x,m}$. However, there is now some $m' > m$ such that no strategy of the form $P_{x,m'}$ is affected by the actions of Q or any of Q's substrategies, and so $P_{x,m'}$ will have one less requirement of stronger priority than $\mathscr{P}_{x,0}$ which might interfere with its measurements and actions. Since any given $P_{x,0}$ will have only finitely many stronger priority strategies of the form Q to deal with, there is some $m'' \in \omega$ such that no strategy $P_{x,m}$ for any $m \geqslant m''$ has its measurements or actions interfered with in the manner we have described. Thus, we may fail to satisfy $\mathscr{P}_{x,m}$ for finitely many m, and in fact the known result that we cannot make A high implies that we cannot possibly satisfy $\mathscr{P}_{x,m}$ for all $x \in \omega$ and for all $m \in \omega$. However, for each fixed $x \in \omega$ we will satisfy cofinitely many $\mathscr{P}_{x,m}$, which is sufficient to satisfy $\mathscr{P}_x$.

Consider next the case in which $P_{x,0}$ has stronger priority than Q. If every substrategy $P_{x,m}$ has finite outcome, then each such substrategy will put only finitely many δ-uses into set A, and will respect the A-restraint desired by (locally) stronger priority substrategies of Q. Therefore each substrategy of Q will be initialized only finitely often, and will have its desired A-restraint respected at every stage after its final initialization.

If some substrategy $P_{x,m}$ has an infinite outcome, then requirement $\mathscr{Q}$ and its (globally weaker priority) subrequirements will be reassigned to strategies which have knowledge of $P_{x,m}$'s infinite outcome and can therefore avoid injury from the δ-uses that $P_{x,m}$ puts into A. So, we may assume that the strategy Q to which we refer is the version which is beneath the infinite outcome of $P_{x,m}$, and that furthermore, the infinite outcome of $P_{x,m}$ was measured accurately (taking $P_{x,m}$ to be the unencumbered $P_{x,m''}$ to which we referred above, if necessary).

We may still have the situation in which some Q-substrategy S_Z has created a link up to Q and now wants to close its gap and restrain $A \restriction \phi(z)$, but believes that some strategy $P_{x,m'}$ with $m' > m$ which lies between Q and S_Z on the tree of strategies is eventually going to put some δ-uses into set A which are lower than the current value of $\phi(z)$. As before, strategy S_Z is allowed to "force" the infinite outcome by putting the δ-uses which are associated with $P_{x,m'}$ into A and changing the corresponding Δ definitions to value 1. However, in this case, we claim that $\mathscr{P}_{x,m'}$ is nevertheless satisfied, by the following reasoning. Assuming that the infinite outcome of $P_{x,m}$ was genuinely measured (as we shall arrange), we have $[m,\infty) \subseteq W_x$, so $[m',\infty) \subseteq W_x$, implying that we really wanted $P_{x,m'}$ to have an infinite outcome, as it would have had if working in isolation. The key in arranging that the infinite outcome of $P_{x,m}$ be measured genuinely is that as links are created from bottom to top, any strategy $P_{x,m}$ not contained in some eventually permanent such link will be allowed to make a new measurement, and therefore cannot have its infinite outcome forced by infinitely many different $\mathscr{S}_Z$-strategies each acting finitely often.

We mention one remaining complication and our method of dealing with it. As we have noted earlier, a link from strategy Q down to a strategy $\mathscr{S}_Z$ which has outcome u along the true path may be in place during cofinitely many stages. If we naively follow the link from Q down to $\mathscr{S}_Z$ at all such stages, it may be that the outcomes of intermediate strategies which we "froze" at the stage of the link creation are not accurate, that is, do not correspond to the outcome that we would have given had we made infinitely many genuine measurements.

We circumvent certain combinatorial problems by using our version of a technique called a "scouting report" which originated in [5] and which is described in [3]. At a stage which is both expansionary and genuine for Q, if Q is the top τ of a link, then we do the following. Before travelling directly to the bottom σ of the link, we check where the apparent true path would lie if the link were not in place.

Now if the apparent true path (with the modification described above) would lie left of σ, then we remove the link (τ, σ) and make $\tau\hat{\ }\langle 0\rangle$ eligible to act next. Otherwise we make σ eligible to act next.

This means that if we have an $\mathscr{S}_Z$-strategy σ with $\sigma\hat{\ }\langle u\rangle$ along the true path, then σ was the bottom of a link which was in place for cofinitely many stages; therefore, the outcomes measured between the top and the bottom of the link may or may not be accurate, since they were really checked only finitely often. However, we will reassign all (sub)requirements which are linked over, and will show that eventually each (sub)requirement needed is assigned to some node at which we make a measurement infinitely often and therefore have an accurate outcome. We will state these intuitions more precisely in the Assignment Lemma 3.12 and in the Truth of Outcome Lemma 5.6.

This concludes our intuitive sketch of the proof. We now proceed to formally define the tree of strategies.

3. The tree of strategies

We fix an effective ω-ordering of all (sub)requirements $\mathscr{P}_{x,m}$, $\mathscr{Q}_{\Phi,U,V,W}$, $\mathscr{N}_{\Phi,U,V,W,\Theta}$, $\mathscr{M}_{\Phi,U,V,W,\Theta,y}$, and $\mathscr{S}_{\Phi,U,V,W,Z}$ such that

(i) $\mathscr{P}_{x,m}$ precedes $\mathscr{P}_{x,m+1}$;

(ii) $\mathscr{Q}_{\Phi,U,V,W}$ precedes $\mathscr{N}_{\Phi,U,V,W,\Theta}$ and $\mathscr{S}_{\Phi,U,V,W,Z}$;

(iii) $\mathscr{N}_{\Phi,U,V,W,\Theta}$ precedes $\mathscr{M}_{\Phi,U,V,W,\Theta,y}$; and

(iv) $\mathscr{M}_{\Phi,U,V,W,\Theta,y}$ precedes $\mathscr{M}_{\Phi,U,V,W,\Theta,y+1}$.

To give a precise definition of the tree of strategies, we will need a number of formal definitions. Before proceeding to these definitions, we make a few general remarks.

The term "link" is used in two distinct senses in this paper. The first sense is that of a dynamic feature of the construction by which we post instructions at the top node τ of the link (τ, σ) to proceed directly to the bottom node σ of the link (assuming that various other conditions are met), rather than making an immediate successor of

τ eligible to act next. This is the sense in which we have used this term up until now, and is the sense in which this term is more commonly used in the literature.

The second sense in which we use this term is to refer to a "link along ξ," where ξ is a node on the tree of strategies. This kind of link is a static feature of any branch of the tree containing ξ, and is the type described in the Lempp–Lerman iterated trees of strategies framework for priority arguments as presented in their proof of the decidability of the existential theory of the poset of r.e. degrees with jump relations [7]. The importance of such a link is that it signifies a change in the manner of satisfaction assumed by nodes below ξ for some requirement assigned to a stronger priority strategy. Such links will be used primarily to coordinate the assignment and reassignment of particular (sub)requirements to particular nodes.

We hope that the sense intended by any particular usage of the word "link" will be clear from the context in which it is used. Of course, the two senses are related, and we comment briefly on the connections between them.

A (static) $\mathscr{Q}$-use-link along the true path (as defined below) corresponds to a (dynamic) link created in the construction which is treated as permanent by those nodes at and below the outcome of the bottom node of the link. $\mathscr{Q}$-gap-links and $\mathscr{N}$-links along the true path correspond to intervals of nodes such that a (dynamic) link between the top node and bottom node is created and removed infinitely often. Static $\mathscr{P}$-links along the true path do not have any (dynamic) links created for their sake in the construction, since we need no special relationship between the timing of action by strategy $P_{x,0}$ and the action of strategy $P_{x,m}$ with $m > 0$.

The first two definitions are intended to help us identify which particular version of a higher level strategy a given substrategy on the tree is working beneath.

Definition 3.1. For a node β working on $\mathscr{N}_{\Phi,U,V,W,\Theta}$, $\mathscr{M}_{\Phi,U,V,W,\Theta,y}$, or $\mathscr{S}_{\Phi,U,V,W,Z}$, define $\mathrm{Top}(\beta)$ to be the node α of greatest length such that $\alpha \subset \beta$ and α works on $\mathscr{Q}_{\Phi,U,V,W}$.

Definition 3.2. For a node β working on $\mathscr{M}_{\Phi,U,V,W,\Theta,y}$, define $\mathrm{Subtop}(\beta)$ to be the node α of greatest length such that $\alpha \subset \beta$ and α works on $\mathscr{N}_{\Phi,U,V,W,\Theta}$.

The remaining definitions are intended to make explicit the ideas about links that we have sketched above.

Definition 3.3. A *$\mathscr{Q}$-use-link along ξ* (*$\mathscr{Q}$-gap-link along ξ*) is an interval of nodes $[\alpha, \beta]$ such that α works on $\mathscr{Q}_{\Phi,U,V,W}$, β works on $\mathscr{S}_Z$ with $\mathrm{Top}(\beta) = \alpha$, and $\beta\hat{\ }\langle u \rangle \subseteq \xi$ (or $\beta\hat{\ }\langle g \rangle \subseteq \xi$, respectively). A *$\mathscr{Q}$-link along ξ* refers to an interval of nodes which is either a $\mathscr{Q}$-use-link or a $\mathscr{Q}$-gap-link.

Definition 3.4. An *$\mathscr{N}$-link along ξ* is an interval of nodes $[\alpha, \beta]$ such that α works on $\mathscr{N}_{\Phi,U,V,W,\Theta}$, β works on $\mathscr{M}_{\Theta,y}$ with $\mathrm{Subtop}(\beta) = \alpha$, and $\beta\hat{\ }\langle 0 \rangle \subseteq \xi$.

Definition 3.5. A *$\mathscr{P}$-link along ξ* is an interval of nodes $[\alpha, \beta]$ such that α works on $\mathscr{P}_{x,0}$, β works on $\mathscr{P}_{x,m}$, and $\beta\hat{\ }\langle 0\rangle \subseteq \xi$, with β $\mathscr{Q}$-use-link-free along ξ (as will be defined in the next definition) and with β the least such.

Definition 3.6. Fix $\mathscr{R} \in \{\mathscr{Q}, \mathscr{N}, \mathscr{P}\}$, and let ξ be any node or branch in $\mathscr{T}$. Say that a node η is *$\mathscr{R}$-link-free along ξ* iff η is not contained in any $\mathscr{R}$-link along ξ. Say that a node η is *free along ξ* iff η is $\mathscr{R}$-link-free for all $\mathscr{R} \in \{\mathscr{Q}, \mathscr{N}, \mathscr{P}\}$.

Definition 3.7. Say that $\mathscr{N}_{\Phi,U,V,W,\Theta}$-strategy η is *refuted* (intuitively, is shown to be partial) along ξ (where ξ may be either a node or a branch) iff either $\eta\hat{\ }\langle 1\rangle \subseteq \xi$ or else there is some $\mathscr{M}_{\Phi,U,V,W,\Theta,y}$-strategy σ with $\sigma\hat{\ }\langle 0\rangle \subseteq \xi$.

Definition 3.8. Let T_0 be any (possibly finite) branch in $\mathscr{T}$. An *$\mathscr{R}$-link along T_0* is an $\mathscr{R}$-link along ξ for some $\xi \subseteq T_0$. A node η is *free along T_0* iff η is free along ξ for every $\xi \subseteq T_0$. $\mathscr{N}_\Theta$-strategy η is *refuted along T_0* iff η is refuted along ξ for some $\xi \subseteq T_0$.

Before defining satisfaction of a requirement along a node, we make two definitions intended to emphasize the connections between our tree structure and the different manners in which a strategy may be considered to be satisfied.

Definition 3.9. Let T_0 be a (possibly finite) branch through the tree $\mathscr{T}$. Say that requirement $\mathscr{P}_x$ has been *resolved in manner* (a) (*or* (b)) *along* T_0 iff η is a $\mathscr{P}_{x,0}$-strategy along T_0 and the condition following (a) (or (b), respectively) applies:

(a) ξ is a $\mathscr{Q}$-use-link-free $\mathscr{P}_{x,m}$-strategy and $\eta \subset \xi\hat{\ }\langle 0\rangle \subseteq T_0$;

(b) there is no such ξ as described in (a).

If $\mathscr{P}_{x,0}$ is resolved in manner (a), say that $\mathscr{P}_{x,0}$ is resolved by node $\xi\hat{\ }\langle 0\rangle$; otherwise, say that $\mathscr{P}_{x,0}$ is resolved by node η.

Definition 3.10. Let T_0 be a (possibly finite) branch through the tree $\mathscr{T}$. Say that requirement $\mathscr{Q}_{\Phi,U,V,W}$ is *resolved in manner* (n) *along* T_0 iff η is a $\mathscr{Q}_{\Phi,U,V,W}$-strategy free along T_0 and the condition following number (n) applies:

(I) $\eta\hat{\ }\langle 1\rangle \subset T_0$;

(II) there is some (longest) node ξ free along T_0 such that $\mathrm{Top}(\xi) = \eta$ and $\mathscr{N}_{\Phi,U,V,W,\Theta}$-strategy ξ is unrefuted along T_0;

(III) there is some (longest) node ξ free along T_0 such that $\mathrm{Top}(\xi) = \eta$ and $\xi\hat{\ }\langle g\rangle \subset T_0$;

(IV) there is some (longest) node ξ free along T_0 such that $\mathrm{Top}(\xi) = \eta$ and $\xi\hat{\ }\langle u\rangle \subset T_0$;

(V) each subrequirement of form $\mathscr{N}_\Theta$ and $\mathscr{S}_Z$ is assigned to some free node $\xi \subset T_0$ with $\mathrm{Top}(\xi) = \eta$.

If (I) applies, say that $\mathscr{Q}$ has been resolved by node $\eta\hat{\ }\langle 1\rangle$; if (II) applies, say that $\mathscr{Q}$ has been resolved by node ξ; if (III) or (IV) applies, say that $\mathscr{Q}$ has been resolved by node $\xi\hat{\ }\langle o\rangle$; and if (V) applies, say that $\mathscr{Q}$ has been resolved by node η. The *manner of resolution* of a free $\mathscr{N}_{\Phi,U,V,W,\Theta}$-strategy ξ along T_0 is either "unrefuted," in which case resolution is achieved by node ξ, or else "refuted," in which case resolution is achieved by the (unique) $\mathscr{M}_{\Theta,y}$-strategy σ with Subtop$(\sigma) = \xi$ and $\sigma\hat{\ }\langle 0\rangle \subset T_0$.

Note that this formal definition refers only to the sequence of nodes along a given branch. We have already hinted, in the previous section, at the intended semantic interpretations of these different manners of resolution. We will show in the Truth of Outcome Lemma 5.6 that these syntactic definitions do indeed correspond to the intended interpretations regarding the manner in which a requirement is satisfied in the course of the construction as inferred from the true path.

Definition 3.11. Define *satisfaction of a requirement along a node* as follows:

(i) $\mathscr{P}_{x,m}$ is satisfied along ξ if there is some $\mathscr{Q}$-use-link-free node $\eta \subset \xi$ already working on $\mathscr{P}_{x,m}$.
We continue assigning $\mathscr{P}_{x,m'}$ with $m' > m$ below $\eta\hat{\ }\langle 0\rangle$ because we may later find that η is contained in a $\mathscr{Q}$-use-link and its infinite outcome is being forced by a stronger-priority $\mathscr{Q}$-strategy.

(ii) $\mathscr{Q}_{\Phi,U,V,W}$ is satisfied along ξ if there is a free node $\eta \subset \xi$ (that is, η is free along ξ) already working on $\mathscr{Q}_{\Phi,U,V,W}$.

(iii) $\mathscr{N}_{\Phi,U,V,W,\Theta}$ is satisfied along ξ if either there is a free node $\eta \subset \xi$ already working on $\mathscr{N}_{\Phi,U,V,W,\Theta}$ (beneath free $\mathscr{Q}_{\Phi,U,V,W}$-strategy Top$(\eta) = \alpha \subset \eta$), or else if $\mathscr{Q}_{\Phi,U,V,W}$ is resolved along ξ in some manner among (I)–(IV).

(iv) $\mathscr{M}_{\Phi,U,V,W,\Theta,y}$ is satisfied along ξ if there is a $\mathscr{Q}$-use-link-free node $\eta \subset \xi$ already working on it with $\mathscr{Q}_{\Phi,U,V,W}$-strategy $\alpha =$ Top(η) free and $\mathscr{N}_{\Phi,U,V,W,\Theta}$-strategy $\beta =$ Subtop(η) free, or if $\mathscr{N}_\Theta$ is already refuted along ξ, or else if $\mathscr{Q}_{\Phi,U,V,W}$ is resolved along ξ in some manner among (I)–(IV).

(v) $\mathscr{S}_{\Phi,U,V,W,Z}$ is satisfied along ξ if either there is a free node $\eta \subset \xi$ already working on it (with the $\mathscr{Q}_{\Phi,U,V,W}$-strategy Top(η) above it free), or else if $\mathscr{Q}_{\Phi,U,V,W}$ is resolved along ξ in some manner among (I)–(IV).

Now define the tree of strategies $\mathscr{T}$ by assigning to each node $\xi \in \mathscr{T}$ the requirement of strongest priority which is not satisfied along ξ. The immediate successors of a node ξ which works on some $\mathscr{S}_Z$ are

$$\xi\hat{\ }\langle h\rangle <_L \xi\hat{\ }\langle g\rangle <_L \xi\hat{\ }\langle u\rangle <_L \xi\hat{\ }\langle f\rangle.$$

The immediate successors of a node ξ which works on any other type of (sub-) requirement are

$$\xi\hat{\ }\langle 0\rangle <_L \xi\hat{\ }\langle 1\rangle.$$

We now prove a lemma which shows that the way we have defined our tree of strategies will allow a satisfactory assignment of any requirement or subrequirement needed, along any infinite branch of the tree $\mathscr{T}$. We prove the lemma in this section rather than in the section containing the rest of the verification in order to emphasize that this lemma concerns only the definition of the tree of strategies, and does not depend in any way on the construction.

Assignment Lemma 3.12. *Let T_0 be any infinite branch through the tree $\mathcal{T}$.*
(i) *Every requirement of the form $\mathcal{P}_{x,m}$ is assigned to some (longest) node η which is $\mathcal{Q}$-use-link-free along T_0.*
(ii) *Every requirement of the form $\mathcal{Q}_{\Phi,U,V,W}$ is assigned to some (longest) node η which is free along T_0. Furthermore, $\mathcal{Q}_{\Phi,U,V,W}$ is resolved in some manner among (I)–(V).*

Proof. We prove (i) and (ii) by the same argument, using induction on the priority of the requirements. Fix a requirement $\mathcal{R}$ and assume that the lemma holds for all stronger priority requirements. Find the least node $\sigma \subset T_0$ long enough so that the manner of resolution for each stronger priority requirement of the form $\mathcal{Q}_{\Phi,U,V,W}$ has already been achieved. Eventually $\mathcal{R}$ will be assigned to some node τ with $\sigma \subseteq \tau \subset T_0$. Now by the way we chose σ, τ will be free along T_0. Examining the definition of satisfaction of a requirement along a node shows that once such a free node assigned to a requirement or subrequirement appears on T_0, no other node assigned to that requirement or subrequirement will appear on T_0, justifying our claim that such a node will be the longest such. □

The Assignment Lemma 3.12 will be applied in the verification with T_0 taken to be the true path TP.

4. The construction

The construction proceeds in stages and substages. At stage $s = 0$, let all sets be the empty sets, let all functionals other than Δ be entirely undefined, and initialize all nodes. We define $\Delta^A(x,s,t) = 0$ for all $x,s,t \in \omega$ so that uses are strictly increasing in each argument, with A initially taken to be the empty set. At the beginning of each stage of the construction, we enumerate whatever new axioms are necessary to reflect changes in the oracle set A, but also following the guideline that once we have defined $\Delta^A(x,s,t) = 1$ at some stage, all new axioms enumerated after that stage will maintain the value of $\Delta^A(x,s,t)$ as 1.

At stage $s+1$, we proceed through substages $t \leqslant s$. At substage $t = 0$, the empty node $\langle\ \rangle$ is eligible to act. At each substage t, a strategy β of length $|\beta| = t$ is eligible to act, and if $t < s$ (and if we are not instructed to end the stage), then we will determine a strategy ξ of length $t+1$ eligible to act next, and we initialize all strategies which are to the right of ξ. To define a parameter *big* means to give it a value larger than any number previously mentioned in the construction.

We now describe the action of the strategy eligible to act at substage t. The values of all sets, computations, functions and parameters are understood as objects in formation, and should be evaluated at the beginning of the substage of the given stage at which the given strategy is eligible to act. The definitions of functionals and their uses are always understood to refer to the particular functional being built at the node which β believes is the top node for satisfaction of the associated requirement.

We first make several definitions which will be used by the strategies.

The A-restraint applied against any node $\beta \in \mathscr{T}$ is

$$r^A(\beta,s) = \max\{\, \tilde{r}^A(\alpha,s) \mid \alpha < \beta \text{ works on } \mathscr{S}_Z \,\},$$

where the function $\tilde{r}^A(\alpha,-)$ for any $\mathscr{S}_Z$-strategy α will be defined in the construction. We will see that if α is an $\mathscr{S}_Z$-strategy, then we will have $\tilde{r}^A(\alpha,s) = s_0$ if α unsuccessfully closed a gap at stage s_0 and this is the most recent such stage and α has not yet reached the halt state (5) in the module, or $\tilde{r}^A(\alpha,s) = 0$ otherwise.

The B-restraint applied against any $\mathscr{S}_{\Phi,U,V,W,Z}$-strategy β is given by

$$r^B(\beta,s) = \max\{\, \tilde{r}^B(\alpha,s) \mid \alpha < \beta \text{ works on } \mathscr{N}_{\Phi,U,V,W,\Theta} \,\},$$

where the function $\tilde{r}^B(\alpha,-)$ for α an $\mathscr{N}_{\Phi,U,V,W,\Theta}$- strategy is defined by

$$\tilde{r}^B(\alpha,s) = \max\{\, \theta_\alpha^{B\oplus V}(y) \mid \Gamma_\alpha^V(y)\downarrow \,\}.$$

Note that such α work below the same $\mathscr{Q}_{\Phi,U,V,W}$-strategy as β.

If α works on $\mathscr{P}_{x,m}$, then define

$$\tilde{q}(\alpha,s) = \min\{\, \delta \mid \delta = \delta(x,s',t) > r^A(\alpha,s) \;\&\; t_l^\alpha < t \leqslant t_m^\alpha \;\&\; \Delta^A(x,s',t)\downarrow = 0 \,\},$$

where t_l^α and t_m^α will be defined in the construction, or simply define $\tilde{q}(\alpha,s) = s$ if the above set is empty. Now for any $\beta \in \mathscr{T}$, define

$$q(\beta,s) = \min\{\, \tilde{q}(\alpha,s) \mid \alpha \text{ works on } \mathscr{P}_{x,m} \text{ and } \alpha\hat{}\langle 0\rangle \subseteq \beta \,\}.$$

Say that a computation is *β-believable* iff its use is less than $q(\beta,s)$.

We now describe the actions of the various strategies.

Case 1: A $\mathscr{P}_{x,m}$-strategy β is eligible to act next.

Let t_l^β be given the value of the largest parameter $t_{m'}^\alpha$ associated with any $\mathscr{P}_{x,m'}$-strategy $\alpha < \beta$ such that α has ever been made eligible to act. Define parameter $t_m = t_m^\beta$ big if this is not already defined.

Let s' be the most recent stage at which β was eligible to act since its most recent initialization, or let $s' = 0$ if there is no such stage. We define the function

$$inc(\beta,s) = \max\{\, i \mid [t_m, t_m + i) \subseteq W_{x,s} \,\}.$$

Note that $inc(\beta,s)$ is nondecreasing in s (after its final initialization).

If $inc(\beta,s) > inc(\beta,s')$ or if there is some $\mathscr{P}_{x,m'}$-strategy α which is $\mathscr{Q}$-use-link-free along β and for which $\alpha\hat{}\langle 0\rangle \subseteq \beta$, then we put into A all uses of the form $\delta(x,s'',t)$ such that $t_l^\beta < t \leqslant t_m^\beta$, $r^A(\beta,s) < \delta(x,s'',t) < s$, and $\Delta^A(x,s'',t)\downarrow = 0$. We then change the definition of Δ^A to $\Delta^A(x,s'',t) = 1$ for all such values, and let $\beta\hat{}\langle 0\rangle$ be eligible to act next.

Otherwise, make $\beta\hat{}\langle 1\rangle$ eligible to act next.

Case 2: A $\mathscr{Q}_{\Phi,U,V,W}$-strategy β is eligible to act next.

Define the length of agreement function

$$\ell(\beta,s)=\max\{\,y \mid \forall x<y[\Phi^A(x)[s]\downarrow = U_s(x)\ \&\ \phi(x) \text{ is } \beta\text{-believable}]\,\}.$$

Then define S to be β-expansionary if $s=0$ or if $\ell(\beta,s)>s'$, the most recent β-expansionary stage.

If s is not β-expansionary, then let $\beta^\frown\langle 1\rangle$ be eligible to act next.

Suppose that s is β-expansionary. If β is not currently the top of any link, then let $\beta^\frown\langle 0\rangle$ be eligible to act next.

Suppose next that β is currently the top of a link with bottom node σ. First we conduct the "scouting report" that we have described in Section 2; that is, we determine which nodes we would make eligible to act if β were not currently the top of a link. If we would visit some node $\xi <_L \sigma$, then remove the link (β,σ) and let $\beta^\frown\langle 0\rangle$ be eligible to act next. Otherwise, let σ be eligible to act next.

Case 3: An $\mathscr{N}_{\Phi,U,V,W,\Theta}$-strategy β is eligible to act next.

Define the length of agreement function

$$\ell(\beta,s)=\max\{\,y \mid \forall x<y[\Theta^{B\oplus V}(x)[s]\downarrow = W_s(x)\ \&\ \theta(x) \text{ is } \beta\text{-believable}]\,\}.$$

Then define $m(\beta,s)=\max\{\,\ell(\beta,s') \mid s'<s\ \beta\text{-expansionary}\}$, and define s to be β-expansionary if $s=0$ or $\ell(\beta,s)>m(\beta,s)$.

If s is not β-expansionary, then let $\beta^\frown\langle 1\rangle$ be eligible to act next.

Suppose that s is β-expansionary. If there is some $\mathscr{M}_{\Theta,y}$-strategy $\sigma\supseteq\beta^\frown\langle 0\rangle$ which has a new infinitary outcome and which currently controls $\Gamma^V_\beta(y)$ with $\Gamma^V_\beta(y)$ defined at the previous β-stage, then let σ be eligible to act next, where σ is the least such, and we remove any link (τ,η) such that $\beta\subset\tau\subset\sigma^\frown\langle 1\rangle\subseteq\eta$. Otherwise, let $\beta^\frown\langle 0\rangle$ be eligible to act next.

Case 4: An $\mathscr{M}_{\Phi,U,V,W,\Theta,y}$-strategy β is eligible to act next.

If there is some strategy α which currently controls $\Gamma^V(y)$ with either $\alpha>\beta$ or else α contained in a $\mathscr{Q}$-use-link along β, then say that α releases control of $\Gamma^V(y)$, and β assumes control of $\Gamma^V(y)$. Now if $\theta^{B\oplus V}$ has increased since the previous β-stage, then let $\beta^\frown\langle 0\rangle$ be eligible to act next. (We refer to this as an "infinitary outcome" of β.)

Otherwise, if $\gamma^V(y)$ is defined and β-believable, then let $\beta^\frown\langle 1\rangle$ be eligible to act next.

Otherwise, define $\gamma^V(y)$ big if this is not already defined (and implicitly, define $\Gamma^V(y)\downarrow = W(y)$ as soon as this use is β-believable), and end all action at this stage.

Case 5: An $\mathscr{S}_{\Phi,U,V,W,Z}$-strategy β is eligible to act next.

The $\mathscr{S}_Z$-strategy β uses the module described below:

(1) If parameter z is currently undefined then choose a new value for z which is big (and hence is larger than the B- restraint imposed on β, and larger than any previous value of z); then wait for z to appear in set Z. If z has not yet entered Z, then make $\beta^\frown\langle f\rangle$ eligible to act next. If and when z appears in Z, then move

to (2) in the module, redefine $\tilde{r}(\beta,s) = 0$, make $\beta\hat{\ }\langle g\rangle$ eligible to act next, and say that we have "opened a gap". Create a link (τ,β) with bottom β and top $\tau = \mathrm{Top}(\beta)$.

(2) Wait for $\ell(\Phi^A, U) > z$. (Note that $z \in Z_{s'}$ implies that $z \leqslant s'$, so that $\ell(\beta,s) > s'$ implies that $\ell(\beta,s) > z$. Therefore by the construction, this will automatically be true by the next genuine β-stage.)

(3) If $\phi(z)$ is no longer β-believable, then "clear" $\phi(z)$ by putting $\delta = \delta(x,s'',t)$ into A for all δ associated with a $\mathscr{P}_{x,m}$-strategy σ such that
   (a) $\mathrm{Top}(\beta) = \tau \subset \sigma\hat{\ }\langle 0\rangle \subseteq \beta$,
   (b) $t_l^\sigma < t \leqslant t_m^\sigma$,
   (c) $\Delta^A(x,s'',t)\downarrow = 0$, and
   (d) $r^A(\sigma,s) < \delta \leqslant \phi(z)$,

where t_l^σ and t_m^σ are defined as in Case 2 of the construction. Return to (2) in the module, and make $\beta\hat{\ }\langle u\rangle$ eligible to act next. Otherwise (that is, if $\phi(z)$ *is* still β-believable), we close the gap, by moving to (4) in the module (during this same stage, and continuing as the module directs us).

(4) Remove the link (τ,β), and let s' be the stage at which the gap was opened by β. Now if $U_s \upharpoonright z \neq U_{s'} \upharpoonright z$, then put z into B, move to (5) in the module, and end all action at this stage. Otherwise (that is, if U has not permitted z), define $\Lambda \upharpoonright z = U \upharpoonright z$ for all such values of Λ not yet defined, redefine restraint $\tilde{r}(\beta,s) = s$, cancel this value of z, move to (1) in the module, and end all action at this stage.

(5) Make $\beta\hat{\ }\langle h\rangle$ eligible to act next. In this (halt) state, the strategy has now permanently established that $\overline{B} \neq Z$ (since now $z \in Z \cap B$), and therefore will never act again.

Now, at any β-substage, we identify the state in which we find the module and allow the module to change states if desired, following the prescribed action associated with that state.

This ends the description of the construction.

5. The verification

Define the apparent true path ATP as a function of the stage so that $ATP(s)$ is the longest node made eligible to act at stage s.

Define the true path TP to be the leftmost path through $\mathscr{T}$ visited infinitely often, that is, by induction on i, define

$$TP(i) = (\mu o \in \Omega)(\exists \beta \supseteq (TP \upharpoonright i)\hat{\ }\langle o\rangle)[\beta \text{ is eligible to act infinitely often}].$$

and define the genuine true path as

$$GTP = \{\, \alpha \in TP \mid \alpha \text{ is eligible to act infinitely often} \,\}.$$

Note that TP is automatically well-defined because the tree is finite-branching.

The first two lemmas argue that our linking procedures do not unduly disrupt the necessary flow of control.

$\mathscr{Q}$-Link Lemma 5.1. *Assume that $\alpha \in \mathscr{T}$ works on $\mathscr{Q}_{\Phi,U,V,W}$. If a $\mathscr{Q}$-link (α,σ) exists at the end of stage s_0 then one of the following two cases holds:*

(i) *The link is removed at some future genuine α-stage s_1 and we make some immediate successor of α eligible to act with link (α,σ) no longer in place at some stage $s_2 \geqslant s_1$; or*

(ii) *The link (α,σ) still exists and is traveled at all future genuine α-stages.*

Proof. This follows from the construction. Note that in part (i), we could have $s_2 = s_1$ only if we remove a link because of the scouting report. $\square$

It is conceivable that because the $\mathscr{N}$-links are created from top to bottom rather than vice versa, nodes along *TP* which are contained in an $\mathscr{N}$-link might not be visited infinitely often. The second lemma will help us to argue later that this is not the case.

$\mathscr{N}$-Link Lemma 5.2. *Assume that $\alpha \in \mathscr{T}$ works on $\mathscr{N}_{\Phi,U,V,W,\Theta}$.*

(i) *If $\mathscr{N}$-link (α,η) is removed at a stage and (α,ξ) is an $\mathscr{N}$-link at the next genuine α-stage, then $\xi < \eta$.*

(ii) *If $\mathscr{N}_\Theta$-strategy α is given infinitely many genuine α-stages then α cannot be the top of an $\mathscr{N}$-link cofinitely often.*

Proof. To prove (i), suppose that $\mathscr{N}$-link (α,η) is removed at stage s and that (α,ξ) is an $\mathscr{N}$-link at the next genuine α-stage. Say that η is an $\mathscr{M}_{\Theta,y}$-strategy, and that ξ is an $\mathscr{M}_{\Theta,y'}$-strategy (since these are the only type of nodes that α can link down to).

First we claim that $y' < y$. We cannot have $y' > y$ because $\Gamma_\alpha^V(y)$ undefined at stage s implies that $\Gamma_\alpha^V(y')$ is undefined for all $y' > y$, from stage s until at least the end of the next genuine α-stage. We cannot have $y' = y$ because only the (unique) leftmost such strategy controls $\Gamma_\alpha^V(y)$.

Furthermore, since η controlled $\Gamma_\alpha^V(y)$ at the α-stage previous to stage s, the node ξ controlling $\Gamma_\alpha^V(y')$ at stage s must have $\xi < \eta$, as claimed.

To prove (ii), suppose that α is the top of an $\mathscr{N}$-link(α,η) at stage s, with η an $\mathscr{M}_{\Theta,y}$-strategy. Then the link will be removed by the next genuine α-stage, if not sooner (either because of a new stronger priority link or else because *ATP* moves left of the link). By the proof of part (i), if α is the top of a new link (α,ξ) at the next genuine α-stage then we have $\xi < \eta$ with ξ an $\mathscr{M}_{\Theta,y'}$-strategy and $y' < y$. However, along any branch in $\mathscr{T}$ beneath α which is left of η, there are only finitely many such ξ. Therefore there are only finitely many possibilities for the bottom node of a new $\mathscr{N}$-link with top node α at the next α-stage, and this number is reduced each time we find a new link with top α and a new bottom node (assuming no interim stage at which α is not the top of a link). This proves the lemma. $\square$

We will prove the next four lemmas by simultaneous induction on $|\beta|$. All of these lemmas are easily verified for $\beta = \langle\ \rangle$. So, we assume that $\beta = \alpha\hat{}\langle o\rangle$.

True Path Lemma 5.3. *Assume that $\beta \subset TP$. Then*

(i) *If $ATP[s] <_L \beta$ infinitely often then there is some node $\eta <_L \beta$ which is made eligible to act infinitely often.*
(ii) *β is initialized only finitely often.*
(iii) *β's work space is infinite, i.e., β's believability function $q(\beta, s)$ is non-decreasing after its final initialization and goes to ∞ as stage s goes to ∞.*
(iv) *Some successor of β is eligible to act infinitely often.*
(v) *$\beta \in GTP \iff \beta$ is not contained in a $\mathscr{Q}$-use-link along TP.*

Proof. (i) If $\alpha = \beta^- \notin GTP$, then by the induction hypothesis of part (v) applied to α, α is contained in some $\mathscr{Q}$-use-link (τ, σ) along TP, which we then travel at cofinitely many of the stages at which $\alpha \subset ATP[s]$, implying that $\sigma \subset TP$, hence $\beta \subseteq \sigma$, and we have $\beta \subset ATP$ at cofinitely many α-stages. By the induction hypothesis applied to $\alpha \subset TP$, ATP cannot be left of α infinitely often, since otherwise we would have $\alpha \not\subset TP$. Therefore ATP cannot be left of β infinitely often.

Next suppose that $\alpha \in GTP$. If α works on $\mathscr{P}_{x,m}$, $\mathscr{M}_{\Theta,y}$ or $\mathscr{S}_Z$ then the claim is clear, since strategies for such requirements make some immediate successor eligible to act next at infinitely many of those stages for which they themselves are eligible to act. If α works on $\mathscr{Q}_{\Phi,U,V,W}$ then we use the $\mathscr{Q}$-Link Lemma 5.1 and induct on the stage. If α works on $\mathscr{N}_\Theta$, then we have that if ATP is left of $\beta = \alpha\hat{\ }\langle 1\rangle$ infinitely often, then $\eta = \alpha\hat{\ }\langle 0\rangle$ is made eligible to act infinitely often, since α cannot be the top of an $\mathscr{N}$-link cofinitely often, by the $\mathscr{N}$-Link Lemma 5.2. This proves (i).

Part (ii) follows from (i) and the fact that β is only initialized at stages when ATP is left of β.

(iii) suppose that τ works on $\mathscr{P}_{x,m}$ and that $\tau\hat{\ }\langle 0\rangle \subseteq \beta$. We must show that $\tilde{q}(\tau, s)$ is nondecreasing after its final initialization and goes to ∞. Note that by the Limit of Restraint Lemma 5.5, we have that $r^A(\tau, s)$ reaches a finite limit in s. The fact that $\tau\hat{\ }\langle 0\rangle \subset TP$, combined with part (v) applied to τ, implies that either τ is eligible to act infinitely often or else $\tau\hat{\ }\langle 0\rangle$ is contained in a $\mathscr{Q}$-use-link at cofinitely many of the stages at which $\tau \subset ATP[s]$. In either case, we will eventually put $\delta = \delta(x, s', t)$ into A for every such δ which is larger than $r^A(\tau, s)$ and has $t_l^\tau < t \leqslant t_m^\tau$, and will change the value of $\Delta^A(x, s', t)$ to 1 when we do so. This proves part (iii).

(iv) By the induction hypothesis applied to α, some successor of α is eligible to act infinitely often. If $\beta \notin GTP$ then this successor of α must also be a successor of β, and we are done. So, assume that $\beta \in GTP$.

If β is the top of a $\mathscr{Q}$-use-link which remains in place for the rest of the construction, then the bottom node of the link will be made eligible to act infinitely often. So assume that this is not the case.

Observe that if β is eligible to act at a stage, then either

(a) we make an immediate successor of β eligible to act at this stage, or
(b) we end all action at this stage, or
(c) β is the top node of a link and we make the bottom node of the link eligible to act next.

If case (a) occurs infinitely often then we are done, so assume otherwise.

Case (b) implies that either $|\beta| = s$, which can only happen once, or else β is an $\mathscr{M}_{\Theta,y}$-strategy waiting for a use to become β-believable, which by part (iii) can delay us from making an immediate successor eligible to act for only finitely many stages, or else β is an $\mathscr{S}_Z$-strategy which has just closed a gap, in which case β will make some immediate successor eligible to act at the next β-stage.

If case (c) applies, then by the $\mathscr{Q}$-Link Lemma 5.1, the $\mathscr{N}$-Link Lemma 5.2, and our assumption that β is not permanently the top of a $\mathscr{Q}$-use-link, therefore we will eventually have a genuine β-stage at which β is not the top of any link.

In either case (b) or (c), β will infinitely often make some immediate successor eligible to act, and therefore will make some immediate successor eligible to act infinitely often, contradicting our assumption that (a) does not hold. This proves (iv).

To prove (v), suppose first that β is contained in some $\mathscr{Q}$-use-link (τ, σ) along TP. Then the link (τ, σ) must be in place at infinitely many stages. Along with the fact that $\sigma \subset TP$, this implies that the link is in place at cofinitely many stages, and therefore $\beta \not\subseteq GTP$. This proves the implication going from left to right.

To prove the implication going from right to left, suppose that β is not contained in any $\mathscr{Q}$-use-link along TP. Then examination of the proof of part (iv), replacing our consideration in (iv) of β making a successor eligible to act with consideration in (v) of $\alpha = \beta^-$ making a successor eligible to act, shows that α will make some immediate successor eligible to act infinitely often, and by virtue of the fact that $\beta \subset TP$, therefore must make β eligible to act infinitely often. This completes the proof of the lemma. □

Respect of Restraint Lemma 5.4. *If $\beta \subset TP$ is never initialized after stage s_0 then the restraint $r^X(\beta, s)$ desired by β is never violated by any strategy $\xi \in \mathscr{T}$ at any stage $s \geqslant s_0$, for $X \in \{A, B\}$.*

Proof. By inductive hypothesis, we can choose a stage late enough so that $\tilde{r}^X(\alpha, s)$ is never thereafter violated for any node $\alpha < \beta$. Therefore it suffices to prove the lemma with $r^X(\beta, s)$ replaced by $\tilde{r}^X(\beta, s)$.

Obviously $\xi = \beta$ does not violate its own desired restraint. We consider the other possible locations of ξ on $\mathscr{T}$ relative to fixed node β. If $\xi > \beta$ then ξ is never allowed to violate $\tilde{r}^X(\beta, s)$. After stage s_0, no node $\xi <_L \beta$ will ever act again, so no node $\xi <_L \beta$ will violate $\tilde{r}^X(\beta, s)$. So, we need only argue that no node $\xi \subset \beta$ ever violates $\tilde{r}^X(\beta, s)$.

First fix $X = A$, and suppose that ξ is a $\mathscr{P}_{x,m}$-strategy (since this is the only type of strategy which puts elements into A). If $\xi\hat{\ }\langle 1 \rangle \subseteq \beta$ then ξ will not put any element into A after stage s_0 (since otherwise β would be initialized again). Suppose on the other hand that $\xi\hat{\ }\langle 0 \rangle \subseteq \beta$. By the Limit of Restraint Lemma 5.5 (i) applied to ξ, $r^A(\xi, s)$ will eventually reach a finite limit. Furthermore, strategy β "knows" what this limit is, in the sense of being initialized every time that $r^A(\xi, s)$ increases. That is, for any given $\mathscr{S}_Z$-strategy $\sigma \subset \xi$, ξ believes (and hence β also believes) that if $\sigma\hat{\ }\langle h\rangle \subseteq \xi$ or $\sigma\hat{\ }\langle g\rangle \subseteq \xi$ then $\lim_s \tilde{r}(\sigma,s)=0$, and if $\sigma\hat{\ }\langle u\rangle \subseteq \xi$ or $\sigma\hat{\ }\langle f\rangle \subseteq \xi$ then $\lim_s \tilde{r}(\sigma,s)$ is equal to the most recent stage at which σ unsuccessfully closed a gap.

Therefore, at any stage $s \geqslant s_0$, β has a correct lower bound on the elements that ξ may put into A. Furthermore, the activity of ξ corresponding to outcome $o=0$ ensures that this lower bound will be no lower than $q(\beta,s)$. However, for $s \geqslant s_0$, β will always have $\tilde{r}^A(\beta,s) < q(\beta,s)$. Therefore $\tilde{r}^A(\beta,s)$ will never be violated by $\xi \subset \beta$.

Next let $X = B$, and suppose that ξ is an $\mathscr{S}_Z$-strategy (since this is the only type of strategy that puts elements into B). Then ξ will only put an element into B if $\xi\hat{\ }\langle h\rangle \subset TP$, but then $\xi\hat{\ }\langle h\rangle \subseteq \beta$, so β was never eligible to act until after the stage s_1 at which ξ put an element into B, and therefore must have $\tilde{r}^B(\beta,s_1)=0$. This proves the lemma for $\xi \subset \beta$, and completes the proof of the lemma. □

Limit of Restraint Lemma 5.5. *Suppose that $\beta \subset TP$, and let S_β denote the set of genuine β-stages.*

(i) $\lim_{s\in S_\beta} r^A(\beta,s) < \infty$ *exists.*

(ii) *If β works on $\mathscr{S}_{\Phi,U,V,W,Z}$ and if B is the set being built by $\mathscr{Q}_{\Phi,U,V,W}$-strategy $\sigma = \mathrm{Top}(\beta)$, then $\lim_{s\in S_\beta} r^B(\beta,s) < \infty$ exists.*

Proof. By inductive hypothesis, it suffices to prove each part of the lemma with $\tilde{r}^X(\beta^-,s)$ in place of $r^X(\beta,s)$. If $\beta \notin GTP$ then S_β is finite, so assume that $\beta \in GTP$.

To prove (i), suppose that $\alpha = \beta^-$ works on $\mathscr{S}_Z$ and $\alpha\hat{\ }\langle g\rangle <_L \beta$, otherwise the claim is true by induction hypothesis. But then α's outcome indicates that the α-module closes a gap unsuccessfully only finitely often, so $\tilde{r}^A(\alpha,s)$ reaches a finite limit.

To prove (ii), first observe that by our tree architecture, any $\mathscr{N}_\Theta$-strategy τ which has stronger priority than β and which has $\mathrm{Top}(\tau) = \mathrm{Top}(\beta)$ will be either contained in a link along β, or else refuted (and thereby is itself the top of an $\mathscr{N}$-link) along β. In either case, τ will have only finitely many substrategies ξ of the form $M_{\Theta,y}$ with $\tau \subset \xi \subset \beta$. For each such ξ, we consider the cases $\xi\hat{\ }\langle 1\rangle \subseteq \beta$ and $\xi\hat{\ }\langle 0\rangle \subseteq \beta$ in turn.

If $\xi\hat{\ }\langle 1\rangle \subseteq \beta$ then the Truth of Outcome Lemma 5.6 (iv) applied to ξ ensures that ξ contributes only a finite amount of restraint to $r^B(\beta,s)$. Suppose that $\xi\hat{\ }\langle 1\rangle \subseteq \beta$ and $\xi \notin GTP$. Then by the True Path Lemma 5.3 (v), ξ is contained in some $\mathscr{Q}$-use-link (η,ρ) along TP, and hence along β. If $\tau = \mathrm{Subtop}(\xi) \subset \eta$ then any infinitary outcome of ξ would remove link (η,ρ), so we may assume that $\eta \subset \tau$. But then any increase in $\theta^V_\tau(y)$ would make $\Gamma^V_\tau(y)$ undefined, and η will not redefine this value as long as the link (η,ρ) prevents τ from being made eligible to act. This shows that ξ contributes only a finite amount of restraint to $r^B(\beta,s)$ in the case where $\xi\hat{\ }\langle 1\rangle \subseteq \beta$.

We next claim that if $\xi\hat{\ }\langle 0\rangle \subseteq \beta$ then $\Gamma^V_\tau(y)$ is undefined at each β-stage. This is true because either ξ was visited at this stage, in which case we made an actual measurement of $\theta^{B\oplus V}(y)$ and the Respect of Restraint Lemma 5.4 gives that an increase in this use must be due to V permission rather than due to B permission, and this V permission also makes $\Gamma^V(y)$ undefined; or else ξ was contained in a link at this stage, but also contained in $ATP(s)$, which implies that $\tau = \mathrm{Subtop}(\xi)$ was contained in the same link at this stage, and therefore did not redefine $\Gamma^V(y)$ since the most recent β-stage at which τ and ξ were not linked over (inducting on the number of β-stages since that stage).

Therefore, the B-restraint that each such τ above β applies against β reaches a finite limit along all β-stages, corresponding to the maximum of the limits of uses of form $\theta^{B\oplus V}(y)$ for stronger priority strategies ξ of form $M_{\Theta,y}$ with $\xi^\frown\langle 1\rangle \subseteq \beta$. □

Truth of Outcome Lemma 5.6. *Assume that $\beta = \alpha^\frown\langle o\rangle \subset TP$ and $\alpha \in GTP$.*

(i) *If α works on $\mathscr{P}_{x,m}$ then parameter t_m reaches a finite limit and*

$$o = 0 \iff [m, \infty) \subseteq W_x.$$

(ii) *If α works on $\mathscr{Q}_{\Phi,U,V,W}$ then*

$$o = 0 \iff \limsup_{s\in S_\alpha} \ell(\Phi^A, U) = \infty.$$

(iii) *If α works on $\mathscr{N}_\Theta$ then*

$$o = 0 \iff \limsup_{s\in S_\alpha} \ell(\Theta^{B\oplus V}, W) = \infty.$$

(iv) *If α works on $\mathscr{M}_{\Theta,y}$ beneath $\mathscr{N}_\Theta$-strategy σ, then*

$$o = 1 \iff \lim_s \theta_\sigma^{B\oplus V}(y) < \infty \text{ exists.}$$

(v) *If α works on $\mathscr{S}_{\Phi,U,V,W,Z}$ then*
  (a) $o = f \Rightarrow B \cup Z \neq \omega$.
  (b) $o = u \Rightarrow z = z(\alpha)$ *reaches a finite limit and* $\phi(z)$ *goes to* ∞.
  (c) $o = g \Rightarrow U = \Lambda$ *is recursive.*
  (d) $o = h \Rightarrow B \cap Z \neq \emptyset$.

Proof. Parameter t_m for $\mathscr{P}_{x,m}$-strategy $\alpha \in GTP$ reaches a finite limit because t_m is rechosen only as often as α is initialized, which by the True Path Lemma 5.3 is only finitely often. Parts (i)–(iv) are now easily verified.

To prove (v), assume that α is an $\mathscr{S}_Z$-strategy. If $o = f$, then α has found a number $z \in \overline{B}\cap\overline{Z}$, so $z \notin B\cup Z$. If $o = u$, then α must have opened a gap which it never closed. Therefore $z(\alpha)$ reaches a finite limit; and since $q(\alpha,s)$ goes to ∞ monotonically by the True Path Lemma 5.3 (iii) and since we infinitely often have $q(\alpha,s) \leqslant d < \phi(z)$ for some δ-use d, $\phi(z)$ also goes to ∞.

If $o = g$ then we must open and close infinitely many gaps; therefore we have dom $\Lambda = \omega$. To claim that $\Lambda = U$, we only need that α's desired A-restraint is never violated after the final stage at which α is initialized, but this follows from the Respect of Restraint Lemma 5.4. Finally, if $o = h$ then α has succeeded in putting some number z into B, whereby $z \in B \cap Z \neq \emptyset$. □

$\mathscr{Q}_{\Phi,U,V,W}$**-Satisfaction Lemma 5.7.** *All requirements of the form $\mathscr{Q}_{\Phi,U,V,W}$ are satisfied.*

Proof. Fix requirement $\mathscr{Q} = \mathscr{Q}_{\Phi,U,V,W}$. By the Assignment Lemma 3.12, there is some longest node $\alpha \subset TP$ which is assigned to $\mathscr{Q}_{\Phi,U,V,W}$ and which is free along TP, hence by the Truth of Outcome Lemma 5.6 (v), has $\alpha \in GTP$. We have furthermore that one of the manners of resolution among (I) – (V) described in Definition 3.10 applies.

If (I) applies, that is, if $\alpha\hat{}\langle 1\rangle \subset TP$, then by the Truth of Outcome Lemma 5.6 (ii), we have $\limsup_{s\in S_\alpha} \ell(\Phi^A, U \oplus W \oplus V) < \infty$; therefore $\mathscr{Q}$ is satisfied.

Suppose next that (II) applies, that is, there is some $\mathscr{N}_\Theta$- strategy $\xi \subset TP$ which is free along TP with $\mathrm{Top}(\xi) = \alpha$ and ξ unrefuted along TP. We claim that in this case, $\Gamma_\xi^V = W$. This is true because by the Truth of Outcome Lemma 5.6 (iii), we have that $\alpha\hat{}\langle 0\rangle \subset TP \Rightarrow \limsup_s \ell(\xi,s) = \infty$, but our assignment procedure along with part (iv) of the Truth of Outcome Lemma 5.6 implies that $\lim_s \theta^{B\oplus V}(y) < \infty$ exists for all $y \in \omega$, whereby Γ_ξ^V will be total. However, a Γ_ξ^V definition can be made incorrect only if ξ's desired B-restraint is injured, and by the Respect of Restraint Lemma 5.4, this will not occur after the final stage at which ξ is initialized. Therefore we have $\Gamma_\xi^V = W$, which satisfies $\mathscr{Q}$.

Suppose next that (III) or (IV) applies, that is, there is some $\mathscr{S}_Z$-strategy $\xi \subset TP$ which is free along TP with $\mathrm{Top}(\xi) = \alpha$, $\xi\hat{}\langle o\rangle \subset TP$, and $o \in \{g, u\}$. Note that ξ free along TP implies that $\xi \in GTP$. Therefore by either part (b) or part (c) of the Truth of Outcome Lemma 5.6 (v), we have that $\mathscr{Q}$ is satisfied.

Finally, assume that (V) applies, that is, each subrequirement of the form $\mathscr{N}_\Theta$ or $\mathscr{S}_Z$ is assigned to some free node $\xi \subset TP$ with $\mathrm{Top}(\xi) = \alpha$. In this case, any $\mathscr{N}_\Theta$-strategy ξ must be "refuted" along TP, that is, we either have $\xi\hat{}\langle 1\rangle \subset TP$, which by the Truth of Outcome Lemma 5.6 (iii) implies that $\Theta^{B\oplus V} \neq W$, or else there is some $\mathscr{M}_{\Theta,y}$-strategy $\eta \in GTP$ with $\xi \subset \eta\hat{}\langle 0\rangle \subset TP$, which by the Truth of Outcome Lemma 5.6 (iv) implies that $\lim_s \theta^{B\oplus V}(y) = \omega$ and therefore $\Theta^{B\oplus V} \neq W$. Likewise, our assignment procedure guarantees that any $\mathscr{S}_Z$-strategy $\xi \subset TP$ with $\mathrm{Top}(\xi) = \alpha$ must have outcome $o \in \{h, f\}$ where $\xi\hat{}\langle o\rangle \subset TP$. Now by the Truth of Outcome Lemma 5.6 (v) parts (a) and (d), this implies that $\overline{B} \neq Z$. We also have $B \leqslant_T U$, since B infinite implies that the set of genuine expansionary Q-stages is recursive, and we only put an element into set B at stage s_1 if that element was permitted by U between s_0 and s_1, where s_0 is the genuine expansionary Q-stage immediately preceeding s_1. Therefore we will have $B \leqslant_T U$ satisfying

$$\forall\Theta(W \neq \Theta^{B\oplus V}) \quad \text{and} \quad \forall Z(\overline{B} \neq Z),$$

so $\mathscr{Q}$ is satisfied. □

$\mathscr{P}_x$-Satisfaction Lemma 5.8. *All requirements of the form $\mathscr{P}_x$ are satisfied.*

Proof. We claim first that for any $x \in \omega$ and for any $t \in \omega$, we will have $\lim_s \Delta^A(x,s,t) < 2$. Note first that by the construction, there will be exactly one $\mathscr{P}_{x,m}$-strategy $\beta \in \mathscr{T}$ which permanently has $t \in (t_l^\beta, t_m^\beta]$, and we must have $\beta \leqslant TP \restriction |\beta|$. If $\beta <_L TP \restriction |\beta|$, then β is given only finitely many chances to change a $\Delta^A(x,s,t)$ definition to 1, so we will have $\lim_x \Delta^A(x,s,t) = 0$. Assume therefore that $\beta \subset TP$.

Recall that by the Limit of Restraint Lemma 5.5 (i), we will have $r = \lim_{s \in S_\beta} r^A(\beta, s)$ finite. Now if $\beta\hat{}\langle 0\rangle \subset TP$, then we will eventually change the definition of $\Delta^A(x,s,t)$ to 1 for all s such that $\delta(x,s,t) > r$; hence, we will have $\lim_s \Delta^A(x,s,t) = 1$. On the other hand if $\beta\hat{}\langle 1\rangle \subset TP$, then $\lim_s \Delta^A(x,s,t) = 0$. We have now shown that for all $x,t \in \omega$, $\lim_s \Delta^A(x,s,t) < 2$ exists, as claimed.

Suppose that W_x is cofinite, that is, $\mathrm{Cof}(x) = 1$. Then there is some $m' \in \omega$ such that $[m', \infty) \subseteq W_x$. By the Assignment Lemma 3.12 and by the Truth of Outcome Lemma 5.6 (i), we must have $\eta\hat{}\langle 0\rangle \subset TP$, where η is the unique $\mathscr{P}_{x,m'}$-strategy which is $\mathscr{Q}$-use-link-free along TP. Now by the construction, for any $\mathscr{P}_{x,m}$-strategy $\xi \supset \eta$, we will also have $\xi\hat{}\langle 0\rangle \subset TP$. Therefore $\lim_t \lim_s \Delta^A(x,s,t) = 1 = \mathrm{Cof}(x)$.

Suppose on the other hand that W_x is coinfinite, that is, $\mathrm{Cof}(x) = 0$. Then for all $m \in \omega$, we have $[m,\infty) \not\subseteq W_x$. We must show that there are only finitely many $\mathscr{P}_{x,m}$-strategies β with $\beta\hat{}\langle 0\rangle \subset TP$. By the Truth of Outcome Lemma 5.6, every $\mathscr{P}_{x,m}$-strategy $\beta \in GTP$ will have $\beta\hat{}\langle 1\rangle \subset TP$. Therefore, if argument $t \in (t_l^\beta, t_m^\beta]$ for some $\mathscr{P}_{x,m}$-strategy β which is either left of TP or else contained in GTP, then we will have $\lim_s \Delta^A(x,s,t) = 0$. So we need only to check $\mathscr{P}_{x,m}$-strategies β such that $\beta\hat{}\langle 0\rangle \subset TP$. Note that such β have $\beta \subset TP$ but $\beta \notin GTP$, and hence each such β is contained in some $\mathscr{Q}$-use-link along TP.

Such a $\mathscr{P}_{x,m}$-strategy β (with $\beta\hat{}\langle 0\rangle \subset TP$) can be contained in a $\mathscr{Q}$-use-link (τ,σ) along TP only if either the top node, $\mathscr{Q}$-strategy τ, has stronger priority than $P_{x,0}$, or else there is some $\mathscr{P}_{x,m'}$- strategy $\xi \in GTP$ with $\xi\hat{}\langle 0\rangle \subset TP$. However, there are only finitely many requirements of the form $\mathscr{Q}$ which have stronger priority than $\mathscr{P}_{x,0}$, and any $\mathscr{P}_{x,m'}$-strategy $\xi \in GTP$ will have $\xi\hat{}\langle 1\rangle \subset TP$, as noted above. Therefore there is some value $m' \in \omega$ and some $\mathscr{P}_{x,m'}$-strategy $\beta \in GTP$ such that for all $t \leqslant t_{m'}^\beta$, we have $\lim_s \Delta^A(x,s,t) = 0$. Therefore $\lim_t \lim_s \Delta^A(x,s,t) = 0 = \mathrm{Cof}(x)$. This proves that requirement $\mathscr{P}_x$ is satisfied, and completes the verification. □

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References

[1] S.B. Cooper, Minimal pairs and high recursively enumerable degrees, J. Symbolic Logic 39 (1974) 655–660.
[2] S.B. Cooper, The jump is definable in the structure of the degrees of unsolvability, Bull. Amer. Math. Soc. 23 (1990) 151–158.
[3] R.G. Downey, S. Lempp and R.A. Shore, Highness and bounding minimal pairs, Math. Logic Quart. 39 (1993) 475–491.
[4] R.G. Downey and R.A. Shore, Degree theoretical definitions of the low_2 recursively enumerable sets, J. Symbolic Logic 60 (1995) 727–756.
[5] R.G. Downey and M. Stob, Minimal pairs in initial segments of the recursively enumerable degrees, to appear.

[6] A.H. Lachlan, A recursively enumerable degree which will not split over all lesser ones, Ann. Math. Logic 9 (1975) 307–365.
[7] S. Lempp and M. Lerman, The decidability of the existential theory of the poset of r.e. degrees with jump relations, Advances in Math. to appear.
[8] D.A. Martin, Classes of recursively enumerable sets and degrees of unsolvability, Z. Math. Logik Grundlag. Math. 12 (1966) 295–310.
[9] R.W. Robinson, Interpolation and embedding in the recursively enumerable degrees, Ann. Math. 93 (2) (1971) 285–314.
[10] G.E. Sacks, On the degrees less than $\mathbf{0}'$, Ann. Math. 77 (2) (1963) 211–231.
[11] R.A. Shore, Defining jump classes in the degrees below $\mathbf{0}'$, Proc. Amer. Math. Soc. 104 (1988) 287–292.
[12] R.A. Shore and T.A. Slaman, Working below a low_2 recursively enumerable degree, Arch. Math. Logic 29 (1990) 201–211.
[13] R.A. Shore and T.A. Slaman, Working below a high recursively enumerable degree, J. Symbolic Logic 58 (1993) 824–859.
[14] R.I. Soare, Automorphisms of the lattice of recursively enumerable sets, Part II: Low sets, Ann. Math. Logic 22 (1982) 69–107.
[15] R.I. Soare, Recursively enumerable sets and degrees, in: Perspectives in Mathematical Logic, Omega Series (Springer, Berlin, 1987).