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Ladder operators for q-orthogonal polynomials ☆

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Abstract

The q-difference analog of the classical ladder operators is derived for those orthogonal polynomials arising from a class of indeterminate moments problem.

Keywords: Difference equations; q-Orthogonal polynomials; Indeterminate moments problem

1. Introduction

This work is a follow up to our work [4] where we derived raising and lowering operators for polynomials orthogonal with respect to absolutely continuous measures μ under certain smoothness assumptions of μ'. This approach goes back to [2,3,11]. The raising and lowering operators derived in these references are differential operators. It was later realized that a similar theory exists for polynomials orthogonal with respect to a measure with masses at the union of at most two geometric progressions, $\{aq^n, bq^n\}$, for some $q \in (0, 1)$ [7]. The corresponding theory for difference operators is in [9]. This material is reproduced in [8]. The raising and lowering operators involve two functions $A_n(x)$ and $B_n(x)$ which satisfy certain recurrence relations. In the case of differential operators we have demonstrated that the knowledge of $A_n(x)$ and $B_n(x)$ determines the polynomials uniquely in the cases of Hermite, Laguerre, and Jacobi polynomials, see [5]. This is done through recovering the properties of the polynomials including the three term recurrence relation which generates the polynomials. This work shows that the corresponding functions determine the polynomials in the cases of Stieltjes–Wigert and q-Laguerre polynomials.

The orthogonal polynomials which arise from indeterminate moment problems have discrete and absolutely continuous orthogonality measures [1]. In many instances it is more convenient to work with absolutely continuous measures [8, Chapter 21].

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In this work we derive raising and lowering operators for polynomials orthogonal with respect to absolutely continuous measures. We shall assume that $\{P_n(x)\}$ are monic orthogonal polynomials, so that

$$\int_0^\infty w(x)P_m(x)P_n(x)\,dx = \zeta_n\delta_{m,n}. \tag{1.1}$$

A weight function w leads to a potential u defined by

$$u(x) = -\frac{D_{q^{-1}}w(x)}{w(x)}, \tag{1.2}$$

where D_q is the q-difference operator

$$(D_q f)(x) = \frac{f(x) - f(qx)}{x - qx}. \tag{1.3}$$

Every monic sequence of orthogonal polynomials satisfies a three term recurrence relation of the form

$$(x - \alpha_n)P_n(x) = P_{n+1}(x) + \beta_n P_{n-1}(x). \tag{1.4}$$

We also write the monic polynomials $P_n(x)$ as follows:

$$P_n(x) = x^n + \mathsf{p}_1(n)x^{n-1} + \cdots$$

and it follows immediately from the three term recurrence relations (1.4) that

$$\alpha_n = \mathsf{p}_1(n) - \mathsf{p}_1(n+1). \tag{1.5}$$

A main result of this work is the following theorem.

Theorem 1.1. *Let*

$$A_n(x) := \frac{1}{\zeta_n}\int_0^\infty \frac{u(qx) - u(y)}{qx - y} P_n(y)P_n(y/q)w(y)\,dy, \tag{1.6}$$

$$B_n(x) := \frac{1}{\zeta_{n-1}}\int_0^\infty \frac{u(qx) - u(y)}{qx - y} P_n(y)P_{n-1}(y/q)w(y)\,dy. \tag{1.7}$$

Then we have the lowering relation

$$D_q P_n(x) = \beta_n A_n(x)P_{n-1}(x) - B_n(x)P_n(x). \tag{1.8}$$

Theorem 1.1 will be proved in Section 2 along with the difference equations satisfied by $A_n(x)$ and $B_n(x)$,

$$B_{n+1}(x) + B_n(x) = (x - \alpha_n)A_n(x) + x(q-1)\sum_{j=0}^{n} A_j(x) - u(qx), \tag{1.9}$$

$$1 + (x - \alpha_n)B_{n+1}(x) - (qx - \alpha_n)B_n(x) = \beta_{n+1}A_{n+1}(x) - \beta_n A_{n-1}(x). \tag{1.10}$$

The identities (1.9)–(1.10) will be referred to as the supplementary conditions.

Theorem 1.1 is the q-analogue of

$$P_n'(x) = \beta_n A_n(x)P_{n-1}(x) - B_n(x)P_n(x)$$

of [4]. See also [5] for a derivation of the supplementary conditions for the $q = 1$ case.

Let

$$L_{1,n} := D_q + B_n(x), \tag{1.11}$$

be the lowering operator. Thus (1.8) is

$$L_{1,n} P_n(x) = \beta_n A_n(x) P_{n-1}(x). \tag{1.12}$$

The raising operator can be found as follows: First replace $\beta_n P_{n-1}(x)$ in (1.8) by $(x - \alpha_n) P_n(x) - P_{n+1}(x)$, using (1.4), and then replace

$$-(x - \alpha_n) A_n(x) + B_n(x)$$

by

$$x(q-1) \sum_{j=0}^{n} A_j(x) - u(qx) - B_{n+1}(x),$$

using (1.9). An easy computation shows that

$$D_q P_n(x) = \left(B_{n+1}(x) + u(qx) - x(q-1) \sum_{j=0}^{n} A_j(x) \right) P_n(x) - A_n(x) P_{n+1}(x).$$

With the replacement of n by $n-1$ in above equation, the raising operator is

$$L_{2,n} := D_q + x(q-1) \sum_{j=0}^{n-1} A_j(x) - u(qx) - B_n(x)$$

and

$$L_{2,n} P_{n-1}(x) = -A_{n-1}(x) P_n(x).$$

It is useful to recall the following analogue of the product rule:

$$D_q\big(f(x)g(x)\big) = \big(D_q f(x)\big) g(x) + f(xq) D_q g(x). \tag{1.13}$$

The following lemma, whose proof is a calculus exercise, will be used in the proofs of our main results.

Lemma 1.2. *If the integrals*

$$\int_0^\infty f(x) g(x) \frac{dx}{x}, \quad \int_0^\infty f(x) g(qx) \frac{dx}{x},$$

exist, then the following q-analogue of integration by parts holds:

$$\int_0^\infty f(x) D_q g(x)\, dx = -\frac{1}{q} \int_0^\infty g(x) D_{q^{-1}} f(x)\, dx. \tag{1.14}$$

An immediate consequence of Lemma 1.2 and (1.1) is

$$\int_0^\infty u(y) P_n(y) P_n(y/q) w(y)\, dy = 0. \tag{1.15}$$

We also have

$$\int_0^\infty u(y) P_{n+1}(y) P_n(y/q) w(y)\, dy = \frac{1 - q^{n+1}}{1-q} q \zeta_n, \tag{1.16}$$

which follows from (1.13), (1.2), (1.14), and the fact that

$$D_q x^n = \frac{1-q^n}{1-q} x^{n-1}.$$

2. Proofs

We shall need the fact [8,13]

$$\zeta_n = \zeta_0\beta_1\beta_2\ldots\beta_n, \tag{2.1}$$

and the Christoffel–Darboux identity [8,13]

$$\sum_{k=0}^{n-1}\frac{P_k(x)P_k(y)}{\zeta_k} = \frac{P_n(x)P_{n-1}(y)-P_n(y)P_{n-1}(x)}{\zeta_{n-1}(x-y)}. \tag{2.2}$$

Proof of Theorem 1.1. Let $D_qP_n(x)=\sum_{k=0}^{n-1}c_{n,k}P_k(x)$. Then

$$\zeta_kc_{n,k}=\int_0^\infty P_k(y)w(y)D_qP_n(y)\,dy.$$

Applying Lemma 1.2, (1.13), we see that

$$\begin{aligned}q\zeta_kc_{n,k} &= -\int_0^\infty P_n(y)\big[\big(D_{q^{-1}}P_k(y)\big)w(y)+P_k(y/q)D_{q^{-1}}w(y)\big]\,dy\\ &= \int_0^\infty P_n(y)P_k(y/q)\left[-\frac{D_{q^{-1}}w(y)}{w(y)}\right]w(y)\,dy,\end{aligned}$$

where the orthogonality property was used in the last step. The definition of u (1.2) yields

$$q\zeta_kc_{n,k}=\int_0^\infty P_n(y)P_k(y/q)u(y)w(y)\,dy=-\int_0^\infty P_n(y)P_k(y/q)\big(u(qx)-u(y)\big)w(y)\,dy,$$

where we again used the orthogonality property in the last step. Therefore by the Christoffel–Darboux identity (2.2)

$$D_qP_n(x)=-\frac{1}{\zeta_{n-1}}\int_0^\infty P_n(y)\frac{u(qx)-u(y)}{qx-y}\big[P_n(x)P_{n-1}(y/q)-P_n(y/q)P_{n-1}(x)\big]w(y)\,dy$$

and (2.1), the theorem follows. □

Proof of (1.9). It is clear that

$$\begin{aligned}B_{n+1}(x)+B_n(x) &= \int_0^\infty \frac{u(qx)-u(y)}{\zeta_n(qx-y)}\big[P_{n+1}(y)P_n(y/q)+\beta_nP_n(y)P_{n-1}(y/q)\big]w(y)\,dy\\ &= I_1+I_2,\end{aligned}$$

where

$$\begin{aligned}I_1 &:= \frac{1}{\zeta_n}\int_0^\infty \frac{u(qx)-u(y)}{qx-y}(y/q-\alpha_n)P_n(y)P_n(y/q)w(y)\,dy,\\ I_2 &:= \frac{1}{\zeta_n}\int_0^\infty \frac{u(qx)-u(y)}{qx-y}\big[P_{n+1}(y)P_n(y/q)-P_n(y)P_{n+1}(y/q)\big]w(y)\,dy,\end{aligned}$$

after $\beta_nP_{n-1}(y/q)$ is replaced by $(y/q-\alpha_n)P_n(y/q)-P_{n+1}(y/q)$. It is easy to see that I_1 is given by

$$I_1 = (x-\alpha_n)A_n(x) - \frac{1}{\zeta_n q}\int_0^\infty \big(u(qx)-u(y)\big)P_n(y)P_n(y/q)w(y)\,dy = (x-\alpha_n)A_n(x) - q^{-n-1}u(qx),$$

where (1.15) and the fact that

$$P_j(y/q) = q^{-j}P_j(y) + \text{lower degree terms} \tag{2.3}$$

were used. To evaluate I_2 first note that (2.3) implies

$$\int_0^\infty P_j(y)P_j(y/q)w(y)\,dy = \zeta_j q^{-j}. \tag{2.4}$$

Next we apply the Christoffel–Darboux formula to

$$P_{n+1}(y)P_n(y/q) - P_n(y)P_{n+1}(y/q),$$

and replace $y - y/q$ by $(y-qx+qx)(1-1/q)$. Therefore we see that

$$\begin{aligned} I_2 &= x(q-1)\sum_{j=0}^n A_j(x) + \frac{1-q}{q}\int_0^\infty \big[u(qx)-u(y)\big]\sum_{j=0}^n \frac{P_j(y)P_j(y/q)}{\zeta_j}w(y)\,dy \\ &= x(q-1)\sum_{j=0}^n A_j(x) + \frac{1-q}{q}u(qx)\sum_{j=0}^n q^{-j} + \frac{1-q}{q}\int_0^\infty \sum_{j=0}^n \frac{P_j(y)P_j(y/q)}{\zeta_j}D_{q^{-1}}w(y)\,dy. \end{aligned}$$

Thus

$$I_2 = x(q-1)\sum_{j=0}^n A_j(x) + \frac{1-q}{q}u(qx)\frac{1-q^{-n-1}}{1-q^{-1}}.$$

Simplifying $I_1 + I_2$ we establish (1.9). □

Proof of (1.10). From the definition of $B_n(x)$ we see that

$$\begin{aligned} (x-\alpha_n)B_{n+1}(x) - (qx-\alpha_n)B_n(x) &= \int_0^\infty w(y)\frac{u(qx)-u(y)}{qx-y} \\ &\quad\times\left[\left(\frac{x-\alpha_n}{\zeta_n}\right)P_{n+1}(y)P_n(y/q) - \left(\frac{qx-\alpha_n}{\zeta_{n-1}}\right)P_n(y)P_{n-1}(y/q)\right]dy \\ &= \int_0^\infty w(y)\big[u(qx)-u(y)\big]\left[\frac{1}{\zeta_n}\left(\frac{1}{q} + \frac{y/q-\alpha_n}{qx-y}\right)P_{n+1}(y)P_n(y/q)\right. \\ &\quad\left. - \frac{1}{\zeta_{n-1}}\left(1 + \frac{y-\alpha_n}{qx-y}\right)P_n(y)P_{n-1}(y/q)\right]dy \\ &= -\frac{1}{\zeta_n q}\int_0^\infty w(y)u(y)P_{n+1}(y)P_n(y/q)\,dy \\ &\quad + \frac{1}{\zeta_n}\int_0^\infty w(y)\frac{u(qx)-u(y)}{qx-y}(y/q-\alpha_n)P_n(y/q)P_{n+1}(y)\,dy \\ &\quad + \frac{1}{\zeta_{n-1}}\int_0^\infty w(y)u(y)P_n(y)P_{n-1}(y/q)\,dy \end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{\zeta_{n-1}} \int_0^\infty w(y) \frac{u(qx) - u(y)}{qx - y} P_n(y) P_{n-1}(y/q)(y - \alpha_n)\, dy \\
&= - \frac{1}{\zeta_n q} \int_0^\infty w(y) u(y) P_{n+1}(y) P_n(y/q)\, dy \\
&+ \frac{1}{\zeta_n} \int_0^\infty w(y) \frac{u(qx) - u(y)}{qx - y} \big[P_{n+1}(y/q) + \beta_n P_{n-1}(y/q)\big] P_{n+1}(y)\, dy \\
&+ \frac{1}{\zeta_{n-1}} \int_0^\infty w(y) u(y) P_n(y) P_{n-1}(y/q)\, dy \\
&- \frac{1}{\zeta_{n-1}} \int_0^\infty w(y) \frac{u(qx) - u(y)}{qx - y} \big(P_{n+1}(y) + \beta_n P_{n-1}(y)\big) P_{n-1}(y/q)\, dy.
\end{aligned}
$$

The result follows after some simplifications using (1.16). □

3. Stieltjes–Wigert polynomials

The computations in this and the next section will show that $A_n(x)$ and $B_n(x)$ are rational in x and consequently (1.9) and (1.10), the supplementary conditions which are identities valid for all x would be particularly useful for the recovery of the recurrence coefficients. As we shall see, systems of apparently non-linear difference equations generated by equating the residues on both sides of (1.9) and (1.10), can be solved explicitly which ultimately determine the recurrence coefficients.

This is example of an indeterminate moment problem associated with the log-normal density. See [6] for a discussion of the associated moment problem. We take the weight to be

$$w(x) = c \exp\big[(\ln x)^2/(2 \ln q)\big], \quad 0 < x < \infty,\ 0 < q < 1,$$

where c is a positive constant which will not appear in subsequent computations.

An easy calculation shows that

$$u(x) = \frac{q}{1-q}\left(\frac{1}{x} - \frac{\sqrt{q}}{x^2}\right),$$

$$A_n(x) = \frac{R_n}{x^2}, \quad \text{where } R_n := \frac{1}{\zeta_n (1-q)\sqrt{q}} \int_0^\infty P_n(y) P_n(y/q) w(y) \frac{dy}{y},$$

$$B_n(x) = \frac{r_n}{x^2} - \frac{1-q^n}{1-q} \frac{1}{x}, \quad \text{where } r_n := \frac{1}{\zeta_{n-1} \sqrt{q}(1-q)} \int_0^\infty P_n(y) P_{n-1}(y/q) w(y) \frac{dy}{y}.$$

From the supplementary conditions, (1.9) and (1.10)

$$\frac{q^{n+1} + q^n - 2}{1-q} = R_n + (q-1) S_n - \frac{1}{1-q}, \tag{3.1}$$

$$r_{n+1} + r_n = -\alpha_n R_n + \frac{1}{\sqrt{q}(1-q)}, \tag{3.2}$$

$$0 = \alpha_n \frac{1 - q^{n+1}}{1-q} - \alpha_n \frac{1-q^n}{1-q} + r_{n+1} - q r_n, \tag{3.3}$$

$$\alpha_n (r_n - r_{n+1}) = \beta_{n+1} R_{n+1} - \beta_n R_{n-1}, \tag{3.4}$$

where $S_n := \sum_{j=0}^{n} R_j$, and

$$R_0 = \frac{1}{\sqrt{q}(1-q)} \frac{\int_0^\infty w(y)/y\,dy}{\int_0^\infty w(y)\,dy} = \frac{1}{1-q}. \tag{3.5}$$

A difference equation satisfied by R_n is found by subtracting (3.1) at "$n-1$" from the same at "n";

$$qR_n - R_{n-1} = -(1+q)q^{n-1}, \tag{3.6}$$

and since the "integrating factor" is q^{-n}, the unique solution is

$$R_n = \frac{q^n}{1-q}. \tag{3.7}$$

Note that (3.3) simplifies to

$$-\alpha_n q^n = r_{n+1} - qr_n. \tag{3.8}$$

Multiplying (3.2) by $1-q$, together with $R_n(1-q) = q^n$ and (3.8) one finds

$$r_n - qr_{n+1} = \frac{1}{\sqrt{q}}, \tag{3.9}$$

and since the "integrating factor" for this is q^n, the unique solution subject to $r_0 = 0$, is

$$r_n = \frac{1-q^{-n}}{(1-q)\sqrt{q}}, \tag{3.10}$$

which with (3.8) immediately gives

$$\alpha_n = \frac{q^{-n}}{\sqrt{q}}\bigl(q^{-n-1} + q^{-n} - 1\bigr). \tag{3.11}$$

Multiplying (3.4) by R_n and replacing $\alpha_n R_n$ with (3.2), we find the resulting first-order difference equation

$$r_{n+1}^2 - \frac{r_{n+1}}{\sqrt{q}(1-q)} - \left(r_n^2 - \frac{r_n}{\sqrt{q}(1-q)}\right) = \beta_{n+1}R_{n+1}R_n - \beta_n R_n R_{n-1},$$

where the solution with the initial conditions $r_0 = \beta_0 = 0$ is

$$r_n^2 - \frac{r_n}{\sqrt{q}(1-q)} = \beta_n R_n R_{n-1}. \tag{3.12}$$

This expresses β_n in terms of the subsidiary quantities r_n and R_n,

$$\beta_n = \frac{r_n}{R_n R_{n-1}}\left(r_n - \frac{1}{\sqrt{q}(1-q)}\right) = q^{-4n} - q^{-3n}. \tag{3.13}$$

In the next section we take a route for the computations of the recurrence coefficients which does not involve the determination of the analogous r_n and R_n.

4. q-Laguerre polynomials

This is also associated with an indeterminate moment problem at a level "higher" than the Stieltjes–Wigert polynomials, in the sense that when an appropriate limit of a parameter is taken, the q-Laguerre polynomials become the Stieltjes–Wigert polynomials. See [10]. We refer the reader to [12] for further information.

We take the weight to be

$$w(x) = \frac{x^\alpha}{(-x;q)_\infty}, \quad 0 < x < \infty,\ \alpha > -1,\ 0 < q < 1, \tag{4.1}$$

where

$$(z;q)_\infty := \prod_{k=0}^{\infty}\bigl(1-zq^k\bigr).$$

This weight leads to

$$u(x)=\frac{q}{1-q}\left(\frac{1-q^{-\alpha}}{x}+\frac{q^{-\alpha}}{x+q}\right),$$
$$\frac{u(qx)-u(y)}{qx-y}=\frac{1}{1-q}\left(\frac{q^{-\alpha}-1}{xy}-\frac{q^{-\alpha}}{(x+1)(y+q)}\right).$$

By definition,

$$\begin{aligned}A_n(x) &= \frac{q^{-\alpha}-1}{\zeta_n(1-q)x}\int_0^\infty P_n(y)P_n(y/q)\frac{w(y)}{y}\,dy-\frac{q^{-\alpha}}{\zeta_n(1-q)(x+1)}\int_0^\infty P_n(y)P_n(y/q)\frac{w(y)}{y+q}\,dy\\ &=: \frac{R_n}{x}-\frac{q^n}{(1-q)(x+1)},\end{aligned} \tag{4.2}$$

$$\begin{aligned}B_n(x) &= \frac{q^{-\alpha}-1}{(1-q)\zeta_{n-1}x}\int_0^\infty P_n(y)P_{n-1}(y/q)\frac{w(y)}{y}\,dy-\frac{q^{-\alpha}}{\zeta_{n-1}(1-q)(x+1)}\int_0^\infty P_n(y)P_{n-1}(y/q)\frac{w(y)}{y+q}\,dy\\ &=: \frac{r_n}{x}-\frac{q^{n-1}\mathsf{p}_1(n)}{x+1},\end{aligned} \tag{4.3}$$

where the evaluation of the second integrals in $A_n(x)$ and $B_n(x)$ will follow later.

Here is a computation of R_0. By definition,

$$R_0 := \frac{q^{-\alpha}-1}{1-q}\frac{\int_0^\infty w(y)\,dy/y}{\int_0^\infty w(y)\,dy}=\frac{q^{-\alpha}-1}{1-q}\frac{I(\alpha)}{I(\alpha+1)}=\frac{1}{1-q},$$

since

$$I(\alpha):=\int_0^\infty \frac{y^{\alpha-1}}{(-y;q)_\infty}\,dy=\frac{(q^{1-\alpha};q)_\infty}{(q;q)_\infty}\frac{\pi}{\sin\pi\alpha}.$$

Also note the identity

$$\frac{1}{(-x;q)_\infty(x+q)}=\frac{1}{q(1+x/q)(-x;q)_\infty}=\frac{1}{q(-x/q;q)_\infty}.$$

Hence,

$$\int_0^\infty P_n(y)P_n(y/q)\frac{w(y)}{y+q}\,dy=\int_0^\infty P_n(y)P_n(y/q)\frac{y^\alpha}{q(-y/q;q)_\infty}\,dy=q^\alpha\int_0^\infty P_n(qy)P_n(y)w(y)\,dy=q^{n+\alpha}\zeta_n,$$

and the result for $A_n(x)$ follows. Similarly,

$$\int_0^\infty P_n(y)P_{n-1}(y/q)\frac{w(y)}{y+q}\,dy=\int_0^\infty P_n(y)P_{n-1}(y/q)\frac{y^\alpha}{q(-y/q;q)_\infty}\,dy=q^\alpha\int_0^\infty P_n(qy)P_{n-1}(y)w(y)\,dy.$$

To complete the evaluation of the above integral, we note the identity

$$\begin{aligned}P_n(qy)&=P_n(qy)+q^nP_n(y)-q^nP_n(y)\\ &=q^nP_n(y)+q^ny^n+q^{n-1}\mathsf{p}_1(n)y^{n-1}+\cdots-q^n\bigl(y^n+\mathsf{p}_1(n)y^{n-1}+\cdots\bigr)\end{aligned}$$

$$= q^n P_n(y) + \mathsf{p}_1(n)\big(q^{n-1} - q^n\big)y^{n-1} + \cdots$$
$$= q^n P_n(y) + \mathsf{p}_1(n)\big(q^{n-1} - q^n\big)P_{n-1}(y) + \cdots.$$

Finally,

$$q^\alpha \int_0^\infty P_n(qy)P_{n-1}(y)w(y)\,dy = q^\alpha \int_0^\infty \big\{q^n P_n(y) + \mathsf{p}_1(n)\big(q^{n-1} - q^n\big)P_{n-1}(y) + \cdots\big\}P_{n-1}(y)w(y)\,dy$$
$$= \left(\frac{1}{q} - 1\right)\mathsf{p}_1(n)q^{n+\alpha}\zeta_{n-1},$$

and the result for $B_n(x)$ follows.

It turns out that for the q-Laguerre weight, the supplementary conditions produce 6 difference equations in contrast with the 4 in the previous example.

Now, by equating the residues for the simple poles at $x = 0$ and $x = -1$ in (1.9), we find

$$r_{n+1} + r_n = -\alpha_n R_n - \frac{1 - q^{-\alpha}}{1 - q}, \tag{4.4}$$

$$\mathsf{p}_1(n+1)q^n + \mathsf{p}_1(n)q^{n-1} = -\frac{1 + \alpha_n}{1 - q}q^n + \frac{1 - q^{n+1}}{1 - q} + \frac{q^{-\alpha}}{1 - q}, \tag{4.5}$$

respectively. We note here another identity involving R_n only, by equating the constant terms of (1.9) at $x = \infty$,

$$R_n - \frac{q^n}{1 - q} + (q - 1)S_n - \frac{1 - q^{n+1}}{1 - q} = 0, \tag{4.6}$$

where $S_n = \sum_{j=0}^n R_j$. A similar consideration on (1.10) shows that

$$\alpha_n(r_n - r_{n+1}) = \beta_{n+1}R_{n+1} - \beta_n R_{n-1}, \tag{4.7}$$

$$-(1 + \alpha_n)q^n \mathsf{p}_1(n+1) + (q + \alpha_n)q^{n-1}\mathsf{p}_1(n) = \frac{\beta_{n+1}q^{n+1} - \beta_n q^{n-1}}{1 - q}, \tag{4.8}$$

$$r_{n+1} - q r_n = -q^n \alpha_n - 1. \tag{4.9}$$

We use the fact that $\alpha_n = \mathsf{p}_1(n) - \mathsf{p}_1(n+1)$ to rewrite (4.5) as a first-order difference equation,

$$\mathsf{p}_1(n+1)q^{n+1} - \mathsf{p}_1(n)q^{n-1} = q^n + q^{n+1} - \big(1 + q^{-\alpha}\big)$$

which has an “integrating factor” q^{n-1} by inspection. Hence the above equation becomes

$$\mathsf{p}_1(n+1)q^{2n} - \mathsf{p}_1(n)q^{2n-2} = (1 + q)q^{2n-1} - \big(1 + q^{-\alpha}\big)q^{n-1}, \tag{4.10}$$

and we find via a telescopic sum and the initial condition $\mathsf{p}_1(0) = 0$,

$$\mathsf{p}_1(n+1)q^{2n} = \frac{1 - q^{2n+2}}{q(1 - q)} - \big(1 + q^{-\alpha}\big)\frac{1 - q^{n+1}}{q(1 - q)}.$$

Therefore,

$$(1 - q)\mathsf{p}_1(n) = -q + \big(1 + q^{-\alpha}\big)q^{-n+1} - q^{-2n-\alpha+1}, \tag{4.11}$$

and Eq. (1.5) gives

$$\alpha_n = \mathsf{p}_1(n) - \mathsf{p}_1(n+1) = q^{-2n-1-\alpha}\big(1 + q - q^{n+1} - q^{n+\alpha+1}\big). \tag{4.12}$$

At this stage R_n can be found from a difference equation obtained by subtracting (4.6) at “n” from the same at “$n + 1$,”

$$q R_{n+1} - R_n = q^{n+1} - q^n, \tag{4.13}$$

with the initial condition $R_0 = 1/(1-q)$. Having now determined α_n in (4.12), r_n can be found from (4.9) with the initial condition $r_0 = 0$. We proceed to the determination of β_n. Multiply (4.8) by the "integrating factor" q^n and by $1-q$,

$$-(1+\alpha_n)q^{2n}(1-q)\mathsf{p}_1(n+1) + (q+\alpha_n)q^{2n-1}(1-q)\mathsf{p}_1(n) = \beta_{n+1}q^{2n+1} - \beta_n q^{2n-1}. \tag{4.14}$$

The left-hand side of the above is simplified to

$$\begin{aligned}&(1-q)q^{2n}\big(\mathsf{p}_1(n) - \mathsf{p}_1(n+1)\big) - \alpha_n(1-q)\big(\mathsf{p}_1(n+1)q^{2n} - \mathsf{p}_1(n)q^{2n-1}\big)\\ &\quad = (1-q)q^{2n}\alpha_n\big[1 - \big(\mathsf{p}_1(n+1) - \mathsf{p}_1(n)/q\big)\big].\end{aligned}$$

With (4.11), the term $\mathsf{p}_1(n+1) - \mathsf{p}_1(n)/q$ simplifies to

$$1 - q^{-2n-\alpha-1},$$

and consequently (4.14) reduces to the first-order difference equation, with the initial condition $\beta_0 = 0$,

$$\beta_{n+1}q^{2n+1} - \beta_n q^{2n-1} = (1-q)q^{-1-\alpha}\alpha_n. \tag{4.15}$$

Taking a telescopic sum, and noting that $\sum_{j=0}^{n-1}\alpha_j = -\mathsf{p}_1(n)$,

$$\beta_n q^{2n-1} = (1-q)q^{-1-\alpha}\sum_{j=0}^{n-1}\alpha_n = -(1-q)q^{-1-\alpha}\mathsf{p}_1(n). \tag{4.16}$$

Finally,

$$\begin{aligned}\beta_n &= -q^{-2n-\alpha}(1-q)\mathsf{p}_1(n) = -q^{-2n-\alpha}\big(-q + \big(1+q^{-\alpha}\big)q^{1-n} - q^{-2n-\alpha+1}\big)\\ &= q^{-4n-2\alpha+1}\big(1-q^n\big)\big(1-q^{n+\alpha}\big).\end{aligned} \tag{4.17}$$

It is interesting to note that in the computations of α_n and β_n, not all the 6 equations are required. We have used only (4.5) and (4.8). However, for an explicit expression of the q-ladder operators and therefore the determination of r_n and R_n we need Eqs. (4.6), (4.9) and α_n in (4.12).

We end with the remark that in the case of the classical Laguerre polynomials, $\alpha_n = 2n+1+\alpha$, and $\beta_n = \sum_{j=0}^{n-1}\alpha_j = n(n+\alpha)$, and this is analogous to (4.12) and (4.16)–(4.17), however, with appropriate modifications in the q-case.

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