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Approximation and numerical realization of 2D contact problems with Coulomb friction and a solution-dependent coefficient of friction

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Abstract

The paper analyzes discrete contact problems with the Coulomb law of friction which involves a solution-dependent coefficient of friction $\mathscr{F}$. Solutions to these problems are defined as fixed points of an auxiliary mapping. It is shown that there exists at least one solution provided that $\mathscr{F}$ is bounded and continuous in $\mathbb{R}^1_+$. Further, conditions guaranteeing uniqueness of the solution are studied. The paper is completed by numerical results of several model examples.

MSC: 74M15; 49M29; 65K10

Keywords: Coulomb friction; Contact problems

0. Introduction

Contact mechanics is a special branch of mechanics of solids analyzing the behavior of loaded deformable bodies which are in mutual contact. Besides unilateral conditions one has to take account the influence of friction on contacting parts. In spite of the fact that Coulomb friction is a classical one, mathematical analysis remained open for a long time for the following reason: the mathematical model leads to a nontrivial *implicit* variational inequality of elliptic type for displacements or to *a quasivariational* inequality for contact stresses [4]. To overcome mathematical difficulties related to this problem, regularized versions such as a nonlocal or a normal compliance friction law were considered [13,14]. The existence of a solution for a local Coulomb friction law was established for the first time in [15] by using *a fixed point approach*. It was shown that for a sufficiently small coefficient of Coulomb friction which does not depend on a solution there exists at least one solution. In [5] the authors used another technique based on a simultaneous *penalization* of unilateral conditions and a *regularization* of the frictional term. This technique is powerful from the theoretical point of view but not very convenient for computations. Indeed, after a discretization one obtains a system of nonlinear algebraic equations which depends on two small parameters. It turns out that the computational process depends strongly on their choice [6]. Nowadays the fixed point approach is preferred as a basis for numerical realization

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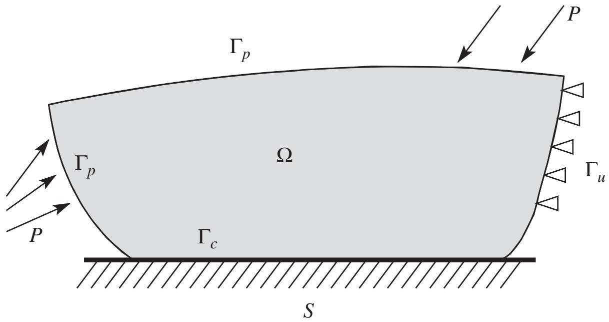

Fig. 1. Geometry of the problem.

of contact problems with Coulomb friction. A possible way how to find fixed points which characterize solutions to contact problems with Coulomb friction is to express the corresponding weak formulation in the form of a generalized equation which can be solved by methods of non-smooth optimization [1,16]. Another way for finding fixed points is a classical *method* of *successive approximations*. Each iterative step leads to a contact problem with given friction whose mathematical model is given by an *elliptic inequality of the second kind* [8,10], i.e. a problem which is much simpler. The efficiency of this approach depends, among others, how efficiently particular iterative steps can be realized. Using their formulation in terms of displacements one gets, after a suitable discretization, a constrained minimization problem for *a non-smooth* function and linear inequality constraints. To avoid difficulties with non-differentiability of the minimized function, a dual formulation in terms of contact stresses which leads to *a smooth* quadratic programming problem with *simple* (*box*) constraints is preferred [10].

A coefficient $\mathscr{F}$ of Coulomb friction is usually assumed to be independent of solutions to the problem. From experiments it is known that $\mathscr{F}$ may depend on the tangential component of contact displacements (or on the tangential velocity in quasistatic problems). Existence of solutions to contact problems with Coulomb friction involving a solution-dependent coefficient $\mathscr{F}$ was proven in [5]. The authors used again the method of a simultaneous penalization and a regularization. The discrete version of this approach was theoretically analyzed in [12]. For the reasons mentioned above we prefer a fixed point approach also in the case when $\mathscr{F}$ depends on a solution. This paper extends results from [11] where the model with given friction was studied. We will focus solely on the discrete case, i.e. no convergence analysis will be done. In Section 1, we introduce definitions of a classical and a weak solution to the problem. Further, we give an equivalent fixed point formulation for a mapping Φ from a convex set X into itself. The set X is a Cartesian product of two positive cones in the trace space defined on the contact part and its dual. Section 2 deals with an appropriate discretization Φ_{hH} of Φ which is based on a mixed finite element approximation of contact problems with given friction and a coefficient which does not depend on a solution. Displacements and contact stresses are approximated by piecewise linear, piecewise constant functions, respectively. Fixed points of Φ_{hH} are considered to be solutions of discrete contact problems with Coulomb friction and a solution-dependent coefficient $\mathscr{F}$. We will prove that fixed points of Φ_{hH} exist for any continuous, positive and bounded function $\mathscr{F}$ in $\mathbb{R}^1_+$. In addition, if $\mathscr{F}$ is small enough and Lipschitz continuous with a sufficiently small modulus of Lipschitz continuity, the mapping Φ_{hH} is *contractive* in the domain of its definition. We also prove that the property "to be contractive" is mesh dependent. Section 3 is devoted to numerical realization of the problem which uses the method of successive approximations. We recall briefly a dual formulation of each iterative step. Finally, numerical results of several model examples will be shown in Section 4.

1. Setting of the problem

A plane elastic body is represented by a bounded domain $\Omega \subset \mathbb{R}^2$ whose Lipschitz boundary $\partial\Omega$ is a union of three non-empty, non-overlapping parts Γ_u, Γ_p and Γ_c: $\partial\Omega = \overline{\Gamma}_u \cup \overline{\Gamma}_p \cup \overline{\Gamma}_c$. The body is fixed on Γ_u, surface tractions of density P act on Γ_p, while the rigid foundation S supports Ω along Γ_c. Next we shall suppose that $S = \mathbb{R}^2_- = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \leqslant 0\}$ and Γ_c is a straight line segment placed on the x_1-axis, i.e. there is no gap between Ω and S (see Fig. 1).

In addition, the influence of friction on Γ_c will be taken into account. Finally, Ω is subject to body forces of density F. Our aim is to find an equilibrium state of Ω.

This state is characterized by a displacement vector $u=(u_1,u_2)$ which satisfies the following system of equations and boundary conditions (a summation convention is adopted):

Equilibrium equations:

$$\frac{\partial \tau_{ij}}{\partial x_j}+F_i=0 \quad \text{in } \Omega;\quad i=1,2. \tag{1.1}$$

A stress tensor $\tau=(\tau_{ij})_{1\leqslant i,j\leqslant 2}$ is related to a linearized strain tensor $\varepsilon:=\varepsilon(u)=(\varepsilon_{ij}(u))_{1\leqslant i,j\leqslant 2}$ by means of a linear Hooke law

$$\tau_{ij}=c_{ijkl}\varepsilon_{kl}(u),\quad i,j,k,l=1,2;\quad \varepsilon_{kl}(u)=\frac{1}{2}\left(\frac{\partial u_k}{\partial x_l}+\frac{\partial u_l}{\partial x_k}\right). \tag{1.2}$$

Elasticity coefficients $c_{ijkl}\in L^{\infty}(\Omega)$ satisfy symmetry and ellipticity conditions

$$c_{ijkl}=c_{jikl}=c_{klij} \quad \text{a.e. in } \Omega,$$

$$\exists \alpha=\text{const.}>0:\ c_{ijkl}\xi_{ij}\xi_{kl}\geqslant \alpha\xi_{ij}\xi_{ij} \quad \forall \xi_{ij}=\xi_{ji}\in\mathbb{R}^1,\ \text{ a.e. in } \Omega.$$

Kinematical boundary conditions:

$$u_i=0 \quad \text{on } \Gamma_u,\ \ i=1,2. \tag{1.3}$$

Compatibility of τ with surface tractions P:

$$T_i:=\tau_{ij}\nu_j=P_i \quad \text{on } \Gamma_p,\ \ i=1,2, \tag{1.4}$$

where $\nu=(\nu_1,\nu_2)$ is the unit outward normal vector to $\partial\Omega$.

Taking into account the geometry of Γ_c and S, unilateral and friction conditions read as follows:

Unilateral conditions:

$$u_2\geqslant 0,\quad T_2\geqslant 0,\quad u_2T_2=0 \ \text{ on } \Gamma_c. \tag{1.5}$$

The Coulomb law of friction:

$$\left.\begin{aligned} u_1(x)=0 &\ \Rightarrow\ |T_1(x)|\leqslant \mathscr{F}(0)T_2(x);\\ u_1(x)\neq 0 &\ \Rightarrow\ T_1(x)=-\mathscr{F}(|u_1(x)|)\,T_2(x)\,\mathrm{sign}\,u_1(x),\quad x\in\Gamma_c. \end{aligned}\right\} \tag{1.6}$$

Here $\mathscr{F}$ denotes a coefficient of Coulomb friction, which depends on a solution u. Throughout this paper the coefficient $\mathscr{F}$ will be represented by a non-negative and bounded function in $\mathbb{R}^1_+$:

$$\exists \mathscr{F}_{\max}>0:\ 0\leqslant\mathscr{F}(t)\leqslant\mathscr{F}_{\max}\ \ \forall t\geqslant 0 \tag{1.7a}$$

satisfying certain smoothness assumptions. In this section dealing with a continuous setting of the problem we will suppose that $\mathscr{F}$ is Lipschitz continuous in $\mathbb{R}^1_+$:

$$\exists l>0:\ |\mathscr{F}(t_1)-\mathscr{F}(t_2)|\leqslant l|t_1-t_2| \quad \forall t_1,t_2\in\mathbb{R}^1_+. \tag{1.7b}$$

By *a classical solution* of a contact problem with Coulomb friction we mean any displacement vector u which satisfies (1.1)–(1.6).

Before we give the definition of a weak solution, we introduce the following sets:[1]

$$V=\{v\in H^1(\Omega)\mid v=0 \text{ on } \Gamma_u\},$$

$$\mathbb{V}=V\times V,$$

[1] To simplify notation we use the symbol $H^{-1/2}(\Gamma_c)$ to denote the dual space to $H^{1/2}(\Gamma_c)$ although in the literature same notation stands for the dual to $H^{1/2}_{00}(\Gamma_c)$.

$$K = \{v = (v_1, v_2) \in \mathbb{V} \mid v_2 \geqslant 0 \text{ a.e. on } \Gamma_c\},$$

$$H^{1/2}(\Gamma_c) = \{\varphi \in L^2(\Gamma_c) \mid \exists v \in V : \; v = \varphi \text{ on } \Gamma_c\},$$

$$H^{-1/2}(\Gamma_c) = (H^{1/2}(\Gamma_c))' \text{ the dual of } H^{1/2}(\Gamma_c),$$

$$H_+^{1/2}(\Gamma_c) = \{\varphi \in H^{1/2}(\Gamma_c) \mid \varphi \geqslant 0 \text{ a.e. on } \Gamma_c\},$$

$$H_+^{-1/2}(\Gamma_c) = \{\mu \in H^{-1/2}(\Gamma_c) \mid \langle \mu, \varphi \rangle \geqslant 0 \; \forall \varphi \in H_+^{1/2}(\Gamma_c)\}.$$

Here and in what follows the symbol $\langle \, , \, \rangle$ stands for a duality pairing between $H^{1/2}(\Gamma_c)$ and $H^{-1/2}(\Gamma_c)$.

Further, denote

$$a(u, v) := \int_\Omega \tau_{ij}(u)\varepsilon_{ij}(v)\,\mathrm{d}x,$$

$$L(v) := \int_\Omega F_i v_i \,\mathrm{d}x + \int_{\Gamma_p} P_i v_i \,\mathrm{d}s \quad u, v \in \mathbb{V},$$

where $F \in (L^2(\Omega))^2$, $P \in (L^2(\Gamma_p))^2$ and $\tau_{ij}(u) = c_{ijkl}\varepsilon_{kl}(u)$.

By *a weak solution* to the problem formulated above we mean any displacement vector u satisfying the following implicit variational inequality:

$$\left.\begin{array}{l}\text{Find } u \in K \text{ such that} \\ a(u, v-u) + \langle \mathscr{F} \circ |u_1|\, T_2(u), |v_1| - |u_1| \rangle \geqslant L(v-u) \quad \forall v \in K,\end{array}\right\} \tag{$\mathscr{P}$}$$

where $T_2(u) = \tau_{2j}(u)v_j$ is the normal contact stress on Γ_c. We now give an equivalent definition of ($\mathscr{P}$), which will be based on a fixed point approach.

Denote $X = H_+^{1/2}(\Gamma_c) \times H_+^{-1/2}(\Gamma_c)$. With any pair $(\varphi, g) \in X$ we associate the auxiliary problem

$$\left.\begin{array}{l}\text{Find } u := u(\varphi, g) \in K \text{ such that} \\ a(u, v-u) + \langle \mathscr{F} \circ \varphi g, |v_1| - |u_1| \rangle \geqslant L(v-u) \quad \forall v \in K.\end{array}\right\} \tag{$\mathscr{P}(\varphi, g)$}$$

It is well-known that ($\mathscr{P}(\varphi, g)$) has *a unique* solution u for any $(\varphi, g) \in X$. Problem ($\mathscr{P}(\varphi, g)$) is a weak formulation of a contact problem with *given* friction and a coefficient $\mathscr{F}_\varphi := \mathscr{F} \circ \varphi$ which *does not* depend on a solution (for more details see [4,10]). This makes it possible to define the mapping $\Phi : X \mapsto X$ by

$$\Phi(\varphi, g) = (|u_{1|_{\Gamma_c}}|, T_2(u)), \quad (\varphi, g) \in X, \tag{1.8}$$

where $u = (u_1, u_2)$ is a solution of ($\mathscr{P}(\varphi, g)$) and $T_2(u) \in H_+^{-1/2}(\Gamma_c)$ is the corresponding normal contact stress. The symbol $u_{1|_{\Gamma_c}}$ stands for the trace of u_1 on Γ_c.

Comparing ($\mathscr{P}$) and ($\mathscr{P}(\varphi, g)$) we see that u is a solution to ($\mathscr{P}$) if and only if a pair $(|u_{1|_{\Gamma_c}}|, T_2(u))$ is *a fixed point* of Φ in X

$$\Phi(|u_{1|_{\Gamma_c}}|, T_2(u)) = (|u_{1|_{\Gamma_c}}|, T_2(u)).$$

Below we recall briefly the mixed formulation of ($\mathscr{P}(\varphi, g)$). To simplify our presentation we will suppose that the non-negative slip bound g belongs to $L^2(\Gamma_c)$ and set $\widetilde{X} = H_+^{1/2}(\Gamma_c) \times L_+^2(\Gamma_c)$ (for more details we refer to [10]). Denote by

$$J(v) = \frac{1}{2}a(v, v) + \int_{\Gamma_c} \mathscr{F} \circ \varphi g |v_1| \,\mathrm{d}x_1 - L(v)$$

the total potential energy functional. It is well-known that a solution u of ($\mathscr{P}(\varphi, g)$) can be also characterized as follows:

$$J(u) = \min_{v \in K} J(v) = \inf_{v \in \mathbb{V}} \sup_{\substack{\mu_1 \in \Lambda_1(\varphi, g) \\ \mu_2 \in \Lambda_2}} \mathscr{L}(v, \mu_1, \mu_2),$$

where

$$\mathscr{L}(v, \mu_1, \mu_2) = \frac{1}{2}a(v, v) - L(v) - \langle \mu_1, v_1 \rangle - \langle \mu_2, v_2 \rangle$$

is the Lagrangian of our problem and

$$\Lambda_1(\varphi, g) = \{\mu \in L^2(\Gamma_c) \mid |\mu| \leqslant \mathscr{F} \circ \varphi g \text{ a.e. on } \Gamma_c\},$$

$$\Lambda_2 = H_+^{-1/2}(\Gamma_c).$$

By *a mixed variational formulation* of $(\mathscr{P}(\varphi, g))$, $(\varphi, g) \in \widetilde{X}$ given, we mean a problem of finding *a saddle-point* of $\mathscr{L}$ on $\mathbb{V} \times \Lambda_1(\varphi, g) \times \Lambda_2$ which is equivalent to

$$\left.\begin{array}{l}\text{Find } (u, \lambda_1, \lambda_2) \in \mathbb{V} \times \Lambda_1(\varphi, g) \times \Lambda_2 \text{ such that} \\ a(u, v) = L(v) + \langle \lambda_1, v_1 \rangle + \langle \lambda_2, v_2 \rangle \quad \forall v \in \mathbb{V}, \\ \langle \mu_1 - \lambda_1, u_1 \rangle + \langle \mu_2 - \lambda_2, u_2 \rangle \geqslant 0 \quad \forall \mu_1 \in \Lambda_1(\varphi, g), \quad \mu_2 \in \Lambda_2. \end{array}\right\} \qquad (\mathscr{M}(\varphi, g))$$

It is known (see [10]) that $(\mathscr{M}(\varphi, g))$ has a unique solution for any $(\varphi, g) \in \widetilde{X}$. In addition, u solves $(\mathscr{P}(\varphi, g))$, $\lambda_1 = T_1(u)$ and $\lambda_2 = T_2(u)$ on Γ_c. This enables us to give an alternative definition of $\widetilde{\Phi} := \Phi_{|\widetilde{X}}$, namely

$$\widetilde{\Phi}(\varphi, g) = (|u_{1|\Gamma_c}|, \lambda_2), \quad (\varphi, g) \in \widetilde{X}, \tag{1.9}$$

where λ_2 is the last component of the solution to $(\mathscr{M}(\varphi, g))$.

Remark 1.1. If the slip bound g belongs to $L_+^2(\Gamma_c)$ then (1.7b) can be replaced by the following continuity assumption on $\mathscr{F}$:

$$\mathscr{F} \in C(\overline{\mathbb{R}}_+^1). \tag{1.7c}$$

2. Discretization of contact problems with Coulomb friction

This part deals with a discretization of the problem formulated in Section 1. We will define an appropriate approximation of the mapping $\Phi : X \mapsto X$ whose fixed points will be considered to be solutions of a discrete contact problem with Coulomb friction and a solution dependent coefficient of friction. To simplify our presentation, we will suppose that Ω is *polygonal*. In this section, we will suppose that $\mathscr{F}$ satisfies (1.7a) and (1.7c).

Let $\mathscr{T}_h$ be a triangulation of $\overline{\Omega}$ and $V_h \subset V$ be the space of continuous, piecewise linear functions over $\mathscr{T}_h$:

$$V_h = \{v_h \in C(\overline{\Omega}) \mid v_{h|_T} \in P_1(T) \ \forall T \in \mathscr{T}_h, \ v_h = 0 \text{ on } \Gamma_u\},$$

$$\mathbb{V}_h = V_h \times V_h.$$

Further, let

$$\mathscr{V}_h = \{\varphi_h \in C(\overline{\Gamma}_c) \mid \exists v_h \in V_h : v_h = \varphi_h \text{ on } \Gamma_c\}$$

be the trace space on Γ_c of functions from V_h and

$$\mathscr{V}_h^+ = \{\varphi_h \in \mathscr{V}_h \mid \varphi_h \geqslant 0 \text{ on } \Gamma_c\}.$$

By $\mathscr{T}_H$ we denote a partition of $\overline{\Gamma}_c$ into segments S_i, $i \in \mathscr{I}$, whose lengths do not exceed H. With $\mathscr{T}_H$ we associate the space L_H of piecewise constant functions over $\mathscr{T}_H$, i.e.

$$L_H = \{\mu_H \in L^2(\Gamma_c) \mid \mu_{H|_{S_i}} \in P_0(S_i) \ \forall i \in \mathscr{I}\}.$$

The set

$$\Lambda_H = \{\mu_H \in L_H \mid \mu_H \geqslant 0 \text{ a.e. on } \Gamma_c\}$$

will be used as a natural discretization of $H_+^{-1/2}(\Gamma_c)$. Finally, let $X_{hH} = \mathscr{V}_h^+ \times \Lambda_H$ be the discretization of X.

For any $(\varphi_h, g_H) \in X_{hH}$ we define the problem

$$\left.\begin{array}{l}\text{Find } (u_h, \lambda_H) \in \mathbb{V}_h \times \Lambda_H \text{ such that}\\ a(u_h, v_h - u_h) + [\mathscr{F} \circ \varphi_h g_H, |v_{h1}| - |u_{h1}|] \geqslant\\ \quad L(v_h - u_h) + [\lambda_H, v_{h2} - u_{h2}] \quad \forall v_h \in \mathbb{V}_h,\\ [\mu_H - \lambda_H, u_{h2}] \geqslant 0 \quad \forall \mu_H \in \Lambda_H,\end{array}\right\} \qquad (\mathscr{M}(\varphi_h, g_H)_h^H)$$

where

$$[\mu_H, z_h] := \int_{\Gamma_c} \mu_H z_h \, dx_1, \quad \mu_H \in L_H, \quad z_h \in V_h.$$

Remark 2.1. Problem $(\mathscr{M}(\varphi_h, g_H))_h^H$ is a mixed finite element approximation of a contact problem with given friction and a coefficient $\mathscr{F}_{\varphi_h} := \mathscr{F} \circ \varphi_h$. The unilateral constraint $u_2 \geqslant 0$ on Γ_c is released by means of Lagrange multipliers from Λ_H. The last inequality in $(\mathscr{M}(\varphi_h, g_H))_h^H$ says that $u_h \in K_{hH}$, where

$$K_{hH} = \left\{ v_h = (v_{h1}, v_{h2}) \in \mathbb{V}_h \,\middle|\, \int_{S_i} v_{h2} \, dx_1 \geqslant 0 \ \forall i \in \mathscr{I} \right\}, \tag{2.1}$$

i.e. the unilateral condition on Γ_c is satisfied in a weak (integral) sense.

In what follows we shall suppose that the following condition is satisfied:

$$\mu_H \in L_H, \quad [\mu_H, z_h] = 0 \quad \forall z_h \in V_h \Rightarrow \mu_H = 0. \tag{2.2}$$

If it is so, $(\mathscr{M}(\varphi_h, g_H))_h^H$ has a unique solution for any $(\varphi_h, g_H) \in X_{hH}$. One of possible ways how to guarantee the satisfaction of (2.2) is to use a partition $\mathscr{T}_H$ which is coarser than $\mathscr{T}_{h|_{\Gamma_c}}$ (see [10]).

Since $(\mathscr{M}(\varphi_h, g_H))_h^H$ has a unique solution (u_h, λ_H) for any $(\varphi_h, g_H) \in X_{hH}$, one can define a mapping $\Phi_{hH} : X_{hH} \mapsto X_{hH}$ by

$$\Phi_{hH}(\varphi_h, g_H) = (r_h |u_{h1|_{\Gamma_c}}|, \lambda_H), \quad (\varphi_h, g_H) \in X_{hH}, \tag{2.3}$$

where r_h is the Lagrange interpolation operator by means of piecewise linear functions over the partition of $\overline{\Gamma}_c$ generated by $\mathscr{T}_{h|_{\Gamma_c}}$. Since λ_H can be viewed to be an approximation of $T_2(u)$ on Γ_c, the mapping Φ_{hH} can be viewed as a discretization of $\tilde{\Phi}$ from (1.9).

Analogously to the continuous setting, any fixed point of Φ_{hH} in X_{hH} will be called a *solution* of a (discrete) contact problem with Coulomb friction and a solution-dependent coefficient of friction.

Next we will show that Φ_{hH} has at least one fixed point for any $\mathscr{F}$ satisfying (1.7a), (1.7c) and we will examine conditions under which the fixed point is unique.

To this end, the space $\mathscr{V}_h \times L_H$ will be equipped with the norm

$$\|(\varphi_h, \mu_H)\| := \|\varphi_h\|_{0,\Gamma_c} + \|\mu_H\|_{-1/2,h}, \quad (\varphi_h, \mu_H) \in \mathscr{V}_h \times L_H, \tag{2.4}$$

where

$$\|\mu_H\|_{-1/2,h} = \sup_{\substack{z_h \in V_h \\ z_h \neq 0}} \frac{[\mu_H, z_h]}{\|z_h\|_{1,\Omega}}. \tag{2.5}$$

Let us observe that in view of (2.2), $\| \ \|_{-1/2,h}$ defined by (2.5) is a mesh dependent dual norm in L_H.

To prove the existence of a fixed point of Φ_{hH} we will use the Brower fixed point theorem. The following result is straightforward:

Lemma 2.1. *The mapping* $\Phi_{hH} : X_{hH} \mapsto X_{hH}$ *defined by* (2.3) *is continuous.*

It remains to show that Φ_{hH} maps a closed, bounded convex subset of X_{hH} into itself. This is what we will do now. Inserting $v_h = 0$ and $2u_h$ into the first inequality in $(\mathscr{M}(\varphi_h, g_H))_h^H$ we obtain

$$\alpha \|u_h\|_{1,\Omega}^2 \leqslant a(u_h, u_h) + [\mathscr{F} \circ \varphi_h g_H, |u_{h1}|] = L(u_h) \leqslant \|L\|_\star \|u_h\|_{1,\Omega} \tag{2.6}$$

as follows from Korn's inequality, the non-negativeness of the frictional term [,] and the fact that $[\lambda_H, u_{h2}] = 0$. The symbol $\|L\|_\star$ denotes the dual norm of L. The trace theorem and (2.6) yield

$$\| |u_{h1}| \|_{0,\Gamma_c} = \|u_{h1}\|_{0,\Gamma_c} \leqslant c_1 \|u_{h1}\|_{1,\Omega} \leqslant \frac{c_1}{\alpha} \|L\|_\star, \tag{2.7}$$

where c_1 is the norm of the trace mapping from V into $L^2(\Gamma_c)$.

Further,

$$\|r_h|u_{h1}|\|_{0,\Gamma_c} \leqslant \|r_h|u_{h1}| - |u_{h1}|\|_{0,\Gamma_c} + \|u_{h1}\|_{0,\Gamma_c} \leqslant ch\|u_{h1}\|_{1,\Gamma_c} + \|u_{h1}\|_{0,\Gamma_c} \leqslant c_2\|u_{h1}\|_{0,\Gamma_c}, \tag{2.8}$$

making use of the approximation properties of r_h and the inverse inequality between $L^2(\Gamma_c)$ and $H^1(\Gamma_c)$ for functions from $\mathscr{V}_h$.

Remark 2.2. If the partition $\mathscr{T}_{h_{|\Gamma_c}}$ belonged to a family of *strongly regular* partitions of $\overline{\Gamma}_c$, the constant c_2 in (2.8) would be independent of h (see [2]).

From (2.7) and (2.8) we obtain the following estimate for $r_h|u_{h1}|$:

$$\|r_h|u_{h1}|\|_{0,\Gamma_c} \leqslant R_1 := \frac{c_1 c_2}{\alpha} \|L\|_\star. \tag{2.9}$$

Let $\overset{\circ}{\mathbb{V}}_h \subset \mathbb{V}_h$ be a subspace of $\mathbb{V}_h$ defined by

$$v_h \in \overset{\circ}{\mathbb{V}}_h \quad \Leftrightarrow \quad v_h = (0, v_{h2}), \quad v_{h2} \in V_h. \tag{2.10}$$

Since

$$a(u_h, v_h) + [\mathscr{F} \circ \varphi_h g_H, |v_{h1}|] \geqslant L(v_h) + [\lambda_H, v_{h2}]$$

holds for every $v_h \in \mathbb{V}_h$, we have

$$a(u_h, v_h) = L(v_h) + [\lambda_H, v_{h2}] \quad \forall v_h \in \overset{\circ}{\mathbb{V}}_h.$$

Therefore,

$$\begin{aligned} \|\lambda_H\|_{-1/2,h} = \sup_{v_{h2} \in V_h} \frac{[\lambda_H, v_{h2}]}{\|v_{h2}\|_{1,\Omega}} &\leqslant \|a\|\|u_h\|_{1,\Omega} + \|L\|_\star \\ &\leqslant R_2 := \left(\frac{\|a\|}{\alpha} + 1\right) \|L\|_\star, \end{aligned} \tag{2.11}$$

making use of (2.6).

We proved the following result.

Lemma 2.2. *The mapping Φ_{hH} maps $X_{hH} \cap B$ into itself, where $B = \{(\varphi_h, \mu_H) \in \mathscr{V}_h \times L_H |\ \|\varphi_h\|_{0,\Gamma_c} \leqslant R_1$, $\|\mu_H\|_{-1/2,h} \leqslant R_2\}$, and R_1, R_2 are the same as in* (2.9), (2.11), *respectively*.

On the basis of Lemmas 2.1 and 2.2 we arrive at the existence result.

Theorem 2.1. *Discrete contact problems with Coulomb friction and a solution-dependent coefficient of friction have at least one solution for any coefficient $\mathscr{F}$ satisfying* (1.7a) *and* (1.7c).

Next, we will analyze under which assumptions on $\mathscr{F}$, the mapping Φ_{hH} is *contractive*. In addition to (1.7a), we will suppose that $\mathscr{F}$ satisfies (1.7b).

Let $(\varphi_h, g_H), (\overline{\varphi}_h, \overline{g}_H) \in X_{hH} \cap B$, where B is the same as in Lemma 2.2, and $(u_h, \lambda_H), (\overline{u}_h, \overline{\lambda}_H)$ be the solutions of $(\mathscr{M}(\varphi_h, g_H))_h^H$, $(\mathscr{M}(\overline{\varphi}_h, \overline{g}_H))_h^H$, respectively. Restricting ourselves to test functions $v_h \in K_{hH}$ we obtain

$$\left.\begin{aligned} &a(u_h, v_h - u_h) + [\mathscr{F} \circ \varphi_h g_H, |v_{h1}| - |u_{h1}|] \geqslant L(v_h - u_h), \\ &a(\overline{u}_h, v_h - \overline{u}_h) + [\mathscr{F} \circ \overline{\varphi}_h \overline{g}_H, |v_{h1}| - |\overline{u}_{h1}|] \geqslant L(v_h - \overline{u}_h). \end{aligned}\right\} \tag{2.12}$$

Inserting $v_h := \overline{u}_h \in K_{hH}$ into $(2.12)_1$ and $v_h := u_h \in K_{hH}$ into $(2.12)_2$ and summing both inequalities we have

$$\begin{aligned} \alpha\|u_h - \overline{u}_h\|_{1,\Omega}^2 &\leqslant a(u_h - \overline{u}_h, u_h - \overline{u}_h) \leqslant \|\mathscr{F} \circ \varphi_h g_H - \mathscr{F} \circ \overline{\varphi}_h \overline{g}_H\|_{0,\Gamma_c} \|u_{h1} - \overline{u}_{h1}\|_{0,\Gamma_c} \\ &\leqslant c_1 \|\mathscr{F} \circ \varphi_h g_H - \mathscr{F} \circ \overline{\varphi}_h \overline{g}_H\|_{0,\Gamma_c} \|u_{h1} - \overline{u}_{h1}\|_{1,\Omega}, \end{aligned} \tag{2.13}$$

where $c_1 > 0$ is the same as in (2.7), and consequently

$$\||u_{h1}| - |\overline{u}_{h1}|\|_{0,\Gamma_c} \leqslant \|u_{h1} - \overline{u}_{h1}\|_{0,\Gamma_c} \leqslant c_1\|u_h - \overline{u}_h\|_{1,\Omega} \leqslant \frac{c_1^2}{\alpha} \|\mathscr{F} \circ \varphi_h g_H - \mathscr{F} \circ \overline{\varphi}_h \overline{g}_H\|_{0,\Gamma_c}. \tag{2.14}$$

The right-hand side of (2.14) can be estimated as follows:

$$\begin{aligned} \|\mathscr{F} \circ \varphi_h g_H - \mathscr{F} \circ \overline{\varphi}_h \overline{g}_H\|_{0,\Gamma_c} &\leqslant \|\mathscr{F} \circ \varphi_h (g_H - \overline{g}_H)\|_{0,\Gamma_c} + \|(\mathscr{F} \circ \varphi_h - \mathscr{F} \circ \overline{\varphi}_h)\overline{g}_H\|_{0,\Gamma_c} \\ &\leqslant \mathscr{F}_{\max} \|g_H - \overline{g}_H\|_{0,\Gamma_c} + l\|\varphi_h - \overline{\varphi}_h\|_{C(\overline{\Gamma}_c)} \|\overline{g}_H\|_{0,\Gamma_c}, \end{aligned} \tag{2.15}$$

making use of (1.7a) and (1.7b). Since $\mathscr{V}_h$ and L_H are finite-dimensional, there exist constants $c_3, c_4 > 0$ such that

$$\left.\begin{aligned} &\|\varphi_h\|_{C(\overline{\Gamma}_c)} \leqslant c_3 \|\varphi_h\|_{0,\Gamma_c} \quad && \forall\, \varphi_h \in \mathscr{V}_h, \\ &\|\mu_H\|_{0,\Gamma_c} \leqslant c_4 \|\mu_H\|_{-1/2,h} \quad && \forall\, \mu_H \in L_H. \end{aligned}\right\} \tag{2.16}$$

This, (2.14) and (2.15) lead to

$$\||u_{h1}| - |\overline{u}_{h1}|\|_{0,\Gamma_c} \leqslant \|u_{h1} - \overline{u}_{h1}\|_{0,\Gamma_c} \leqslant \frac{c_1^2 c_4}{\alpha} \mathscr{F}_{\max} \|g_H - \overline{g}_H\|_{-1/2,h} + \frac{c_1^2 c_3 c_4}{\alpha} l R_2 \|\varphi_h - \overline{\varphi}_h\|_{0,\Gamma_c} \tag{2.17}$$

using that $\|\overline{g}_H\|_{-1/2,h} \leqslant R_2$.

Since r_h enjoys the monotonicity property, one can easily verify that

$$|r_h(|u_{h1}| - |\overline{u}_{h1}|)| \leqslant r_h |u_{h1} - \overline{u}_{h1}| \quad \text{on } \Gamma_c.$$

Hence

$$\begin{aligned} \|r_h|u_{h1}| - r_h|\overline{u}_{h1}|\|_{0,\Gamma_c} &\leqslant \|r_h|u_{h1} - \overline{u}_{h1}|\|_{0,\Gamma_c} \leqslant ch\|u_{h1} - \overline{u}_{h1}\|_{1,\Gamma_c} + \|u_{h1} - \overline{u}_{h1}\|_{0,\Gamma_c} \\ &\leqslant c_2 \|u_{h1} - \overline{u}_{h1}\|_{0,\Gamma_c} \end{aligned}$$

arguing as in (2.8). This, together with (2.17) imply the following estimate:

$$\|r_h|u_{h1}| - r_h|\overline{u}_{h1}|\|_{0,\Gamma_c} \leqslant \frac{c_1^2 c_2 c_4}{\alpha} \mathscr{F}_{\max} \|g_H - \overline{g}_H\|_{-1/2,h} + \frac{c_1^2 c_2 c_3 c_4}{\alpha} l R_2 \|\varphi_h - \overline{\varphi}_h\|_{0,\Gamma_c}. \tag{2.18}$$

Inserting $v_h \in \mathring{\mathbb{V}}_h$ into $(\mathscr{M}(\varphi_h, g_H))_h^H$ and $(\mathscr{M}(\overline{\varphi}_h, \overline{g}_H))_h^H$ we have

$$a(u_h, v_h) = L(v_h) + [\lambda_H, v_{h2}],$$

$$a(\overline{u}_h, v_h) = L(v_h) + [\overline{\lambda}_H, v_{h2}].$$

Subtracting these two equations we obtain

$$\begin{aligned} \|\lambda_H - \overline{\lambda}_H\|_{-1/2,h} &= \sup_{\substack{v_{h2} \in V_h \\ v_{h2} \neq 0}} \frac{[\lambda_H - \overline{\lambda}_H, v_{h2}]}{\|v_{h2}\|_{1,\Omega}} \leqslant \|a\| \|u_h - \overline{u}_h\|_{1,\Omega} \\ &\leqslant \frac{\|a\| c_1 c_4}{\alpha} \mathscr{F}_{\max} \|g_H - \overline{g}_H\|_{-1/2,h} + \frac{\|a\| c_1 c_3 c_4}{\alpha} l R_2 \|\varphi_h - \overline{\varphi}_h\|_{0,\Gamma_c} \end{aligned} \tag{2.19}$$

using (2.14), (2.15) and the last inequality in (2.17).

We now are able to announce the main result of this section.

Theorem 2.2. *Let $\Phi_{hH} : X_{hH} \mapsto X_{hH}$ be the mapping defined by* (2.3) *and let $\mathscr{F}$ satisfy* (1.7a), (1.7b). *Then there exists a positive number q such that*

$$\|\Phi_{hH}(\varphi_h, g_H) - \Phi_{hH}(\overline{\varphi}_h, \overline{g}_H)\| \leqslant q\|(\varphi_h - \overline{\varphi}_h, g_H - \overline{g}_H)\| \tag{2.20}$$

holds for any $(\varphi_h, g_H), (\overline{\varphi}_h, \overline{g}_H) \in X_{hH} \cap B$, where the norm used in (2.20) *is defined by* (2.4) *and B is the same as in Lemma 2.2. In addition, if $\mathscr{F}_{\max}$ and l from* (1.7a) *and* (1.7b), *respectively are small enough, the constant q in* (2.20) *is less than* 1, *i.e. Φ_{hH} is contractive.*

Proof. It follows directly from the definition of Φ_{hH}, (2.18) and (2.19) by setting

$$q = \max\left\{\left(\frac{c_1^2c_2c_4}{\alpha} + \frac{\|a\|c_1c_4}{\alpha}\right)\mathscr{F}_{\max}, \quad \left(\frac{c_1^2c_2c_3c_4}{\alpha}R_2 + \frac{\|a\|c_1c_3c_4}{\alpha}R_2\right)l\right\}. \tag{2.21}$$

For $\mathscr{F}_{\max}$ and l small enough, the number q is less than 1. □

Corollary 2.1. *Let $\mathscr{F}_{\max}$ and l be small enough. Then there exists a unique fixed point of Φ_{hH} in $X_{hH} \cap B$. In addition, the method of successive approximations*

$$\left.\begin{array}{l} (\varphi_h^{(0)}, g_H^{(0)}) \in X_{hH} \text{ given;} \\ \quad \text{for } k = 1, 2, \ldots \text{ set} \\ (\varphi_h^{(k+1)}, g_H^{(k+1)}) = \Phi_{hH}(\varphi_h^{(k)}, g_H^{(k)}) \end{array}\right\} \tag{2.22}$$

converges for any choice of $(\varphi_h^{(0)}, g_H^{(0)}) \in X_{hH}$.

Remark 2.3. Suppose that the Babuška–Brezzi condition is satisfied, i.e.

$$\exists \beta = \text{const.} > 0 \quad \text{such that} \quad \sup_{\substack{z_h \in V_h \\ z_h \neq 0}} \frac{[\mu_H, z_h]}{\|z_h\|_{1,\Omega}} \geqslant \beta\|\mu_H\|_{-1/2,\Gamma_c}$$

holds for every $\mu_H \in L_H$, where β does not depend on H, h and $\| \ \|_{-1/2,\Gamma_c}$ is the norm in $H^{-1/2}(\Gamma_c)$. Then

$$\beta\|\mu_H\|_{-1/2,\Gamma_c} \leqslant \|\mu_H\|_{-1/2,h} \leqslant \|\mu_H\|_{-1/2,\Gamma_c} \quad \forall \mu_H \in L_H$$

and the mesh dependent norm $\| \ \|_{-1/2,h}$ in the previous estimates can be replaced by the dual norm $\| \ \|_{-1/2,\Gamma_c}$. In addition, the inverse inequality between $L^2(\Gamma_c)$ and $H^{-1/2}(\Gamma_c)$ for $\mu_H \in L_H$ implies that the constant c_4 in $(2.16)_2$ behaves as $1/\sqrt{H}$ provided that $\mathscr{T}_H$ belongs to a family $\{\mathscr{T}_H\}$, $H \to 0+$ of strongly regular partitions of $\overline{\Gamma}_c$. In addition, if $\mathscr{T}_h$ satisfies the locally inverse assumption on $\overline{\Gamma}_c$, the constant c_3 in $(2.16)_1$ behaves as $1/\sqrt{h}$ (see [7]). From this, Remark 2.2 and (2.21) we see that to ensure $q < 1$ the parameters $\mathscr{F}_{\max}$ and l have to decay as $\sqrt{Hh}$, $H, h \to 0+$. A similar result has been proven for contact problems with Coulomb friction whose coefficient $\mathscr{F}$ is independent of the solution (see [9]).

3. Numerical realization

For numerical realization of contact problems with the Coulomb law of friction involving a coefficient $\mathscr{F}$ which depends on a solution the method of successive approximations (2.22) will be used. Let us recall that each iterative step leads to a contact problem with given friction and a coefficient which already does not depend on the solution. The iterative procedure (2.22) updates the slip bound g and the coefficient $\mathscr{F}$ by using data from the previous iteration. Since $(\mathscr{M}(\varphi_h, g_H))_h^H$ is a central part of our algorithm we shall describe in more details its numerical realization.

In order to satisfy the condition (2.2) it must be $\dim \mathscr{V}_h \geqslant \dim L_H$. Below we show the construction of L_H for which $\dim \mathscr{V}_h = \dim L_H$. Let $\mathscr{N}_h = \{x^{(i)}\}_{i=0}^q$ be the set of all nodes of $\mathscr{T}_h$ which are placed on $\overline{\Gamma}_c$ and denote by

$x^{(i+1/2)}$ the midpoint of the segment $[x^{(i)}, x^{(i+1)}]$, $i = 0, \ldots, q-1$. The definition of the partition $\mathscr{T}_H = \{S_i\}_{i \in \mathscr{I}}$ of $\overline{\Gamma}_c$ depends on the mutual position of Γ_u and Γ_c:

- if $\overline{\Gamma}_u \cap \overline{\Gamma}_c = \emptyset$ then $\dim \mathscr{V}_h = q+1$ and $S_0 = [x^{(0)}, x^{(1/2)}]$, $S_i = [x^{(i-1/2)}, x^{(i+1/2)}]$, $i = 1, \ldots, q-1$, $\quad S_q = [x^{(q-1/2)}, x^{(q)}]$;
- if $\overline{\Gamma}_u \cap \overline{\Gamma}_c = \{x^{(0)}\}$ then $\dim \mathscr{V}_h = q$ and $S_1 = [x^{(0)}, x^{(3/2)}]$, $S_i = [x^{(i-1/2)}, x^{(i+1/2)}]$, $i = 2, \ldots, q-1$, $\quad S_q = [x^{(q-1/2)}, x^{(q)}]$;

  (analogously if $\overline{\Gamma}_u \cap \overline{\Gamma}_c = \{x^{(q)}\}$);
- if $\overline{\Gamma}_u \cap \overline{\Gamma}_c = \{x^{(0)}, x^{(q)}\}$then $\dim \mathscr{V}_h = q-1$ and $S_1 = [x^{(0)}, x^{(3/2)}]$, $S_i = [x^{(i-1/2)}, x^{(i+1/2)}]$, $i = 2, \ldots, q-2$, $\quad S_{q-1} = [x^{(q-3/2)}, x^{(q)}]$.

From the construction of S_i, $i \in \mathscr{I}$ we see that with any S_i one can associate exactly one node $x^{(i)} \in S_i$.

To evaluate the frictional term we first replace $|v_{h1}|$ by its linear Lagrange interpolant $r_h|v_{h1}|$

$$\int_{\Gamma_c} \mathscr{F} \circ \varphi_h g_H |v_{h1}| \, dx_1 \approx \int_{\Gamma_c} \mathscr{F} \circ \varphi_h g_H \, r_h|v_{h1}| \, dx_1. \tag{3.1}$$

Next, the integral on the right of (3.1) will be evaluated by the rectangular formulae with the nodes at $x^{(i)} \in S_i$, $i \in \mathscr{I}$:

$$\int_{\Gamma_c} \mathscr{F} \circ \varphi_h g_H \, r_h|v_{h1}| \, dx_1 \approx \sum_{i \in \mathscr{I}} \mathscr{F} \circ \varphi_h(x^{(i)}) g_H^{(i)} |v_{h1}(x^{(i)})| \operatorname{meas} S_i, \tag{3.2}$$

where $g_H^{(i)} = g_{H|_{S_i}}$ using also that $r_h|v_{h1}(x^{(i)})| = |v_{h1}(x^{(i)})| \ \forall i \in \mathscr{I}$. The same integration formulae will be used for the evaluation of the duality term

$$[\mu_H, v_{h2}] = \int_{\Gamma_c} \mu_H v_{h2} \, dx_1 \approx \sum_{i \in \mathscr{I}} \mu_H^{(i)} v_{h2}(x^{(i)}) \operatorname{meas} S_i. \tag{3.3}$$

Thus $(\mathscr{M}(\varphi_h, g_H))_h^H$ reads as follows:

$$\left.\begin{aligned}
&\text{Find } (u_h, \lambda_H) \in \mathbb{V}_h \times \Lambda_H \text{ such that} \\
&a(u_h, v_h - u_h) + \sum_{i \in \mathscr{I}} \mathscr{F} \circ \varphi_h(x^{(i)}) g_H^{(i)} \operatorname{meas} S_i (|v_{h1}(x^{(i)})| - |u_{h1}(x^{(i)})|) \\
&\quad \geqslant L(v_h - u_h) + \sum_{i \in \mathscr{I}} \lambda_H^{(i)} (v_{h2}(x^{(i)}) - u_{h2}(x^{(i)})) \operatorname{meas} S_i \quad \forall v_h \in \mathbb{V}_h, \\
&\sum_{i \in \mathscr{I}} (\mu_H^{(i)} - \lambda_H^{(i)}) \operatorname{meas} S_i u_{h2}(x^{(i)}) \geqslant 0 \quad \forall \mu_H \in \Lambda_H.
\end{aligned}\right\} \tag{3.4}$$

Substituting $v_h := w_h + u_h$, $w_h \in \overset{\circ}{\mathbb{V}}_h$ into the first inequality in (3.4) we obtain

$$a(u_h, w_h) = L(w_h) + \sum_{i \in \mathscr{I}} \lambda_H^{(i)} \operatorname{meas} S_i \, w_{h2}(x^{(i)}) \quad \forall w_h \in \overset{\circ}{\mathbb{V}}_h. \tag{3.5}$$

From the last inequality in (3.4) we see that $u_{h2}(x^{(i)}) \geqslant 0 \ \forall i \in \mathscr{I}$ so that $u_{h2} \geqslant 0$ on Γ_c. In other words, the rectangular formulae in (3.3) leads to the *inner* approximation K_h of K, where

$$K_h = \{v_h = (v_{h1}, v_{h2}) \in \mathbb{V}_h \mid v_{h2} \geqslant 0 \text{ on } \Gamma_c\}.$$

The first component u_h of the solution to (3.4) solves the following minimization problem:

$$u_h = \operatorname*{argmin}_{K_h} \left\{ \frac{1}{2} a(v_h, v_h) - L(v_h) + \sum_{i \in \mathscr{I}} \mathscr{F} \circ \varphi_h(x^{(i)}) g_H^{(i)} \operatorname{meas} S_i \, |v_{h1}(x^{(i)})| \right\}. \tag{3.6}$$

For numerical realization of (3.6) we use again a duality approach.

To this end, we introduce the convex sets

$$\boldsymbol{\Lambda}_1(\varphi_h, g_H) = \{\boldsymbol{\mu} \in \mathbb{R}^{\operatorname{card} \mathscr{I}} \mid |\mu^{(i)}| \leqslant \mathscr{F} \circ \varphi_h(x^{(i)}) g_H^{(i)} \operatorname{meas} S_i \ \forall i \in \mathscr{I}\},$$

$$\boldsymbol{\Lambda}_2 = \mathbb{R}_+^{\operatorname{card} \mathscr{I}},$$

where $\mu^{(i)}$ stands for the i-th component of $\boldsymbol{\mu}$.

Problem (3.6) is equivalent to the following mixed type formulation:

$$\left.\begin{aligned}
&\text{Find } (u_h, \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) \in \mathbb{V}_h \times \boldsymbol{\Lambda}_1(\varphi_h, g_H) \times \boldsymbol{\Lambda}_2 \text{ such that} \\
&a(u_h, v_h) = L(v_h) + \sum_{i \in \mathscr{I}} \lambda_1^{(i)} v_{h1}(x^{(i)}) + \sum_{i \in \mathscr{I}} \lambda_2^{(i)} v_{h2}(x^{(i)}) \quad \forall v_h \in \mathbb{V}_h, \\
&\sum_{i \in \mathscr{I}} (\mu_1^{(i)} - \lambda_1^{(i)}) u_{h1}(x^{(i)}) + \sum_{i \in \mathscr{I}} (\mu_2^{(i)} - \lambda_2^{(i)}) u_{h2}(x^{(i)}) \geqslant 0 \\
&\quad \forall \boldsymbol{\mu}_1 \in \boldsymbol{\Lambda}_1(\varphi_h, g_H), \ \boldsymbol{\mu}_2 \in \boldsymbol{\Lambda}_2.
\end{aligned}\right\} \tag{3.7}$$

Inserting $v_h \in \mathring{\mathbb{V}}_h$ into the first equation in (3.7), we obtain

$$a(u_h, v_h) = L(v_h) + \sum_{i \in \mathscr{I}} \lambda_2^{(i)} v_{h2}(x^{(i)}) \quad \forall v_h \in \mathring{\mathbb{V}}_h.$$

From this and (3.5) we see that

$$\lambda_2^{(i)} = \lambda_H^{(i)} \operatorname{meas} S_i \quad \forall i \in \mathscr{I}. \tag{3.8}$$

Next, we present an algebraic form of (3.7). Let $\mathbf{v} \in \mathbb{R}^n$, $n = \dim \mathbb{V}_h$, be a nodal displacement vector and $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^{\operatorname{card} \mathscr{I}}$ be its subvectors whose components are displacements at the contact nodes $x^{(i)}$, $i \in \mathscr{I}$ in the x_1, x_2-direction, respectively. Further, let $\mathbb{B}_1, \mathbb{B}_2$ be the matrix representations of the linear mappings

$$\mathbf{v} \mapsto \mathbf{v}_1, \quad \mathbf{v} \mapsto \mathbf{v}_2,$$

respectively. Then (3.7) is equivalent to

$$\left.\begin{aligned}
&\text{Find } \mathbf{u} \in \mathbb{R}^n, \ \boldsymbol{\lambda}_1 \in \boldsymbol{\Lambda}_1(\varphi_h, g_H), \ \boldsymbol{\lambda}_2 \in \boldsymbol{\Lambda}_2 \text{ such that} \\
&\mathbb{A}\mathbf{u} = \mathbb{L} + \mathbb{B}_1^\top \boldsymbol{\lambda}_1 + \mathbb{B}_2^\top \boldsymbol{\lambda}_2 \\
&(\boldsymbol{\mu}_1 - \boldsymbol{\lambda}_1, \mathbb{B}_1\mathbf{u}) + (\boldsymbol{\mu}_2 - \boldsymbol{\lambda}_2, \mathbb{B}_2\mathbf{u}) \geqslant 0 \quad \forall \boldsymbol{\mu}_1 \in \boldsymbol{\Lambda}_1(\varphi_h, g_H) \ \forall \boldsymbol{\mu}_2 \in \boldsymbol{\Lambda}_2,
\end{aligned}\right\} \tag{3.9}$$

where $\mathbb{A}$ is the stiffness matrix, $\mathbb{L}$ is the load vector, $(\ ,\ )$ is the scalar product in $\mathbb{R}^{\operatorname{card} \mathscr{I}}$, and $\mathbb{B}_1^\top, \mathbb{B}_2^\top$ are the transposes of $\mathbb{B}_1, \mathbb{B}_2$, respectively.

For numerical realization of (3.9) we use a dual approach. From the first equation in (3.9) one can express

$$\mathbf{u} = \mathbb{A}^{-1}(\mathbb{L} + \mathbb{B}_1^\top \boldsymbol{\lambda}_1 + \mathbb{B}_2^\top \boldsymbol{\lambda}_2). \tag{3.10}$$

Inserting (3.10) into the inequality in (3.9) we obtain the following *quadratic programming problem* with *simple* (box) constraints in terms of the Lagrange multipliers, only:

$$\left.\begin{aligned}
&\text{Find } \boldsymbol{\lambda}_1 \in \boldsymbol{\Lambda}_1(\varphi_h, g_H), \ \boldsymbol{\lambda}_2 \in \boldsymbol{\Lambda}_2 \text{ such that} \\
&\mathscr{S}(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) \leqslant \mathscr{S}(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \quad \forall \boldsymbol{\mu}_1 \in \boldsymbol{\Lambda}_1(\varphi_h, g_H) \ \forall \boldsymbol{\mu}_2 \in \boldsymbol{\Lambda}_2,
\end{aligned}\right\} \tag{3.11}$$

where

$$\mathscr{S}(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2) = \frac{1}{2}(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \begin{pmatrix} \mathbb{Q}_{11} & \mathbb{Q}_{21} \\ \mathbb{Q}_{21} & \mathbb{Q}_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix} - (\mathbf{h}_1, \mathbf{h}_2) \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}$$

with

$$\mathbb{Q}_{ij} = \mathbb{B}_i \mathbb{A}^{-1} \mathbb{B}_j^\top, \quad \mathbf{h}_i = \mathbb{B}_i \mathbb{A}^{-1} \mathbb{L}, \quad i, j \in \{1, 2\}.$$

The iterative process (2.22) which uses the dual formulation (3.11) reads as follows:

$$\left.\begin{array}{l} \text{let } (\varphi_h^{(0)}, g_H^{(0)}) \in X_{hH} \text{ be given;} \\ \text{For } (\varphi_h^{(k)}, g_H^{(k)}) \in X_{hH},\ k=1,2,\ldots \text{ known, solve :} \\ \quad (\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) = \operatorname{argmin}\{\mathscr{S}(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2),\ \boldsymbol{\mu}_1 \in \boldsymbol{\Lambda}_1(\varphi_h^{(k)}, g_H^{(k)}),\ \boldsymbol{\mu}_2 \in \boldsymbol{\Lambda}_2\}; \\ \quad \text{set } {g_H}_{|S_i}^{(k+1)} = \lambda_2^{(i)}/\text{meas } S_i \quad \forall i \in \mathscr{I}; \\ \qquad \varphi_h^{(k+1)}(x^{(i)}) = |u_1^{(i)}| \quad \forall i \in \mathscr{I}; \\ \text{until stopping criterion.} \end{array}\right\} \tag{3.12}$$

The symbol $u_1^{(i)}$ in (3.12) denotes the i-th component of $\mathbf{u}_1$. To have $\mathbf{u}_1$ at disposal, it is not necessary to compute the whole vector $\mathbf{u}$ from (3.10). Indeed, it is easy to show (see [11]) that $\mathbf{u}_1$ is related to the Lagrange multipliers which release the constraint $\boldsymbol{\mu}_1 \in \boldsymbol{\Lambda}_1(\varphi_h^{(k)}, g_H^{(k)})$ in (3.11). The minimization of the function $\mathscr{S}$ in (3.12) was realized by a conjugate gradient method with proportioning [3].

4. Model examples

An elastic body is represented by a rectangle $\Omega = (0, 5) \times (0, 1)$ (in m). The used material is characterized by the Young modulus $E = 21.19e10[Pa]$ and Poisson's ratio $\sigma = 0.277$. The body is fixed along $\Gamma_u = \{0\} \times (0, 1)$ and linearly distributed surface tractions of density $P = (P_1, P_2)$ are applied on $\Gamma_p = \Gamma_p^1 \cup \Gamma_p^2$ (see Fig. 2), where

$$P_1 = (1-\lambda)P_x^1 + \lambda P_x^2, \quad \lambda \in [0,1], \quad P_2 = 0 \quad \text{on } \Gamma_p^1,$$

$$P_1 = 0, \quad P_2 = (1-\lambda)P_y^1 + \lambda P_y^2, \quad \lambda \in [0,1] \quad \text{on } \Gamma_p^2,$$

$P_x^1 = 2.e6[N]$, $P_x^2 = 4.e6[N]$, $P_y^1 = -10.e6[N]$, $P_y^2 = 1.e6[N]$. The coefficient of friction $\mathscr{F}$ is defined by

$$\mathscr{F}(t) = \begin{cases} 0.3 - 0.1t/\text{param} & t \in \langle 0, \text{param}), \\ 0.2 & t \in \langle \text{param}, \infty). \end{cases} \tag{4.1}$$

Three different values of param were considered, namely param $= 9.e-5$, $6.e-5$, and $3.e-5$ (see Fig. 3).

The displacement vector is approximated by continuous, piecewise linear functions over five triangulations $\mathscr{T}_h$ of $\overline{\Omega}$. The total number n_p of the primal variables is $n_p = 1560, 6000, 13\,320, 23\,520$ and $36\,600$, respectively. For the discretization of Lagrange multipliers we use the space L_H, whose construction is described in Section 3. Recall that $\dim \mathscr{V}_h = \dim L_H$. In what follows the symbol n_d stands for the number of the dual variables. The stopping criterion is the same in all examples, namely

$$\frac{\|\boldsymbol{\varphi}^{(k)} - \boldsymbol{\varphi}^{(k-1)}\|}{\|\boldsymbol{\varphi}^{(k)}\|} + \frac{\|\mathbf{g}^{(k)} - \mathbf{g}^{(k-1)}\|}{\|\mathbf{g}^{(k)}\|} < 10^{-6},$$

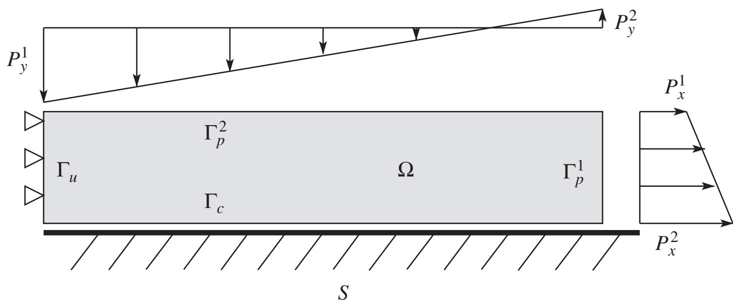

Fig. 2. Geometry of the problem.

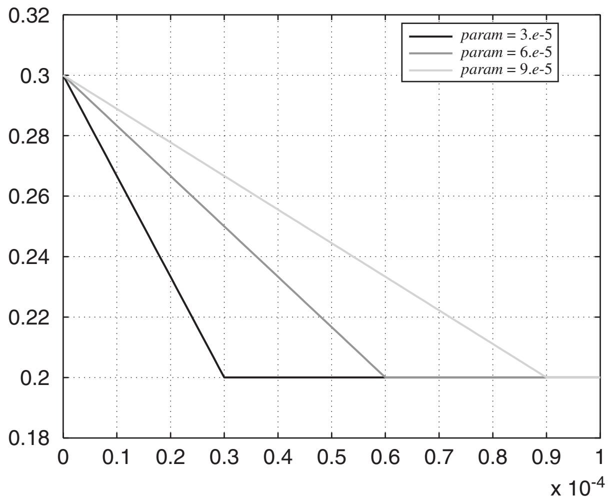

Fig. 3. Function $\mathscr{F}$.

Table 1

n_p	n_d	*CG it*	*it*
1560	120	334/332	10/9
6000	240	431/441	10/9
13 320	360	680/716	10/9
23 520	480	780/823	10/9
36 600	600	1034/960	11/9

Table 2

param	*CG it*	*it*
$3.e-5$	1104	12
$6.e-5$	1034	11
$9.e-5$	947	10
$\mathscr{F}=0.3$	960	9

where $\boldsymbol{\varphi}^{(k)}, \mathbf{g}^{(k)} \in \mathbb{R}^{\operatorname{card}\mathscr{I}}$ are vectors whose components are $g_{H_{|S_i}}^{(k)}, \varphi_h^{(k)}(x^{(i)}), i \in \mathscr{I}$ computed in (3.12) and $\| \ \|$ stands for the Euclidean norm.

Table 1 shows how the total number of conjugate gradient iterations (*CGit*) and the number of fixed point iterations (*it*) depend on n_p and n_d. Results for param $= 6.e-5$ are represented by the first integer in the respective column and they are compared with the ones for $\mathscr{F}=0.3$, i.e. the case when $\mathscr{F}$ does not depend on u (the second integer).

In Table 2 we illustrate how (*CG it*) and (*it*) depend on $\mathscr{F}$ and results are again compared with a solution independent coefficient $\mathscr{F}=0.3$. Computations were done for $n_p = 36\,600$.

The following figures depict a typical behavior of contact stresses and displacements. Results for $\mathscr{F}=0.3$ and $\mathscr{F}$ defined by (4.1) with param $= 6 \cdot e - 5$ are compared. Figs. 4 and 5 show the distribution of contact stresses and displacements along Γ_c. From Fig. 5(b) we see that the tangential displacements on Γ_c are higher for a solution-dependent coefficient $\mathscr{F}$ which is a decreasing function of $|u_t|$.

From Fig. 6 which compares $-T_t(u)$ with the product $\mathscr{F}(|u_t|)T_n(u)$ one can verify the satisfaction of friction conditions (1.6). Figs. 7(a), (b) show a detail in a vicinity of Γ_u. We see that a small part of Γ_c is stuck to the rigid foundation S and the value $T_t(u)$ is less than the product $\mathscr{F}(|u_t|)T_n(u)$. Finally, Fig. 8 illustrates the function $\mathscr{F} \circ |u_t|$: $x \mapsto \mathscr{F}(|u_t(x)|), x \in \Gamma_c$, i.e. the distribution of the coefficient $\mathscr{F}$ along Γ_c for param $= 6.e-5$.

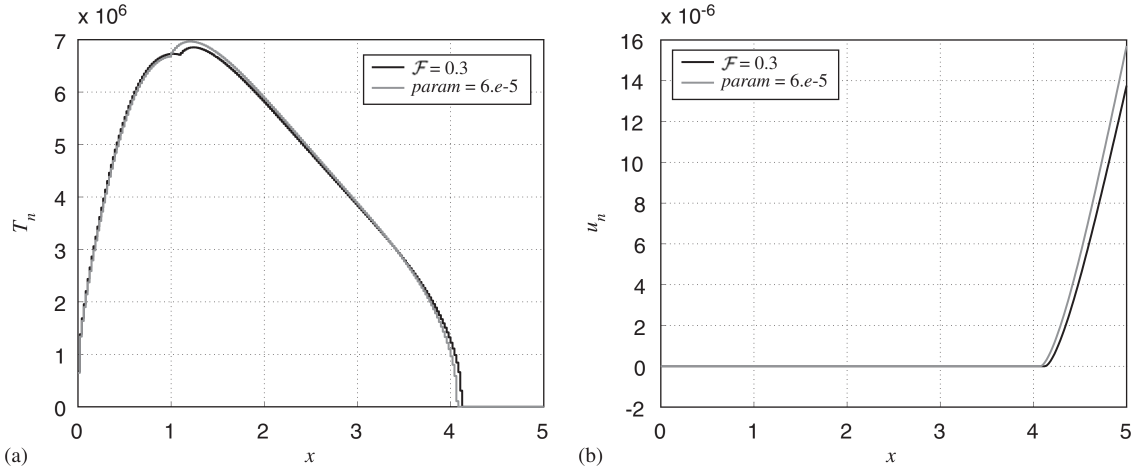

Fig. 4. (a) Normal contact stresses, (b) normal contact displacements.

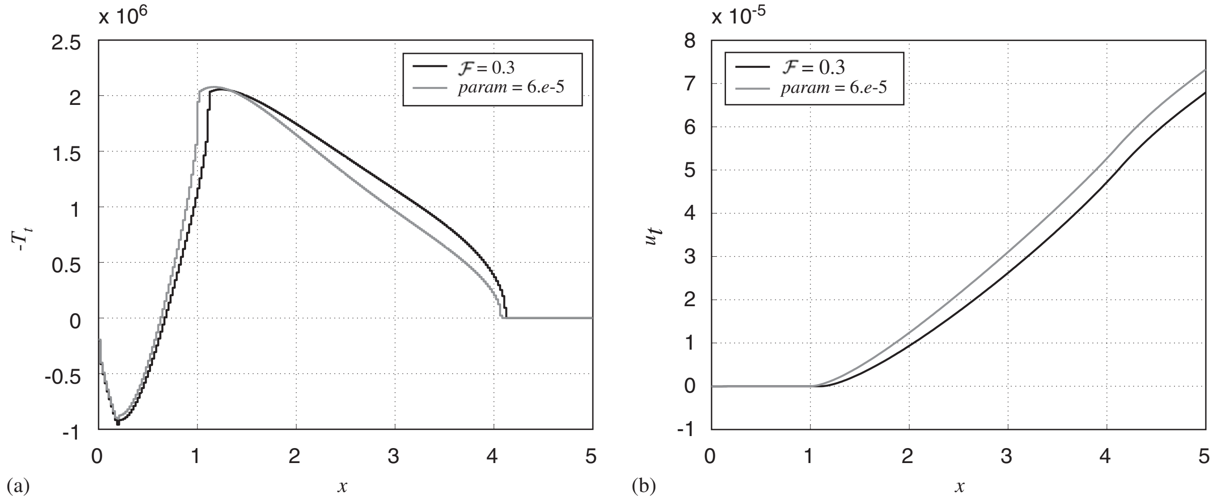

Fig. 5. (a) Tangential contact stresses, (b) tangential contact displacements.

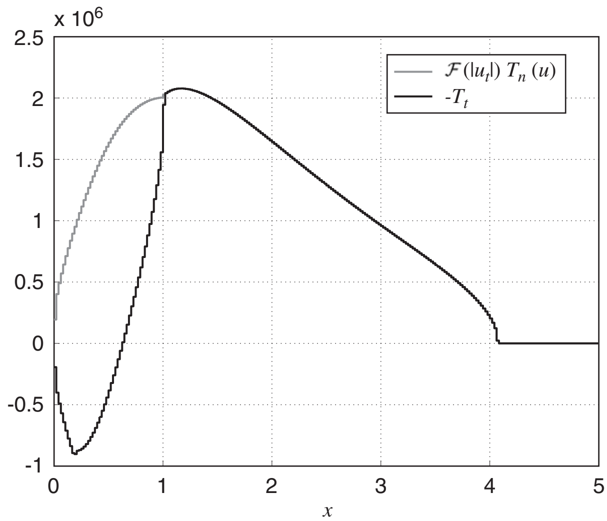

Fig. 6. Comparison of $-T_t(u)$ and $\mathscr{F}(|u_t|)T_n(u)$ $(param = 6.e - 5)$.

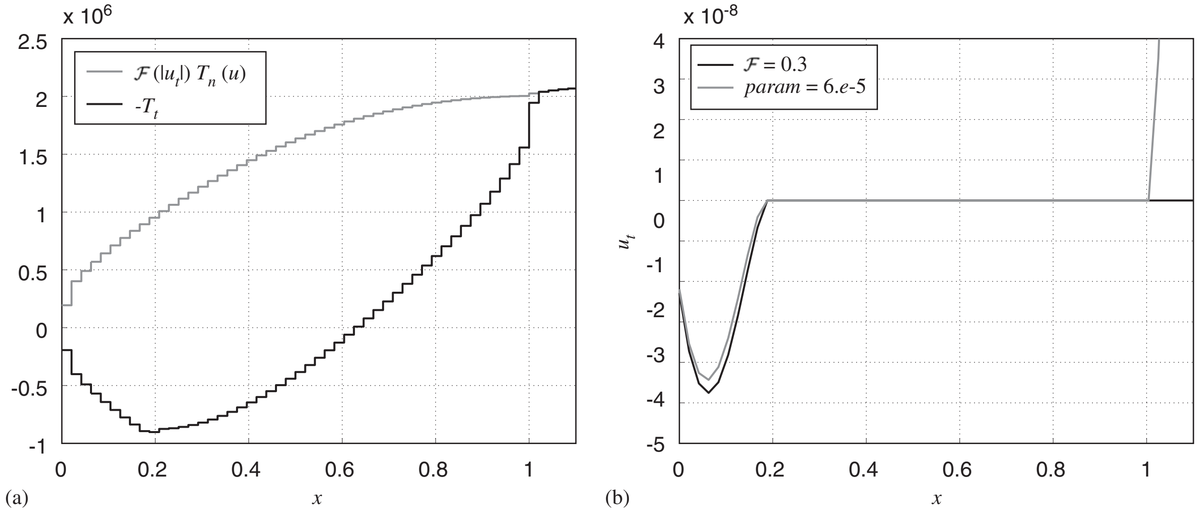

Fig. 7. (a) Detail of Fig. 6, (b) detail of Fig. 5(b).

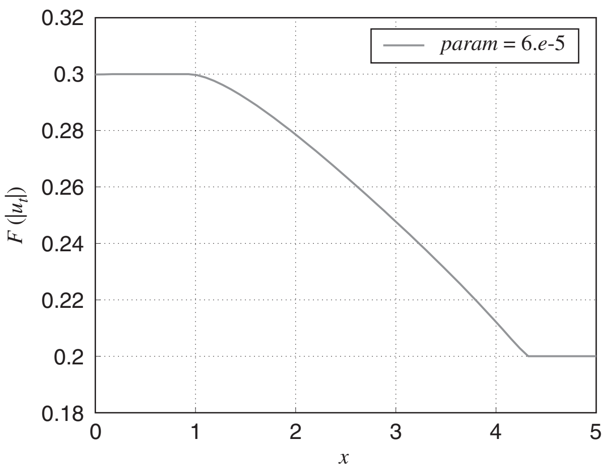

Fig. 8. Distribution of the coefficient $\mathcal{F}$ along Γ_c.

5. Conclusions

The paper deals with a discretization and numerical realization of 2D contact problems with Coulomb friction and a coefficient of friction $\mathcal{F}$ which depends on a solution. Solutions to these problems are defined as fixed points of an auxiliary mapping Φ_{hH}. This mapping was constructed by means of a mixed finite element approximation of contact problems with given friction and a coefficient of friction which is independent of solutions. We proved the existence of at least one solution for any $\mathcal{F}$ which is defined by a bounded, positive and continuous function and we established conditions under which the solution is unique. The method of successive approximations was proposed for finding fixed points of Φ_{hH}. Model examples with several coefficients of friction were computed. It turned out that the number of iterations of the method of successive approximations which is necessary to get a solution with the required accuracy is small and practically it does not depend on the slope of $\mathcal{F}$. Each iterative step was realized by a conjugate gradient method *without* preconditioning. This explains the increase of the conjugate gradient iterations for finer meshes. We focused on the static case, only, because our main goal was to test the efficiency and the reliability of the fixed point approach. More realistic quasistatic case leads, after a time discretization, to a sequence of static problems studied in this paper.

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