Journal of Algebra 349 (2012) 98–116

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

# Extremal and maximal normal abelian subgroups of a maximal unipotent subgroup in groups of Lie type ☆

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ARTICLE INFO

*Article history:*
Received 16 January 2011
Available online 28 October 2011
Communicated by E.I. Khukhro

*Keywords:*
Group of Lie type
Unipotent subgroup
Maximal abelian normal subgroup
Extremal subgroup
Large abelian subgroup

ABSTRACT

We describe all maximal abelian normal subgroups in the unipotent radical $U$ of a Borel subgroup in a group of Lie type $G$ over a field $K$. This gives a new description of the extremal subgroups in $U$ which were studied by C. Parker and P. Rowley. For a finite field $K$, we prove that either each large abelian subgroup in $U$ is $G$-conjugate to a normal subgroup in $U$ or $G$ is of certain exceptional Lie type.



## Introduction

Let $G$ be a group of Lie type over a field $K$, and let $U$ be the unipotent radical of a Borel subgroup in $G$. The present paper is devoted to studying certain abelian normal subgroups in $U$ and some related problems.

The study of these questions has been under active investigation since 1970s. J. Gibbs [5] described the lower and upper central series, the characteristic subgroups and the automorphisms of $U$ with $char\, K \neq 2, 3$. A description for an arbitrary field $K$ was completed in [13], and it solves the problem (1.5) from [7]. The approach of [13] uses a description of maximal abelian normal subgroups of the unitriangular group and close structural connections of $U$ and its associated Lie ring, cf. [10,12, 8,9,16].

The theorems announced in [15] and Theorems 4.1 and 4.6 about the normal structure use the concept of *corners* of subsets in $U$ (for notation see Section 1). Thus, the *extremal subgroups* from [18]

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0021-8693/$ – see front matter 
doi:10.1016/j.jalgebra.2011.10.025

are the normal abelian subgroups in $U$ with a simple corner. For the application to symplectic amalgams [21] and the revision of the classification of finite simple groups, C. Parker and P. Rowley studied the groups $U$ with an extremal subgroup and the possible simple corners of such a subgroup [18–20].

Theorems 3.1, 4.8 and 5.1 of the present paper and [15, Theorem 5] (for the classical types) describe all maximal abelian normal subgroups in $U$. Therefore, we have a new solution to the Parker–Rowley problem. Theorem 2.1 gives a clarification of some assertions from [18,19] when $U$ is of type $D_4$ and ${}^2D_4$.

In Section 6 we consider an application to description of the large abelian and normal large abelian subgroups in the finite groups $U$. For the exceptional types, this problem was pointed out in A.S. Kondratiev's survey [7, Problem (1.6)] (for the classical types, see [1,2,28,29]). Using a computer approach as well as a generalization of A.I. Mal'tsev's method [17], E.P. Vdovin [26, Table 4] determined the orders of large abelian subgroups of $U$.

Given a group-theoretic property $\mathcal{P}$, we recall that every $\mathcal{P}$-subgroup of largest order in a finite group is a *large $\mathcal{P}$-subgroup*. Theorem 6.1 and [16, Table 2] (for the classical types) give the list of all large normal abelian subgroups in the finite groups $U$. Using the approach of [17] and [26] we show that the identical list gives the normal large abelian subgroups (Theorem 6.4). (In general, there exists a large normal $\mathcal{P}$-subgroup, which is not a large $\mathcal{P}$-subgroup, cf. Section 6.) It allows us to clarify some orders of large abelian subgroups in $U$ which were found in [26, Table 4], cf. Remark in Section 6.

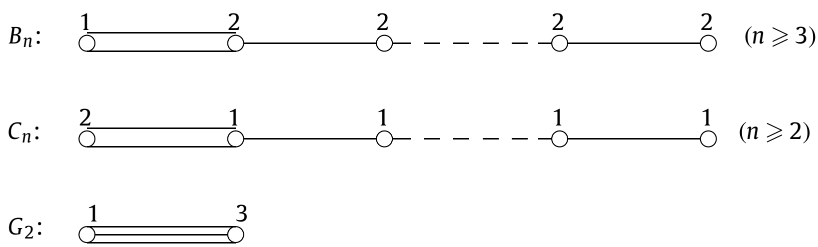
Finally, in Section 6 we show that either each large abelian subgroup in $U$ is $G$-conjugate to a normal subgroup in $U$ or $G$ is of certain exceptional type and there exists a normal large abelian subgroup in $U$ which is not extremal.

## 1. Preliminary remarks and notation

Along with the usual notation of [22,4,23] we use notation from [13], which simplifies our proofs.

Let $\Phi(K)$ denotes a Chevalley group with the root system $\Phi$ over a field $K$. This group is generated by the root elements $x_r(t)$ $(t \in K,\ r \in \Phi)$. Let $\Pi = \Pi(\Phi)$ be a basis for simple roots in $\Phi$, and let $\Phi^+$ be the set of positive roots of $\Phi$ with respect to $\Pi$. We set $p(\Phi) = \max\{(r,r)/(s,s) \mid r, s \in \Pi(\Phi)\}$.

A *Coxeter graph* of $\Phi$ is defined in J.-P. Serre [22, V.12]. (This concept coincides with the concept of the Dynkin diagram discussed by R. Carter [4, § 3.4].) The nodes of this graph are all roots from $\Pi$. By [22, V.15], it gives a *Dynkin diagram* of $\Phi$ if the numbers $p(\Phi)$ and 1 put into correspondence with the long and short roots $r \in \Pi$, respectively. For example, we get the following different Dynkin diagrams




The twisted group ${}^m\Phi(K)$ is the centralizer in $\Phi(K)$ of a *twisting automorphism* $\theta \in Aut\,\Phi(K)$ of order $m = 2$ or 3. According to [23, § 11], $\theta$ is the composition of a graph automorphism $\tau$ and a non-trivial automorphism $\sigma : t \to \bar{t}$ $(t \in K)$ of $K$ satisfying the condition $p(\Phi)\sigma^m = 1$. We also denote by $\bar{}$ the symmetry of Coxeter graph. For certain extension of the symmetry $\bar{}$ of order $m$ on the Coxeter graph to the root system $\Phi$, we have $\theta(X_r) = \tau(X_r) = X_{\bar{r}}$ $(r \in \Phi,\ X_r = x_r(K))$.

As usual, the "root" elements of ${}^m\Phi(K)$ are given by the subgroups $X_S^1 = {}^m\Phi(K) \cap \langle X_r \mid r \in S\rangle$ for certain equivalence classes $S$ of $\Phi$, cf. [23,4]. We now associate the root elements with the $\bar{}$-orbits.

A mapping of a root system to another one is called a *homomorphism* if it can be extended to a homomorphism of the root lattices of these root systems. By [11, Lemma 7], for $p(\Phi)=1$ there exists a homomorphism $\zeta$ of $\Phi$ onto a root system such that $\zeta(r)=\zeta(s)$ if and only if either $r=s$ or $\bar{r}=s$ or $\bar{s}=r$. Therefore, if either $(\Phi,m)=(D_4,3)$ or $m=2$ and $\Phi$ is of type $E_6$, $D_{n+1}$, $A_{2n-1}$ or $A_{2n}$ then $\zeta(\Phi)$ is of type $G_2$, $F_4$, $B_n$, $C_n$ or $BC_n$ [22, V.16], respectively, cf. [4, Remark 13.3.8] and [11, Lemma 8].

When $S$ is an $^-$-orbit in $\Phi$, $S$ has type $A_1$, $A_1\times A_1$ or $A_1\times A_1\times A_1$, by Propositions 13.6.3 and 13.6.4 in [4]. Then $X_S^1=x_S(F)\simeq F^+$, where $F$ is the subfield $\{t\in K\mid \bar{t}=t\}=\ker(1-\sigma)$, $K$ or $K$, respectively for each type, and $F^+$ is the additive group of $F$. If $S=\{r,\bar{r},r+\bar{r}\}$ has type $A_2$ then $\Phi$ is of type $A_{2n}$ and

$$X_S^1=\bigl\{x_S(t,u)\bigm| x_S(t,u)=x_r(t)x_{\bar{r}}(\bar{t})x_{r+\bar{r}}(u),\ u,t\in K,\ u+\bar{u}=\pm t\bar{t}\bigr\}.$$

For the $^-$-orbits $\{r+\bar{r}\}$ and $\{r,\bar{r}\}$, we denote, respectively, $x_{r+\bar{r}}(\ker(1+\sigma))$ by $X_{2R}$, where $2R=\zeta(r+\bar{r})$, and $x_R(K)$ by $X_R$, where $R=\zeta(r)$, and $X_R$ is the system of representatives $x_R(t)=x_r(t)x_{\bar{r}}(\bar{t})x_{r+\bar{r}}(\tilde{t})$ (for all $t\in K$) of cosets in $X_S^1$ by the subgroup $X_{2R}$, and $\tilde{}$ is a transformation of $K$. In the remaining cases, $S$ has type $B_2$ or $G_2$ (see [4, Proposition 13.6.4]), and ${}^m\Phi(K)$ is of type ${}^2G_2$, ${}^2B_2$ or ${}^2F_4$. Then $S$ is the union of $^-$-orbits having representatives $r$, $r+\bar{r}$ (and also $2r+\bar{r}$ for type $G_2$). We now use the root subsets $\alpha(K)=X_R$, $\beta(K)=X_{2R}$, and $\gamma(K)=X_{3R}$, which were defined in Proposition 13.6.4 (vi) and (vii) in [4].

Thus, the $^-$-orbit $\alpha$ of each root $r\in\Phi$ uniquely determines a root subset $X_\alpha$ in ${}^m\Phi(K)$. The set of all such $\alpha$ will be denoted by ${}^m\Phi$. If $\alpha$ is of order 1 then $\alpha$ is said to be of the *first type*. Choosing all $\alpha$ with $r\in\Pi(\Phi)$ we get a basis $\Pi({}^m\Phi)$ for ${}^m\Phi$. If $p(\Phi)=1$ then ${}^m\Phi=\zeta(\Phi)$, and $\Pi({}^m\Phi)=\zeta(\Pi(\Phi))$. Thus, for type ${}^3D_4$, the root system $\zeta(\Phi)$ is of type $G_2$ with $r,q\in\Pi(\Phi)$, $q=\bar{q}$, and we have

$$X_a=x_a(K),\quad a=\zeta(r)\ \bigl(x_a(t):=x_r(t)x_{\bar{r}}(\bar{t})x_{\bar{\bar{r}}}(\bar{\bar{t}}),\ t\in K\bigr),$$
$$X_b=x_q\bigl(\ker(1-\sigma)\bigr),\quad b=\zeta(q)\ \bigl(x_b(t):=x_q(t),\ t=\bar{t}\bigr).$$

By analogy with [13], $G(K)$ denotes a group of Lie type associated either with the system $G={}^m\Phi$ or $G=\Phi$. We fix a basis $\Pi$ for $G$ and the set $G^+$ of all *positive roots* with respect to $\Pi$. We define a unipotent subgroup $U$ by $U=UG(K)\colon=\langle X_s\mid s\in G^+\rangle$, cf. [4,23,13].

Let $\{r\}^+$ be the family of $s\in G^+$ with nonnegative coefficients in the linear expression of $s-r$ by $\Pi$. We set

$$T(r):=\bigl\langle X_s\bigm| s\in\{r\}^+\bigr\rangle,\qquad Q(r):=\bigl\langle X_s\bigm| s\in\{r\}^+\setminus\{r\}\bigr\rangle\quad(r\in G).$$

If $H\subseteq T(r_1)T(r_2)\cdots T(r_m)$ and the inclusion fails under every substitution of $T(r_i)$ by $Q(r_i)$ then $\mathcal{L}(H)=\{r_1,r_2,\dots,r_m\}$ is said to be the *set of corners* of $H$.

As in [4, § 4.4], take the $K$-algebra $\mathcal{L}_K$ with Chevalley basis $\{e_r\ (r\in\Phi),\dots\}$. Denote by $N\Phi(K)$ the subalgebra in $\mathcal{L}_K$ with the basis $\{e_r\mid r\in\Phi^+\}$. The Lie products $e_r*e_s=c_{rs}e_{r+s}$ ($c_{rs}=0$ for $r+s\notin\Phi$) define the structure constants of Chevalley basis in $N\Phi(K)$. Chevalley's commutator formula gives $[X_r,X_s]=x_{r+s}(c_{rs}K)$ *mod* $Q(r+s)$. Using also relations from [13, § 4 (I)] and [16, Theorem 2] for the twisted groups, we easily get

**Lemma 1.1.** *Let $U=UG(K)$ and $r,s,r+s\in G^+$. Then either $[X_r,X_s]=X_{r+s}$ mod $Q(r+s)$ or $G=\Phi$, $c_{rs}K=0=p(\Phi)!K$, and $[X_r,X_s]\subseteq Q(r+s)$.*

It is well known that every element $\gamma\in U$ is uniquely represented as the product of root elements $x_r(\gamma_r)$, $r\in G^+$, arranged according to a fixed order in $G$, cf. [23, Lemma 18] (we call such representation as *the canonical decomposition* of $\gamma$). The coefficient $\gamma_r$ is said to be an *r-projection* of $\gamma$. Putting

$$\pi(\gamma) := \sum_{r\in\Phi^+} \gamma_r e_r\big(\gamma \in U\Phi(K)\big), \qquad \alpha \circ \beta := \pi\big(\pi^{-1}(\alpha)\pi^{-1}(\beta)\big) \quad \big(\alpha, \beta \in N\Phi(K)\big),$$

we define an adjoint group $(N\Phi(K), \circ)$, which is isomorphic to the group $U\Phi(K)$. Similar representation of $U^m\Phi(K)$ for $p(\Phi) = 1$ as an adjoint group of certain $K_\sigma$-module $N^m\Phi(K)$ is used in [13] and [16].

The set of $r$-projections of all elements in a subset $H \subseteq UG(K)$ is called an *r-projection of H*. If an $s$-projection of $\gamma \in H$ is the product of its $r$-projection and a fixed non-zero scalar, not depending on a choice of $\gamma$, then $r$, $s$ are said to be *connected in H*. If also there exist $p, r + p, s + p \in G^+$ then $r$ and $s$ are said to be *p-connected in H*. It is easy to prove the following

**Lemma 1.2.** *Let $H \trianglelefteq U\Phi(K)$, $p(\Phi)!K = K$, $r$ be a corner in $H$, $s \in \{r\}^+$, and $s \neq r$. Then $H$ possesses a subgroup with a corner $s$ and with the $s$-projection $K$.*

The *highest root* in $G^+$ is denoted by $\rho$. If $r \in G$ then $r = \sum_{\alpha\in\Pi} c_\alpha\alpha$ with $c_\alpha \in \mathbb{Z}$. The *height* of $r$ is defined by $ht(r) = \sum_{\alpha\in\Pi} c_\alpha$. For every system $G$, the *Coxeter number* $h$ is defined by $ht(\rho) + 1 = h(G) = h$. The highest roots of root systems and $h$ are described in [3, Tables I–IX]. When $G$ is of type ${}^2F_4$, ${}^2B_2$, ${}^2G_2$ or ${}^2A_{2n}$, we have $h = 9, 3, 4$ or $2n$, respectively.

The subgroups $U_i = \langle X_r \mid r \in G^+,\ ht(r) \geqslant i\rangle$ form the *standard central series* $U = U_1 \supset U_2 \supset \cdots \supset U_h = 1$ in $U$, by [4, Theorem 5.3.3] and [13]. We shall use some property of the hypercenters (Lemma 1.3). Some subgroups $A$ and $B$ in a group are said to be *incident* if $A \subseteq B$ or $B \subseteq A$. Under the conditions of the following lemma the upper central (or hypercentral) series $1 = Z_0 \subset Z_1 \subset Z_2 \subset \cdots$ is standard, by [13]. Set $t(U) = 6, 3$ or $1$ for $G = E_8, E_6, A_n$, respectively,

$$t(U) = 4 \quad \text{for } G = G_2, F_4, {}^2F_4, {}^2E_6, E_7, \text{ or } 2K = K \text{ and } G = {}^3D_4,$$

and $t(U) = 2$ in the other cases. By [14, Lemma 3], we have

**Lemma 1.3.** *Let $U = UG(K)$, and let $p(\Phi)!K = K$ for $G = \Phi$. Then each normal subgroup of $U$ is incident with every hypercenter $Z_i$, $0 \leqslant i \leqslant t(U)$.*

The centralizer $C(T(r))$ of $T(r)$ in $U$ was determined in [13]. For $G = \Phi$, we distinguish also some subgroups of the following form:

$$\alpha(K)\big(C\big(T(r)\big) \cap C\big(T\big(r'\big)\big)\big), \quad \alpha(t) := x_r(t)x_{r'}(t)\ (t \in K),\ r + r' = \rho; \tag{1}$$

$$\beta(K)\big(C\big(T(r)\big) \cap C\big(T\big(r'\big)\big)\big)\big\{x_r(t)x_{r'}(t)x_{r+p}(ct)\ \big|\ t \in K\big\} \quad (c \in K),$$

$$\beta(t) := x_{r+p}(t)x_{r'+p}(t),\ r + r' + p = \rho. \tag{2}$$

The group $U$ of type $A_n$ (denoted by $UA_n(K)$) is isomorphic to the unitriangular group $UT(n + 1, K)$. By [10, Theorem 3] (for a finite field $K$ of odd order, see also [27, Theorem 7]), we get

**Lemma 1.4.** *Up to conjugation by a diagonal automorphism, every maximal abelian normal subgroup of $UA_n(K)$ is either $T(p)$, or (1), or (2) for $2K = 0$, $n \geqslant 3$ and some $r, r' \in \Phi^+$, $p \in \Pi$.*

## 2. Extremal subgroups

Let $U = UG(K)$. According to [18] and [19], a normal abelian subgroup $A$ in $U$ is said to be *extremal* if $A \not\subseteq U_2$. Therefore, there exists a simple corner $p$ in $A$, i.e., $A \not\subseteq \langle X_r \mid r \in G^+,\ r \neq p\rangle$ (see also [4, § 8.1]). For the purpose of application to the revision of the classification of finite simple groups and etc., C. Parker and P. Rowley [18–20] studied the groups $U$, having extremal subgroups, and simple corners of such subgroups.

Now, we correct some flaws in [18] and [19]. For $UD_4(K)$ over a field $K$ of characteristic 2, the example in [18, pp. 396–397] gives some extremal subgroups with three simple corners (see also [18, Theorem 1.3]). By [19, Theorem 1.2], if $U^2D_4(K)$ has an extremal subgroup with two simple corners then $2K = 0$. But we now show that if $U^2D_4(K)$ and $UD_4(K)$ were chosen as above, then, in fact, $|K| = 4$ and $|K| = 2$, respectively.

Let $\Phi$ be a root system of type $D_4$, and let $\bar{}$ be a symmetry of order 3 of the Coxeter graph of $\Phi$. We consider simple roots $r$, $\bar{r}$, $\bar{\bar{r}}$, and $q = \bar{q}$. Clearly, $UD_4(K)$ and $U^2D_4(K)$ contain the element

$$\vartheta := x_r(1)x_{\bar{r}}(1)x_{\bar{\bar{r}}}(1)x_{s-r}(1)x_{s-\bar{r}}(1)x_{s-\bar{\bar{r}}}(1) \quad (s := q + r + \bar{r} + \bar{\bar{r}}). \tag{3}$$

**Theorem 2.1.** *The groups $UD_4(K)$ for $|K| > 2$ and $U^2D_4(K)$ for $|K| > 4$ have no extremal subgroups with $\geqslant 3$ or $\geqslant 2$ simple corners, respectively. The normal closure of* (3) *in $UD_4(2)$, and $U^2D_4(4)$ is an extremal subgroup with three and two simple corners, respectively.*

**Proof.** Note that if $U$ is of type $D_4$ and ${}^2D_4$ then every its extremal subgroup contains $U_4$, by Lemma 1.3, and also $U_3 = C(U_3)$.

Let $U = UD_4(K)$. Suppose that $r$, $q$, $s$ are chosen as above. Assume that there exists an extremal subgroup $M$ in $U$ with $\geqslant 3$ simple corners. Then we have

$$U_4 \subset M \subset C(U_4) = T(r)T(\bar{r})T(\bar{\bar{r}}), \qquad \mathcal{L}(M) = \{r, \bar{r}, \bar{\bar{r}}\},$$
$$U/T(r) \simeq U/T(\bar{r}) \simeq U/T(\bar{\bar{r}}) \simeq UT(4, K).$$

By [10, Theorem 3], all corners in $M$ are $q$-connected and $2K = 0$. Setting

$$\xi(t) := x_r(t)x_{\bar{r}}(t)x_{\bar{\bar{r}}}(t), \qquad \eta(t) := x_{q+r}(t)x_{q+\bar{r}}(t)x_{q+\bar{\bar{r}}}(t), \qquad \kappa_p(t) := x_{s-p}(t)x_{s-\bar{p}}(t),$$

up to conjugation of $M$ by a diagonal automorphism we easily obtain

$$M = \xi(F) \ mod\ U_2, \qquad M \cap U_2 = [M, X_q] = \eta(K) \ mod\ U_3,$$
$$M \cap U_3 = \bigl[\eta(K), U\bigr] = U_4 \cdot \prod_{p \in \Pi \setminus \{q\}} \kappa_p(K),$$

where $F$ is an additive subgroup $F$ of $K$ and $F \supseteq GF(2)$. Therefore, for some map $\tilde{}: F \to K$ and $v_r, v_{\bar{r}}, v_{\bar{\bar{r}}} \in K$, every $\gamma \in M$ may be written modulo $M \cap U_3$ in the form

$$\gamma = \xi(f)\bigl(x_{q+r}(v_r)x_{q+\bar{r}}(v_{\bar{r}})x_{q+\bar{\bar{r}}}(v_{\bar{\bar{r}}})\bigr)x_{s-r}(\tilde{f}) \quad (f \in F).$$

Since $s + q$ is equal to the highest root $\rho$ and $[\xi(F), \kappa_p(K)] = 1$, we obtain

$$\bigl[\gamma, \kappa_p(K)\bigr] = \bigl[x_{q+r}(v_r)x_{q+\bar{r}}(v_{\bar{r}})x_{q+\bar{\bar{r}}}(v_{\bar{\bar{r}}}), \kappa_p(K)\bigr] = x_\rho\bigl((v_p + v_{\bar{p}})K\bigr) = 1$$

and therefore $v_r = v_{\bar{r}} = v_{\bar{\bar{r}}}$. Consequently,

$$\gamma = \xi(f)x_{s-r}(\tilde{f}) \ mod\ M \cap U_2.$$

Also we note that every $\omega \in M \cap U_2$ may be written modulo $M \cap U_3$ as $\omega = \eta(t)x_{s-r}(t')$ for some $t, t' \in K$.

Now, taking into account that $U_3$ is abelian, we obtain

$$\begin{aligned} 1 = [\gamma, \omega] &= \bigl[\gamma, x_{s-r}\bigl(t'\bigr)\bigr]\bigl[\xi(f), \eta(t)\bigr]\bigl[x_{s-r}(\tilde{f}), \eta(t)\bigr] \\ &= x_s\bigl(t'f\bigr)x_\rho(\tilde{f}t)\bigl[\xi(f), \eta(t)\bigr] = x_s\bigl(t'f + f^2t\bigr)x_\rho\bigl(\tilde{f}t + ft^2\bigr). \end{aligned}$$

When $f = 1$, the equality $t'f + f^2t = 0$ implies $t' = t$ for every $t \in K$.

Analogously, for all $f \in F$ and $t \in K$, we obtain $f = \tilde{f}$, $t^2 + t = 0$, and hence $|K| = 2 = |F|$. Consequently, $M$ coincides with the normal closure

$$\bigl\{\bigl(U_4 \times \bigl\langle\bigl[\vartheta, x_{q+r}(1)\bigr], \bigl[\vartheta, x_{q+\bar{r}}(1)\bigr]\bigr\rangle\bigr) \rtimes \bigl\langle\bigl[\vartheta, x_q(1)\bigr]\bigr\rangle\bigr\} \rtimes \langle\vartheta\rangle \qquad (4)$$

of the element $\vartheta$ from (3) in $UD_4(2)$. Moreover, (4) is the unique extremal subgroup in $UD_4(2)$ with three simple corners.

Let $M$ be an extremal subgroup in $U = U^2D_4(K)$ possessing at least two simple corners. Take the twisted automorphism $\theta \in Aut\, D_4(K)$ of order 2 such that $\theta(x_r(1)) = x_{\bar{r}}(1)$, $\theta(X_{\bar{\bar{r}}}) = X_{\bar{\bar{r}}}$. Then the system $\zeta(\Phi)$ is of type $B_3$ and $\mathcal{L}(M) = \{a, b\}$, where $a = \zeta(r)$, $b = \zeta(\bar{\bar{r}})$.

Up to conjugation by a diagonal automorphism, we obtain $\vartheta \in U_2M$. Using the argument of previous case, we get

$$x_{a+\zeta(q)+b}(K_\sigma)U_4 = \bigl[[\vartheta, X_{\zeta(q)}], X_b\bigr] \subset M, \quad |K_\sigma| = 2,$$

and, finally, $M$ coincides with the subgroup (4) in $UD_4(2) \cap U^2D_4(4)$. This completes the proof of Theorem 2.1. □

A description of maximal abelian normal subgroups of $U$ in Sections 3–5 and [15, Theorem 5] (for the classical types) gives also a description of extremal subgroups and hence a new solution to the Parker–Rowley problem.

## 3. The case of Lie rank ⩽ 2

Let $U$ be the group $UG(K)$ of exceptional type over a field $K$. In this section we prove the following theorem.

**Theorem 3.1.** *If $U$ is of rank $\leqslant 2$ then all maximal abelian normal subgroups in $U$ are exhausted by the following subgroups*:

(a) $\langle\gamma\rangle U_2$ $(\gamma \in U \setminus U_2)$ *for* $G = {}^2B_2$;
(b) $U_2$ *for* $G = {}^2G_2$ (*or* $G = G_2$ *and* $3K = 0$);
(c) $U_3$ *for* $G = G_2$ *if* $6K = K$, *and, additionally,* $\beta_c(K) \cdot U_4$ $(c \in K)$ *for* $2K = 0$, *and also* $\langle\alpha\rangle \times \langle\beta_1(1)\rangle$ *for* $|K| = 2$, *where*

$$\alpha = x_a(1)x_{2a+b}(1), \qquad \beta_c(t) = x_{a+b}(t)x_{2a+b}(tc);$$

(d) *$U_3$ for $G = {}^3D_4$, and, when $2K = 0$, additionally, up to conjugation by a diagonal automorphism, $\beta_c(K_\sigma)x_{2a+b}(K^{1+\sigma}) \cdot U_4$ ($c \in K$), and also*

$$\langle\alpha\rangle \times \big\langle\beta_1(1)\big\rangle \times x_{2a+b}\big(K^{1+\sigma}\big) \quad \text{if } |K_\sigma| = 2.$$

**Proof.** Consider an arbitrary maximal abelian normal subgroup $M$ of $U$. Note that the Coxeter number $h$ is even and $U_{h/2}$ is an abelian normal subgroup for every root system $\Phi$ of type $\neq A_n$.

The Coxeter number of a root system of type $G_2$ is equal to 6. Therefore, the normal subgroup $U_3$ (i.e., $T(2a+b)$) is abelian in the group $U$ of type $G_2$ or ${}^3D_4$. For $M \nsubseteq U_3$, the intersection $M \cap U_2$ has the corner $a+b$ and

$$U_4 = [X_a, M \cap U_2]U_5 \subseteq M \subseteq C(U_4) = T(a).$$

Thus, up to conjugation of $M$ by a diagonal automorphism, there exist some additive subgroups $F$, $Q$, $P$ of $K$ ($1 \in Q$, $1 \in F$ or $F = 0$) and a map $\tilde{}: Q \to K$ such that

$$M = x_a(F) \bmod U_2, \qquad M \cap U_2 = \beta(Q)x_{2a+b}(P)U_4,$$

where $\beta(v) := x_{a+b}(v)x_{2a+b}(\tilde{v}) \in M$ ($v \in Q$).

Suppose that $U = UG_2(K)$. If $6K = K$ then $U_3$ is a self-centralizing subgroup and each normal subgroup $H$ of the group $UG_2(K)$ is incident with $U_3$ by Lemma 1.3. It follows that $M = U_3$. Since $[M \cap U_2, M] = x_{2a+b}(2FK) \bmod U_4$, we have $2F = 0$. In particular, $T(a+b)$ (i.e., $U_2$) is a unique maximal abelian normal subgroup for $3K = 0$.

When $2K = 0$, the relations

$$\big[\beta(Q), x_{2a+b}(P)\big] = x_{3a+b}(3QP) \bmod U_5, \qquad \big[\beta(u), \beta(v)\big] = x_{3a+2b}\big(3(u\tilde{v} + v\tilde{u})\big)$$

show that $P = 0$ and $\tilde{v} = vd$ ($v \in Q$) for a fixed $d = \tilde{1} \in K$. Consequently, the intersection $M \cap U_2$ is contained into the abelian normal subgroup

$$\mathcal{M}_{c,d} = \big\{x_{a+b}(ct)x_{2a+b}(td) \bigm| t \in K\big\}U_4 \quad \big((c,d) \neq (0,0)\big)$$

for $c = 1$. Assume that $M \nsubseteq U_2$. Then $1 \in F$ and $\alpha = x_a(1)x_{2a+b}(f) \in M$ for $f \in K$. Since $[\alpha, X_b]U_4 \subseteq M \cap U_2$, we obtain

$$M \cap U_2 = \mathcal{M}_{1,1}, \qquad 1 = [\alpha, \mathcal{M}_{1,1}] = x_{3a+2b}\big(\big\{t^2 + tf \bigm| t \in K\big\}\big).$$

Hence, $f = 1$ and $|K| = 2$. On the other hand, $\langle x_a(1)x_{2a+b}(1)\rangle\mathcal{M}_{1,1}$ is an abelian normal subgroup of order $|K|^4 = 2^4$ for $|K| = 2$. If $|K| > 2$ then $M = U_3 = \mathcal{M}_{0,1}$ or $M = \mathcal{M}_{1,d}$ for an arbitrary $d \in K$.

For $U$ of type ${}^3D_4$, the ideal $K^{1+\sigma+\sigma^2} = \{t + \bar{t} + \bar{\bar{t}} \mid t \in K\}$ of the subfield $K_\sigma$ is non-zero (see also [19, Lemma 2.3]), and hence $K_\sigma = K^{1+\sigma+\sigma^2}$. Since $K_\sigma \cap K^{1+\sigma} = 2K_\sigma$, we get

$$K \supseteq K^{1+\sigma} + K^{1+\sigma+\sigma^2} \supseteq K^{\sigma^2} = K, \qquad K = K^{1+\sigma} + K_\sigma;$$
$$1 = \big[\big[X_a, \beta(1)\big], \beta(1)\big] = \big[x_{2a+b}\big(K^{1+\sigma}\big), \beta(1)\big] = x_{3a+2b}\big(\big(K^{1+\sigma}\big)^{1+\sigma+\sigma^2}\big).$$

Hence, $0 = 2K^{1+\sigma+\sigma^2} = 2K_\sigma = 2K$, whence the sum $K^{1+\sigma} + K_\sigma$ is direct and $P = K^{1+\sigma} + (P \cap K_\sigma)$. Taking into account the relations

$$1 = \big[x_{2a+b}(P \cap K_\sigma), \beta(1)\big] = x_{3a+2b}\big((P \cap K_\sigma)^{1+\sigma+\sigma^2}\big),$$

we deduce $0 = 3(P \cap K_\sigma) = P \cap K_\sigma$ and hence $P = K^{1+\sigma} = P^\sigma$. Also, $M \cap U_3$ centralizes $M \cap U_2$ and $x_{a+b}(K_\sigma)U_3$. Therefore,

$$1 = \big[\beta\big(Q \cap K^{1+\sigma}\big), x_{2a+b}(P)\big] = x_{3a+2b}\big(\big(P\big(Q \cap K^{1+\sigma}\big)\big)^{1+\sigma+\sigma^2}\big),$$

$$\big((\bar{v} + \bar{\bar{v}})\big(Q \cap K^{1+\sigma}\big)\big)^{1+\sigma+\sigma^2} = 0, \qquad (v + \bar{v} + \bar{\bar{v}})\big(Q \cap K^{1+\sigma}\big)^{1+\sigma+\sigma^2} = 0 \quad (v \in K).$$

Summarizing the last two equalities, we get $(v(Q \cap K^{1+\sigma}))^{1+\sigma+\sigma^2} = 0$ for all $v \in K$. Consequently, $Q \cap K^{1+\sigma} = 0$ (otherwise $K^{1+\sigma+\sigma^2} = 0$) and $Q \subseteq K_\sigma$.

Choose a system $\beta(Q)$ of coset representatives of $M \cap U_3$ in $M \cap U_2$ such that $\tilde{Q} \subseteq K_\sigma$. Using the isomorphism $U^3D_4(K) \cap UD_4(K_\sigma) \simeq UG_2(K_\sigma)$ we obtain $\tilde{v} = dv$ for $d = \tilde{1}$. Therefore, $M \cap U_2$ coincides with

$$\mathcal{M}_d = \big\{x_{a+b}(v)x_{2a+b}(dv) \bigm| v \in K_\sigma\big\}x_{2a+b}\big(K^{1+\sigma}\big)U_4.$$

For $F \neq 0$, $\alpha = x_a(1)x_{2a+b}(f) \in M$ may be chosen with $f \in K_\sigma$. The subgroup $\langle\alpha\rangle\beta(Q)U_4$ is normal in $U^3D_4(K) \cap UD_4(K_\sigma)$. As above, $\langle\alpha\rangle\beta(Q)U_4$ is abelian if and only if $f = 1$ and $|K_\sigma| = 2$. Note that $\langle x_a(1)x_{2a+b}(1)\rangle\mathcal{M}_1$ is an abelian normal subgroup in $U^3D_4(K)$ for $|K| = 8$. If $|K_\sigma| > 2$ then either $M = U_3$ or $M$ coincides with $\mathcal{M}_d$ for an arbitrary $d \in K$.

If $K$ possesses an automorphism $\sigma$ such that $3\sigma^2 = 1$ then $U^2G_2(K)$ is represented by the elements $(t, u, v)$ and

$$(t, u, v)\big(t', u', v'\big) = \big(t + t', u + u' - t\big(t'\big)^{3\sigma}, v + v' - ut' + t\big(t'\big)^{3\sigma+1} - t^2\big(t'\big)^{3\sigma}\big)$$

(see [5, 13.6.4 (viii)] and [23]). The subgroups $(0, 0, F)$, $(0, F, K)$, and $(F, K, K)$ in $U^2G_2(K)$ exhaust all normal subgroups by Lemma 1.3, where $F$ is an additive subgroup of $K$. Obviously, $U_2$ is abelian and $(F, K, K)$ with $F \neq 0$ are not abelian.

In [5, 13.6.4 (vii)], $U^2B_2(K)$ is represented as

$$U^2B_2(K) = \big\{(t, u) \bigm| t, u \in K\big\}, \quad (t, u)\big(t', u'\big) = \big(t + t', u + u' + (\bar{t})^2 t'\big), \tag{5}$$

where $K$ possesses a non-trivial automorphism $^-$ such that $\bar{\bar{x}}^2 = x$ $(x \in K)$. The center $Z_1$ of $U^2B_2(K)$ is equal to $(0, K)$ and, by Lemma 1.3, every normal subgroup is of the form either $(0, F)$ or $(F, K)$ for an arbitrary additive subgroup $F$ of $K$. For the commuting elements $(t, u)$ and $(t', u')$, we have $(\bar{t})^2 t' = \bar{t}'^2 t$. When $t' \neq 0$, up to conjugation by a diagonal element, we may assume that $t' = 1$. In this case $t = (\bar{t})^2 = (\bar{\bar{t}})^4 = t^2$, whence either $t = 0$ or $t = 1$. Therefore, the maximal abelian normal subgroups of $U^2B_2(K)$ are exhausted by the centralizers of the elements of order 4; they have the form $(F, K)$ with $|F| = 2$. Thus, Theorem 3.1 is proved. □

## 4. The normal structure

In this section, we consider the normal structure of $UG(K)$ and describe the maximal abelian normal subgroups of groups $UE_n(K)$, $n = 6, 7, 8$.

Let $U = UG(K)$ and $H \subseteq U$. Since $H \subseteq \prod_{s \in \mathcal{L}(H)} T(s)$, there exists a subset $\mathcal{F}(H)$ in $\prod_{s \in \mathcal{L}(H)} X_s$ such that $\mathcal{F}(H) = H$ *mod* $\prod_{s \in \mathcal{L}(H)} Q(s)$. As in [15], $\mathcal{F}(H)$ is said to be a *frame of* $H$. The following theorem holds.

**Theorem 4.1.** *Let $H$ be a subgroup in the group $U$ of classical type or of type $E_n$ over a field $K$. Assume that $2K = K$ or $U$ is of type $A_n$ or $^2A_n$. Then $H \trianglelefteq U$ if and only if $\mathcal{F}([H, X_p]) \subseteq H$ for each $p \in \Pi(G)$.*

Let us consider the idea of the proof.

Using the representation $\pi$ from Section 1 of $U$ we define a frame of a subset $\pi(H)$ in $(NG(K), \circ)$ by the rule $\mathcal{F}(\pi(H)) := \pi(\mathcal{F}(H))$. The concept of frame and the representation $\pi$ allow us to apply linear methods, cf. [12,13,15,16]. The multiplication $\circ$ and the addition on the frame $\mathcal{F}(\pi(H))$ coincide modulo $\sum_{r\in\mathcal{L}(H)} \pi(Q(r))$. Also, we may consider an arbitrary frame in the module $NG(K)$ as a submodule. When $G = \Phi$, we get

**Lemma 4.2.** *Let $H \subseteq U\Phi(K)$, $\pi(H)$ be a subgroup in the adjoint or additive group of $N\Phi(K)$, and let $p \in \Phi^+$. Then $\pi(\mathcal{F}([H, X_p]))$ is a $K$-submodule in $N\Phi(K)$ coinciding with the frame of $\pi(H) * Ke_p$.*

**Lemma 4.3.** *Let $U = UG(K)$, $H \subseteq U$ and $p \in G^+$. Then $|\mathcal{L}([H, X_p])| \leqslant 3$.*

**Proof.** The standard commutator relations show that every corner in $[H, X_p]$ can be written in the form $s + p$ for $s \in \bigcup_{r\in\mathcal{L}(H)}\{r\}^+$. Evidently, $|\mathcal{L}(H)| \leqslant rank\, G$. By the well known classification of root systems, for $G = \Phi$, the minimal root subsystem of $\Phi$ containing $\mathcal{L}([H, X_p]) \cup \{p\}$ has a connected Coxeter graph of rank $\leqslant 4$. Therefore, $|\mathcal{L}([H, X_p])| \leqslant 3$. Using the root system $\zeta(\Phi)$ we get this inequality for $G = {}^m\Phi$, $p(\Phi) = 1$. □

Now let $U = UG(K)$, $G = {}^2\Phi$, $p(\Phi) = 1$, $r, s, r + s \in G^+$, and let

$$x_r(F) \subseteq X_r, \qquad x_s(V) \subseteq X_s \quad \text{for some } F, V \subseteq K,\ FV \neq 0.$$

**Lemma 4.4.**

(i) *If $[x_r(F), x_s(V)] \subseteq Q(r + s)$ then $r + s$ is of the first type, $r$ and $s$ are not of the first type, and, up to conjugation by a diagonal automorphism, either $F \subseteq K_\sigma$, $V \subseteq K^{1-\sigma}$ or $G = {}^2A_{2n}$, $F, V \subseteq K_\sigma$.*

(ii) *If $[x_r(F), X_s]$ does not coincide with 0, $X_{r+s}$ modulo $Q(r + s)$ then $s$ is of the first type, $r$, $r + s$ are not of the first type, and $FK_\sigma$ is a 1-dimensional $K_\sigma$-module.*

**Proof.** Firstly, assume that either $r$ (or $s$) is of the first type or $r + s$ is not of the first type. Then the basic relations of the twisted group $U$ (cf. [4,23] and [16, Theorem 2]) show that $[x_r(u), x_s(v)] = x_{r+s}(\pm\eta)$ *mod* $Q(r + s)$ for $\eta = uv, \bar{u}v, u\bar{v}$ or $\bar{u}\bar{v}$, and hence $r + s$ is a corner of the commutator $[x_r(F), x_s(V)]$.

Thus, the assumption $[x_r(F), x_s(V)] \subseteq Q(r + s)$ shows that $r + s$ is of the first type, $r$ and $s$ are not of the first type, and $\eta = 0$ for all $u \in F$, $v \in V$, where either $\eta = uv + \bar{u}\bar{v}$ $(u\bar{v} + \bar{u}v)$ or $\eta = u\bar{v} - \bar{u}v$ when $G = {}^2A_{2n}$. Up to conjugation by a diagonal automorphism, we may assume that $1 \in F$. It immediately follows that either $V \subseteq Ker(1 + \sigma) = K^{1-\sigma}$, $F \subseteq K_\sigma$ or $G = {}^2A_{2n}$, $V, F \subseteq K_\sigma$.

When $[x_r(F), X_s]Q(r + s)$ does not coincide with $Q(r + s)$ and $T(r + s)$, we easily infer that $s$ is of the first type, $r + s$ and $r$ are not of the first type, and $FK_\sigma$ is a 1-dimensional $K_\sigma$-module. □

Using Lemma 1.1, Lemma 4.4, and (ii) we obtain the following lemma.

**Lemma 4.5.** *Let $H \trianglelefteq UG(K)$ and $\mathcal{L}(H) = \{r\}$. Then either $H = Q(r)\mathcal{F}(H)$ or* (a) *$G = {}^2\Phi$, $p(\Phi) = 1$, $r$ is not of the first type, $r$-projection of $H$ generates a 1-dimensional $K_\sigma$-module and there exists $s \in \Pi(G)$ of the first type with $r + s \in G^+$, or* (b) *$G = \Phi$, $p(\Phi)!K = 0$ or $G = {}^3D_4$, $2K = 0$.*

It is well known that for $G = {}^2A_{2n}$ every $s \in \Pi(G)$ is not of the first type. Using Lemmas 4.4 and 4.5 repeatedly we get the following theorem from [15].

**Theorem 4.6.** *Let $UG(K)$ be of type $B_n$, $C_n$ for $2K = K$ or of type $A_n$, ${}^2A_n$. A subgroup $H$ is normal if and only if for each corner $r$ of $H$ and $p \in \Pi(G)$ with $r + p \in G$ either*

(A) $\mathcal{F}([H, X_p])Q(r + p) \subseteq H$

*or* $G = B_n$ *and*

(B) *for some* $q \in \Pi(G)$ *two corners in* $[H, X_p]$ *are* $q$*-connected, two corners in* $[H, X_q]$ *are connected, and* $\mathcal{F}([H, X_p])\mathcal{F}([H, X_q])Q(r + p, r + p + q) \subset H$.

For the group $U$ of type $E_n$, the analogue of this theorem is not satisfied [25]. By [15, Theorems 3 and 5], for $U$ of type $D_n$ and ${}^2D_n$ there exists a normal subgroup $M$ such that the height of commutator $[[\ldots[[M, U], U]\ldots], U]$ grows unboundedly together with the grows of $n$, where the commutator is not generated by the root elements of $M$. To finish the consideration of remaining groups $U$ in Theorem 4.1 we use the normal closures of subgroups which are similar to the subgroups from Theorem 2.1, and we get

**Lemma 4.7.** *If* $H \trianglelefteq UG(K)$ *for type* $D_n$ *(or* ${}^2D_n$*) and* $\mathcal{F}([H, X_p]) \nsubseteq H$ *for some* $p \in \Pi(G)$ *then there exist simple corners* $r, \bar{r}$ *(respectively,* $\zeta(r)$*) and a* $p$*-connected corner in* $H$ *which have the projections of order* 2.

Our description of abelian normal subgroups uses a specific notation.

For every $\Psi \subseteq G^+$, we set $X_\Psi = \langle X_r \mid r \in \Psi\rangle$. A subset $\Psi$ in $G^+$ is called *normal* if $\{s\}^+ \subseteq \Psi$ for all $s \in \Psi$, and hence $X_\Psi \trianglelefteq UG(K)$. By [17], a subset $\Psi$ in $\Phi^+$ is called *abelian* if $r + s \notin \Phi$ for all $r, s \in \Psi$. Then $X_\Psi$ is the direct product of some root subgroups. For $H \subseteq UG(K)$, put

$$\Psi(H) = \{r \in G^+ \mid H \cap X_r \neq 1\}. \tag{6}$$

Denote by $\widehat{\Psi}(H)$ the set of all corners of the elements in $H$, which are not in $\Psi(H)$, and also all sums in $G^+$ of such corners. Thus, for the subgroup $H$ in $U\Phi(K)$ of the shape (1) or (2) from Lemma 1.4, $\widehat{\Psi}(H)$ is $\{r, r', \rho\}$ or $\{r, r', r + p, r' + p, \rho\}$, respectively.

Further, we use the elements $\alpha(t)$ and $\beta(t)$ from (1) and (2). By [15], for $2K = 0$, $UD_n(K)$ has a unique maximal abelian normal subgroup $M_0$ possessing some simple corners $r$ and $r' = \bar{r}$ with $\alpha(1) \in M_0$ and $\widehat{\Psi}(M_0) = \{r\}^+ \cup \{r'\}^+$. For $n = 4$ and some $p, q \in \Pi(\Phi)$, $M_0$ is of the shape

$$\alpha(K)\beta(K)\{x_{r+p+q}(t)x_{r'+p+q}(t) \mid t \in K\}\bigl(C\bigl(T(r)\bigr) \cap C\bigl(T(r')\bigr)\bigr). \tag{7}$$

**Theorem 4.8.** *Let* $M$ *be a maximal abelian normal subgroup of the group* $U = U\Phi(K)$, $\Psi = \Psi(M)$ *and* $p(\Phi)!K = K$. *Then* $X_\Psi \subseteq M$ *and for* $M \neq X_\Psi$*, up to conjugation by diagonal automorphism, there are two cases*:

(i) *M is of the form* (1) *and* $X_{\widehat{\Psi}} \simeq UT(3, K)$;
(ii) $2K = 0$, $p(\Phi) = 1$, $X_{\widehat{\Psi}} \cap M$ *has* $p$*-connected corners for a simple root* $p$.

*Moreover, in* (ii) *one of the following subcases holds*:

(a) *M is of the form* (2) *and* $X_p X_{\widehat{\Psi}} \simeq UT(4, K)$,
(b) $U = UD_4(2) = X_{\widehat{\Psi}} X_p$,
(c) *U is of type* $D_n$, $E_m$, *and* $X_{\widehat{\Psi}} \times X_s \simeq [UD_4(K), UD_4(K)]$ *for some* $s \in \Psi$,
(d) *M is of the form* $M_0$ *or* (7)*, respectively, for types* $D_n$, $E_m$.

**Proof.** Using Lemmas 1.2 and 4.5, we easily find that $\Psi$ and $\Psi \cup \{r\}^+$ are commutative normal sets in $\Phi^+$ for $r \in \mathcal{L}(M)$. The subgroup $X_\Psi$ centralizes $M$, and hence $X_\Psi \subseteq M$. Obviously, $M = X_\Psi$ if and only if $\Psi$ is a maximal commutative normal set in $\Phi^+$. Let $\widehat{\Psi} = \widehat{\Psi}(M)$. Assuming $M \neq X_\Psi$ we get

$$\mathcal{L}(X_{\widehat{\Psi}}) = \mathcal{L}(M \cap X_{\widehat{\Psi}}), \quad X_\Psi = \bigcap_{r \in \mathcal{L}(X_{\widehat{\Psi}})} C\big(T(r)\big), \; M = (M \cap X_{\widehat{\Psi}}) \times X_{\Psi \setminus (\Psi \cap \widehat{\Psi})}.$$

Each root $r$ in $\widehat{\Psi} \setminus (\Psi \cap \widehat{\Psi})$ does not commute with at least one root of $\widehat{\Psi} \setminus (\Psi \cap \widehat{\Psi})$, since $X_r$ centralizes no $M$. Therefore, each corner in $M \cap X_{\widehat{\Psi}}$ is connected with another corner in $M \cap X_{\widehat{\Psi}}$.

If there exist corners $r$ and $r'$ in $M \cap X_{\widehat{\Psi}}$ which are not commuting then $[M, X_r][M, X_{r'}] \subset M$, and the root systems from [3] give

$$X_{\widehat{\Psi}} \simeq UT(3, K), \quad \widehat{\Psi} = \{r, r', r + r'\}, \qquad X_\Psi = C\{T(r)T(r')\}, \quad r + r' = \rho \in \Psi.$$

Then, by Lemma 1.4, $M$ is conjugate by a diagonal automorphism to (1).

In the other cases, for a simple root $p$, there exist some $p$-connected corners $r$ and $r'$ in $M \cap X_{\widehat{\Psi}}$ and $\{r, r', r+p, r'+p, r+r'+p\} \subseteq \widehat{\Psi}$ holds. If this inclusion turns into an equality then $p(\Phi) = 1$, $2K = 0$, $M$ is reduced to the form (2), and

$$r' + r + p = \rho, \quad X_p X_{\widehat{\Psi}} \simeq UT(4, K).$$

In the other cases, for type $D_n$ and $|\mathcal{L}(X_{\widehat{\Psi}})| = 2$, we have $X_{\widehat{\Psi}} = T(r)T(\bar{r})$ by [15]. Up to conjugation by a diagonal automorphism, the subgroup $M$ in $U$ is of the shape (7) if $U$ is of type $E_m$.

The case $|\mathcal{L}(X_{\widehat{\Psi}})| = 3$ is possible when $U$ is of type $E_m$ and $D_n$. Then two of three corners $r_1$, $r_2$, $r_3$ in $M \cap X_{\widehat{\Psi}}$ are $p$-connected, two of them are $q$-connected, and $X_{\widehat{\Psi}} \times X_s \simeq [UD_4(K), UD_4(K)]$ for some $s \in \Psi$ and some simple roots $p$, $q \neq p$. In this case, $M$ has the form

$$\big\{x_{r_1}(t)x_{r_2}(t)x_{r_3}(t)x_{r_2+p}(ct) \bigm| t \in K\big\}\big\{x_{r_1+p}(t)x_{r_2+p}(t) \bigm| t \in K\big\}\big\{x_{r_1+q}(t)x_{r_3+q}(t) \bigm| t \in K\big\}X_\Psi. \qquad (8)$$

In the remaining cases, for $U$ of type $D_n$, $M$ has three simple corners and $U = UD_4(2) = X_{\widehat{\Psi}} X_q$ (see Theorem 2.1 and [15, Theorem 5]). □

We now list the maximal commutative normal sets $\Psi \subseteq \Phi$ and all subgroups (1)–(8) in $U$ of type $E_m$. For $UE_6(K)$, this enumeration is given up to a graph automorphism. For a root system $\Phi$ of type $E_m$ corresponding to $m = 6, 7$ or $8$, the Coxeter number is equal to $h = 12, 18$ or $30$; moreover,

$$Z_k = U_{h-k} \subseteq M \subseteq C(Z_k), \quad k = 4, 6 \text{ or } 10.$$

Choose some simple roots $\alpha_i$ ($1 \leqslant i \leqslant m$) as in [3, Tables V–VII]. When $M$ has a corner of height $\leqslant 4$, using Lemma 1.2 we infer that either $U$ is of type $E_7$ and $M = T(\alpha_7)$ or $U$ is of type $E_6$ and $M$ is one of the subgroups $T(\alpha_1)$ and $T(\alpha_6)$ or $M \subseteq (U_4 \cap (T(\alpha_1)T(\alpha_6)))U_5$. We set

$$\begin{matrix} acde\ldots f \\ b \end{matrix} = \big(ac[db]'e\ldots f\big) := a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4 + e\alpha_5 + \cdots + f\alpha_m.$$

*A) The maximal commutative normal sets $\Psi$*

Type $E_6$: $\{11[10]'10\}^+ \cup \{01[21]'21\}^+$, $\{11[10]'11\}^+ \cup \{\tilde{\mu}_4 + \alpha_1\}^+ \cup \{\tilde{\mu}_4 + \alpha_6\}^+$, $\{\alpha_1\}^+$, $\{\tilde{\mu}_4\}^+$, where $\tilde{\mu}_4 = (01[21]'10)$ (the highest root of subsystem of type $D_4$ with the root $\alpha_4$);

Type $E_7$: $\{\alpha_7\}^+$, $\{12[32]'210\}^+ \cup \{00[11]'111\}^+$, $\{12[31]'210\}^+ \cup \{01[21]'111\}^+$, $\{12[21]'210\}^+ \cup \{12[21]'111\}^+ \cup \{01[21]'211\}^+$, $\{12[21]'110\}^+ \cup \{01[21]'221\}^+$, $\{11[21]'210\}^+ \cup \{01[21]'211\}^+$, $\{12[21]'100\}^+$, $\{01[21]'210\}^+$;

Type $E_8$: $\{12[32]'2100\}^+$, $\{12[31]'3210\}^+$, $\{12[32]'3210\}^+ \cup \{12[31]'3211\}^+$, $\{12[32]'2210\}^+ \cup \{12[31]'3321\}^+$, $\{12[42]'3210\}^+ \cup \{12[31]'2221\}^+$, $\{13[42]'3210\}^+ \cup \{12[21]'2221\}^+$, $\{23[42]'3210\}^+ \cup \{11[21]'2221\}^+$, $\{12[32]'3210\}^+ \cup \{12[32]'2221\}^+ \cup \{12[31]'3221\}^+$, $\{01[21]'2221\}^+$.

*B*) *The roots r defining the subgroup* (1)

Type $E_6$: $(11[11]'00)$, $(11[11]'10)$, $\tilde{\mu}_4$;

Type $E_7$: $(11[10]'111)$, $(12[21]'100)$, $(12[21]'110)$, $(11[21]'210)$, $(11[21]'111)$, $(11[11]'111)$;

Type $E_8$: $(12[32]'2111)$, $(12[32]'2211)$, $(12[31]'3211)$, $(12[32]'3211)$, $(12[21]'2221)$, $(11[21]'2221)$, $(01[21]'2221)$.

*C*) *The pairs* $\{r, p\}$ *defining the subgroup* (2)

Type $E_6$: $\{(11[11]'00), \alpha_5\}$, $\{(11[11]'10), \alpha_6\}$, $\{(11[11]'10), \alpha_4\}$;

Type $E_7$: $\{(12[21]'100), (11[21]'211)\}$, $\{(12[21]'110), (11[21]'210)\}$, $\{(12[21]'110), (11[21]'111)\}$, $\{(11[21]'210), (11[21]'111)\}$, $\{(12[21]'210), (11[11]'111)\}$, $\{(12[31]'210), (11[10]'111)\}$;

Type $E_8$: $\{(12[31]'3221), (12[32]'2111)\}$, $\{(12[31]'3211), (12[32]'2211)\}$, $\{(12[31]'2221), (12[31]'3211)\}$, $\{(12[31]'2221), (12[32]'2211)\}$, $\{(12[21]'2221), (12[32]'3211)\}$, $\{(11[21]'2221), (12[42]'3211)\}$.

*D*) *The corners* $\{r, r'\}$ *defining the subgroup* (7) *with q-connected corners in the commutator group* $[M, X_p]$

For types $E_6$, $E_7$, and $E_8$ such corners are $\{(11[10]10), (01[10]'11)\}$, $\{(01[21]'210), (01[21]'111)\}$, and $\{(12[31]'3210), (12[32]'2210)\}$, respectively.

*E*) *The pairwise p-connected or q-connected corners* $\{r_1, r_2, r_3\}$ *of the subgroup* (8)

Type $E_8$: $\{(12[31]'2221), (12[31]'3211), (12[32]'2211)\}$;

Type $E_7$: $\{(12[21]'110), (11[21]'210), (11[21]'111)\}$;

Type $E_6$: $\{(11[11]'10), \tilde{\mu}_4, (01[11]'11)\}$.

## 5. The groups $U$ of types $F_4$, $^2F_4$ and $^2E_6$

For the root system $\Phi$ of type $F_4$, we need notation from [13].

By [3, Tables I–IV] and [13], the positive roots of systems of types $A_{n-1}$, $B_n$, $C_n$, $BC_n$, and $D_n$ may be written as

$$\varepsilon_i - m\varepsilon_j = p_{i,mj}, \quad 1 \leqslant j \leqslant i \leqslant n,\ m = 0, 1, -1.$$

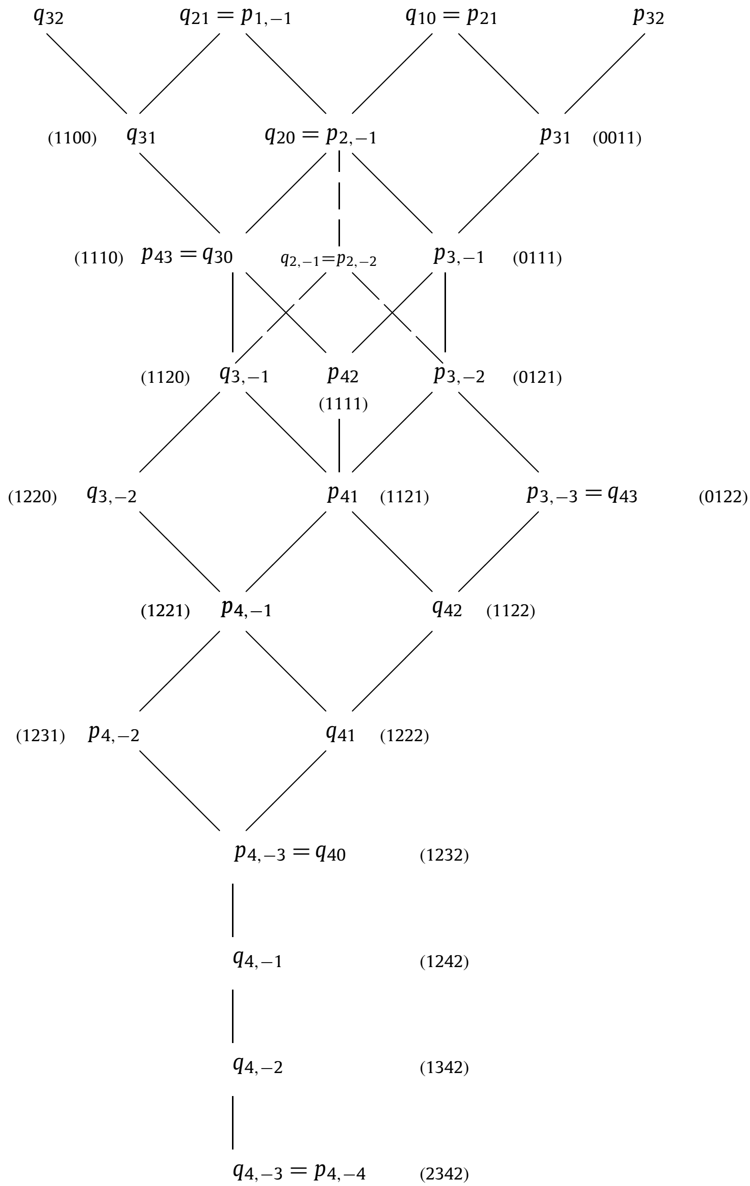
Set $T_{iv} = T(p_{iv})$ for exception the case $T_{i1} = T(p_{i,-1})T(p_{i1})$ for type $D_n$. If $UG(K)$ is a group of classical type distinct from $A_n$ then, by [13, Lemma 6 (II)], the centralizer $C(T_{iv})$ in $UG(K)$ coincides with $T_{1,-v-1}$, when either $i < n$ or $G = {}^2A_m$ or $2K = K$, $G = C_n$; in the remaining cases, we have $C(T_{nv}) = T_{1,-v-1}T_{nn-1}$.

Let $C_n^+ = \{p_{iv} \mid 0 < |v| \leqslant i \leqslant n,\ v \neq i\}$, as above. For type $B_n$, we set $\varepsilon_i - m\varepsilon_j = q_{i,mj}$. By analogy with [13], we represent the positive system $F_4^+$ as the union $C_4^+ \cup B_4^+$ with the given intersection

$$B_4^+ \cap C_4^+ = \{q_{i0}, p_{i,-i}\ (1 \leqslant i \leqslant 4)\}, \qquad B_4^+ = \{q_{ij} \mid 0 \leqslant |j| < i \leqslant 4\}.$$

Also, we use the following diagram from [13]. (The roots are accompanied by the notation (*abcd*) from [3, Table VIII].) The substitution $^- : \Phi \mapsto \Phi$ is defined by the simple rule: $\bar{p}_{ij} = q_{ij}$, $\bar{q}_{ij} = p_{ij}$ $(1 \leqslant |j| < i \leqslant 4)$.

*The positive roots of the system $F_4$*




Consider the "root elements" of $U^2F_4(K)$, cf. Section 1. Let $r = q_{ij}$. Put $R_{ij}(t) = x_r(t)x_{\bar{r}}(\bar{t})$ if either $(i, j) = (2, -1), (3, 2), (3, -2)$ or $i = 4$, $j \in \{-3, -2, -1, 1, 2\}$. When $(i, j) = (2, 1), (3, 1), (3, -1)$ or $(4, 3)$, according to [4], $\{r, \bar{r}, r + \bar{r}, r + 2\bar{r}\}$ is a class of type $B_2$ and we set

$$R_{ij}(t) = x_{\bar{r}}(\bar{t})x_r(t)x_{r+\bar{r}}(t\bar{t}) \quad (t \in K).$$

By [13, § 4 (I)], $U_k$ in $U^2F_4(K)$ is generated by the elements $R_{ij}(t)$ corresponding to the columns with number $\geqslant k$ in the following table:

$$R_{21}\quad R_{2,-1}\quad R_{3,-1}\quad R_{3,-2}$$
$$R_{32}\quad R_{31}\quad R_{43}\quad R_{42}\quad R_{41}\quad R_{4,-1}\quad R_{4,-2}\quad R_{4,-3}.$$

Recall that the system ${}^2\Phi$ of type ${}^2E_6$ is associated with a root system of type $F_4$. Choose the following subgroups in $UF_4(K)$ and $U^2E_6(K)$ with $F=K$ and $F=K_\sigma$, respectively:

$$T(q_{43})U_6,\quad T(p_{4,-1})T(q_{3,-2}),\quad T(p_{4,-1})\bigl\{x_{q_{3,-2}}(t)x_{q_{42}}(t)\bigm| t\in F\bigr\};\tag{9}$$

$$T(p_{42})X_{q_{43}},\quad T(p_{42})X_{p_{43}},\quad T(p_{3,-2}),\quad T(p_{3,-2})^\tau,\quad T(q_{3,-2})X_{p_{41}}X_{p_{3,-2}};\tag{10}$$

$$\bigl\{x_{p_{3,-2}}(t)x_{p_{42}}(t)\bigm| t\in K\bigr\}S,\quad S=T(q_{43})T(p_{41})\text{ or }T(q_{3,-2})X_{p_{41}};\tag{11}$$

$$\bigl\{x_{q_{3,-2}}(t)x_{q_{42}}(t)\bigm| t\in K\bigr\}T(p_{4,-1})X_{p_{41}}S,\quad S=X_{p_{43}}X_{p_{42}}\text{ or }X_{p_{3,-2}};\tag{12}$$

$$\bigl\langle x_{p_{43}}(1)x_{q_{43}}(d)\bigr\rangle T(p_{42})\quad\bigl(d\in K^*\bigr);\tag{13}$$

$$\bigl[\bigl\langle x_{p_{3,-2}}(t)x_{p_{42}}(t)\bigm| t\in K\bigr\rangle\times\bigl\langle x_{q_{3,-2}}(t)x_{q_{42}}(dt)\bigm| t\in K\bigr\rangle\bigr]T(p_{4,-1})X_{p_{41}}.\tag{14}$$

The main theorem of this section is the following one.

**Theorem 5.1.** *Up to conjugation by a diagonal automorphism, the maximal abelian normal subgroups in $UF_4(K)$ and $U^2E_6(K)$ are exhausted by the subgroups* (9) *for $2K=K$; when $2K=0$, they are exhausted by the subgroups* (10)–(14) *and, respectively, by* (9), $(T(p_{3,-2})\cap E_6(K_\sigma))U_7$, *and*

$$\bigl\{x_{p_{41}}(t)x_{p_{4,-1}}(ft)\bigm| t\in F\bigr\}x_{p_{4,-1}}(K_\sigma)T(q_{43})U_7\quad(f\in K\setminus K_\sigma).\tag{15}$$

*In $U^2F_4(K)$, they are exhausted by the subgroups*

$$\bigl\langle R_{43}(1)\bigr\rangle R_{42}(K)U_5,\qquad\bigl\{R_{3,-2}(t)R_{42}(ct)\bigm| t\in K\bigr\}U_5\quad(c\in K).\tag{16}$$

**Proof.** Note that if the roots $r$, $s$, and $r+s$ from $F_4^+$ do not lie simultaneously in one of the subsystems $B_4^+$ or $C_4^+$ then they lie in one of the following subsystems of type $B_2^+$:

$$\{p_{3,-v},q_{3,2v},p_{4,2v},q_{4v}\},\qquad\{p_{3,2v},q_{3,-v},p_{4v},q_{4,2v}\},$$
$$\{p_{3v},q_{3v},p_{4,2v},q_{4,2v}\},\qquad\{p_{3,2v},q_{3,-2v},p_{4,-v},q_{4v}\},\quad|v|=1.$$

Also we have $U^2E_6(K)=\langle x_{p_{iv}}(K),x_{q_{iv}}(K_\sigma)\ (1\leqslant|v|<i\leqslant4)\rangle$.

Consider an arbitrary maximal abelian normal subgroup $M$ in $U$ of type $F_4$ and ${}^2E_6$. When $p_{41}$-projection of $M$ is zero, we get

$$T(q_{3,-2})T(q_{43})\supset M=C(M)\supset C\bigl(T(q_{3,-2})T(q_{43})\bigr)\supseteq T(p_{4,-1}).$$

Let $F=K$ or $F=K_\sigma$ as in the theorem. Since $X_{q_{42}}X_{q_{3,-2}}X_{q_{4,-3}}\simeq UT(3,K)$ by Lemma 1.4, we obtain the subgroups (9).

Further, we may assume that the $p_{41}$-projection in $M$ is non-zero. Then the $p_{41}$-projection $P$ of the intersection $M\cap U_5$ is also non-zero because of $M\trianglelefteqslant U$. Up to conjugation of $M$ by a diagonal automorphism, we have $1\in P$. Commuting $M\cap U_5$ firstly with $T(p_{1,-1})$ and then with $U$, we find the subgroup $x_{p_{4,-1}}(FP)T(p_{4,-2})$ in $M$ (see the diagram). Since the centralizer of this subgroup coincides with $T(p_{2,-1})$, we obtain $M\subseteq T(p_{2,-1})$ and $2K=0$, because of the equality $[x_{p_{4,-1}}(FP),M\cap U_5]=1$. Thus, if $2K=K$ then $M$ is one of the subgroups (9).

Note that $U^2E_6(K)\cap E_6(K_\sigma)\simeq UF_4(K_\sigma)$. For type ${}^2E_6$ we also infer that the $K_\sigma$-module $FP$ is one-dimensional, and $1\in P\subseteq K_\sigma$. The $p_{4,-1}$-projection of the subgroup $M\cap(T(p_{4,-1})T(q_{43}))$ is contained in $K_\sigma$, since $M$ is an abelian subgroup. Taking into account the normality of $M$, we obtain

$$T(p_{3,-2}) \supseteq M \supseteq C\big(T(p_{3,-2})\big), \qquad M \cap T(p_{41}) = \alpha(P)x_{p_{4,-1}}(K_\sigma)T(q_{42})U_7$$

where $\alpha(t) = x_{p_{41}}(t)x_{p_{4,-1}}(\tilde{t})$ for a suitable mapping $\tilde{}: P \to K$. Set $f = \tilde{1}$ and $t_0 = \tilde{t} + ft$. Using $[\alpha(t), \alpha(1)] = 1$ we find

$$\tilde{t} + \bar{\tilde{t}} + f\bar{t} + \bar{f}t = 0, \qquad t_0 = \bar{t}_0, \qquad \tilde{t} = t_0 + ft \in ft + K_\sigma \quad (t \in P).$$

Clearly, $(T(p_{3,-2}) \cap E_6(K_\sigma))U_7$ is an abelian normal subgroup. Consequently, if the $p_{3,-2}$-projection in $M$ is zero then we have $f \in K \setminus K_\sigma$. Therefore, $P = K_\sigma$, and $M$ is the second subgroup in (15). Similarly

$$M = \beta(P)x_{p_{41}}(K_\sigma)x_{p_{4,-1}}(K_\sigma)T(q_{42})U_7, \quad \beta(t) = x_{p_{3,-2}}(t)x_{p_{4,-1}}(ft)$$

in the case when $M$ has the corner $p_{3,-2}$. But in the latter case the condition $[\beta(t), \beta(1)] = 1$ gives $f\bar{t} + \bar{f}t = 0$ $(t \in P)$. Therefore, $f \in K_\sigma$, and $M$ coincides with the subgroup $(T(p_{3,-2}) \cap E_6(K_\sigma))U_7$.

In $UF_4(K)$, the subgroup $X_{p_{41}}T(p_{4,-1})$ centralizes $U_5$. Using the normality of $M$, we also find the corner $p_{41}$ of the intersection $M \cap T(p_{41})$ for the case $M \nsubseteq U_5$. Therefore, the $p_{2,-1}$-projection and $p_{3,-1}$-projection in $M$ are zero, i.e., $M \subseteq T(p_{43})T(q_{2,-1})$. If either the $q_{2,-1}$-projection or the $q_{3,-1}$-projection in $M$ is non-zero then $M \cap T(q_{41})$ has the corner $q_{41}$, and $M \cap T(q_{41})$ does not centralize $M$, a contradiction. It follows that

$$T(p_{4,-1})X_{p_{41}} \subseteq M \subseteq X_{p_{43}}T(p_{42})T(p_{3,-2})T(q_{3,-2}).$$

Since $1 = [[M, T(q_{32})], M]$, the $q_{3,-2}$-projection should be zero if the $q_{43}$-projection in $M$ is non-zero. Similarly, the $p_{3,-2}$-projection in $M$ is zero if the $p_{43}$-projection is non-zero. For the center $Z$ of $U$, the subgroup $B = X_{p_{43}}X_{q_{43}}Z$ has a direct complement $D$ in $X_{p_{43}}X_{q_{43}}T(p_{42})$, and

$$Z \times D = T(p_{42}) \subseteq M \subseteq B \times D, \qquad M = (M \cap B) \times D, \qquad B \simeq UB_2(K).$$

If $p_{43}$ and $q_{43}$ are corners in $M$ then they are connected. By [15, Theorem 5], the projections on these corners have order 2. Thus, $M \cap B$ is a maximal abelian normal subgroup in $B$, and $M$ is the subgroup (13).

The other cases for the non-zero $p_{43}$-projection or $q_{43}$-projection give one of the subgroups $T(p_{3,-2})$, $T(q_{3,-2})X_{p_{43}}X_{p_{42}}X_{p_{41}}$ (i.e., $T(p_{3,-2})^\tau$, when $K$ is perfect and hence there exists a graph automorphism), $T(p_{42})X_{q_{43}}$, $T(p_{42})X_{p_{43}}$ and the first of subgroups in (11) and (12). If $M \subseteq T(p_{42})T(p_{3,-2})$ then $M$ coincides with one of $T(p_{3,-2})$, $T(p_{42})T(q_{43})$ or (11).

Considering the subgroups $M$ with the corners $p_{3,-2}$ and $q_{3,-2}$ we get the subgroups $T(q_{3,-2})X_{p_{41}}X_{p_{3,-2}}$, (14) and the remaining subgroups in (11) and (12).

By Lemma 1.3 every normal subgroup in $U^2F_4(K)$ is incident with the abelian normal subgroup $Z_4 = U_5$. Therefore,

$$Z_4 = U_5 \subseteq M \subseteq C(Z_4) = R_{43}(K)Z_5.$$

The defining relations for the twisted group $U^2F_4(K)$ in terms of generators $R_{iv}(t)$ $(t \in K)$ were described in [13, Lemma 4]. In particular,

$$\big[R_{43}(a), R_{3v}(b)\big] = R_{4v}(ab) \quad \big(|v| \leqslant 2\big), \qquad \big[R_{4v}(a), R_{3,-v}(b)\big] = R_{4,-3}(ab) \quad (v = \pm 2).$$

Also, we obtain the isomorphic embeddings $t \to R_{iv}(t)$ of the additive group $K^+$ into $R_{iv}(K)$ for all $(i, v)$ such that $(i, v) \notin \{(2, 1), (4, 3), (3, 1), (3, -1)\}$. For the remaining cases, using the representation (5), we get the following isomorphic embeddings of the group $U^2B_2(K)$ into $U^2F_4(K)$:

$$(t, u) \to R_{iv}(t)R_{i,-v}(u) \quad \bigl((i, v) = (2, 1), (4, 3)\bigr), \qquad (t, u) \to R_{3v}(t)R_{4,2v}(u) \quad \bigl(|v| = 1\bigr).$$

The subgroup $T(R_{42})$ centralizes $R_{43}(K)T(R_{42})$. For $M \subseteq Z_5$, the isomorphism

$$R_{42}(K)R_{3,-2}(K)R_{4,-3}(K) \simeq UT(3, K)$$

and Lemma 1.4 give the equality $M = \{R_{3,-2}(t)R_{42}(td) \mid t \in K\}Z_4$ for a fixed $d \in K$.

Let $M_{iv}$ be an $R_{iv}$-projection of $M$. Since

$$1 = \bigl[M, \bigl[M, R_{32}(1)\bigr]\bigr] = \bigl[M, R_{42}(M_{43})\bigr] = R_{4,-3}(M_{3,-2}M_{43}),$$

we get $M_{3,-2}M_{43} = 0$. If $M_{3,-2} = 0$ and hence $M \subseteq T(R_{43})$ then the description of the abelian normal subgroups in $U^2B_2(K)$ implies $M = T(R_{42})\langle\alpha\rangle$ for an arbitrary $\alpha \in T(R_{43}) \setminus T(R_{42})$. Thus, Theorem 5.1 is proved. □

## 6. Some large $\mathcal{P}$-subgroups

In this section, we consider some application to the problem (1.6) from [7] of description of the large abelian and normal large abelian subgroups in a finite group $U$ of exceptional Lie type. Under notation of Theorems 3.1, 4.8 and 5.1, as a consequence, we obtain

**Theorem 6.1.** *Let $U = UG(K)$ for a finite field $K$. Then the large normal abelian subgroups in $U$ are the following*:

(a) $T(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6)$ *in* $UE_8(K)$, $T(\alpha_1)$ *and* $T(\alpha_6)$ *in* $UE_6(K)$, $T(\alpha_7)$ *in* $UE_7(K)$;
(b) $\langle\gamma\rangle U_2$ $(\gamma \in U \setminus U_2)$ *for* $G = {}^2B_2$, $\langle R_{43}(c)\rangle R_{42}(K)U_5$ $(c \neq 0)$ *for* $G = {}^2F_4$;
(c) $T(q_{43})U_6$ *for* $2K = K$, $G = F_4$ *or* ${}^2E_6$;
(d) $U_3$ *in* $UG_2(K)$ *for* $6K = K$ *and in* $U^3D_4(K)$ *for* $2K = K$;
(e) $U_2$ *for* $3K = 0$, $G = G_2$ *or* ${}^2G_2$, $\langle\alpha\rangle \times \langle\beta_1(1)\rangle$ *in* $UG_2(2)$, *and* $U_3$ *and* $\beta_c(K)U_4$ $(c \in K)$ *in* $UG_2(K)$ *for* $2K = 0$, $|K| > 2$;
(f) *when* $2K = 0$, *up to conjugation by a diagonal automorphism,*

$$\bigl(T(p_{3,-2}) \cap E_6(K_\sigma)\bigr)U_7 \quad \text{in } U^2E_6(K), \tag{17}$$

$$\beta_c(K_\sigma)x_{2a+b}\bigl(K^{1+\sigma}\bigr) \cdot U_4 \quad (c \in K) \quad \text{and} \quad U_3 \quad \text{in } U^3D_4(K) \text{ for } |K_\sigma| > 2,$$

$$\langle\alpha\rangle \times \bigl\langle\beta_1(1)\bigr\rangle \times x_{2a+b}\bigl(K^{1+\sigma}\bigr) \quad \text{in } U^3D_4(8), \tag{18}$$

$$T(p_{3,-2})^\tau, \quad X_{p_{43}}T(p_{42}), \quad X_{p_{43}}X_{p_{42}}X_{p_{41}}\bigl\{x_{q_{3,-2}}(t)x_{q_{42}}(t) \bigm| t \in K\bigr\}T(p_{4,-1}),$$

$$T(p_{3,-2}), \quad X_{q_{43}}T(p_{42}), \quad \bigl\{x_{p_{3,-2}}(t)x_{p_{42}}(t) \bigm| t \in K\bigr\}X_{q_{43}}T(p_{41})$$

*and, in addition,* $\langle x_{p_{43}}(1)x_{q_{43}}(1)\rangle T(p_{42})$ *for* $|K| = 2$ *in* $UF_4(K)$.

Now we show that the large normal abelian subgroups in $U$ are large abelian subgroups.

In general, a large normal $\mathcal{P}$-subgroup of a finite group is not a normal large $\mathcal{P}$-subgroup. In fact, the center of $SL(n, K)$ is a large normal cyclic subgroup but this group has no a normal large cyclic subgroup.

We have to prove the inequality $\mathbf{a}(U) \leqslant \mathbf{b}(U)$, where $\mathbf{a}(U)$ (and $\mathbf{b}(U)$) is the largest order of all (respectively, normal) abelian subgroups in $U$. This fact is well known for the groups of Lie type of rank 1 or of classical type, [7] and [16]. Theorem 6.1 explicitly gives the number $\mathbf{b}(U)$ for every $U$ of exceptional Lie type.

Further, we use the notion of a *regular ordering of roots*, which agrees with the height function on roots [4, Lemma 5.3.1]. Taking into account the representation $\zeta$ in Section 1 we may use similar ordering for the twisted system.

Now, in the canonical decomposition of every $\alpha \in U = UG(K)$, the first non-unit cofactor corresponds to the first corner in $\alpha$. Evidently, if $M \subseteq U$ then for every corner $r$ in $M$ the $r$-projection $F_M(r)$ of $M$ does not depend on the choice of ordering in $G$. The following lemma is immediate.

**Lemma 6.2.** *Let $M$ be a subgroup in $UG(K)$, and let $\mathcal{L}_1(M)$ be the set of first corners of all elements in $M$. Then $|M| = \prod_{r\in\mathcal{L}_1(M)} |F_M(r)|$.*

By Lemma 1.1, a subgroup $X_\Psi$ in $U\Phi(K)$ (with $p(\Phi)!K = K$) is abelian if and only if $\Psi$ is an abelian subset in $\Phi^+$, and hence $\{e_r \mid r \in \Psi\}$ is a basis for an abelian subalgebra in $N\Phi(K)$. According to E.P. Vdovin [26], a subset $\Psi$ of $\Phi^+$ is said to be *p-abelian* if in the algebra $N\Phi(K)$ over a field $K$ of characteristic $p$ we have $e_r * e_s = 0$ for all $r, s \in \Psi$. For $p(\Phi)!K = K$, this gives $r + s \notin \Psi$, i.e., $\Psi$ is an abelian subset. Clearly, every abelian subset in $\Phi^+$ is always $p$-abelian for every prime $p$. The largest order of abelian and $p$-abelian subsets in $\Phi^+$ is denoted by $\mathbf{a}(\Phi)$ and $\mathbf{a}(\Phi, p)$, respectively.

An application of the first corner of the elements in $U$ and Lemma 6.2 give a simplified proof of the following statement (see [26, § 2]).

**Lemma 6.3.** *Let $A$ be an abelian subgroup in $U\Phi(K)$. Then $\mathcal{L}_1(A)$ is a $p$-abelian subset in $\Phi^+$, and $|A| \leqslant |K|^{\mathbf{a}(\Phi,p)}$.*

A.I. Mal'tsev [17] described the abelian subsets of largest order in $\Phi^+$. His description shows that there exists a normal abelian subset $\Psi$ of order $\mathbf{a}(\Phi)$. For $U\Phi(K)$ with $p(\Phi)!K = K$, $X_\Psi$ is a normal large abelian subgroup of order $|K|^{\mathbf{a}(\Phi)}$, and hence $\mathbf{a}(U) = \mathbf{b}(U) = |K|^{\mathbf{a}(\Phi)}$.

Analogously, if $p(\Phi) = char K = p \geqslant 2$ then there exists a normal $p$-abelian subset $\Psi$ in $\Phi^+$ of order $\mathbf{a}(\Phi, p)$ and $\mathbf{a}(U) = \mathbf{b}(U) = |K|^{\mathbf{a}(\Phi,p)}$. For type $C_n$, this result follows from the description in [2]. By E.P. Vdovin [26], for types $G_2$ and $F_4$ we get, respectively,

$$X_\Psi = U_2, \quad \mathbf{a}(\Phi, 3) = 4, \quad \text{and} \quad X_\Psi = T(p_{3,-2}), \quad \mathbf{a}(\Phi, 2) = 11.$$

Also, if $\Phi$ is of type $G_2$ then $\{a, a+b, 3a+b, 3a+2b\}$ is a unique 2-abelian subset in $\Phi^+$ of order $> 3$ and $a(\Phi, 2) = 4$. Every abelian subgroup $A$ in $UG_2(K)$ $(2K = 0)$ either is of order $|A| \leqslant |U_3| = |K|^3$ or

$$A = \langle x_a(t)x_{2a+b}(st), x_{a+b}(s)x_{2a+b}(st)\rangle U_4 \quad (s, t \in K^*),\ |A| = 4 \cdot |K|^2. \tag{19}$$

The subgroup (19) is of order $\geqslant |K|^3$ if and only if $|K| = 2$ or 4. If this inequality is strict then $|K| = 2$ and (19) is a normal subgroup. Therefore, $\mathbf{a}(U) = \mathbf{b}(U)$ holds for all $U\Phi(K)$.

The same holds for the groups $U$ of type ${}^2F_4$, ${}^2B_2$, and ${}^2G_2$, since a corner projection of every their root set $X_r$ coincides with $K$.

For the remaining groups $U^m\Phi(K)$ of type ${}^3D_4$ and ${}^2E_6$, E.P. Vdovin [26] suggested to use the description from [17] of abelian subsets in $\Phi$ of type $D_4$ and $E_6$. Simplifying this approach, we use a description of $p$-abelian subsets of the associated root systems $\zeta(\Phi)$.

Consider $U$ of type ${}^2E_6$ in detail. Then $\zeta(\Phi)$ is of type $F_4$, and $U \cap UE_6(K_\sigma) \simeq UF_4(K_\sigma)$. Let $A$ be an arbitrary large abelian subgroup in $U$. By Lemma 4.4, if $|F_A(r)| > |K_\sigma|$ for some root $r \in \mathcal{L}_1(A)$ then $r + s \notin \mathcal{L}_1(A)$ for all $s \in \mathcal{L}_1(A)$. For $2K = K$, according to Lemma 6.3, $\mathcal{L}_1(A)$ is an abelian subset in $\zeta(\Phi)^+$ of order $\leqslant \mathbf{a}(\zeta(\Phi)) = 9$. By [17], $\mathcal{L}_1(A)$ doesn't contain more than six classes of every fixed type, and also it doesn't contain more than three classes of type $A_1 \times A_1$, i.e., $\mathcal{L}_1(A)$ possesses no the roots $p_{iv}$ with $1 \leqslant |v| < i$ in the diagram of Section 5. By Lemma 6.2, we obtain

$$\mathbf{a}(U) = |A| \leqslant |K_\sigma|^6 \cdot |K|^3 = |K_\sigma|^{12} = \mathbf{b}(U).$$

If $2K = 0$ then $\mathcal{L}_1(A)$ is a 2-abelian subset, and $|\mathcal{L}_1(A)| \leqslant \mathbf{a}(\zeta(\Phi), 2) = 11$. Also, the description from [26] and Lemma 4.4 show that the number of all $r \in \mathcal{L}_1(A)$ with $|F_A(r)| > |K_\sigma|$ is less than 3. Since the number $\mathbf{b}(U)$ coincides with the order $|K_\sigma|^{13}$ of the abelian normal subgroup (17), we get

$$\mathbf{a}(U) = |A| \leqslant |K|^2 \cdot |K_\sigma|^9 = |K_\sigma|^{13} = \mathbf{b}(U).$$

Thus, $\mathbf{a}(U) = \mathbf{b}(U)$ holds for all $U$. We arrived at the following

**Theorem 6.4.** *Let $U = UG(K)$ for a finite field $K$. Then a subgroup in $U$ is a large normal abelian subgroup if and only if it is a normal large abelian subgroup.*

**Remark.** For the groups $UG_2(2)$, $U^3D_4(8)$, and $U^2E_6(K)$ with $2K = 0$, Theorem 6.4 allows us to refine the values $\mathbf{a}(U)$, which by [26] might be $2^3$, $2^5$ or $|K_\sigma|^{12}$, respectively. The subgroups (19), (18) and (17) of orders $2^4$, $2^6$ and $|K_\sigma|^{13}$, respectively, were omitted in [26].

Now it is easy to show that if all normal large abelian subgroups in a finite group $U$ are extremal then all large abelian subgroups in $U$ are normal. We note that for every finite group $G$ of Lie type the authors have the proof of the following theorem. (See also [16, Theorem 4] for the classical types [24], and the question in [6, § 1].)

**Theorem 6.5.** *In every finite group $U$, either each large abelian subgroup is $G$-conjugate to a normal subgroup in $U$ or $G$ is of type $G_2$, ${}^3D_4$, $F_4$ or ${}^2E_6$.*

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