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ELSEVIER

Linear Algebra and its Applications 374 (2003) 159–174

www.elsevier.com/locate/laa

Characterizations of classes of stable matrices ☆

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Received 3 December 2002; accepted 30 April 2003

Submitted by D. Hershkowitz

Abstract

This paper extends some results on the structure of subsets of the set of stable matrices. For these subsets, different characterizations are obtained using the set product, defined in this paper, as well as inertia and algebraic characterizations for low dimensions (2×2 and 3×3 matrices). Some inclusion relations that hold for these classes of matrices are proved and some open questions mentioned.

Keywords: Matrix stability; Liapunov equation; Diagonal stability; Inertia

1. Introduction

All the sets of matrices considered in this paper are subsets of the set of *positive stable* matrices, denoted $\mathscr{S}$ and defined as $\{A \in \mathbb{R}^{n\times n} : \mathrm{Spec}(A) \subset \mathbb{C}_+\}$, where $\mathrm{Spec}(A)$ denotes the spectrum or set of eigenvalues of the matrix A, and $\mathbb{C}_+$ the open right half plane. It should be noted that, in most applications, the set of stable matrices is defined as $-\mathscr{S}$, and also referred to as the set of Hurwitz or Hurwitz-stable matrices with spectrum in the open left half plane. From a mathematical point of view, it is more convenient to consider the set of positive stable matrices, and it is a simple matter to translate the results obtained for this class to the class of Hurwitz matrices by introducing a negative sign wherever appropriate.

☆ This research was partially supported by the Brazilian Government agencies CAPES, CNPq, MCT, PRONEX, FINEP e FAPERJ program.

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doi:10.1016/S0024-3795(03)00558-5

A recent article by Duan and Patton [9] made the observation that every Hurwitz-stable matrix can be expressed as the product of a symmetric positive definite matrix and a generalized negative definite matrix and used this fact to write the set of Hurwitz matrices as the set product of two convex open cones, namely the sets of symmetric positive definite and generalized negative definite matrices, respectively.

Another recent paper, by Cain et al. [4], unifies a number of stability concepts by using the following general definition. Given a set $\mathscr{B} \subset \mathbb{R}^{n\times n}$, a matrix $A \in \mathbb{R}^{n\times n}$ is said to be $\mathscr{B}$-stable if $\mathscr{B}A := \{BA : B \in \mathscr{B}\} \subset \mathscr{S}$. It should be pointed out that [4] also considers the general case of bounded operators in a (possibly) infinite-dimensional complex Hilbert space, whereas this paper will only deal with real matrices.

Motivated by these papers, as well as some other developments in the theory of diagonally stable and D-stable matrices, this paper derives several other relations between various sets of stable matrices and expresses these relationships in terms of the set product. The use of the set product means that the concept of $\mathscr{B}$-stability in which a set $\mathscr{B}$ multiplies a matrix A is now being viewed from the perspective of all matrices in one set multiplying all the matrices in another set.

For example, the class of matrices that can be made diagonally dominant by premultiplication by a positive definite matrix, denoted $\mathscr{W}_{\mathrm{dom}}$ in this paper, arises in the stability analysis of variable structure systems and has not yet been characterized [17, p. 90 ff.]—this paper gives a result on the structure of this set.

1.1. Definitions and preliminaries

All matrix classes defined below are subsets of $\mathbb{R}^{n\times n}$ and are denoted by calligraphic letters. Given a calligraphic letter denoting a subset of $\mathbb{R}^{n\times n}$, its subsets are denoted by the same letter adorned with appropriate subscripts.

Note that $\mathrm{diag}(p_1, \ldots, p_n)$ denotes a diagonal matrix with diagonal elements $p_1, p_2, \ldots, p_n$, whereas diagonal(A) denotes the matrix of diagonal elements of a matrix $A = (a_{ij})$, i.e. $\mathrm{diag}(a_{11}, \ldots, a_{nn})$. If all components of a vector v are positive, this is denoted as $v \succ 0$. Similarly, if all elements of a matrix A are positive, this is denoted $A \succ 0$.

A basic fact, due to Liapunov, about the set of positive stable matrices defined in the introduction, is that it can be characterized by a linear matrix equation (or inequality) as in the theorem below:

Theorem 1.1. *The matrix $A \in \mathbb{R}^{n\times n}$ is positive stable (i.e. $A \in \mathscr{S}$) if and only if there exists $P \in \mathscr{P}_{\mathrm{sym}}$ such that the Liapunov equation*

$$A^{\mathrm{T}}P + PA = Q \tag{1}$$

is satisfied for some $Q \in \mathscr{P}_{\mathrm{sym}}$. The notation $A \in \mathscr{L}(P)$ is used to denote that the matrix A satisfies the Liapunov equation (1) with solution P. The term linear matrix inequality (LMI) is also used to denote the matrix inequality $A^{\mathrm{T}}P + PA > 0$, *where the matrix P is a positive definite matrix that makes the left-hand side positive definite.*

Definition 1.2

(i) $\mathscr{P} := \{A : A + A^{\mathrm{T}} > 0\}$ is the set of *generalized positive definite* matrices.
(ii) $\mathscr{P}_{\text{sym}} := \{A \in \mathbb{R}^{n\times n} : A = A^{\mathrm{T}} > 0\}$ is the set of *symmetric positive definite* matrices. Thus, $\mathscr{P}_{\text{sym}} \subset \mathscr{P}$.
(iii) $\mathscr{P}_{\text{diag}} := \{A : A \text{ diagonal}, A > 0\}$ is the set of *positive diagonal* matrices.
(iv) $\mathscr{S}_{\mathbb{D}} := \{A \in \mathbb{R}^{n\times n} : AD \in \mathscr{S} \ \ \forall D \in \mathscr{P}_{\text{diag}}\}$ is the set of *D-stable* matrices.
(v) $\mathscr{S}_{S\mathbb{D}} := \{A \in \mathbb{R}^{n\times n} : \exists W \in \mathscr{P}_{\text{sym}}, \forall D \in \mathscr{P}_{\text{diag}}, W(AD) \in \mathscr{P}\}$ is the set of *simultaneously D-stable* matrices.
(vi) $\mathscr{S}_{\mathscr{D}} := \{A \in \mathbb{R}^{n\times n} : \exists P \in \mathscr{P}_{\text{diag}}, PA \in \mathscr{P}\}$ is the set of *diagonally stable* matrices. The notation $A \in \mathscr{S}_{\mathscr{D}}(P)$ is used to indicate the diagonal solution P to the *Liapunov equation* $A^{\mathrm{T}}P + PA = Q > 0$, whenever necessary.
(vii) $\mathscr{W} := \{A \in \mathbb{R}^{n\times n} : \exists W \in \mathscr{P}_{\text{sym}}, WA \in \mathscr{P}_{\text{diag}}\}$.
(viii) $\mathscr{W}_1 := \{A \in \mathbb{R}^{n\times n} : \exists W \in \mathscr{P}, WA \in \mathscr{P}_{\text{diag}}\}$.
(ix) $\mathscr{Q}_{\text{dom}} := \{A : A \text{ is row-diagonally quasidominant, with diagonal}(A) > 0\}$, i.e. the set of matrices A such that there exists a positive diagonal matrix $P = \text{diag}(p_1, \ldots, p_n)$ such that $a_{ii}p_i \geqslant \sum_{j\neq i} |a_{ij}|p_j, \forall i$, is the set of *row-diagonally quasidominant* matrices. If these inequalities are strict, the matrix A is referred to as strictly row-sum quasidominant. If P can be chosen as the identity matrix, then the matrix is called *row-diagonally dominant* and the set of such matrices is denoted $\mathscr{R}$.
(x) $\mathscr{W}_{\text{dom}} := \{A \in \mathbb{R}^{n\times n} : \exists W \in \mathscr{P}_{\text{sym}}, WA \in \mathscr{Q}_{\text{dom}}\}$.
(xi) $\mathscr{M} := \{A \in \mathbb{R}^{n\times n} : \exists P \in \mathscr{R} \cap \mathscr{P}_{\text{sym}}, PA \in \mathscr{P}\}$.
(xii) $\mathscr{H} := \{A \in \mathbb{R}^{n\times n} : \forall P \in \mathscr{P}_{\text{sym}}, PA \in \mathscr{S}\}$ is the set of *$\mathscr{H}$-stable* matrices.

In view of the above definitions, it should be noted that an alternative statement of Theorem 1.1 is that $A \in \mathscr{S}$ if and only if there exists $P \in \mathscr{P}_{\text{sym}}$ such that $PA \in \mathscr{P}$.

Some points are worth noting about the definitions of dominance and quasidominance. Clearly A is row-sum quasidominant with the scaling factors given by the diagonal entries of the matrix $P \in \mathscr{P}_{\text{diag}}$ if and only if AP is row dominant. Actually, row-sum and column-sum quasidominance are equivalent, although strict row and column dominance are not. Of course, there is no requirement that the *same* diagonal matrix work for rows and columns. In other words, if A is quasidominant, there exist positive diagonal matrices P, Q such that AP and QA are row- and column-dominant respectively. Thus, it is usual to refer to a matrix as simply being *quasidominant*. Notice also that the definition only applies to a matrix which has positive diagonal entries. If this is not the case, quasidominance is defined by some authors using the absolute value sign on the diagonal entries a_{ii}, i.e., $d_i|a_{ii}| > \sum_{j\neq i} d_j|a_{ij}|$. This definition is not used in this paper. Also, it is a well known consequence of Gershgorin's theorem that row diagonal dominance defined as above implies stability: i.e., $\mathscr{R} \subset \mathscr{S}$.

The class $\mathscr{M}$ was introduced by Liu and Michel (see [13]) and studied further in [6,7].

Note that $\mathscr{P}_{\text{sym}}$ and $\mathscr{P}_{\text{diag}}$ are closed under inversion. In other words, the inverse of a positive definite matrix is also positive definite and the same applies for positive diagonal matrices. As a result, the set $\mathscr{W}_{\text{dom}}$ can also be defined as:

$$\mathscr{W}_{\text{dom}} = \{A : A = VQ, V \in \mathscr{P}_{\text{sym}}, Q \in \mathscr{Q}_{\text{dom}}\}.$$

The following definition is crucial to Section 3, which contains the main contributions of this paper.

Definition 1.3. Let $\mathscr{M}_1$ and $\mathscr{M}_2$ be two subsets of $\mathbb{R}^{n\times n}$. Their *set product*, denoted $\mathscr{M}_1 \otimes \mathscr{M}_2$, is defined as follows:

$$\mathscr{M}_1 \otimes \mathscr{M}_2 := \{M : M = M_1 M_2, M_1 \in \mathscr{M}_1, M_2 \in \mathscr{M}_2\}.$$

Given sets of matrices X, Y, Z, note the following obvious properties of the set product:

(i) Associativity: $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$.
(ii) Transitivity: If $Y \subset Z$, then $X \otimes Y \subset X \otimes Z$.

It is also obvious that the set product is not, in general, commutative. Some exceptions to this, for the sets considered in this paper, are given in Theorem 3.1 in Section 3. Note, however, that the determinantal identity $\det(I - AB) = \det(I - BA)$ [5] shows that $\text{Spec}(A \otimes B) = \text{Spec}(B \otimes A)$.

Definition 1.4. The inertia of a matrix A of order n, denoted by $\text{In}(A)$, is the triplet $(\pi(A), \nu(A), \zeta(A))$, where $\pi(A)$, $\nu(A)$, and $\zeta(A)$ are, respectively, the number of eigenvalues of A with positive, negative, and zero real parts, counting multiplicities.

Lemma 1.5. *$A \in \mathscr{S}$ if and only if $A^{\mathrm{T}} \in \mathscr{S}$.*

Theorem 1.6 [15]

(i) *A necessary and sufficient condition that there exists a Hermitian matrix X such that*

$$XA + A^* X = M > 0 \tag{2}$$

is that $\zeta(A) = 0$.

(ii) *If X is Hermitian and satisfies* (2), *then* $\text{In}(A) = \text{In}(X)$.

Lemma 1.7 [1]. *$A \in \mathscr{S}_{\mathscr{D}}(P)$ implies $A^{-1} \in \mathscr{S}_{\mathscr{D}}(P)$, as well as $A^{\mathrm{T}} \in \mathscr{S}_{\mathscr{D}}(P^{-1})$.*

Lemma 1.8 [10]. *Given a matrix $A \in \mathbb{R}^{n\times n}$ with nonpositive offdiagonal elements, the following are equivalent*:

- *There exists a vector $v \succ 0$ such that $Av \succ 0$.*
- *There exists a vector $w \succ 0$ such that $A^{\mathrm{T}}w \succ 0$.*
- *There exists a matrix $P \in \mathscr{P}_{\mathrm{diag}}$ such that $A \in \mathscr{S}_{\mathscr{D}}(P)$.*

The following definition is needed in the next lemma:

Definition 1.9. A signature matrix is any diagonal matrix $S = \mathrm{diag}(s_1, \ldots, s_n)$ whose diagonal entries are $+1$ or -1.

Lemma 1.10. *If* diagonal$(A) > 0$, *then A is row quasidominant iff it is column quasidominant*. *Quasidominant matrices are diagonally stable*, *i.e.*, $\mathscr{Q}_{\mathrm{dom}} \subset \mathscr{S}_{\mathscr{D}}$.

Proof. Let A be row quasidominant with $\mathrm{diag}(A) > 0$. This means that there exists a matrix $P = \mathrm{diag}(p_1, \ldots, p_n) \in \mathscr{P}_{\mathrm{diag}}$ such that AP is row dominant, i.e.:

$$\exists P \in \mathscr{P}_{\mathrm{diag}},\ \forall i,\ p_i a_{ii} > \sum_{j\neq i}^{n} p_j |a_{ij}|. \tag{3}$$

Note that (3) can be written in matrix form as

$$\exists p \in \mathbb{R}^n,\, p \succ 0,\, C(A)p \succ 0, \tag{4}$$

where $C(A) = (c_{ij})$ is the comparison matrix of A, defined as $c_{ii} = a_{ii}$ and $c_{ij} = -|a_{ij}|$, $i \neq j$. By Lemma 1.8, there also exists a vector $q \succ 0$ satisfying $C(A)^{\mathrm{T}}q \succ 0$. By the same reasoning that led to (4), this implies that $A^{\mathrm{T}}Q$ is dominant or, equivalently, A is column quasidominant. Consequently $C(A) \in \mathscr{S}_{\mathscr{D}}(R)$ for $R = \mathrm{diag}(q_i/p_j)$. The second assertion is proved in [11]. □

2. Relations between the classes $\mathscr{S}_{\mathbb{D}}, \mathscr{S}_{\mathscr{D}}, \mathscr{W}, \mathscr{W}_1, \mathscr{W}_{\mathrm{dom}}$

This section derives some of the basic relations that hold between the various sets defined in the previous section. It prepares the ground for the next section, in which these relationships are expressed in terms of the set product.

In terms of the notation above, the following relationships hold:

Theorem 2.1
(1) $\mathscr{W} \subset \mathscr{S}_{\mathscr{D}} \subset \mathscr{S}_{\mathbb{D}} \subset \mathscr{S}$.
(2) $\mathscr{S}_{\mathscr{D}} = \mathscr{W}_1$.

(3) $\mathscr{P} \subset \mathscr{S}_{\mathscr{D}}$.
(4) $\mathscr{W} \subset \mathscr{S}_{S\mathbb{D}} \subset \mathscr{S}_{\mathbb{D}}$.

The book [11] contains many other results on $\mathscr{S}_{\mathbb{D}}$ and $\mathscr{S}_{\mathscr{D}}$.

Proof. *Item* (1) ($\mathscr{W} \subset \mathscr{S}_{\mathscr{D}}$): $A \in \mathscr{W}$ means that there exist $W \in \mathscr{P}_{\text{sym}}$ and $D \in \mathscr{P}_{\text{diag}}$ such that $WA = D$. Thus $A = W^{-1}D$, so that

$$A^{\mathrm{T}}D + DA = DW^{-1}D + DW^{-1}D = 2DW^{-1}D.$$

However, W in $\mathscr{P}_{\text{sym}}$ implies W^{-1} in $\mathscr{P}_{\text{sym}}$, so that $DW^{-1}D$ is also in $\mathscr{P}_{\text{sym}}$, being congruent to W^{-1}. Thus $A^{\mathrm{T}}D + DA$ is in $\mathscr{P}_{\text{sym}}$ and $A \in \mathscr{S}_{\mathscr{D}}$.

($\mathscr{S}_{\mathscr{D}} \subset \mathscr{S}_{\mathbb{D}}$): $A \in \mathscr{S}_{\mathscr{D}}$ implies that there exists $P \in \mathscr{P}_{\text{diag}}$ such that $A^{\mathrm{T}}P + PA > 0$. Pre- and post-multiplying by any $K \in \mathscr{P}_{\text{diag}}$ yields

$$KPAK + KA^{\mathrm{T}}PK = KQK > 0,$$

which can be rewritten as $P_1(AK) + (AK)^{\mathrm{T}}P_1 > 0$, where $P_1 := PK = KP \in \mathscr{P}_{\text{diag}}$.

($\mathscr{S}_{\mathbb{D}} \subset \mathscr{S}$): Direct consequence of the definition.

Item (2) ($\mathscr{S}_{\mathscr{D}} \subset \mathscr{W}_1$): Let $A \in \mathscr{S}_{\mathscr{D}}(P)$. That is, P positive diagonal is such that

$$PA + A^{\mathrm{T}}P > 0.$$

Define $W = PA^{-1}$. Then W is not symmetric, but twice the symmetric part of W is

$$W + W^{\mathrm{T}} = PA^{-1} + (A^{-1})^{\mathrm{T}}P > 0,$$

by item (2), so that the symmetric part of W is indeed in $\mathscr{P}_{\text{sym}}$. Also clearly $WA = P$ positive diagonal.

($\mathscr{W}_1 \subset \mathscr{S}_{\mathscr{D}}$): Let $A \in \mathscr{W}_1$. Then $W + W^{\mathrm{T}}$ is in $\mathscr{P}_{\text{sym}}$. Also $W = PA^{-1}$, so substituting

$$W + W^{\mathrm{T}} = PA^{-1} + (A^{-1})^{\mathrm{T}}P > 0.$$

This means that $A^{-1} \in \mathscr{S}_{\mathscr{D}}$, whence by Lemma 1.7, $A \in \mathscr{S}_{\mathscr{D}}$.

Item (3) ($\mathscr{P} \subset \mathscr{S}_{\mathscr{D}}$): Direct consequence of the definition, since $A \in \mathscr{P}$ means that $A + A^{\mathrm{T}} > 0$, which means that $A \in \mathscr{S}_{\mathscr{D}}(I)$.

Item (4) The inclusion $\mathscr{W} \subset \mathscr{S}_{S\mathbb{D}}$ is proved as follows:

Let $A \in \mathscr{W}$. Then, since WA is positive diagonal, it follows that for any positive diagonal D, $WAD + DA^{\mathrm{T}}W$ is a symmetric positive definite matrix for all positive diagonal matrices D. Writing this as

$$W(AD) + (AD)^{\mathrm{T}}W > 0,$$

it can be seen that W is a solution to the Liapunov equation in AD for all positive diagonal D.

The inclusion $\mathscr{S}_{S\mathbb{D}} \subset \mathscr{S}_{\mathbb{D}}$ is a direct consequence of the definitions. □

The following lemma establishes a relationship between the multiplicative inverse eigenvalue problem and the problem of determining whether a matrix belongs to the set $\mathscr{W}_{\rm dom}$.

Lemma 2.2. *$A \in \mathscr{W}_{\rm dom}$ implies $\exists D \in \mathscr{P}_{\rm diag}$ such that $AD \in \mathscr{S}$.*

Proof. By definition, $A \in \mathscr{W}_{\rm dom}$ means that there exists W in $\mathscr{P}_{\rm sym}$ such that WA is row-diagonally dominant, with positive diagonal elements. Since diagonal dominance is a special case of quasidominance, by Lemma 1.8, $WA \in \mathscr{S}_{\mathscr{D}}$. This means that there exists a positive diagonal matrix P such that

$$P(WA) + (WA)^{\rm T}P > 0.$$

Applying the congruence transformation determined by P^{-1} to both sides of this Liapunov LMI, the result is the LMI

$$W(AP^{-1}) + (P^{-1}A^{\rm T})W > 0,$$

which shows that $AP^{-1} \in \mathscr{S}$, proving the lemma with $D = P^{-1}$. □

Remark. Lemma 2.2 may be of interest in conjunction with results on the multiplicative inverse eigenvalue problem, since if we can show, for example, that the latter is not solvable for $\text{Spec}(AD) \subset \mathbb{C}_+$, then we can conclude that $A \notin \mathscr{W}_{\rm dom}$ [14].

Lemma 2.3. *There exists $P \in \mathscr{P}_{\rm diag}$ such that $AP \in \mathscr{S}$ if and only if there exists $W \in \mathscr{P}_{\rm sym}$ such that $WA \in \mathscr{S}_{\mathscr{D}}$.*

Proof. If there exists a positive diagonal matrix P such that AP is positive stable, then there exists W in $\mathscr{P}_{\rm sym}$ such that $WA \in \mathscr{S}_{\mathscr{D}}$.

The Liapunov LMI for AP is

$$WAP + PA^{\rm T}W > 0, \text{ for some } W \text{ in } \mathscr{P}_{\rm sym}.$$

Applying the congruence transformation determined by P^{-1} to both sides of this LMI, the result is the LMI

$$P^{-1}(WA) + (WA)^{\rm T}P^{-1} > 0,$$

which is a Liapunov LMI showing that $WA \in \mathscr{S}_{\mathscr{D}}(P^{-1})$. □

3. Relations between stable subsets in terms of the set product

In terms of the set product, the following theorem holds, in which the results are according to the type of relation that is valid.

Theorem 3.1

(I) *Equalities*:

(1) $\mathscr{Q}_{\rm dom} = \mathscr{P}_{\rm diag} \otimes \mathscr{Q}_{\rm dom}$.

(2) $\mathscr{S} = \mathscr{P} \otimes \mathscr{P}_{\rm sym} = \mathscr{P}_{\rm sym} \otimes \mathscr{P}$.

(3) $\mathscr{S}_{\mathscr{D}} = \mathscr{P} \otimes \mathscr{P}_{\rm diag} = \mathscr{P}_{\rm diag} \otimes \mathscr{P}$.

(4) $\mathscr{W} = \mathscr{P}_{\rm sym} \otimes \mathscr{P}_{\rm diag} = \mathscr{P}_{\rm diag} \otimes \mathscr{P}_{\rm sym}$.

(II) *Inclusions*:

(1) $\mathscr{P}_{\rm sym} \otimes \mathscr{P}_{\rm sym} \subset \mathscr{S}$.

(2) $\mathscr{W}_{\rm dom} = \mathscr{P}_{\rm sym} \otimes \mathscr{Q}_{\rm dom} \subset \mathscr{P}_{\rm sym} \otimes \mathscr{S}_{\mathscr{D}} = \mathscr{P}_{\rm sym} \otimes \mathscr{P}_{\rm diag} \otimes \mathscr{P} = \mathscr{P}_{\rm diag} \otimes \mathscr{S}$.

(3) $\mathscr{W}_{\rm dom} = \mathscr{P}_{\rm sym} \otimes \mathscr{Q}_{\rm dom} \subset\subset \mathscr{S} = \mathscr{P} \otimes \mathscr{P}_{\rm sym}$ *and* $\mathscr{W}_{\rm dom}$ *is a strict subset of* $\mathscr{S}$.

(III) *Negative results*:

(1) $\mathscr{P} \otimes \mathscr{P} \not\subset \mathscr{P}$

(2) $\mathscr{P}_{\rm sym} \otimes \mathscr{P}_{\rm diag} \not\subset \mathscr{P}$

(3) $\mathscr{S} \otimes \mathscr{S} \not\subset \mathscr{S}$

(4) $\mathscr{S}_p \otimes \mathscr{S}_p \not\subset \mathscr{S}$ *where* $\mathscr{S}_p := \{M = AB \in \mathbb{R}^{n\times n} : A, B \in \mathscr{L}(P)\}$.

In [9], only $\mathscr{S} = \mathscr{P} \otimes \mathscr{P}_{\rm sym}$ was proved.

Proof $(of\,({\rm I})Equalities)$

Item (1): Clearly if $A \in \mathscr{Q}_{\rm dom}$, then so is αA, for all $\alpha > 0$. Since $A = (1/\alpha)I(\alpha A)$, clearly A can be written as an element of $\mathscr{P}_{\rm diag} \otimes \mathscr{Q}_{\rm dom}$.

Conversely, let $A = PQ$, where $P = \mathrm{diag}(p_1, \ldots, p_n) \in \mathscr{P}_{\rm diag}$ (so $p_i > 0$) and $Q = (q_{ij}) \in \mathscr{Q}_{\rm dom}$. Clearly, $q_{ii} > \sum_{i\neq j} |q_{ij}|$ implies $p_i q_{ii} > \sum_{i\neq j} p_i |q_{ij}|$, which in turn implies that $PQ \in \mathscr{Q}_{\rm dom}$.

Item (2): Let $A \in \mathscr{S}$. Then, by Lemma 1.5, $A^{\rm T} \in \mathscr{S}$, so by Theorem 1.1, there exists $P \in \mathscr{P}_{\rm sym}$ such that $AP + PA^{\rm T} > 0$. In other words, $AP = Q \in \mathscr{P}$, so that $A = QP^{-1}$, showing that $A \in \mathscr{P} \otimes \mathscr{P}_{\rm sym}$, since $P^{-1} \in \mathscr{P}_{\rm sym}$. Conversely, let $A = QP$, with $Q \in \mathscr{P}$ and $P \in \mathscr{P}_{\rm sym}$. This means that $AP^{-1} = Q \in \mathscr{P}$, i.e. $AP^{-1} + P^{-1}A^{\rm T} > 0$, showing that $A^{\rm T} \in \mathscr{L}(P^{-1})$. This means that $A^{\rm T} \in \mathscr{S}$ and hence, by Lemma 1.5, $A \in \mathscr{S}$.

For the equality $\mathscr{S} = \mathscr{P}_{\rm sym} \otimes \mathscr{P}$, the arguments are as follows. Let $A \in \mathscr{S}$. Then, by Theorem 1.1, there exists $P \in \mathscr{P}_{\rm sym}$ such that $PA = Q$, where $Q \in \mathscr{P}$. But $P \in \mathscr{P}_{\rm sym}$ implies $P^{-1} \in \mathscr{P}_{\rm sym}$ and $A = P^{-1}Q$, showing that A is an element of $\mathscr{P}_{\rm sym} \otimes \mathscr{P}$. On the other hand, if $A = PQ$, where $P \in \mathscr{P}_{\rm sym}$ and $Q \in \mathscr{P}$, then $P^{-1}A = Q \in \mathscr{P}$, which means that $A \in \mathscr{L}(P^{-1})$ and is therefore positive stable.

Item (3): Special case of Item (2).

Item (4): Let A be in $\mathscr{W}$. Then, there exists $W \in \mathscr{P}_{\rm sym}$ such that $WA = D \in \mathscr{P}_{\rm diag}$. This means that $A = W^{-1}D$, where $W^{-1} \in \mathscr{P}_{\rm sym}$, and A has therefore been written as an element of $\mathscr{P}_{\rm sym} \otimes \mathscr{P}_{\rm diag}$. In the other direction, if $A = WD$, with $W \in \mathscr{P}_{\rm sym}$ and $D \in \mathscr{P}_{\rm diag}$, then $W^{-1}A = D \in \mathscr{P}_{\rm diag}$, where $W^{-1} \in \mathscr{P}_{\rm sym}$, and this, by definition, implies that $A \in \mathscr{W}$.

For the equality, $\mathscr{W} = \mathscr{P}_{\text{diag}} \otimes \mathscr{P}_{\text{sym}}$, note that $WA = D \in \mathscr{P}_{\text{diag}} \Leftrightarrow A^{\mathrm{T}}W = D \in \mathscr{P}_{\text{diag}}$, then using the same idea to prove the first equality of this item we prove this equality. $\square$

Proof (*of* (II)*Inclusions*)

Item (1): Let $A, B \in \mathscr{P}_{\text{sym}}$. The following calculation shows that $AB \in \mathscr{L}(A^{-1})$: $BAA^{-1} + A^{-1}AB = 2B > 0$, so that $AB \in \mathscr{S}$.

Item (2): By definition, $\mathscr{W}_{\text{dom}} = \mathscr{P}_{\text{sym}} \otimes \mathscr{Q}_{\text{dom}}$. Since $\mathscr{Q}_{\text{dom}} \subset \mathscr{S}_{\mathscr{D}}$, then, by the transitivity property, $\mathscr{P}_{\text{sym}} \otimes \mathscr{Q}_{\text{dom}} \subset \mathscr{P}_{\text{sym}} \otimes \mathscr{S}_{\mathscr{D}} = (\mathscr{P}_{\text{sym}} \otimes \mathscr{P}_{\text{diag}}) \otimes \mathscr{P}(\text{item}(3)) = (\mathscr{P}_{\text{diag}} \otimes \mathscr{P}_{\text{sym}}) \otimes \mathscr{P}(Item(3)) = \mathscr{P}_{\text{diag}} \otimes (\mathscr{P}_{\text{sym}} \otimes \mathscr{P}) = \mathscr{P}_{\text{diag}} \otimes \mathscr{S}(\text{item}(2))$.

Item (3): The following example from [17] shows that $\mathscr{W}_{\text{dom}} = \mathscr{P}_{\text{sym}} \otimes \mathscr{Q}_{\text{dom}}$ is a strict subset of $\mathscr{S} = \mathscr{P} \otimes \mathscr{P}_{\text{sym}}$: the matrix $A_7 := \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ belongs to the set $\mathscr{S}$, but there does not exist any $P \in \mathscr{P}_{\text{sym}}$ such that $PA_7 \in \mathscr{Q}_{\text{dom}}$. On the other hand, it is easily checked that $PA_7 \in \mathscr{S}, \forall P \in \mathscr{P}_{\text{sym}}$, so that $A_7 \in \mathscr{H}$. Also, $A_7 + A_7^{\mathrm{T}} = \text{diag}(2, 2)$ so that $A_7 \in \mathscr{P}$. $\square$

Proof (*of* (III) *Negative results*). The negative results are proved by examples, as follows:

Item (1): An example shows that $A, B \in \mathscr{P}$ does not imply $AB \in \mathscr{P}$. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}.$$

Then the matrices $A + A^{\mathrm{T}}$ and $B + B^{\mathrm{T}}$ are clearly in $\mathscr{P}_{\text{sym}}$, so that $A, B \in \mathscr{P}$, whereas the eigenvalues of $AB + B^{\mathrm{T}}A^{\mathrm{T}}$ are -0.1623, 6.1623, so that $AB \notin \mathscr{P}$. Note, however, that $AB \in \mathscr{S}$.

Item (2): Consider the matrix A below

$$A = \begin{bmatrix} 3 & -2 \\ -4 & 2.8 \end{bmatrix}.$$

A can be written as:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 1.4 \end{bmatrix},$$

i.e., $A \in \mathscr{P}_{\text{sym}} \otimes \mathscr{P}_{\text{diag}} = \mathscr{W}$, but the eigenvalues of $M = A + A^{\mathrm{T}}$ are equal to 11.8033 and -0.2033, thus $W \not\subset \mathscr{P}$.

Item (3): The example below shows that $A \in \mathscr{S}$, $B \in \mathscr{S}$ does not imply $AB \in \mathscr{S}$.

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}; \quad B = \begin{bmatrix} -1 & -1 \\ 3 & 2 \end{bmatrix}.$$

The eigenvalues of AB are $\{-0.5000 \pm 1.6583\mathrm{i}\}$, so that $AB \notin \mathscr{S}$. One way of understanding this example is to note that, although $B \in \mathscr{S}$, $B \notin \mathscr{S}_{\mathbb{D}}$, so that, on being multiplied by $A \in \mathscr{P}_{\text{diag}}$, the product AB is no longer in $\mathscr{S}$.

Item (4): Considering A, B given by

$$A = \begin{bmatrix} -0.5 & -4 \\ 2 & 2 \end{bmatrix}; \quad B = \begin{bmatrix} -1 & -4 \\ 2 & 2 \end{bmatrix}$$

$A, B \in \mathscr{L}(P)$ where P is equal to

$$P = \begin{bmatrix} 35.0988 & 24.2357 \\ 24.2357 & 60.2590 \end{bmatrix}$$

but $AB = \begin{bmatrix} -7.5 & -6 \\ 2 & -4 \end{bmatrix}$ does not belong to $\mathscr{S}$. □

4. Low dimensional characterizations

4.1. 2×2 Matrices

For two by two matrices, algebraic characterizations of some of the sets studied in section 1 are available [4] and are listed below. Consider a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Then we have the characterizations below:

(i) $A \in \mathscr{S} \Leftrightarrow \operatorname{tr} A > 0, \det A > 0.$
(ii) $A \in \mathscr{S}_{\mathbb{D}} \Leftrightarrow A \in \mathscr{S}, a_{11} \geqslant 0, a_{22} \geqslant 0.$
(iii) $A \in \mathscr{S}_{\mathscr{D}} \Leftrightarrow a_{11} > 0, a_{22} > 0, \det A > 0.$
(iv) $A \in \mathscr{H} \Leftrightarrow A \in \mathscr{S}, a_{11} \geqslant 0, a_{22} \geqslant 0, 4a_{11}a_{22} - (a_{12} + a_{21})^2 \geqslant 0.$
(v) $A \in \mathscr{P} \Leftrightarrow a_{11} > 0, 4a_{11}a_{22} - (a_{12} + a_{21})^2 > 0.$
(vi) $A \in \mathscr{M} \Leftrightarrow A \in \mathscr{S}, a_{11} + |a_{21}| > 0, a_{22} + |a_{12}| > 0.$

The proof of (i)–(v) is in [4]. The equivalence (vi) is new and restated and proved as the theorem below:

Theorem 4.1. *A matrix* $A \in \mathbb{R}^{2\times 2}$ *belongs to* $\mathscr{M}$ *if and only if*

(a) *A is Hurwitz stable.*
(b) $a_{11} + |a_{21}| > 0$ *and* $a_{22} + |a_{12}| > 0.$ (5)

Proof. A matrix A belongs to $\mathscr{M}$ if and only if there exists $P > 0$ diagonally dominant, i.e.

$$\begin{aligned} &p_{11} > |p_{12}|, \\ &p_{22} > |p_{21}| = |p_{12}| \end{aligned} \tag{6}$$

such that

$$A^{\mathrm{T}}P + PA > 0. \tag{7}$$

Then obviously, condition (a) is satisfied.

Defining

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

and substituting in (7) we have the following inequality:

$$\begin{bmatrix} 2(a_{11}p_{11} + a_{21}p_{12}) & p_{12}(a_{11} + a_{22}) + a_{21} \\ * & 2(a_{12}p_{12} + a_{22}p_{22}) \end{bmatrix} > 0.$$

The matrix above is positive definite and is in $\mathbb{R}^{2\times 2}$, thus it is easy to verify

$$\begin{aligned} &a_{11}p_{11} + a_{21}p_{12} > 0, \\ &a_{12}p_{12} + a_{22}p_{22} > 0. \end{aligned} \tag{8}$$

Since p_{11} and p_{22} are positive, divide the first inequality by p_{11} and the second by p_{22}. Using (6) we get

$$\begin{aligned} &a_{11} + |a_{21}| \geqslant a_{11} + \left|a_{21}\frac{p_{12}}{p_{11}}\right| \geqslant a_{11} + a_{21}\frac{p_{12}}{p_{11}} > 0, \\ &a_{22} + |a_{12}| \geqslant a_{22} + \left|a_{12}\frac{p_{12}}{p_{22}}\right| \geqslant a_{22} + a_{12}\frac{p_{12}}{p_{22}} > 0, \end{aligned} \tag{9}$$

showing that (b) is satisfied.

To prove the sufficiency of conditions (a) and (b), consider the matrix A to be Hurwitz, then (8) is satisfied, and it is easy to verify that

$$\begin{aligned} &-a_{11}p_{11} < |a_{21}||p_{12}|, \\ &-a_{22}p_{22} < |a_{12}||p_{12}|. \end{aligned} \tag{10}$$

If item (b) of (10) is false, then there are two possibilities:

(a) $-a_{11} \geqslant |a_{21}| \Rightarrow p_{11} < |p_{12}|$, showing that P cannot be dominant.
(b) $-a_{22} \geqslant |a_{12}| \Rightarrow p_{22} < |p_{12}|$, showing that P cannot be dominant.

Thus the only case in which $A \in \mathscr{M}$ is when condition (b) is true, consequently (a) and (b) are necessary and sufficient conditions for $A \in \mathscr{M}$. □

Using the characterizations listed above we can also prove a negative result by an example.

Theorem 4.2. $\mathscr{S}_{\mathscr{D}} = \mathscr{P}_{\text{sym}} \otimes \mathscr{P}_{\text{diag}} \not\subset \mathscr{S}_{S\mathbb{D}}$.

Proof. Take the following matrix:

$$B = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \in \mathscr{S}_{\mathscr{D}}.$$

Now consider matrices $W \in \mathscr{P}_{\text{sym}}$, $D \in \mathscr{P}_{\text{diag}}$ given by

$$W = \begin{bmatrix} w_1 & w_3 \\ w_3 & w_2 \end{bmatrix} D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}.$$

The matrix $Q = W(BD) = \begin{bmatrix} w_1 d_1 & (3w_1 + w_3)d_2 \\ w_3 d_1 & (3w_3 + w_2)d_2 \end{bmatrix}$. We prove that $Q \notin \mathscr{P}$, $\forall W \in \mathscr{P}_{\text{sym}}$. In fact Q does not satisfy the characterization (v) for some $W \in \mathscr{P}_{\text{sym}}$, i.e.

$$\begin{aligned} 4q_{11}q_{22} - (q_{12} + q_{21})^2 &= (12w_1w_3 + 4w_1w_2)d_1d_2 \\ &\quad -((3w_1 + w_3)d_2 + w_3d_1)^2 \leqslant 0 \end{aligned} \tag{11}$$

$\forall W \in \mathscr{P}_{\text{sym}}$. Developing the expression above we have

$$\begin{aligned} &(12w_1w_3 + 4w_1w_2)d_1d_2 - ((3w_1 + w_3)d_2 + w_3d_1)^2 \\ &\quad = (12w_1w_3 + 4w_1w_2)d_1d_2 - (((3w_1 + w_3)^2(d_2)^2) \\ &\qquad +((6w_1w_3 + 2w_3^2)d_1d_2) + w_3^2(d_1)^2) \\ &\quad = (6w_1w_3 + 4w_1w_2 - 2w_3^2)d_1d_2 - ((3w_1 + w_3)^2(d_2)^2 + w_3^2(d_1)^2) \end{aligned} \tag{12}$$

Fixing d_2 and letting $d_1 \to 0$ the expression above tends to $-(3w_1 + w_3)^2 d_2^2 < 0$; thus $\forall W \in \mathscr{P}_{\text{sym}}$, there always exists $D \in \mathscr{P}_{\text{diag}}$ such that the expression (11) is less or equal to zero, so that $B \notin \mathscr{S}_{S\mathbb{D}}$. □

4.2. 3×3 matrices

In this section, for completeness, we mention the available results for matrices of dimension 3.

The algebraic characterizations for 3×3 matrices are known only for the sets $\mathscr{S}_{\mathscr{D}}$ and $\mathscr{S}_{\mathbb{D}}$ as follows:

Theorem 4.3 [8]. *A matrix $A \in \mathscr{S}_{\mathscr{D}}$ if and only if its principal minors are positive and defining $w_i = \sqrt{a_{ii}b_{ii}}$, where b_{ii} is the ith diagonal element of the inverse matrix of A,* $\max(1, w_1, w_2, w_3) < 1/2(1 + w_1 + w_2 + w_3)$.

Theorem 4.4 [8]. *A matrix $A \in \mathscr{S}_{\mathbb{D}}$ if and only if its principal minors m_1, m_2, m_3 are nonnegative, at least one minor of each order is positive and*

(1) $\sum_{i=1}^{3} \sqrt{\frac{a_{ii}m_{ii}}{\det(A)}} > 1$ *or*

(2) $\sum_{i=1}^{3} \sqrt{\frac{a_{ii}m_{ii}}{\det(A)}} = 1$ *and there exists an index* $j \in \{1, 2, 3\}$ *such that* $a_{jj}m_{jj} = 0$, *with* $a_{jj} \neq 0$ *or* $m_{jj} \neq 0$.

Considering $A \in \mathscr{S}$, a necessary and sufficient condition for a 4×4 matrix to belong to $\mathscr{S}_{\mathbb{D}}$ is given in [16].

5. The set inclusions and equalities that hold for the Venn diagram

Using results showed in the previous section, the definitions of the sets and numerical tests, we have two possible relations between the sets defined here, are shown below extending the Venn diagram in [4]. The only relation that is not defined completely is between the sets $\mathscr{S}_{\mathbb{D}}$ and $\mathscr{M}$. An example is given below of a matrix $K \in \mathscr{M}$, that does not belong to $\mathscr{S}_{\mathbb{D}}$. However, it is not known whether there exists $G \in \mathscr{S}_{\mathbb{D}}$ that does not belong to $\mathscr{M}$.

Regions: I—$\mathscr{P}_{\text{diag}}$, II—$\mathscr{P}_{\text{sym}}$, III—$\mathscr{W}$, IV—$\mathscr{P}$, V—$\mathscr{S}_{\mathscr{D}}$, VI—$\mathscr{S}_{\mathbb{D}}$, VII—$\mathscr{M}$, VIII—$\mathscr{S}$ (Figs. 1 and 2).

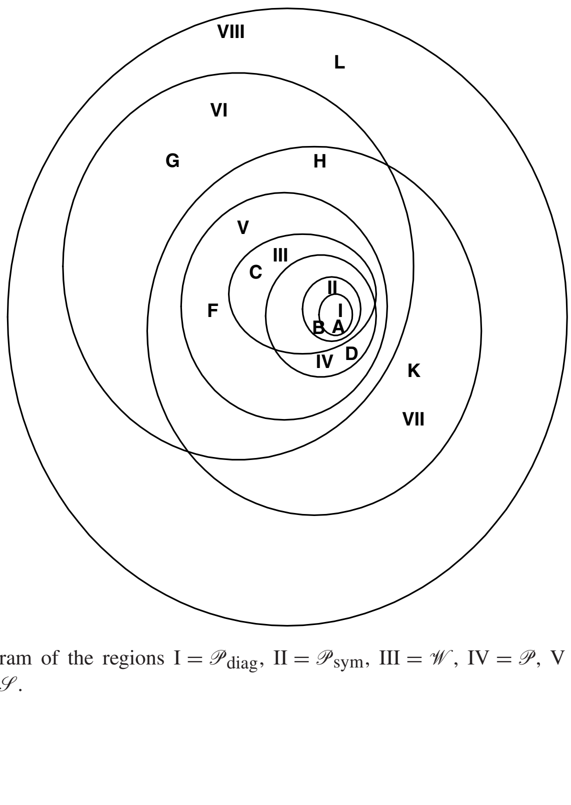

Fig. 1. Venn diagram of the regions I = $\mathscr{P}_{\text{diag}}$, II = $\mathscr{P}_{\text{sym}}$, III = $\mathscr{W}$, IV = $\mathscr{P}$, V = $\mathscr{S}_{\mathscr{D}}$, VI = $\mathscr{S}_{\mathbb{D}}$, VII = $\mathscr{M}$, VIII = $\mathscr{S}$.

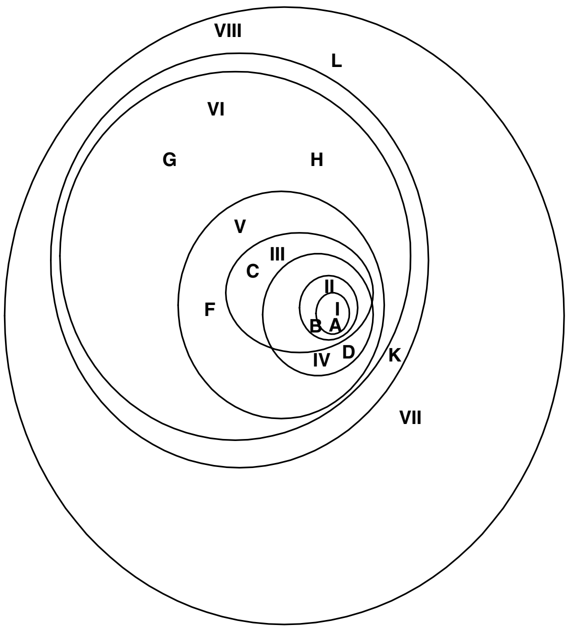

Fig. 2. A second possibility for the Venn diagram. Regions as in Fig. 1.

Matrices:

$$A=\begin{bmatrix}5 & 0\\ 0 & 1\end{bmatrix},\quad B=\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix},\quad C=\begin{bmatrix}3 & -2\\ -4 & 2.8\end{bmatrix},$$

$$D=\begin{bmatrix}3 & -2\\ -4 & 2.8\end{bmatrix},\quad E=\begin{bmatrix}2 & 1/2\\ 1 & 1\end{bmatrix},\quad F=\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix},$$

$G=$ not found,

$$H=\begin{bmatrix}2 & -1\\ 2 & 0\end{bmatrix},\quad K=\begin{bmatrix}-1 & -4\\ 2 & 2\end{bmatrix},\quad L=\begin{bmatrix}5 & -3\\ 3 & -4\end{bmatrix}.$$

A relevant observation is that for $n=2$, using the characterizations (2) and (5), we see that $\mathscr{S}_{\mathbb{D}} \subset \mathscr{M}$.

Now, considering the system with saturation below [12]:

$$\dot{x}(t) = h[T(x(t))], \tag{13}$$

where $x(t) \in D^n := \{x \in \mathbb{R}^n : -1 \leqslant x_i \leqslant 1, i = 1, 2, \ldots, n\}$ and

$$h(Tx) = \left[h_1\left(\sum_{j=1}^n t_{1j}x_j\right) \quad \ldots \quad h_n\left(\sum_{j=1}^n t_{nj}x_j\right)\right],$$

where

$$h_i\left(\sum_{j=1}^n t_{ij}x_j\right) = \left\{\begin{matrix}0, & |x_i| = 1 \text{ and } \left(\sum_{j=1}^n t_{ij}x_j\right) \geqslant 0, \\ \sum_{j=1}^n t_{ij}x_j, & \text{otherwise.}\end{matrix}\right\}$$

It is proved in [12] that $T \in \mathscr{M}$ is a sufficient condition for (13) to be globally asymptotically stable. An open question here is whether $T \in \mathscr{S}_{\mathbb{D}}$ is a sufficient condition for global asymptotic stability of (13). If so this would allow us to infer the relation between subsets $\mathscr{M}$ and $\mathscr{S}_{\mathbb{D}}$.

6. T-inertia preserving matrices

Some of the sets defined in 1.1 can be characterized using the definitions used in [3].

Definition 6.1. The inertia of a square matrix A is a triple:

$$\text{In}\, A = \left\{i_+(A), i_0(A), i_-(A)\right\},$$

where $i_+(A)$ denotes the number of eigenvalues with a positive real part, $i_0(A)$ is the number of pure imaginary eigenvalues and $i_-(A)$ is the number of eigenvalues with a negative part.

Definition 6.2. A matrix A preserves the inertia of a matrix G if $\text{In}(G) = \text{In}(AG)$. The matrix is called *$\mathscr{T}$-inertia preserving* if A preserves the inertia of a set of matrices $\mathscr{T}$, i.e., preserves the inertia of every matrix $G \in \mathscr{T}$.

Using the concept and some results in [2,3] we can rewrite some of the results obtained above as follows:

(i) $A \in \mathscr{S} \Leftrightarrow A$ preserves the inertia of I, the identity matrix. Then we can conclude that if $I \in \mathscr{T}$, then $\mathscr{T}$-inertia preserving matrices belong to $\mathscr{S}$.
(ii) $A \in \mathscr{P} \Leftrightarrow A$ is $\mathscr{P}_{\text{sym}}$-inertia preserving. This affirmation is proved in Theorem 9 of [3].
(iii) $A \in \mathscr{S}_{\mathbb{D}} \Leftrightarrow A$ is $\mathscr{P}_{\text{diag}}$-inertia preserving.
(iv) $A \in \mathscr{S}_{\mathscr{D}} \Rightarrow A$ is $\mathscr{P}^1_{\text{diag}}$-inertia matrix, where $\mathscr{P}^1_{\text{diag}}$ is the set of real diagonal matrices, see Theorem 1 and 3.4 in [2].
(v) $A \in \mathscr{S}_{S\mathbb{D}} \Leftrightarrow \exists W \in \mathscr{P}_{\text{sym}}, \forall D \in \mathscr{P}_{\text{diag}}, W(AD)$ is $\mathscr{P}_{\text{sym}}$-inertia preserving, by (ii).
(vi) $A \in \mathscr{M} \Leftrightarrow \exists P \in \mathscr{R} \cap \mathscr{P}_{\text{sym}}, PA$ is $\mathscr{P}_{\text{sym}}$-inertia preserving, by (ii).
(vii) $A \in \mathscr{H} \Leftrightarrow \forall P \in \mathscr{P}_{\text{sym}}, PA$ preserves the inertia of the identity matrix, by (i).

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