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# Classification of rotations on the torus $\mathbb{T}^2$

Nicolas Bedaride*

*Fédération de recherches des unités de mathématiques de Marseille, Laboratoire d'Analyse topologie, et probabilités UMR 6632, Case cours A, Avenue Escadrille Normandie Niemen, 13397 Marseille cedex 20, France*

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**Abstract**

We consider rotations on the torus $\mathbb{T}^2$, and we classify them with respect to the complexity functions. In dimension one, a minimal rotation can be coded by a sturmian word. A sturmian word has complexity $n + 1$ by the Morse–Hedlund theorem. Here we make a generalization in dimension two.


*Keywords:* Billiard; Symbolic dynamic; Words; Complexity; Sturmian words

## 1. Introduction

Sturmian words are infinite words over a two-letter alphabet that have exactly $n + 1$ factors of length $n$ for each integer $n$. The number of factors, of a given length, of an infinite word is called the complexity function. These words have been introduced by Morse–Hedlund, [14]. They admit several equivalent definitions. One possibility is to consider the rotation of angle $\alpha$ on the torus $\mathbb{T}^1$. Consider a two-letter alphabet corresponding to the intervals $(0; 1-\alpha)$ and $(1 - \alpha; 1)$. Then, the orbit of any point under the rotation is coded by an infinite word. This word is a sturmian word if and only if $\alpha$ is an irrational number. If $\alpha$ is rational, then the rotation is periodic and the word is periodic.

A sturmian word can also be defined by the billiard map: A billiard ball, i.e. a point mass, moves inside a polyhedron $P$ with unit speed along a straight line until it reaches the boundary $\partial P$, then it instantaneously changes direction according to the mirror law, and continues along the new line. Label the faces of $P$ by symbols from a finite alphabet $\mathcal{A}$ whose cardinality equals the number of faces of $P$. In the case of the square, we can code the parallel faces by the same letter, and the orbit of a point is coded by an infinite word on two letters. Since the work of Coven–Hedlund, [8], we know that this word is a sturmian word (under suitable hypothesis on the direction). Thus we have three equivalent definitions for a sturmian word: An infinite word with $n + 1$ factors of length $n$; an infinite word which codes the orbit of a point under an irrational rotation on the torus $\mathbb{T}^1$; an infinite word which codes the orbit of a point under the billiard orbit inside the square.

Several attempts have been made to extend sturmian words to words over alphabets of more than two letters. An

* Corresponding address: Laboratoire d'Analyse Topologie Probabilites (LATP), Department of Mathematics, Universite Paul Cezanne, case cours A, 32 Rue Guibal, 13397 Marseille cedex 20, France. Tel.: +33 0615982123.

*E-mail addresses:* nicolas.bedaride@univ-cezanne.fr, nicolas.bedaride@univ.u-3mrs.fr.

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approach has been initiated by Rauzy [17], and developed by Arnoux, Rauzy [3]. Here we present another approach. We consider a rotation on the torus of dimension two, with a natural partition of the torus, see [16]. The orbit of a point is coded by an infinite word. As in the one-dimensional case, this map can be seen as a billiard orbit inside the cube coded with three letters. Thus the infinite word can be viewed as the coding of the billiard orbit of a point inside the cube. The computation of the complexity has been made when the direction satisfies some algebraic conditions. Under these assumptions the complexity equals $n^2 + n + 1$. The first proof was given in [1,2], and a general proof in dimension $s \geq 3$ appears in [4]. Moreover, we give a new proof of the three-dimensional result in [5], and we remark that the proof of [1,2] is false: there exists a minimal direction with a complexity less than $n^2 + n + 1$.

In this paper we give a complete characterization of the complexity of two-dimensional rotations, and obtain the complexity of a billiard word in the non-totally irrational cases. Moreover our study allows us to describe the geometry of a non minimal rotation orbit. In most of the cases it reduces to a one-dimensional translation. The proof could be generalized to higher dimension, the scheme of the proof is the same using $d$-dimensional translations instead of one-dimensional translations.

### 1.1. Outline of the paper

In this paper, we consider a rotation on the torus $\mathbb{T}^2$. This rotation is related to a billiard orbit inside the cube. We consider a point, and its orbit under the rotation. It gives an infinite word; its complexity is function of the angle of the rotation. We classify these complexities under the hypothesis fulfilled by the angle of rotation. Since the angle of rotation is linked to the direction of the billiard orbit, we express the hypothesis in terms of the direction.

In Section 2, we recall some definitions of combinatorics, of the billiard map, and some results about the complexity of billiard words. In Section 3 we give the statement of the theorem. In Section 4 we prove our result. We split the proof in different propositions which represent the different cases of the theorem. Most of the time, the proof consists in a reduction to the one-dimensional case.

## 2. Background

### 2.1. Combinatorics

**Definition 1.** Let $\mathcal{A}$ be a finite set called the *alphabet*. By a language $L$ over $\mathcal{A}$, we always mean a factorial extendable language: a language is a collection of sets $(L_n)_{n\geq 0}$ where the only element of $L_0$ is the empty word, and each $L_n$ consists of words of the form $a_1 a_2 \dots a_n$ where $a_i \in \mathcal{A}$ and such that for each $v \in L_n$ there exist $a, b \in \mathcal{A}$ with $av, vb \in L_{n+1}$, and for all $v \in L_{n+1}$ if $v = au = u'b$ with $a, b \in \mathcal{A}$ then $u, u' \in L_n$.

**Remark 2.** This definition is closely related to the lamination language defined in [9].

**Definition 3.** Let $\mathcal{L}$ be an extendable, factorial language. The complexity function of the language $\mathcal{L}$ is defined by

$$p : \mathbb{N} \to \mathbb{N}$$
$$p(n) = card(L_n).$$

**Definition 4.** An infinite word $v$ over the alphabet $\mathcal{A}$ is a sequence $(v_n)_{n\in\mathbb{N}}$ such that $v_n \in \mathcal{A}$ for every integer $n$. A subword $w$ of $v$ of length $n$ is a finite word such that there exists $n_0 \in \mathbb{N}$ and $w = v_{n_0} v_{n_0+1} \dots v_{n_0+n-1}$. The set of subwords of length $n$ is denoted by $\mathcal{L}_n$. If $v$ is an infinite word defined over a finite alphabet, then the union $L = \bigcup \mathcal{L}_n$ forms a language. The complexity of $u$ is by definition the complexity of $L$.

### 2.2. Billiard map

We recall some facts from billiard theory. Additional details can be found in [18] or [15].

**Definition 5.** Let $C$ be the cube $[0; 1]^3$, we denote by $(e_i)_{0\leq i\leq 3}$ the orthonormal basis of $\mathbb{R}^3$, and by $\langle e_i, e_j \rangle$ the square generated by the vectors $e_i, e_j$.

$$\langle e_i, e_j \rangle = \{\lambda e_i + \mu e_j, 0 \leq \lambda \leq 1; 0 \leq \mu \leq 1\}.$$

Then $\partial C$ is the union of the following six squares:

$$\langle e_1, e_2\rangle + (0,0,0); \qquad \langle e_1, e_2\rangle + (0,0,1);$$
$$\langle e_1, e_3\rangle + (0,0,0); \qquad \langle e_1, e_3\rangle + (0,1,0);$$
$$\langle e_2, e_3\rangle + (0,0,0); \qquad \langle e_2, e_3\rangle + (1,0,0).$$

It is the boundary of the cube.

A billiard ball, i.e. a point mass, moves inside $C$ with unit speed along a straight line until it reaches the boundary $\partial C$ (see Definition 5), then instantaneously changes direction according to the mirror law, and continues along the new line. More precisely, the billiard map $T$ is defined on a subset $X$ of $\partial C \times \mathbb{RP}^2$ by the following method (where $\mathbb{RP}^2$ is the projective plane):

First we define the set $X' \subset \partial C \times \mathbb{RP}^2$. A point $(m, \omega)$ belongs to $X'$ if and only if one of the two following conditions holds:

(1) The line $m + \mathbb{R}[\omega]$ intersects an edge of $C$, where $[\omega]$ is a vector of $\mathbb{R}^3$ which represents $\omega$.
(2) The line $m + \mathbb{R}[\omega]$ is included inside the face of $C$ which contains $m$.

Then we define $X$ as the set

$$X = (\partial C \times \mathbb{RP}^2) \setminus X'.$$

Now we define the map $T$: Consider $(m, \omega) \in X$, then we have $T(m, \omega) = (m', \omega')$ if and only if $mm'$ is collinear to $[\omega]$, and $[\omega'] = s[\omega]$, where $s$ is the linear reflection over the face which contains $m'$.

$$T : X \to \partial C \times \mathbb{RP}^3$$
$$T : (m, \omega) \mapsto (m', \omega').$$

**Remark 6.** In the sequel we identify $\mathbb{RP}^2$ with the unit vectors of $\mathbb{R}^3$ (i.e. we identify $\omega$ and $[\omega]$).

### 2.3. *Notations for the billiard map*

Label the faces of $C$ by three symbols from a finite alphabet $\mathcal{A}$ such that two parallel faces of the cube are coded by the same symbols. To the orbit of a point in a direction $\omega$, we associate a word in the alphabet $\mathcal{A}$ defined by the sequence of faces of the billiard trajectory:

**Definition 7.** The set of points $(m, \omega)$ such that for all integers $n$, $T^n(m, \omega) \in X$ is denoted by $X_\infty$. The infinite word associated with a point $(m, \omega)$ in $X_\infty$ is denoted by $v_{m,\omega}$.

We finish by three definitions used in the proof of Lemma 37.

**Definition 8.** An edge parallel to the axis $Ox$, respectively, $Oy$, $Oz$ is called of type 1, (type 2, type 3, respectively).

**Definition 9.** We label the three different faces of the cube by $(v_i)_{i=1\dots3}$.

**Definition 10.** Consider the billiard map $T$ inside the cube, and a point $(m, \omega) \in X_\infty$. We define the complexity $p(n, m, \omega)$ by the complexity of the infinite word $v_{m,\omega}$ (see Definition 4). We call it the *directional complexity*.

### 2.4. *Unfolding: Definition and example*

The *unfolding* is a very useful tool in the study of billiards behavior. Consider a billiard trajectory in a polyhedron. To draw the orbit, we must reflect the line each time it hits a face of the polyhedron. The unfolding consists in reflecting the polyhedron through the face and continuing on the same line.

Although we deal with the cube, the figures are made in the case of the square.

**Example 11.** Example of the cube.
The billiard orbit of $(m, \omega)$ appears to be as the sequence of intersections of the line $m + \mathbb{R}\omega$ with the lattice $\mathbb{Z}^3$, see Fig. 1. In the left picture, we represent one billiard orbit inside the square on dash points. It is unfolded in the line which intersects with $\mathbb{Z}^2$.

On the right picture, we see that the study of the billiard orbit can be made on the big square where we identify the opposite sides. Then we obtain a torus, and the map is a translation on this torus.

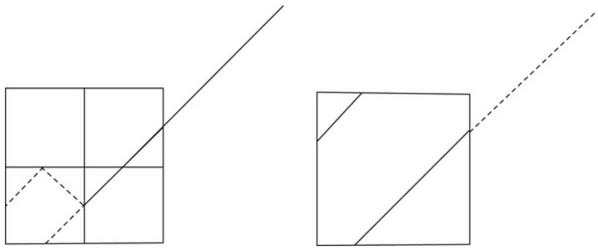

Fig. 1. Unfolding.

**Definition 12.** For $\omega \in \mathbb{R}^3$, a translation $T_\omega$ of the torus is a map defined as follows.

$$\mathbb{R}^3/\mathbb{Z}^3 \to \mathbb{R}^3/\mathbb{Z}^3$$
$$T_\omega : (x, y, z) \mapsto (x, y, z) + \omega.$$

Fig. 1 explains the following result:

**Lemma 13.** *Let $\omega \in \mathbb{R}^3$, and consider the billiard map $T$ in a cube. Then it is equivalent to study the orbit $(T^n(m, \omega))_n$ or the orbit $(T^n_\omega(m))_n$.*

*2.5. Minimality*

**Definition 14.** A direction $\omega \in \mathbb{RP}^2$ is called a minimal direction if for all point $m$, the sequence $(T^n(m, \omega) \cap \partial C)_{n\in\mathbb{N}}$ is dense in $X_\infty$.

The following lemma deals with minimality of billiard words in the one-dimensional case. This minimality depends on algebraic properties of the translation direction. It will be used in the proof of the last cases of our main theorem.

**Lemma 15.** *Let $\omega = (a, b)$ be an unit vector of $\mathbb{R}^2$. Consider the billiard map in the square. Then*
(i) *The direction $\omega$ is a minimal direction if and only if $a, b$ are rationally independent over $\mathbb{Q}$.*
(ii) *If the direction is not a minimal one, then for all point $(m, \omega) \in X_\infty$, the billiard orbit of $(m, \omega)$ is periodic.*

**Lemma 16.** *In the cube, a direction $\omega$ is a minimal direction if and only if the numbers $(\omega_i)_{i\le 3}$ are independent over $\mathbb{Q}$.*

The proof of this lemma is based on Kronecker's lemma, see [13].

*2.6. Billiard complexity*

We consider the coding of the billiard map defined in Section 2.3.

**Definition 17.** For any $n \ge 1$ let $s(n) := p(n+1) - p(n)$. For $v \in \mathcal{L}(n)$ let

$$m_l(v) = card\{a \in \mathcal{A}, va \in \mathcal{L}(n+1)\},$$
$$m_r(v) = card\{b \in \mathcal{A}, bv \in \mathcal{L}(n+1)\},$$
$$m_b(v) = card\{a \in \mathcal{A}, b \in \mathcal{A}, bva \in \mathcal{L}(n+2)\}.$$

A word is called *right special* if $m_r(v) \ge 2$, *left special* if $m_l(v) \ge 2$ and *bispecial* if it is right and left special. Let $\mathcal{BL}(n)$ be the set of bispecial words of length $n$.

Cassaigne [7] has proved the following result, which can be also found in [10]:

**Lemma 18.**

$$s(n+1) - s(n) = \sum_{v\in\mathcal{BL}(n)} [m_b(v) - m_r(v) - m_l(v) + 1].$$

**Lemma 19** *([18]). For a minimal direction, the directional complexity is independent of the initial point $m$.*

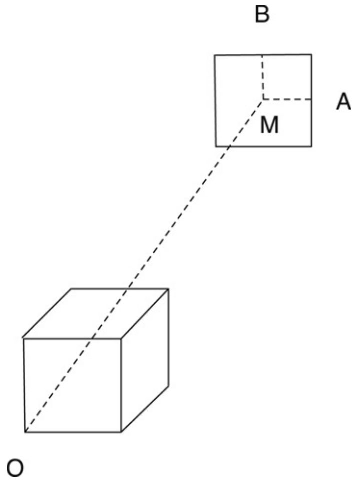


Fig. 2. Generalized diagonal in the cube.

**Notations.** This result implies that we can omit the initial point in the notation $p(n, m, \omega)$, with the assumption that the orbit of $(m, \omega)$ is dense in $X$. In other cases we will denote by $p(n, \omega)$ the maxima of $p(n, m, \omega)$ over all admissible points $m$, see Definitions 7 and 10.

**Definition 20.** In a polyhedron, a generalized diagonal of direction $\omega$ between two edges is the union of all the billiard trajectories of direction $\omega$ between two points of these edges. We say it is of length $n$ if each billiard trajectory hits $n$ faces between the two points.

If we fix the initial edge, we can describe the edges of length $n$ by the following result.

**Lemma 21** ([6]). *Fix an edge $A$ of the initial cube. The edge $B$ is at length $n$ from the edge $A$ if and only if for all point $(b_1, b_2, b_3)$ of $B$, we have*

$$\lfloor b_1 \rfloor + \lfloor b_2 \rfloor + \lfloor b_3 \rfloor = n.$$

We recall the result of [5] which will be useful in the following.

**Proposition 22.** *Assume the cube is coded with three letters such that two parallel faces correspond to the same letter. Let $\omega$ be a unit vector, which is minimal for the cubic billiard, then for all integer $n$, we have*

$$s(n+1, \omega) - s(n, \omega) = N(n, \omega),$$

*where $N(n, \omega)$ is the number of generalized diagonals of direction $\omega$ and length $n$.*

With the same hypothesis, the next lemma proves that we can construct at most two diagonals of combinatorial length $n$ in this direction.

**Lemma 23.** *If $\omega$ is minimal for the billiard map inside a cube, then we have*

$$N(n, \omega) \leq 2 \quad \forall n \in \mathbb{N}^*.$$

**Proof.** Let $O$ be a vertex of the cube and consider the segment of direction $\omega$ which starts from $O$ and ends at a point $M$ after it passes through $n$ cubes (see Fig. 2). $M$ is a point of a face of an unfolding cube, and if we translate $M$ with a direction parallel to one of the two directions of the face, we obtain a point $A$ on an edge, and if we call $C$ the point such that $\vec{OC} = \vec{MA}$, then $CA$ is a generalized diagonal, and we have another one, $DB$ in the figure, arising from the second translation.

The symmetries of the cube imply that these diagonals are the only ones. It remains to prove that the two generalized diagonals are of combinatorial length $n$.

The first thing to remark is that the condition of total irrationality implies that a generalized diagonal cannot begin and end on two parallel edges.

To see that the combinatorial length is at most $n$, we can remark that the sum of the length of the projections is twice the length of the trajectory, so we just have to prove it for the projection, i.e. billiard in the square, where it follows from the symmetry. □

The following lemma recalls some usual results from [11,18]. It deals with minimality of billiard word in the one-dimensional case, depending on algebraic properties of the translation direction.

**Lemma 24.** *Consider a square coded with two letters.*

(i) *If $\theta$ is a minimal direction, then we obtain $p(n, m, \theta) = n + 1$ for all $m$.*

(ii) *The orthogonal projection of a cubic billiard trajectory on a face of the cube is a billiard trajectory inside a square.*

To conclude this section, we recall a complexity result for a linear flow inside a polygon.

**Lemma 25** (*[12]*). *Consider a minimal linear flow on a polygon with parallel opposite sides. Code the flow with a letter by sides; then the orbit of a point is coded by an infinite word of sub-linear complexity. Moreover, the complexity does not depend on the initial point and on the direction.*

## 3. Statement of the theorem

**Theorem 26.** *Fix an orthonormal basis of $\mathbb{R}^3$ such that the edges of the cube are parallel to the coordinate axis. Let $\omega = (\omega_1, \omega_2, \omega_3)$ be a unit vector of $\mathbb{R}^3$ such that $\omega_i \neq 0$ for all $i$. Denote $\alpha = \frac{\omega_2}{\omega_1}$, $\beta = \frac{\omega_3}{\omega_1}$. Then assume one of the following holds:*

(1) *If $\alpha, \beta$ are rational numbers, then there exists $C > 0$, $n_0 \in \mathbb{N}$ such that $p(n, \omega) = C$ for all integer $n \geq n_0$.*
(2) *If $\alpha$ is an irrational number, and $\beta$ is a rational number, then there exists $C$ such that $p(n, \omega) \leq Cn$.*
(3) *If $\alpha, \beta$ are irrational numbers such that $1, \alpha, \beta$ are linearly dependent over $\mathbb{Q}$, then there exists $C$ such that $p(n, \omega) \leq Cn$ for all $n$.*
(4) *If $\alpha, \beta, 1$ are linearly independent over $\mathbb{Q}$, and if $\alpha^{-1}, \beta^{-1}, 1$ are linearly dependent over $\mathbb{Q}$, then there exists $C \in ]0; 1[$ such that $p(n, \omega) \sim Cn^2$.*
(5) *$p(n, \omega) = n^2 + n + 1$, in all other cases.*

**Remark 27.** The cases (2) and (3) correspond to the same algebraic condition. We separate them, since in (2) an orthonormal projection on a face gives a periodic word.

Except in the first case it is clear that the obtained billiard words are not ultimately periodic, thus we have $p(n, \omega) \geq n + 1$.

**Remark 28.** In the two last cases, the complexity is independent of the initial point $m$, by Lemma 19. In the first cases, we use the notations explained after Lemma 19.

We study the dependence of the complexity function when the parameters vary.

**Corollary 29.** *In case* (2)*, two directions with the same value of $\beta$ have the same complexity.*

This statement is a consequence of the last sentence of the proof of Case 4.2.

**Corollary 30.** *For the third case, two directions in the same plane have the same complexity. It means if two directions $\omega, \theta$ satisfy $a\omega_1 + b\omega_2 + c\omega_3 = a\theta_1 + b\theta_2 + c\theta_3 = 0$ with $a, b, c \in \mathbb{Z}$, then $p(n, \omega) = p(n, \theta)$.*

This point is a consequence of the last sentence of Case 4.3.

**Corollary 31.** *In case* (4)*, we can compute the constant $C$. If $(\omega_i)$ satisfy the equation $\frac{A}{\omega_1} = \frac{B}{\omega_2} + \frac{C}{\omega_3}$, with $A, B, C \in \mathbb{N}$, then we obtain*

$$C = 1 - \frac{1}{A(\alpha + \beta + 1)}.$$

*The other cases are obtained by permutation.*

The last point is a consequence of Lemmas 40 and 39, since $p = f_0$.

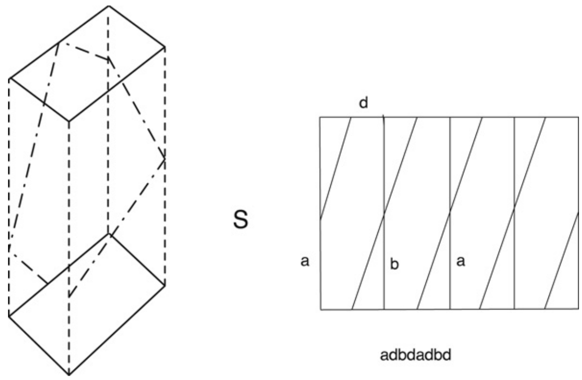


Fig. 3. Billiard trajectory inside $S$, and unfolding.

## 4. Proof of the theorem

By Lemma 13, we will study the orbit of $(m, \omega)$ under the map $T_\omega$, see Definition 12. Each case of the theorem will be treated separately; the first case reduces to a periodic one-dimensional coding, the second case to a billiard word inside a square, the third case reduces to a linear flow inside a polygon with parallel opposite sides, allowing to apply a result from Hubert, see Lemma 25. The fourth case studies the complexity function by studying bispecial factors.

### *4.1. First case*

We prove the following result

**Proposition 32.** *Assume the direction $\omega$ is such that the numbers $\alpha, \beta$ are rational numbers. Then there exists $C > 0, n_0$ such that $p(n, \omega) = C$ for all integer $n \geq n_0$.*

**Proof.** We study the orbit of the point $m_0 = (x, y, z)$ under $T_\omega$. By unfolding, we must compute the intersections of the line $m_0 + \mathbb{R}\omega$ with the three sort of faces. The computation is similar in any case; thus we treat only the case of the intersection with the face $Y = k$ (same thing for the faces $X = l$ or $Z = m$ with $m, l, k \in \mathbb{Z}$). Suppose that there exists $\lambda$ such that $\big(x + \lambda,\quad y + \lambda\alpha,\quad z + \lambda\beta\big)$ belongs to the face $Y = k$. We obtain $\lambda = \frac{k-y}{\alpha}$, we deduce that the intersection point is $(x + \frac{k-y}{\alpha}, k, z + \frac{k-y}{\alpha}\beta)$. The point of the cube which corresponds in the unfolding to this point is $(x + \frac{k-y}{\alpha} \mod 1, 0, z + \frac{k-y}{\alpha}\beta \mod 1)$.

To obtain the sequence coding the orbit of $(x, y, z)$ by $T_\omega$, it remains to make $k$ vary in $\mathbb{Z}$. Since $\alpha, \beta$ are two rational numbers, we deduce that the sequence is periodic. Thus the trajectory is periodic, and the complexity is an eventually constant function. $\square$

### *4.2. Case number 2*

**Proposition 33.** *Assume $\alpha$ is an irrational number, and $\beta$ is a rational number. Then there exists $C$ such that $p(n, \omega) \leq Cn$, for all integer $n$.*

**Proof.** Consider the projection on the plane $Oxz$. Since $\beta$ is a rational number, we have a periodic trajectory in the square (see Lemma 24). Denote by $(a_i)$ the periodic sequence of points inside the square, such that $a_p = a_1$, denote $b_i$ the points of the cube such that $(a_i b_i)$ is parallel to the axis $Oz$. Consider the union $S$ of the intervals

$$[a_i, a_{i+1}], [b_i, b_{i+1}], [a_i, b_i] \quad i \leq p - 1.$$

The trajectory of $(m_0, \omega)$ is included in $S$, as can be seen by projection: see the left part of Fig. 3. Now unfold the trajectory. The unfolding of $S$ is a rectangle. This rectangle is partitioned into several rectangles of the same shape. The trajectory is a translation in this rectangle: see right part of Fig. 3. This translation is coded with three letters and it is minimal by hypothesis on $\alpha$. If the translation were coded by two letters, we would obtain a sturmian word. The computation of the complexity is reduced to the computation of the complexity of a translation: it is clearly sub-linear. Moreover, remark that the rectangle $S$ only depends on $\beta$ by construction. $\square$

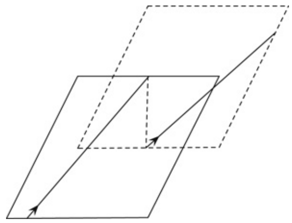

Fig. 4. Billiard map inside the union of polygons.

### 4.3. Case number 3

Let $(x, y, z) \in \mathbb{R}^3$, and $a, b, c \in \mathbb{N}$. Consider the plane $P$ of equation $cX + aY + bZ = cx + ay + bz$. Then consider the canonical projection

$$\pi : \mathbb{R}^3 \mapsto \mathbb{R}^3/\mathbb{Z}^3.$$

The plane intersects the cubes of $\mathbb{Z}^3$ into polygons, and we use this projection to translate the polygons inside the initial cube. Indeed, the initial cube can be identified with $\mathbb{R}^3/\mathbb{Z}^3$.

**Lemma 34.** *The set $\pi(P)$ is the union of a finite number of polygons.*

**Proof.** Consider the intersection $P$ of the plane with the initial cube. The other polygons are obtained by translating the intersection of $P$ with another cube. Thus a study of the edges of the polygons in the face $Z = 0$ can be made by looking at the edges in the face $Z = k$, when $k$ takes values in $\mathbb{Z}$. Consider the intersection of $P$ with the face $Z = k$. We obtain a line of equations

$$\begin{cases} Z = k \\ cX + aY = cx + ay + bz - bk. \end{cases}$$

The slope of this line is $\frac{-c}{a}$. It is a rational number. When $k$ changes this slope is constant; thus all the edges in this face are parallel. Moreover the intersections of this line with the edges of the cube are obtained by replacing $Y$ or $X$ by an integer $l$. For example, we obtain

$$X_{k,l} = \frac{cx + a + bz - bk - al}{c} = \frac{cx + ay + bz}{c} - \frac{bk + al}{c} \mod 1.$$

The set of all points is obtained by taking the union of $k, l$ in $\mathbb{Z}$. This gives a finite number of points, since these numbers are rational. Thus, in each face there are a finite number of parallel edges. Moreover inside two parallel faces the edges are parallel. □

**Proposition 35.** *Assume that: $\alpha, \beta$ are irrational numbers such that $1, \alpha, \beta$ are linearly dependent over $\mathbb{Q}$. Then, there exists $C > 0$ such that $p(n, \omega) \leq Cn$.*

**Proof.** Consider the relation $a\alpha + b\beta + c = 0$ with $a, b, c \in \mathbb{Z}$. We will study the orbit of $m_0 = (x, y, z)$ under $T_\omega$. A point on this line has coordinates $(\lambda + x, \lambda\alpha + y, \lambda\beta + z)$. Thus it is in the plane $cX + aY + bZ = cx + ay + bz$. This plane intersects each unity cube of the lattice $\mathbb{Z}^3$ in a polygon. By a translation, each polygon is shifted to the initial cube. This union of polygons contains the orbit of a point, see Fig. 4. By Lemma 34, there is a finite number of polygons. Now, the orbit of a point is included inside this finite union of polygons. The opposite sides of these polygons are parallel. Thus the billiard flow becomes a linear flow inside a polygon with parallel opposites sides. We apply the result of Lemma 25. Here we remark that several edges can be coded by the same letter; thus the complexity can be less than the initial one. To end the proof, we remark that the complexity only depends on the polygon. Hence it only depends on $a, b, c$. □

### 4.4. Case number 4

In this section, we will show that the number of generalized diagonals in the direction $\omega$ can be strictly less than two. First of all we recall the following lemma.

**Lemma 36.** *Consider three numbers $a, b, c$ linearly independent over $\mathbb{Q}$. Assume that the following equation*

$$x/a + y/b + z/c = 0,$$

*has an integer solution $(x, y, z)$ with $x \neq 0$. Then the rational solutions of the equation are:*

$$r\left(x', \frac{yx'}{x}, \frac{zx'}{x}\right) \quad x', r \in \mathbb{Q}.$$

**Proof.** Consider two rational solutions:

$$\begin{cases} x/a + y/b + z/c = 0 \\ x'/a + y'/b + z'/c = 0 \end{cases} \quad \begin{cases} x/a + y/b + z/c = 0 \\ (yx' - xy')/b + (zx' - xz')/c = 0. \end{cases}$$

Since $b/c$ is an irrational number, we deduce

$$\begin{cases} x/a + y/b + z/c = 0 \\ yx' = xy' \\ zx' = xz' \end{cases}$$

$$\begin{cases} y' = yx'/x \\ z' = zx'/x. \end{cases} \qquad \square$$

**Lemma 37.** *Assume there exists n such that $N(n, \omega) < 2$; then:*

$$s(n + 1, \omega) - s(n, \omega) = 0.$$

*Moreover there exists a line of direction $\omega$ which intersects the three types of edges (see Definition 8) and these three edges are in a fixed order, given by the direction.*

**Proof.** First recall that the minimality of $\omega$ implies that the edges of a diagonal in direction $\omega$ cannot be parallel. In the rest of the proof, we can assume that the edges of the generalized diagonal of direction $\omega$ are of type 1 and 3, see Definition 8.

(1) Consider a trajectory in direction $\omega$ between two edges of types 1 and 3, consider the orthogonal reflection over the plane $X = Z$. This map exchanges the edges of type 1 and 3, but it leaves invariant edges of type 2. That implies that $N(n, \omega)$ is an even number, thus we cannot have $N(n, \omega) = 1$. Hence we have $N(n, \omega) = 0$. Proposition 22 finishes the first part of the proof.

(2) By applying a translation, we can always assume that the intersection points of the line $m + \mathbb{R}\omega$ with the edges of the cube have for coordinates

$$(x, 0, 0); (a, y, b); (c, d, z),$$

with $x, y, z$ real numbers and $a, b, c, d$ integers.

We obtain the system

$$\begin{cases} x + \lambda\omega_1 = a \\ \lambda\omega_2 = y \\ \lambda\omega_3 = b \\ x + \mu\omega_1 = c \\ \mu\omega_2 = d \\ \mu\omega_3 = z \end{cases}$$

where $\lambda$, $\mu$ are real numbers.

$$\begin{cases} \lambda\omega_2 = y \\ \lambda\omega_3 = b \\ \mu\omega_2 = d \\ \mu\omega_3 = z \\ x = a - b\frac{\omega_1}{\omega_3} \\ \frac{a-c}{\omega_1} = \frac{b}{\omega_3} - \frac{d}{\omega_2}. \end{cases}$$

By hypothesis on $\omega$, we have a relation of the form

$$\frac{A}{\omega_1} + \frac{B}{\omega_2} + \frac{C}{\omega_3} = 0,$$

where $A, B, C \in \mathbb{Z}$ are relatively prime.

The last equation of the system is of the same form:

$$\frac{a-c}{\omega_1} + \frac{-b}{\omega_3} + \frac{d}{\omega_2} = 0.$$

We apply Lemma 36, and one deduces that $A$ is a divisor of $a - c$, and:

$$\begin{cases} d = B\frac{a-c}{A} \\ -b = C\frac{a-c}{A}. \end{cases}$$

Finally the system becomes

$$\begin{cases} \lambda\omega_2 = y \\ \lambda\omega_3 = b \\ \mu\omega_2 = d \\ \mu\omega_3 = z \\ x = a - b\frac{\omega_1}{\omega_3} \\ d = B\frac{a-c}{A} \\ -b = C\frac{a-c}{A}. \end{cases}$$

This system has at least one solution. Hence the existence of the line is proved.

This system allows us to make several remarks. First, the coordinates $\omega_i$ are positive numbers. This implies that $A, B, C$ cannot all be positive numbers. Assume that we have $A < 0, B > 0, C > 0$ (the other cases are similar). We deduce that $a - c$ and $d$ are of opposite sign, and that $a - c$ and $b$ are of same sign.

(3) Assume $a - c \geq 0$; this implies

$$d \leq 0, b \geq 0.$$

From the system, we deduce

$$\lambda \geq 0, \mu \leq 0.$$

This implies that the edges appear in the order 3; 1; 2.

(4) On the other hand, if $a - c \leq 0$, by a similar argument, we have that the order is 2; 1; 3.

Moreover, the two orders are correlated: it depends on the direction that it used to move along the line. Hence, we can reduce to one order. □

**Corollary 38.** *Assume $\omega$ is a minimal direction and satisying*

$$\frac{A}{\omega_1} = \frac{B}{\omega_2} + \frac{C}{\omega_3} \quad A, B, C \in \mathbb{N}^*.$$

*Then, for all integer $n$, we have the dichotomy: either the billiard orbit of the origin, at step $n$, meets a face labelled by $v_1$ (see Definition 9), and if A divides $n$, then $s(n+1, \omega) - s(n, \omega) = 0$, or $s(n+1) - s(n) = 2$.*

**Proof.** First we claim that there exist an infinite number of integers $n$ such that $s(n) = 0$. Indeed, in the last system obtained in the proof of Lemma 37, we can modify the values of $a, c$ such that $A$ divides $a - c$ . Now we can assume that the order related to the edges is 3; 1; 2 see Lemma 37. Consider the orbit of the origin, and the intersection with a face (of a cube of $\mathbb{Z}^3$) parallel to $X = 0$. With the method of Lemma 23, we deduce that the only possibility for a generalized diagonal is a trajectory between edges 3 and 2. Denote by $n$ the length of the diagonal; from the previous system we deduce that if $A$ divides $n$, the trajectory between 3 and 2 passes through the edge 1. We deduce $s(n + 1, \omega) = s(n, \omega)$. The first part is proved.

Assume now that we meet another face at step $n$, for example the face parallel to $Z = 0$. Then the two associated diagonals have for order 1; 2 and 2; 1. We prove by contradiction that we cannot have $N(n, \omega) \leq 1$. Since the order is unique, see Lemma 37, the only possibility to obtain a third edge is to start from the edge labelled 1. Then the diagonal which starts form 3 does not intersect another edge. This implies $N(n, \omega) = 1$, but this is a contradiction with the first part of Lemma 37. □

This corollary implies that the sequence $(s(n, \omega))_{n\in\mathbb{N}}$ can take only two values. Due to the next lemma, to finish the proof it remains to obtain the frequency of each value.

**Lemma 39.** *Assume that the sequence $(s(n+1, \omega) - s(n, \omega))_{n\in\mathbb{N}}$ has a value in $\{0; 1; 2\}$, and that the numbers 0; 1; 2 have respectively the frequencies $l, m, p$. Then the complexity satisfies*

$$p(n) \sim \frac{m + 2p}{2} n^2.$$

**Lemma 40.** *Assume the direction satisfies the hypothesis $\frac{A}{\omega_1} = \frac{B}{\omega_2} + \frac{C}{\omega_3}$, with $A, B, C \in \mathbb{N}$. Then the frequency $l$ of 0 in the sequence $(s(n + 1, \omega) - s(n, \omega))_{n\in\mathbb{N}}$ is:*

$$l = \frac{\omega_1}{A(\omega_1 + \omega_2 + \omega_3)}.$$

**Proof.** By Corollary 38, it is equivalent to considering the intersection of the orbit of the origin with the planes parallel to $X = 0$. A point in the orbit of the origin has for coordinates:

$$(\lambda\omega_1, \lambda\omega_2, \lambda\omega_3).$$

It meets the face $X = iA$ at the point

$$\left(iA, \frac{iA}{\omega_1}\omega_2, \frac{iA}{\omega_1}\omega_3\right).$$

Then we must compute the number of $i$ such that this point is at combinatorial length less than $n$. By Lemma 21, it remains to compute

$$card \left\{ i \,\middle|\, iA + \left[\frac{iA}{\omega_1}\omega_2\right] + \left[\frac{iA}{\omega_1}\omega_3\right] \leq n \right\} = \frac{n\omega_1}{A(\omega_1 + \omega_2 + \omega_3)} + o(n).$$

We deduce the value of the frequency

$$l = \frac{\omega_1}{A(\omega_1 + \omega_2 + \omega_3)}. \quad \square$$

**Proposition 41.** *Assume the numbers $\alpha, \beta, 1$ are linearly independent over $\mathbb{Q}$, and $\alpha^{-1}, \beta^{-1}, 1$ are linearly dependant over $\mathbb{Q}$. Then there exists $C \in ]0; 1[$ such that $p(n, \omega) \sim Cn^2$.*

**Proof.** The proof is a consequence of Corollary 38, Lemmas 39 and 40.

### *4.5. Last case*

The proof can be found in [5] or in [4] for the $s$-dimensional case. In the first article, the main object of the proof consists in stating that $N(n, \omega) = 2$ for every integer $n$. In the second article, the main step of the proof consists in stating that $p(n, \omega)$ does not depend on the direction $\omega$.

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## References

[1] P. Arnoux, C. Mauduit, I. Shiokawa, J. Tamura, Complexity of sequences defined by billiard in the cube, Bull. Soc. Math. France 122 (1) (1994) 1–12.
[2] P. Arnoux, C. Mauduit, I. Shiokawa, J. Tamura, Rauzy's conjecture on billiards in the cube, Tokyo J. Math. 17 (1) (1994) 211–218.
[3] P. Arnoux, G. Rauzy, Représentation géométrique de suites de complexité $2n + 1$, Bull. Soc. 1 Math. France 119 (2) (1991) 199–215.
[4] Yu. Baryshnikov, Complexity of trajectories in rectangular billiards, Comm. Math. Phys. 174 (1) (1995) 43–56.
[5] N. Bedaride, Billiard complexity in rational polyhedra, Regul. Chaotic Dyn. 8 (1) (2003) 97–104.
[6] N. Bedaride, P. Hubert, Billiard complexity in the cube, Ann. Inst. Fourier (2007) (in press).
[7] J. Cassaigne, Complexité et facteurs spéciaux, Bull. Belg. Math. Soc. Simon Stevin 4 (1) (1997) 67–88. Journées Montoises (Mons, 1994).
[8] Ethan M. Coven, G.A. Hedlund, Sequences with minimal block growth, Math. Syst. Theory 7 (1973) 138–153.
[9] T Coulbois, A. Hilion, M. Lustig, Non-unique ergodicity, observers' topology and the dual lamination for r-trees, Ilinois J. Math. (2007) (in press).
[10] J. Cassaigne, P. Hubert, S. Troubetzkoy, Complexity and growth for polygonal billiards, Ann. Inst. Fourier (Grenoble) 52 (3) (2002) 835–847.
[11] Pytheas Fogg, in: V. Berthé, S. Ferenczi, C. Mauduit, A. Siegel (Eds.), Substitutions in Dynamics, Arithmetics and Combinatorics, in: Lecture Notes in Mathematics, vol. 1794, Springer-Verlag, Berlin, 2002.
[12] P. Hubert, Complexité de suites définies par des billards rationnels, Bull. Soc. Math. France 123 (2) (1995) 257–270.
[13] G.H. Hardy, E.M. Wright, An Introduction to the Theory of Numbers, fifth ed., The Clarendon Press Oxford University Press, New York, 1979.
[14] M. Morse, G.A. Hedlund, Symbolic dynamics II. Sturmian trajectories, Amer. J. Math. 62 (1940) 1–42.
[15] H. Masur, S. Tabachnikov, Rational billiards and flat structures, in: Handbook of Dynamical Systems, vol. 1A, North-Holland, Amsterdam, 2002, pp. 1015–1089.
[16] G. Rauzy, Nombres algébriques et substitutions, Bull. Soc. Math. France 110 (2) (1982) 147–178.
[17] G. Rauzy, Suites à termes dans un alphabet fini, in: Seminar on Number Theory, 1982–1983 (Talence, 1982/1983), pages Exp. No. 25, 16. Univ. Bordeaux I, Talence, 1983.
[18] S. Tabachnikov, Billiards, in: Panoramas et Synthèses, vol. 1, SMF, Paris, 1995.