Wavelets: Properties and Approximate Solution of a Second Kind Integral Equation

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Abstract—In this paper, we show that, under some conditions, a wavelet basis of $L^2(R)$ can be used as a tool for the uniform approximation in the space $C^a(R) \cap L^2(R)$, $\alpha > 0$, where $C^a(R)$ denotes the Hölder space of exponent $\alpha$. As a result of this property, we give a numerical application of wavelets. This application is a wavelet-based method for the numerical solution of a Fredholm equation of the second kind with solution lying in $C^a(R)$, the Hölder space of compactly supported functions with Hölder exponent $\alpha > 1/2$. © 2003 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Consider a basis of $L^2(R)$ given by a two-parameter family of wavelets,

$$\psi_{jk}(x) = 2^{-j/2} \psi(2^{-j}x - k), \quad x \in \mathbb{R}, \quad j, k \in \mathbb{Z},$$

(1.1)

generated by dilations and translations of a single mother wavelet $\psi \in L^2(R)$. Then any function $f \in L^2(R)$ can be expanded in the double wavelet series

$$f(x) = \sum_{j,k \in \mathbb{Z}} \langle f, \psi_{jk} \rangle \psi_{jk}(x),$$

(1.2)

which converges in the $L^2$-norm; the scalar product is defined by

$$\langle f, \psi_{jk} \rangle = \int_R f(x) \overline{\psi_{jk}(x)} \, dx.$$

Moreover, if

$$\int_R \psi_{jk}(x) \psi_{j'k'}(x) \, dx = \delta_{jj'} \delta_{kk'}, \quad \forall j, j', k, k' \in \mathbb{Z},$$

(1.3)
and the family \( (\psi_{jk})_{jk} \) satisfies the stability condition

\[
\forall f \in L^2(R), \quad c_1 \|f\|^2_2 \leq \sum_{j,k \in \mathbb{Z}} |\langle f, \psi_{jk} \rangle|^2 \leq c_2 \|f\|^2_2,
\]

for some constants \( c_1, c_2 > 0 \), then \( (\psi_{jk})_{jk} \) is said to be an orthonormal wavelet basis of \( L^2(R) \).

In general, a compactly supported wavelet is associated with a compactly supported real scaling function \( \phi \), a solution of the refinement equation

\[
(1.5)
\]

where \( (\alpha_n)_n \) is an appropriate finite sequence of real numbers. The translates of \( \phi \) have to satisfy the orthonormality condition

\[
\langle \phi(\cdot - j), \phi(\cdot - k) \rangle = \int_R \phi(x - j)\overline{\phi(x - k)} \, dx = \delta_{jk}.
\]

The mother wavelet \( \psi \) is simply given by

\[
\psi(x) = \sum_{n=-\infty}^{\infty} \alpha_n \phi(2x - n).
\]

A biorthogonal wavelet basis is a pair of families of dual wavelets \( \psi_{jk} \) and \( \tilde{\psi}_{jk} \) derived from two mother wavelets \( \phi \) and \( \tilde{\phi} \), respectively. In this case, any function \( f \in L^2(R) \) can be written in the forms

\[
(1.7)
\]

For more details on the construction and the regularity of compactly supported orthonormal and biorthogonal wavelet bases, the reader is referred to [1-3].

It is well known that wavelets have many applications in applied mathematics and numerical analysis [4-6], etc. In this work, the focus is on the interesting property of uniform approximation by wavelets in the space \( C^\alpha(R) \cap L^2(R) \) for any positive real number \( \alpha > 0 \). More precisely, we show that if the mother scaling function \( \phi \) and the wavelet \( \psi \) belong to the Hölder space \( C_0^\alpha(R) \) and if \( f \in C^\alpha(R) \cap L^2(R), 0 < \alpha < \tau \), then the wavelet series expansion of \( f \) converges to \( f \) in the \( || \cdot ||_\infty \) norm. As a result, wavelets can be used to build numerical schemes for solving problems with solutions lying in \( C^\alpha(R) \cap L^2(R), 1/2 < \alpha \). An example of such a scheme will be provided in this paper.

This work is organized as follows. In Section 2, we review the Sobolev regularity of the mother scaling function \( \phi \) and the stability of the associated wavelet basis. Also, we use an idea from iterative interpolation theory to derive an estimate of the Sobolev and the Hölder regularity of an orthonormal wavelet basis. The results of this section will be needed in the remaining sections of the paper. In Section 3, we prove the quality of approximation by wavelets in the \( || \cdot ||_\infty \) norm and provide the reader with an error bound for the approximation of a function by its truncated wavelet series expansion. Section 4 is devoted to a wavelet-based numerical scheme for the solution of a Fredholm equation of the second kind with solution lying in \( C^\alpha(R), \alpha > 1/2 \). Also, we provide the reader with a bound of the condition number of the matrix associated with the proposed numerical scheme. Finally, in Section 5, we give a wavelet-based quadrature method for the computation of the wavelet coefficients.

Note that, in this work, we restrict ourselves to the use of one-dimensional compactly supported orthonormal wavelet bases. Nonetheless, the techniques and the results of this paper can be easily extended to biorthogonal wavelet bases. Moreover, the one-dimensional schemes of this
paper can be easily generalized to higher dimensions. For the construction and the regularity of multidimensional wavelet bases, the reader is referred to [7–10].

2. STABILITY AND REGULARITY ESTIMATE OF AN ORTHONORMAL WAVELET BASIS

In this section, we show that the Sobolev regularity of the scaling function $\phi$ given by (1.5) implies the stability of the associated wavelet basis. Also, we adapt a method from iterative interpolation theory to derive a sharp estimate of the Sobolev regularity and an estimate of a wavelet basis. Note that some of the results of this section have been derived differently in the literature.

2.1. Sobolev Regularity and Stability of Wavelet Bases

We first define a Sobolev space of order $s > 0$.

**Definition 1.** A Sobolev space of order $s > 0$, denoted by $H^s(\mathbb{R})$, is a subspace of $L^2(\mathbb{R})$, given by

$$H^s(\mathbb{R}) = \{ f \in L^2(\mathbb{R}); \left| \hat{f}(\xi) \left(1 + |\xi|^2\right)^{s/2} \right| \in L^2(\mathbb{R}) \}.$$ 

In order to prove the relationship between the Sobolev regularity and the stability of a wavelet basis, we need the following lemma.

**Lemma 1.** Suppose that the scaling function $\phi$ satisfies

$$\sup_{\xi \in \mathbb{R}} \sum_{k \in \mathbb{Z}} \left| \phi(\xi + 2k\pi) \right|^{2-\sigma} < +\infty, \quad (2.1)$$

for some $\sigma > 0$, and

$$\sup_{\xi \in \mathbb{R}} (1 + |\xi|)^{\sigma} \left| \phi(\xi) \right| < +\infty. \quad (2.2)$$

Then there exists a constant $c$ such that, for all $f \in L^2(\mathbb{R})$,

$$\sum_{j,k \in \mathbb{Z}} |(f, \psi_{jk})|^2 \leq c \| f \|_2^2. \quad (2.3)$$

**Proof.** See [2].

The first result of this section is given by the following theorem.

**Theorem 1.** Assume that the sequence $(\alpha_n)_n$ given by (1.5), is finite and satisfies the condition

$$\sum_n \alpha_{n-2k} \alpha_{n-2l} = \delta_{-k}, \quad (2.4)$$

and assume that for some $\epsilon > 0$, we have

$$|\hat{\psi}(\xi)| < c (1 + |\xi|^2)^{-\epsilon - 1/4} \quad (2.5)$$

where $\hat{\psi}$ denotes the Fourier transform of the mother wavelet $\psi$. Finally, assume that the mother scaling function $\phi \in H^s(\mathbb{R})$ for some positive number $s$. Then there exist two positive constants $C_1(s)$ and $C_2(s)$ such that

$$\forall f \in L^2(\mathbb{R}), \quad c_1(s) \| f \|_2^2 \leq \sum_{j,k \in \mathbb{Z}} |(f, \psi_{jk})|^2 \leq c_2(s) \| f \|_2^2. \quad (2.6)$$
PROOF. To prove the upper bound of (2.6), it suffices to check that if $\phi \in H^s(R)$ for some $s > 0$, then conditions (2.1) and (2.2) of Lemma 1 are satisfied. To get (2.1), we first prove that there exists $0 < \alpha < 1$ such that
\[ \int |\phi(\xi)|^{2-2\alpha} \, d\xi < +\infty. \tag{2.7} \]
Since
\[ \int |\phi(\xi)|^{2-2\alpha} \, d\xi \leq \int [(1 + |\xi|)^{2s} \, |\hat{\phi}(\xi)|^2]^{1-\alpha} \left[ \int (1 + |\xi|)^{2s(\alpha-1)} \, d\xi \right]^{\alpha}, \]
and by using Hölder's inequality, one gets
\[ \int |\phi(\xi)|^{2-2\alpha} \, d\xi \leq C \left[ \int (1 + |\xi|)^{2s(\alpha-1)} \, d\xi \right]^{\alpha} \]
Hence, if $0 < \alpha < 1/(1 + 1/2s)$, then (2.7) holds. To get (2.1), it suffices to use the following inequalities which can be found in [2]:
\[ \sum_{k\in\mathbb{Z}} |\phi(\xi + 2k\pi)|^{2-2\alpha} < \int \frac{d}{d\xi} \left( |\hat{\phi}(\xi)|^{2-\alpha}(\xi) \right) \, d\xi \]
\[ \leq (2 - \alpha) \int \frac{d\hat{\phi}}{d\xi} \left( |\hat{\phi}(\xi)|^{1-\alpha} \right) \, d\xi \]
\[ \leq 2 \left[ \int \left( \frac{d\hat{\phi}}{d\xi} \right)^2 \, d\xi \right]^{1/2} \left[ \int |\hat{\phi}(\xi)|^{2-2\alpha} \, d\xi \right]^{1/2}. \tag{2.8} \]
Since $(\alpha_n)_n$ is finite, it follows that the associated scaling function $\phi$ is compactly supported. Moreover, $\phi \in H^s(R)$ for some $s > 0$ implies that $\phi \in L^2(R)$. It becomes clear that the first factor of the last inequality is proportional to the $L^2$ norm of $x\phi(x)$ which is finite, and the second factor is finite whenever $0 < \alpha < 1/(1 + 1/(2s))$. To prove (2.2), we consider a point $\xi$ such that $|\xi| \in [2^{-n-1}\pi, 2^n\pi], n \geq 1$. Then the techniques used to get (2.8) give us
\[ |\phi(\xi)|^2 \leq c \left[ \int_{2^{-n-1}\pi \leq |\xi| \leq 2^n\pi} |\phi(\xi)|^2 \, d\xi \right]^{1/2}. \tag{2.9} \]
Since
\[ \int_{2^{-n-1}\pi \leq |\xi| \leq 2^n\pi} |\phi(\xi)|^2 \, d\xi = \int_{2^{-n-1}\pi \leq |\xi| \leq 2^n\pi} \left[ \left( \frac{1}{2} + |\xi| \right)^{2s} \, |\phi(\xi)|^2 \right] \frac{1}{(1/2 + |\xi|)^{2s}} \, d\xi, \]
therefore,
\[ \int_{2^{-n-1}\pi \leq |\xi| \leq 2^n\pi} |\phi(\xi)|^2 \, d\xi \leq c_1 \left( \frac{1}{2} + 2^{n-1}\pi \right)^{-2s} \]
\[ \leq c_1 \left( \frac{1}{2} + \frac{|\xi|}{2} \right)^{-2s} \]
\[ \leq c_2(s)(1 + |\xi|)^{-2s}. \]
Consequently, there exists a constant $c_3(s)$ depending only on $s$ such that
\[ |\phi(\xi)| \leq c_3(s)(1 + |\xi|)^{-s}, \quad \forall \xi \in R. \]
Collecting everything together, one concludes that for any arbitrary real number $\alpha$ satisfying

$$0 < \alpha < \min \left( s, \frac{1}{1 + 1/(2s)} \right),$$

the scaling function $\phi$ satisfies conditions (2.1) and (2.2). Consequently, the upper bound of (2.6) is proven. To prove the lower bound of (2.6), we first mention that under conditions (2.4) and (2.5), the wavelet series expansion of an $L^2$ function $f$ converges to $f$ in the $L^2$-sense, that is,

$$\forall f \in L^2(R), \quad f(x) = \sum_{j,k \in \mathbb{Z}} \langle f, \psi_{jk} \rangle \psi_{jk}(x), \quad (2.10)$$

where the equality holds in the $L^2$-sense. For the proof of this last result, the reader is referred to [11]. From (2.4) and (2.10), one concludes that, for all $f \in L^2(R),$

$$\|f\|^2 = \lim_{N \to +\infty} \sum_{j=-N}^{N} \sum_{k \in \mathbb{Z}} |\langle f, \psi_{jk} \rangle|^2$$

$$\leq \sum_{j,k \in \mathbb{Z}} |\langle f, \psi_{jk} \rangle|^2$$

$$\leq \sqrt{c_2(s)} \|f\|_{2} \left[ \sum_{j,k \in \mathbb{Z}} |\langle f, \psi_{jk} \rangle|^2 \right]^{1/2}$$

Hence,

$$\|f\|^2 \leq \frac{1}{c_2(s)} \sum_{j,k \in \mathbb{Z}} |\langle f, \psi_{jk} \rangle|^2,$$

which proves the lower bound of (2.6) and concludes the proof of the theorem.

### 2.2. An Estimate of the Hölder Regularity of Wavelets

In this paragraph, we study a practical method for estimating the Hölder regularity of a wavelet basis. We first define a Hölder space. This definition is to be used in the different sections of this paper.

**Definition 2.** If $\alpha = n + r$, $n \in \mathbb{N}$, $0 < r < 1$, then the Hölder space of order $\alpha$ denoted by $C^\alpha(R)$ is defined by

$$C^\alpha(R) = \left\{ f \in L^\infty; \sup_{x,h} \frac{|D^n(f)(x+h) - D^n(f)(x)|}{|h|^r} < \infty \right\}.$$ 

If $\alpha = n \in \mathbb{N}^*$, then

$$C^n(R) = \left\{ f \in L^\infty; \sup_{x,h} \frac{|D^{n-1}(f)(x+h) + D^{n-1}(f)(x-h) - 2D^{n-1}(f)(x)|}{|h|} < \infty \right\}.$$ 

In [12], the authors have given a practical and sharp method for estimating the Hölder regularity of the solution of a symmetric iterative interpolation process. This method is described as follows. Let $F(t)$ be the $L^1$-solution of the iterative interpolation process

$$G(t) = \sum_{-M}^{M} a_n G(2t - n), \quad \sum_{-M}^{M} a_n = 2.$$
Define the trigonometric polynomial $M_0(\xi)$ by $M_0(\xi) = \sum_{n=-M}^{M} \alpha_n e^{in\xi}$. We assume that $M_0(\xi)$ factors in the form

$$M_0(\xi) = \left(\cos\left(\frac{\xi}{2}\right)\right)^{2N} \sum_{n=-M+N}^{M-N} \beta_n e^{in\xi}.$$ 

Define a symmetric matrix $A$ by $A = [\beta_{n-2m}]_{-M+N \leq n,m \leq M-N}$. Let $r$ denote the spectral radius of $A$. Under the condition that

$$M_0(\xi) \geq 0, \quad \forall \xi \in \mathbb{R}, \quad (2.11)$$

it is shown in [12] that the solution $F(\cdot)$ of the above iterative process satisfies

$$\int_{\mathbb{R}} |\xi|^\alpha |\hat{F}(\xi)| \, d\xi < +\infty$$

and consequently, it belongs to the Hölder space $C^\alpha(\mathbb{R})$ for all $\alpha < 2N - \log(r)/\log 2$. Moreover, this Hölder estimate is sharp. Note that in the case of wavelets, condition (2.11) is not satisfied, and consequently, the above method cannot be used as it is, for checking the regularity of wavelets. Nonetheless, as will be shown in the proof of the following proposition, by a minor modification of the above scheme, one gets a practical method for estimating the Sobolev and the Hölder regularities of compactly supported wavelets.

**Proposition 1.** Assume that $\phi$ is an $L^1$ bounded solution of the refinement equation

$$\phi(t) = \sum_n \alpha_n \phi(2t_n), \quad \sum_n \alpha_n = 2. \quad (2.12)$$

Here, we assume that the sequence $(\alpha_n)_n$ is finite in length. Define the matrix $B$ by $B = [\beta_{n-2m}]_{n,m}$, where $\beta_k = \sum \gamma_n \gamma_{n-k}$ and where the finite sequence $(\gamma_n)_n$ is defined by

$$\sum_{n,m} \alpha_n \alpha_m e^{i(n-m)\xi} = 2 \left(\frac{1 + e^{i\xi}}{2}\right)^N \left(\frac{1 + e^{-i\xi}}{2}\right)^N \sum_{n,m} \gamma_n \gamma_{n-k} e^{ik\xi}.$$ 

If $r$ denotes the spectral radius of $B$, then the scaling function $\phi$, and the associated wavelets belong to the Sobolev space $H^s(\mathbb{R})$ and to the Hölder space $C^{s-1/2}$ for all $s < 2N - \log(r)/\log 2$.

**Proof.** By applying the Fourier transform to both sides of (2.12), one gets

$$\hat{\phi}(\xi) = m_0 \left(\frac{\xi}{2}\right) \hat{\phi} \left(\frac{\xi}{2}\right). \quad (2.13)$$

If we define the function $g(\cdot)$ by $g(t) = \phi(-t)$, then

$$\left(\phi * g\right)(\xi) = m_0 \left(\frac{\xi}{2}\right) m_0 \left(-\frac{\xi}{2}\right) \left(\phi * g\right) \left(\frac{\xi}{2}\right). \quad (2.14)$$

It is clear that the convolution function $(\phi * g)$ is an $L^1$ function. Applying the inverse Fourier transform to (2.14), one concludes that

$$(\phi * g)(x) = \sum_k \beta_k (\phi * g)(2x - k), \quad \beta_{-k} = \beta_k = \sum_n \alpha_n \alpha_{n-k}.$$ 

Note that $m_0(0) = 1$ implies that $\sum_k \beta_k = 2$ and $(\phi * g)$ is an $L^1$ symmetric solution of an equation of the type (2.12). Since

$$M_0(\xi) = 2m_0(\xi)m_0(-\xi)$$
satisfies condition (2.11), since
\[ \hat{\phi} \ast \hat{g}(\xi) = |\hat{\phi}(\xi)|^2, \]
and by applying the method of [12], one concludes that
\[ \int_R |\xi|^\alpha |\hat{\phi}(\xi)|^2 d\xi < +\infty, \quad \forall \alpha < 2N - \frac{\log r}{\log 2}, \quad (2.15) \]
where \( r \) is the spectral radius of \( B \). Since \( \phi \) is a bounded \( L^1 \) function, it follows that \( \phi \in L^2(R) \). Hence, (2.15) implies that \( \phi \in H^{s}(R), \forall s < 2N - \log r/\log 2 \). Finally, to prove the Hölder regularity of the scaling function, we consider an arbitrary small \( \epsilon > 0 \). Then by using the Cauchy-Schwartz inequality, one concludes that
\[ \left[ \int (1 + |\xi|^2)^{\delta/2 - 1/4 - \epsilon/2} |\hat{\phi}(\xi)| \, d\xi \right]^2 \leq \int \frac{d\xi}{(1 + |\xi|^2)^{1/2 + \epsilon}} \int (1 + |\xi|^2)^{\delta} |\hat{\phi}(\xi)|^2 \, d\xi. \]
Hence, if \( \phi \in H^\delta(R) \), then \( \int |\xi|^{\delta - 1/2 - \epsilon} |\hat{\phi}(\xi)| \, d\xi < +\infty \), and consequently, \( \phi \in C^{\delta - 1/2 - \epsilon} \) for any \( \epsilon > 0 \).

The transition operator based technique has been extensively studied and used for checking the stability, Sobolev, and Hölder regularity of a wavelet basis; see [13–15], etc. This technique is described as follows. Let \( m_0(\xi) \) be the wavelet filter given by (2.13); the associated transition operator is defined by
\[ T_{m_0}(f)(\xi) = \left| m_0 \left( \frac{\xi}{2} \right) \right|^2 f \left( \frac{\xi}{2} \right) + \left| m_0 \left( \frac{\xi}{2} + \pi \right) \right|^2 f \left( \frac{\xi}{2} + \pi \right). \]
Let \( V \) be a finite-dimensional subspace generated by trigonometric polynomials such that \( T_{m_0}(V) \subseteq V \). Let \( T_{m_0/V} \) denote the restriction of \( T_{m_0} \) on \( V \). If \( 1 \) is a simple eigenvalue of \( T_{m_0/V} \) and all the other eigenvalues are inside the unit circle, then \( m_0(\xi) \) generate stable basis of wavelets. Moreover, if \( |\lambda_V| \) denotes the largest eigenvalue (in absolute value) different from \( 1 \) of \( T_{m_0/V} \), then \( \phi \in H^\delta(R), \forall s < -\log |\lambda_V|/2\log 2 \). Note that this technique gives results similar to those of the technique given in [12] and used in the previous proposition. It is important to mention that the method given in [12] is easy to use in the wavelet framework and it has preceded the transition operator based technique. The result of this proposition has been derived in [1, p. 231] by the use of the transition operator technique.

3. UNIFORM APPROXIMATION OF WAVELET EXPANSIONS IN HÖLDER SPACES

In this section, we prove the uniform convergence of a wavelet series expansion and state some properties of the projection operator associated with a wavelet basis. As will be seen, the results of this section depend on the use of a wavelet basis with some Hölder regularity. The results of the previous section can be used to check this last condition. The following theorem gives conditions for the uniform convergence of the wavelet series expansion of a function \( f \).

**Theorem 2.** Let \( f \) be a function belonging to \( (C^\alpha \cap L^2)(R) \), \( \alpha > 0 \), and \( \phi \) and \( \psi \) be a mother scaling function and a mother wavelet. Assume that \( \phi, \psi \in C^\tau \) for some \( 0 < \alpha < \tau \). Also, assume that \( \phi, \psi \) satisfy the following decay condition:
\[ |\phi(x)| \leq c(1 + |x|)^{-1-\epsilon}, \quad |\psi(x)| \leq c(1 + |x|)^{-1-\epsilon}, \quad (3.1) \]
for some constants \( c, \epsilon > 0 \). For \( J > 0 \), define the projection operator \( P_J \) by
\[ P_J(f)(x) = \sum_{k \in \mathbb{Z}} \langle f, \phi_{0k} \rangle \phi_{0k}(x) + \sum_{j=-J+1}^{-1} \sum_{k \in \mathbb{Z}} \langle f, \psi_{jk} \rangle \psi_{jk}(x). \quad (3.2) \]
Then \( P_J(f) \) converges to \( f \) in the \( \| \cdot \|_\infty \) norm. Moreover, \( \| P_J(f) - f \|_\infty \leq c2^{-J\alpha} \) for some constant \( c \) depending only on \( f \).

**Proof.** We first note that if \( f \in L^2(R) \), then the wavelet expansion (3.2) converges to \( f \) in the \( L^2 \)-norm; see [2]. Moreover, \( f \in C^\alpha(R) \), implies that the wavelet coefficients satisfy the inequality
\[
|\langle f, \psi_{jk} \rangle| \leq c'2^{j(\alpha + 1/2)}, \quad \forall j < 0,
\]
for some constant \( c' \); see [16]. Now, define the sequence of functions \( \{g_j(\cdot), j < 0\} \) by
\[
g_j(x) = \sum_{k \in \mathbb{Z}} \langle f, \psi_{jk} \rangle \psi_{jk}(x).
\]
Hence, if \( x \in \text{supp } f \), where \( \text{supp } f \) denotes the support of \( f \), then
\[
\sum_{k \in \mathbb{Z}} |\langle f, \psi_{jk} \rangle| |\psi_{jk}(x)| \leq c'2^{j(\alpha + 1/2)} \sum_{k \in \mathbb{Z}} |\psi_{jk}(x)|
\leq c'2^{j(\alpha + 1/2)} \sum_{k \in \mathbb{Z}} 2^{-j/2} (1 + |2^{-j}x - k|)^{-1-\epsilon}
\leq c_i 2^{j\alpha}.
\]
Consequently, the sequence of functions \( \{g_j, j < 0\} \) is uniformly convergent. Moreover, since for \( k \in \mathbb{Z} \) and \( j < 0 \), \( \psi_{jk} \) is continuous, then \( g_j(x) \) is continuous for all integer \( j < 0 \). Similarly, one shows that
\[
g_0(x) = \sum_{k \in \mathbb{Z}} \langle f, \phi_{0k} \rangle \phi_{0k}
\]
is also a continuous function. Since there exists a constant \( c_2 > 0 \) such that
\[
\sup_{j \leq 0} \sup_x |g_j(x)| \leq c_2 2^{j\alpha},
\]
it follows that \( \sum_{j \leq 0} g_j(x) \) is uniformly convergent, and consequently, the equality
\[
f(x) = \sum_{k \in \mathbb{Z}} \langle f, \phi_{0k} \rangle \phi_{0k}(x) + \sum_{j < 0} \sum_{k \in \mathbb{Z}} \langle f, \psi_{jk} \rangle \psi_{jk}(x)
\]
holds pointwise. Finally, to prove that \( P_J(f) \) converges to \( f \) in the \( \| \cdot \|_\infty \) norm, we remark that for \( x \in \text{supp } f \), we have
\[
|f(x) - P_J(f)(x)| = \left| \sum_{j \leq -J} \sum_{k \in \mathbb{Z}} \langle f, \psi_{jk} \rangle \psi_{jk}(x) \right|
\leq \sum_{j \leq -J} c2^{j(\alpha + 1/2)} \sum_{k \in \mathbb{Z}} 2^{-j/2} (1 + |2^{-j}x - k|)^{-1-\epsilon}
\leq c_1 \sum_{j \leq -J} 2^{-J\alpha} = c'2^{-J\alpha}.
\]
Hence, \( \| P_J(f) - f \|_\infty \leq c'2^{-J\alpha} \), and consequently, \( P_J(f) \) converges uniformly to \( f \).

**Remark 1.** Note that condition (3.1) is satisfied whenever \( \phi \) and \( \psi \) are continuous and compactly supported.

We shall mention that the operator \( P_J \) is stable under perturbation as shown by the following corollary.
COROLLARY 1. Let \( f \) and \( P_J \) be as defined by Theorem 2. Let \( \tilde{f} \) be a function belonging to \((C^\alpha \cap L^2)(R)\) and satisfying \( \|f - \tilde{f}\|_\infty \leq \epsilon \) for some \( \epsilon > 0 \). Then we have
\[
\|f - P_J\left(\tilde{f}\right)\|_\infty \leq c2^{-J\alpha} + \epsilon,
\]
for some constant \( c \) depending only on \( f \) and \( \tilde{f} \).

PROOF. Since the operator \( P_J \) is linear, one concludes that
\[
\|f - P_J\left(\tilde{f}\right)\|_\infty \leq \|f - P_J(f)\|_\infty + \|P_J(f) - P_J\left(\tilde{f}\right)\|_\infty.
\]
Using the multiresolution analysis [16], we can write equality (3.2) in the following form:
\[
P_J(f)(x) = \sum_{k \in Z} \langle f, \phi_{-Jk} \rangle \phi_{-Jk}(x).
\]

Since the different wavelet coefficients \( s_k^J = \langle f, \phi_{-Jk} \rangle \) cannot be computed exactly, an extra error is involved in the approximation of \( f \) by its projection \( P_J \). If \( f \in C^\alpha \), \( \alpha > 0 \) is a compactly supported function with a positive Hölder exponent and if \( s_k^{num} \) denotes the numerical approximations to the wavelet coefficients \( s_k^J \), then the following proposition gives an error bound for the approximation involving the use of the \( s_k^{num} \).

PROPOSITION 2. Let \( f \) be a function belonging to \( C^\alpha(R) \), \( \alpha > 0 \), and let
\[
\epsilon = \sup_k |s_k^{num} - s_k^J|.
\]
Let \( P_J^{num}(f) = \sum_k s_k^{num} \phi_{-Jk}(\cdot) \) be a numerical approximation to \( P_J(f) \). Then
\[
\|f - P_J^{num}(f)\|_\infty \leq c \left( 2^{-J\alpha} + 2^{J/2} \epsilon \right),
\]
for some constant \( c \).

PROOF. We first note that since \( f \) has compact support, therefore, the set \((s_k^J)_k\) is finite. Consequently, \( \epsilon = \sup_k |s_k^{num} - s_k^J| = \max_k |s_k^{num} - s_k^J| \). The proof of the error bound goes as follows:
\[
\|f - P_J^{num}(f)\|_\infty \leq \|f - P_J(f)\|_\infty + \|P_J f - P_J^{num}(f)\|_\infty
\]
\[
\leq \|f - P_J(f)\|_\infty + \left\| \sum_k (s_k^{num} - s_k^J) \phi_{-Jk} \right\|_\infty.
\]
By using the result of Theorem 2, one concludes that there exists a constant \( c_f \) depending on \( f \) such that
\[
\|f - P_J^{num} f\|_\infty \leq c_f 2^{-J\alpha} + \epsilon 2^{J/2} \sup_k \left| \phi(2^J x - k) \right|.
\]
Using hypothesis (3.1), one concludes that \( \sum_k \sup_x \left| \tilde{\phi}(2^J x - k) \right| \) is uniformly bounded. Consequently, there exists a positive constant \( c \) such that
\[
\|f - P_J^{num} f\|_\infty \leq c \left( 2^{-J\alpha} + 2^{J/2} \epsilon \right).
\]

A possible application of the above results is a wavelet-based scheme for the numerical solution of some Fredholm equations. This is the subject of the following section.
4. WAVELETS AND SOLUTIONS OF FREDHOLM EQUATIONS OF THE SECOND KIND

A Fredholm equation of the second kind is an integral equation of the type

\[ f(t) = \lambda \int_a^b K(t, s)f(s) \, ds + g(t), \quad a \leq t \leq b. \]  

(4.1)

Here \( f(t) \) is the unknown function, \( g(t) \) is a known function, and \( K(t, s) \) is the kernel of the equation. For simplicity, we assume that \([a, b]\) is compact and \( \lambda = 1 \). Note that most numerical schemes for solving equation (4.1) use \( m \) points quadrature rule to approximate the integral \( \int_a^b K(t, s)f(s) \, ds \). Thus, equation (4.1) is replaced by the following semidiscrete analogue:

\[ \tilde{f}(t) = \sum_{j=1}^m \tau_j K(t, s_j)\tilde{f}(s_j) + g(t), \quad t \in [a, b], \]  

(4.2)

where the \( \tau_j \) are the weights of the \( m \) points quadrature rule. Evaluating equality (4.2) at the quadrature points, one obtains the following set of \( m \) linear equations in the \( m \) unknown \( \tilde{f}(t_j) \):

\[ \tilde{f}(t_j) = \sum_{j=1}^N \tau_j K(t_j, s_j)\tilde{f}(s_j) + g(t_j), \quad j = 1, \ldots, m. \]  

(4.3)

By solving the above system of equations, one obtains an approximation to the solution at the quadrature points \( t_j \).

In this section, we give a wavelet based method for solving equation (4.1). This scheme has the advantage of handling (4.1) with solutions lying in \( C^\alpha([a, b]) \) with \( 1/2 < \alpha \). This scheme is contained in the following theorem.

**THEOREM 3.** Consider the Fredholm equation of the second kind

\[ f(t) = \int_a^b K(t, s)f(s) \, ds + g(t), \quad -\infty < a \leq t \leq b < +\infty. \]  

(4.4)

Assume that \( K(t, s) \) is continuous on the square \([a, b]^2\) and the solution \( f \) of the above equation lies in \( C^\alpha([a, b]) \) for some \( \alpha > 1/2 \). Let \( \phi \in C^r, r > \alpha, \) be a compactly supported scaling function with support \([-N_1, N_2], N_1, N_2 \in \mathbb{Z}\). Let \( J \) be a positive integer and define the finite set of integers \( S_J \) by

\[ S_J = \{[-N_2 + 2^J a + 1], [-N_2 + 2^J a + 1] + 1, \ldots, [N_1 + 2^J b]\}, \]

where \([x]\) denotes the integer part of \( x \). Define the finite set of real numbers \( \{\beta_k, k \in S_J\} \) to be the solution of the linear system of equations

\[ \sum_{k \in S_J} \beta_k \phi_+ - J_k(t_j) = \int_a^b K(t_j, s) \sum_{k \in S_J} \beta_k \phi_+ - J_k(s) \, ds + g(t_j), \quad j \in S_J. \]  

(4.5)

Here the points \( t_j \) are chosen in such a way that the matrix

\[ A_J = \left[ \phi_+ - J_k(t_j) - \int_a^b K(t_j, s) \phi_+ - J_k(s) \, ds \right]_{j, k \in S_J} \]

is invertible. Under the above assumptions and notation, we have

\[ \sup_{t \in [a, b]} \left| f(t) - \sum_{k \in S_J} \beta_k \phi_+ - J_k(t) \right| \leq c 2^{-J \alpha} \left( 1 + 2^{J/2} \|A_J^{-1}\|_\infty \right). \]
PROOF. Let \( P_J(f)(t) = \sum_{k \in S_J} (f, \phi_{-Jk}) \phi_{-Jk}(t) \) be the projection of \( f \) defined by Theorem 2 and let \( P^\text{num}_J(f)(t) = \sum_{k \in S_J} \beta_{Jk} \phi_{-Jk}(t) \) be defined by the above theorem. By introducing \( P_J(f) \) in equation (4.4), the latter can be written in the following form:

\[
P_J(f)(t) + [f(t) - P_J(f)(t)] = \int_a^b K(t, s) [(f(s) - P_J(f)(s)) + P_J(f)(s)] ds + g(t), \quad t \in [a, b].
\]

Consequently, \( P_J(f) \) satisfies the following equation:

\[
P_J(f)(t) = \int_a^b K(t, s)P_J(f)(s) ds + g(t) + P_J(f)(t) - f(t) + \int_a^b K(t, s)(f(s) - P_J(f)(s)) ds. \tag{4.6}
\]

If we define the function \( \tilde{g} \) by

\[
\tilde{g}(t) = g(t) + P_J(f)(t) - f(t) + \int_a^b K(t, s)(f(s) - P_J(f)(s)) ds,
\]

then (4.6) becomes

\[
P_J(f)(t) = \int_a^b K(t, s)P_J(f)(s) ds + g(t). \tag{4.7}
\]

Moreover, we have

\[
P^\text{num}_J(f)(t_j) = \int_a^b K(t, s)P^\text{num}_J(f)(s) ds + g(t_j), \quad j \in S_J \tag{4.8}
\]

By combining equations (4.7) and (4.8), one obtains the following vector equalities:

\[
\begin{align*}
(\beta_{Jk})_{k \in S_J} &= A_{J}^{-1} \left( [\tilde{g}(t_j)]_{j \in S_J} \right), \\
(\beta^\text{num}_{Jk})_{k \in S_J} &= A_{J}^{-1} \left( [g(t_j)]_{j \in S_J} \right).
\end{align*}
\]

Consequently,

\[
\sup_{k \in S_J} |\beta_{Jk} - \beta^\text{num}_{Jk}| \leq \|A_{J}^{-1}\|_\infty \sup_{j \in S_J} |g(t_j) - \tilde{g}(t_j)|.
\]

By using the hypothesis and the result of Theorem 2, one concludes that

\[
\sup_{j \in S_J} |g(t_j) - \tilde{g}(t_j)| \leq \sup_{t \in [a, b]} |g(t) - \tilde{g}(t)| \\
\leq \sup_{t \in [a, b]} |f(t) - P_J(f)(t)| + \sup_{t \in [a, b]} \int_a^b |K(t, s)| |f(s) - P_J(f)(s)| ds \\
\leq c_J 2^{-J \alpha} + c_K (b - a) 2^{-J \alpha} = c' 2^{-J \alpha}.
\]

Hence,

\[
\sup_{k \in S_J} |\beta_{Jk} - \beta^\text{num}_{Jk}| = \|\beta_{J} - \beta^\text{num}_{J} \|_\infty \leq c' \|A_{J}^{-1}\|_\infty 2^{-J \alpha}. \tag{4.9}
\]

Moreover, for all \( t \in [a, b] \), we have

\[
|P_J(f)(t) - P^\text{num}_J(f)(t)| = \left| \sum_{k \in S_J} (\beta_{Jk} - \beta^\text{num}_{Jk}) \phi_{-Jk}(t) \right| \\
\leq \|\beta_{Jk} - \beta^\text{num}_{Jk}\|_\infty \sum_{k \in S_J} |\phi_{-Jk}(t)| \\
\leq c' 2^{-J \alpha} \|A_{J}^{-1}\|_\infty 2^{1/2} \sum_{k \in S_J} (1 + |2^j t - k|)^{-1-\varepsilon} \\
\leq c 2^{-J (\alpha - 1/2)} \|A_{J}^{-1}\|_\infty.
\]
Hence,
\[
\|f - P_j^{\text{num}}(f)\|_\infty \leq \|f - P_j(f)\|_\infty + \|P_j(f) - P_j^{\text{num}}(f)\|_\infty \\
\quad \leq c_j 2^{-J\alpha} + c_2 2^{-J(\alpha - 1/2)} \|A^{-1}_J\|_\infty = c_2^{-J\alpha} \left(1 + 2^{J/2} \|A^{-1}_J\|_\infty\right).
\]

A serious drawback of the scheme of Theorem 3 is the error bound that contains the quantity \(\|A^{-1}_J\|_\infty\). Until now, we did not have any idea how large this infinity norm could be and, consequently, how large the condition number of \(A_J\) is. We should mention that the magnitude of the condition number of \(A_J\) plays a major role in finding an accurate approximation to the solution of the linear system (4.5) in the unknowns \(\beta_J\): the wavelet coefficients of the solution of our Fredholm equation. The smaller the condition number is, the better is the numerical solution of (4.5). By using an extra condition, the following proposition provides us with a bound of \(\|A^{-1}_J\|_\infty\) as well as a bound for the condition number of \(A_J\).

**Proposition 3.** Assume that there exists \(J' \geq J\) such that
\[
\max_{j \in S_J} \sum_{k \in S_J} \left|2^{(J-J')/2} \phi(2^{J}t_j - k) - \int_a^b K(t_j, s) 2^{(J-J')/2} \phi(2^{J}s - k) \, ds - \delta_{jk}\right| = C_{J'} < 1.
\]
Then
\[
\|A^{-1}_J\|_\infty \leq \frac{2^{-J'/2}}{1 - C_{J'}}.
\]
Moreover, a bound for the condition number of \(A_J\) is given by
\[
\kappa_\infty = \|A_J\|_\infty \|A^{-1}_J\|_\infty \leq c \frac{2^{(J-J')/2}}{1 - C_{J'}},
\]
for some constant \(c\).

**Proof.** We first give a bound for \(\|A^{-1}_J\|_\infty\). To this end, we remark that
\[
\max_{j \in S_J} \sum_{k \in S_J} \left|2^{(J-J')/2} \phi(2^{J}t_j - k) - \int_a^b K(t_j, s) 2^{(J-J')/2} \phi(2^{J}s - k) \, ds - \delta_{jk}\right| = \|2^{-J'/2} A_J - I_{S_J}\|_\infty,
\]
where \(I_{S_J}\) denotes the identity matrix of order \(|S_J|\). It is known that if \(\|B\| < 1\), where \(B\) is a square matrix, then \(\|(I + B)^{-1}\| \leq 1/(1 - \|B\|)\). Hence, if we let
\[
B_{J'} = 2^{-J'/2} A_J - I_{S_J},
\]
then
\[
\|B_{J'}\|_\infty = C_{J'} < 1.
\]
Moreover,
\[
\|(I_{S_J} + B_{J'})\|_\infty = \left\|\left(I_{S_J} + 2^{-J'/2} A_J - I_{S_J}\right)\right\|_\infty \\
\quad = \left\|\left(2^{-J'/2} A_J\right)^{-1}\right\|_\infty = 2^{J'/2} \|A^{-1}_J\|_\infty \\
\quad \leq \frac{1}{1 - \|B_{J'}\|_\infty} = \frac{1}{1 - C_{J'}}.
\]
Consequently,
\[
\|A_j^{-1}\|_\infty \leq \frac{2^{-j/2}}{1 - C_j}.
\]

Now, to bound the condition number of \(A_j\), we need to bound \(\|A_j\|_\infty\). This is done as follows. Since
\[
\|A_j\|_\infty = \max_{j \in S_j} \sum_{k \in S_j} \left| \phi_{-j_k}(t_j) - \int_a^b K(t_j, s) \phi_{-j_k}(s) \, ds \right|
\]
\[
\leq \max_{j \in S_j} \left( \sum_{k \in S_j} 2^{j/2} \left| \phi(2^j t_j - k) \right| + 2^{j/2} \int_a^b |K(t_j, s)| \left( \sum_{k \in S_j} \left| \phi(2^j s - k) \right| \right) \right),
\]
and since \(\text{supp } \phi = [-N_2, N_1]\), it follows that
\[
\sum_{k \in S_j} |\phi_{-j_k}(t_j)| = \sum_{k \in S_j} 2^{j/2} \left| \phi(2^j t_j - k) \right| \leq 2^{j/2}(N_1 + N_2 + 1) \sup_t |\phi(t)| = 2^{j/2}M_1.
\]
Consequently,
\[
\|A_j\|_\infty \leq 2^{j/2}M_1(1 + M_2) = c2^{j/2},
\]
where \(M_2 = \max_{j \in S_j, k} |K(t_j, s)| \, ds\). Finally, by using the previous bound of \(\|A_j^{-1}\|_\infty\), one concludes that
\[
\kappa_\infty(A_j) = \|A_j\|_\infty \|A_j^{-1}\|_\infty \leq c2^{j/2} \frac{2^{-j/2}}{1 - C_j}.
\]

5. WAVELET-BASED QUADRATURE METHODS

Approximations with high accuracy of the different values of \(\phi_{-j_k}(t_j)\) and \(\int_a^b K(t_j, s) \phi_{-j_k}(s) \, ds\) are the subject of this section.

For reasons of simplicity, we assume that the scaling function \(\phi\) to be used in this section is supported on the interval [0, \(N\)]. Trivial modifications are needed to extend the results of this section to other types of compactly supported scaling functions and wavelets. In this section, we provide the reader with a wavelet-based quadrature method for approximating the wavelet coefficients \(\int K(t_j, s) \phi_{-j_k}(s) \, ds\). The application of this method requires the computation of the values of \(\phi\) at some discrete sets of dyadic numbers. The following algorithm can be used to compute \(\phi\) at dyadic points.

**Algorithm.** Computation of \(\phi\) at dyadic points.

**Step 1.** Use the cascade algorithm given in [1] and get a vector \(\Phi^0\) a first approximation to \(\Phi = (\phi(1), \ldots, \phi(N - 1))\).

**Step 2.** Get an approximation of \(\Phi\) to machine precision by applying the iterations: \(\Phi^{n+1} = A \Phi^n\), where \(\Phi^0\) is given by Step 1 and
\[
A = \begin{bmatrix}
\alpha_1 & \alpha_0 & 0 & 0 & \cdots & 0 \\
\alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & \cdots & 0 \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
\alpha_{2k-1} & \cdots & \alpha_{2k-N+1} & \cdots & \cdots & \alpha_{N-1}
\end{bmatrix}.
\]

**Step 3.** Use the refinement equation (1.5) together with the values of \(\phi\) at the integers and get the values of \(\phi\) at the desired dyadic points.
Next, note that by using an appropriate change of variables, the computation of the wavelet coefficient $s_k^j = \langle K(t_j, \cdot), \phi_J \rangle$ can be easily brought to the computation of an inner product of the type

$$S_J = \left\langle \tilde{K}_J, \phi \right\rangle = \int_0^N \tilde{K}_J(s) \phi(s) \, ds. \quad (5.1)$$

Many wavelet-based quadrature methods have been proposed to approximate (5.1) in the case where the function $f$ is smooth enough; see [4,17]. It is known that if $f$ is sufficiently smooth, then by iteratively computing the moments of $\phi$ given by $\mu_k = \int_0^N x^k \phi(x) \, dx$, $k = 0, \ldots, m$, one gets a wavelet-based quadrature method of precision $m$. Hence, if $\forall j \in S_J$, the kernel $K(t_j, s)$ is sufficiently smooth, then the coefficients $\int_0^1 K(t_j, s) \phi_{-Jk}(s) \, ds$ can be computed with high precision. Next, assume that for some $j \in S_J$, the kernel $K(t_j, s)$ has a homogeneous singularity at $s = s_j$ in the sense that $K(t_j, s) = L_j(s)/|s - s_j|^{\gamma}$, $0 < \gamma < 1$ and $L_j$ is sufficiently smooth. Also, assume that the scaling function $\phi \in C^n$ for some $n \geq 1$. Under the above assumptions and by using an $m_1 = 2^p + 1$ points Newton-Cotes formula composed over $m_2 = 2^q$ subintervals of $[0, N]$ with weight $w_j(s) = 1/|s - s_j|^{\gamma}$, one obtains the following wavelet-based singular quadrature method:

$$\int_0^N \tilde{K}_J(s) \phi(s) \, ds = \sum_{k=0}^{N-2^p-1} \sum_{l=0}^{2^p} \sum_{i=0}^{2^q} \omega_{ik}^{l} \phi \left( s_{ik}^{l} \right) \left[ \left| s_{ik}^{l} - s_j \right|^{\gamma} L_j \left( s_{ik}^{l} \right) \right] + E, \quad (5.2)$$

where

$$\omega_{ik}^{l} = \int_{k+(l+1)/2^q}^{k+l/2^q} \frac{1}{|s - s_j|^{\gamma}} \prod_{m=0, m \neq i}^{2^p} \frac{s - s_{mk}^{l}}{(s_{ik}^{l} - s_{mk}^{l})} \, ds$$

and $s_{ik}^{l} = k + il/2^p + q$, $i = 0, 2^p$, $l = 0, 2^q - 1$. Note that in this case, an error bound of the above wavelet-based singular quadrature method is given by

$$|E| \leq c \left( \frac{1}{2^p+q} \right)^n,$$

for some constant $c$.

REFERENCES