ELSEVIER Journal of Computational and Applied Mathematics 65 (1995) 215–231

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

# Results on the associated classical orthogonal polynomials[1]

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Received 4 November 1994; revised 3 March 1995

**Abstract**

Let $\{P_k(x)\}$ be any system of the classical orthogonal polynomials, and let $\{P_k(x;c)\}$ be the corresponding associated polynomials of order $c$ ($c \in \mathbb{N}$). Second-order recurrence relation (in $k$) is given for the *connection coefficient* $a_{n-1,k}^{(c)}$ in

$$P_{n-1}(x;c) = \sum_{k=0}^{n-1} a_{n-1,k}^{(c)} P_k(x).$$

This result is obtained thanks to a new explicit form of the fourth-order differential equation satisfied by $P_{n-1}(\cdot;c)$.

*Keywords:* Classical orthogonal polynomials; Associated polynomials of higher order; Fourth-order differential equation; Connection coefficients; Recurrence relations

*AMS Classification:* Primary 33A65; Secondary 65D30

## 1. Introduction

Let $\{P_k(x)\}$ be any system of the classical orthogonal polynomials (i.e., associated with the names of Jacobi, Hermite, Laguerre and Bessel),

$$\int_I \rho(x) P_k(x) P_l(x) \, dx = \delta_{kl} h_k \quad (k, l = 0, 1, \dots), \tag{1.1}$$

where $h_k \neq 0$ ($k = 0, 1, \dots$); the support $I$ of the weight function $\rho$ is $[-1,1]$, $(-\infty, \infty)$, $[0, \infty)$ and $\{z \in \mathbb{C}: |z| = 1\}$, respectively. Besides the three-term recurrence relation

$$xP_k(x) = \xi_0(k) P_{k-1}(x) + \xi_1(k) P_k(x) + \xi_2(k) P_{k+1}(x)$$
$$(k = 0, 1, \dots; \ P_{-1}(x) \equiv 0, P_0(x) \equiv 1), \tag{1.2}$$

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[1] Supported by Komitet Badań Naukowych (Poland) under grant 2 1029 91 01.


*SSDI* 0377-0427(95)00112-3

these polynomials enjoy a number of similar properties which in turn provide characterizations of them ([6, pp. 150–152] or [1], or [4]). We shall need three of those properties.

First, the weight function $\rho$ satisfies a differential equation of the Pearson type,

$$(\sigma\rho)' = \tau\rho, \tag{1.3}$$

where $\sigma$ is a polynomial of degree at most 2, and $\tau$ is a first-degree polynomial.

Second, for arbitrary $n$, the polynomial $P_n$ obeys the second-order differential equation

$$\mathbb{L}_n P_n(x) \equiv [\sigma\mathbb{D}^2 + \tau\mathbb{D} + \lambda_n\mathbb{I}]P_n(x) = 0, \tag{1.4}$$

where $\mathbb{D} := \mathrm{d}/\mathrm{d}x$, $\mathbb{I}$ is the identity operator, and

$$\lambda_n := -\tfrac{1}{2}n[(n-1)\sigma'' + 2\tau']. \tag{1.5}$$

Third, we have the so-called structure relation

$$\sigma(x)\mathbb{D}P_k(x) = d_0(k)P_{k-1}(x) + d_1(k)P_k(x) + d_2(k)P_{k+1}(x). \tag{1.6}$$

Recently, Yáñez et al. [16] (see also [17]) have shown that the coefficients, $\xi_i(k)$ and $d_i(k)$ of the relations (1.2) and (1.6), respectively, can be expressed in terms of the coefficients $\sigma$ and $\tau$ of Eq. (1.3).

The *associated polynomials* $\{P_k(x;c)\}$ *of order* $c$ ($c \in \mathbb{N}$) are defined recursively by

$$xP_k(x;c) = \xi_0(k+c)P_{k-1}(x;c) + \xi_1(k+c)P_k(x;c) + \xi_2(k+c)P_{k+1}(x;c)$$
$$(k = 0, 1, \ldots;\ P_{-1}(x;c) \equiv 0,\ P_0(x;c) \equiv 1). \tag{1.7}$$

They are known to belong to the Hahn–Laguerre class of orthogonal polynomials (see, e.g., [3]). For arbitrary $n$ and $c$, $P_{n-1}(\cdot\,;c)$ satisfies a fourth-order differential equation

$$\mathbb{M}_n^{(c)} f \equiv \sum_{i=0}^{4} p_i(\cdot\,;c)\mathbb{D}^i f = 0, \tag{1.8}$$

where $p_i(\cdot\,;c)$ are polynomials of degree independent of $n$ and $c$ (see [3, 12]). For the history of searching this equation see [18]. For the Jacobi case (even for the arbitrary real positive $c$) it was obtained in [15] using MACSYMA. For each of the four classical polynomial families it was given in [3, 12]. Recently, Zarzo et al. [18] used MATHEMATICA to obtain the coefficients $p_i(\cdot\,;c)$ in terms of $\sigma$ and $\tau$.

In this paper we show that the fourth-order differential operator $\mathbb{M}_n^{(c)}$ can be written in the form

$$\mathbb{M}_n^{(c)} = \mathbb{M}_n^{(1)} + \vartheta_n^{(c)}\mathbb{Q}_n,$$

where $\mathbb{M}_n^{(1)}$ is the well-known fourth-order operator corresponding to the special case $c = 1$ [11], and where

$$\vartheta_n^{(c)} := (1-c)[(n+c-2)\sigma'' + 2\tau'],$$
$$\mathbb{Q}_n := 2\sigma\mathbb{D}^2 + 3\sigma'\mathbb{D} - (n^2-1)\sigma''\mathbb{I};$$

see Theorem 4.3. This result will help us to construct a second-order recurrence relation of the form

$$\mathscr{L}_2^{(c)} a_{n-1,k}^{(c)} \equiv A_0(k)a_{n-1,k-1}^{(c)} + A_1(k)a_{n-1,k}^{(c)} + A_2(k)a_{n-1,k+1}^{(c)} = 0 \quad (1 \leqslant k \leqslant n-1) \tag{1.9}$$

for the *connection coefficients* $a_{n-1,k}^{(c)}$ in

$$P_{n-1}(x;c) = \sum_{k=0}^{n-1} a_{n-1,k}^{(c)} P_k(x). \tag{1.10}$$

The coefficients of this recurrence relation are expressed in terms of $\sigma$, $\tau$, $\xi_i(k)$ and $d_i(k)$; see Corollary 4.7. In Section 5, we apply the results obtained to each of the four classical families.

## 2. Identities involving the Fourier coefficients

We shall need certain properties of the Fourier coefficients of a function $f$, defined by

$$a_k[f] := \frac{1}{h_k} b_k[f], \tag{2.1}$$

where

$$b_k[f] := \int_I \rho(x) P_k(x) f(x)\, \mathrm{d}x \quad (k = 0, 1, \dots), \tag{2.2}$$

i.e., the coefficients in the formal expansion

$$f \sim \sum_{k=0}^{\infty} a_k[f] P_k.$$

Let $\mathscr{X}$ and $\mathscr{D}$ be the difference operators defined by

$$\mathscr{X} := \xi_0(k)\mathscr{E}^{-1} + \xi_1(k)\mathscr{T} + \xi_2(k)\mathscr{E}, \tag{2.3}$$

$$\mathscr{D} := d_0(k)\mathscr{E}^{-1} + d_1(k)\mathscr{T} + d_2(k)\mathscr{E} \tag{2.4}$$

(cf. (1.2) and (1.6)) where $\mathscr{T}$ is the identity operator, and $\mathscr{E}^m$ the $m$th shift operator: $\mathscr{T} b_k[f] = b_k[f]$, $\mathscr{E}^m b_k[f] = b_{k+m}[f]$ $(m \in \mathbb{Z})$. For the sake of simplicity, we write $\mathscr{E}$ in place of $\mathscr{E}^1$.

Further, let us define the differential operator $\mathbb{U}$ by

$$\mathbb{U} := \sigma(x)\mathbb{D} + \tau(x)\mathbb{I}. \tag{2.5}$$

We prove the following lemma.

**Lemma 2.1.** *The coefficients* (2.2) *obey the identities*:

$$b_k[xf] = \mathscr{X} b_k[f], \tag{2.6}$$

$$\mathscr{D} b_k[\mathbb{D} f] = \lambda_k b_k[f], \tag{2.7}$$

$$b_k[\mathbb{U} f] = -\mathscr{D} b_k[f], \tag{2.8}$$

$$b_k[\mathbb{L}_n f] = (\lambda_n - \lambda_k) b_k[f]. \tag{2.9}$$

**Proof.** In view of (1.2) and (2.3), identity (2.6) is obviously true.

We will prove the identity (2.7). Using (1.6), integrating by parts, and then using the equation $(\sigma\rho P_k')' = -\lambda_k \rho P_k$, (cf. (1.4) and (1.3)), we get

$$\mathscr{D}b_k[\mathbb{D}f] = \mathscr{D}\int_I \rho P_k f' = \int_I \rho\sigma P_k' f'$$
$$= \int_I (\rho\sigma P_k' f)' - \int_I (\rho\sigma P_k')' f = \lambda_k \int_I \rho P_k f = \lambda_k b_k[f].$$

In a similar way, using $(\sigma\rho P_k)' = \rho(\sigma P_k' + \tau P_k)$, we obtain

$$b_k[\sigma f'] = \int_I \rho\sigma P_k f' = \int_I (\rho\sigma P_k f)' - \int_I (\sigma\rho P_k)' f$$
$$= -\int_I \rho(\sigma P_k' + \tau P_k) f = -\int_I \rho \mathscr{D} P_k f - \int_I \rho\tau P_k f = -\mathscr{D}b_k[f] - b_k[\tau f].$$

Hence follows the identity (2.8).

As we can write $\mathbb{L}_n = \mathbb{U}\mathbb{D} + \lambda_n \mathbb{I}$, we have, using (2.8) and (2.7),

$$b_k[\mathbb{L}_n f] = b_k[\mathbb{U}f'] + \lambda_n b_k[f] = -\mathscr{D}b_k[f'] + \lambda_n b_k[f] = (\lambda_n - \lambda_k) b_k[f].$$

This proves the validity of (2.9). □

**Remark 2.2.** Identity (2.6) can be easily generalized to the form

$$b_k[qf] = q(\mathscr{X}) b_k[f], \tag{2.10}$$

where $q$ is any polynomial.

## 3. First associated polynomials

As we have already remarked, the case $c = 1$ plays an exceptional role in these considerations. The operators $\mathbb{M}_n^{(c)}$ and $\mathscr{L}_2^{(c)}$, defining the left-hand sides of the Eqs. (1.8) and (1.9), respectively, are given in terms of the operators $\mathbb{M}_n^{(1)}$ and $\mathscr{L}_2^{(1)}$, associated with this special case.

The *first associated polynomials* (or *numerator polynomials*) $\{P_k(x; 1)\}$ are defined recursively by (1.7) with $c = 1$, i.e.,

$$xP_k(x;1) = \xi_0(k+1)P_{k-1}(x;1) + \xi_1(k+1)P_k(x;1) + \xi_2(k+1)P_{k+1}(x;1)$$
$$(k = 0, 1, \ldots;\ P_{-1}(x;1) \equiv 0,\ P_0(x;1) \equiv 1). \tag{3.1}$$

### 3.1. Differential equation

We have the following.

**Theorem 3.1** (Ronveaux [11]). *For arbitrary n, $P_{n-1}(x; 1)$ satisfies the fourth-order differential equation*

$$\mathbb{M}_n^{(1)} P_{n-1}(x;1) = 0, \tag{3.2}$$

*where*

$$\mathbb{M}_n^{(1)} := \mathbb{N}_n \mathbb{L}_n^*, \tag{3.3}$$

*and*

$$\mathbb{L}_n^* := \sigma\mathbb{D}^2 + (2\sigma' - \tau)\mathbb{D} + (\lambda_n + \sigma'' - \tau')\mathbb{I}, \tag{3.4}$$

$$\mathbb{N}_n := \sigma\mathbb{D}^2 + (\sigma' + \tau)\mathbb{D} + (\lambda_n + \tau')\mathbb{I}. \tag{3.5}$$

*The coefficients* $p_i(\cdot\,; 1)$ *in*

$$\mathbb{M}_n^{(1)} = \sum_{i=0}^{4} p_i(\cdot\,; 1)\mathbb{D}^i$$

*are given by*

$$p_4(\cdot\,; 1) = \sigma^2, \tag{3.6}$$

$$p_3(\cdot\,; 1) = 5\sigma\sigma', \tag{3.7}$$

$$p_2(\cdot\,; 1) = (\sigma' + \tau)(3\sigma' - \tau) + 2\sigma(\lambda_n + 3\sigma'' - \tau'), \tag{3.8}$$

$$p_1(\cdot\,; 1) = 3\tau(\sigma'' - \tau') + 3\sigma'(\lambda_n + \sigma''), \tag{3.9}$$

$$p_0(\cdot\,; 1) = \lambda_{n-1}\lambda_{n+1}. \tag{3.10}$$

**Remark 3.2.** Let us note that $P_{n-1}(x; 1)$ satisfies also the following differential equations of order two, three and five, respectively (see, e.g., [11, 13]:

$$\mathbb{L}_n^* P_{n-1}(x; 1) = \omega\mathbb{D}P_n(x) \quad \left(\omega := \frac{(\sigma'' - 2\tau')}{\int_I \rho}\right), \tag{3.11}$$

$$\mathbb{U}\mathbb{L}_n^* P_{n-1}(x; 1) = -\lambda_n\omega P_n(x), \tag{3.12}$$

$$\mathbb{T}_n^{(1)} P_{n-1}(x; 1) = 0 \quad (\mathbb{T}_n^{(1)} := \mathbb{L}_n\mathbb{U}\mathbb{L}_n^*). \tag{3.13}$$

(By the way, it can be checked that $\mathbb{L}_n\mathbb{U} = \mathbb{U}\mathbb{N}_n$, hence $\mathbb{T}_n^{(1)} = \mathbb{U}\mathbb{M}_n^{(1)}$.)

### *3.2. Recurrence relation for the connection coefficients*

Let us construct a recurrence relation for the Fourier coefficients $a_k[P_{n-1}(\cdot\,; 1)]$, i.e., the connection coefficients $\{a_{n-1,k}^{(1)}\}$ in

$$P_{n-1}(x; 1) = \sum_{k=0}^{n-1} a_{n-1,k}^{(1)} P_k(x). \tag{3.14}$$

Let us denote

$$b_{n-1,k}^{(1)} := b_k[P_{n-1}(\cdot\,; 1)] = h_k a_{n-1,k}^{(1)}. \tag{3.15}$$

**Theorem 3.3.** *The coefficients* $b^{(1)}_{n-1,k}$ *satisfy the second-order recurrence relation*

$$\mathscr{M}^{(1)}_2 b^{(1)}_{n-1,k} = 0 \quad (1 \leqslant k \leqslant n-1), \tag{3.16}$$

*where*

$$\mathscr{M}^{(1)}_2 := \mathscr{D}(\mu_k \mathscr{T}) + 2\lambda_k[\sigma'(\mathscr{X}) - \tau(\mathscr{X})], \tag{3.17}$$

*whereas* $\mu_k := \lambda_n - \lambda_k - \sigma'' + \tau'$.

**Proof.** By virtue of Theorem 3.1, we have

$$b_k[\mathbb{M}^{(1)}_n f] = 0 \tag{3.18}$$

with $f = P_{n-1}(\cdot; 1)$. Writing the operator (3.5) in the form $\mathbb{N}_n = \mathbb{D}(\mathbb{U} + \lambda_n \mathbb{I})$, we obtain — in view of (2.7) and (2.8) — the identity

$$\mathscr{D} b_k[\mathbb{M}^{(1)}_n f] = \mathscr{D} b_k[\mathbb{N}_n \mathbb{L}^*_n f] = (\lambda_n - \lambda_k)\mathscr{D} b_k[\mathbb{L}^*_n f]. \tag{3.19}$$

Similarly, writing the operator (3.4) in the form $\mathbb{L}^*_n = \mathbb{L}_n + 2\mathbb{D}([\sigma' - \tau]\mathbb{I}) + (\tau' - \sigma'')\mathbb{I}$, and using (2.9), (2.7), (2.10), we obtain

$$\mathscr{D} b_k[\mathbb{L}^*_n f] = \mathscr{M}^{(1)}_2 b_k[f], \tag{3.20}$$

where $\mathscr{M}^{(1)}_2$ is the difference operator (3.17). This together with (3.18) and (3.19) implies the theorem. □

**Remark 3.4.** It should be remarked that the result given in the above theorem can also be obtained using any of the differential equations (3.11)–(3.13). Indeed, (3.11) implies the identity $\mathscr{D} b_k[\mathbb{L}^*_n f] = \omega \mathscr{D} b_k[\mathbb{D} P_n]$ with $f = P_{n-1}(\cdot; 1)$, which by (3.20) and (2.7) reduces to $\mathscr{M}^{(1)}_2 b_k[f] = \lambda_k \omega b_k[P_n]$. Now, it remains to observe that $b_k[P_n] = h_n \delta_{kn}$.

As for using Eqs. (3.12) and (3.13), it suffices to observe that

$$b_k[\mathbb{U}\mathbb{L}^*_n f] = -\mathscr{D} b_k[\mathbb{L}^*_n f] = -\mathscr{M}^{(1)}_2 b_k[f],$$
$$b_k[\mathbb{T}^{(1)}_n f] = b_k[\mathbb{U}\mathbb{M}^{(1)}_n f] = -\mathscr{D} b_k[\mathbb{M}^{(1)}_n f] = (\lambda_k - \lambda_n)\mathscr{M}^{(1)}_2 b_k[f].$$

**Corollary 3.5.** *The connection coefficients* $a^{(1)}_{n-1,k}$ *obey the second-order recurrence relation*

$$\mathscr{L}^{(1)}_2 a^{(1)}_{n-1,k} \equiv B_0(k) a^{(1)}_{n-1,k-1} + B_1(k) a^{(1)}_{n-1,k} + B_2(k) a^{(1)}_{n-1,k+1} = 0 \quad (1 \leqslant k \leqslant n-1), \tag{3.21}$$

*with the initial conditions*

$$a^{(1)}_{n-1,n-1} = \frac{\xi_2(0)}{\xi_2(n-1)}, \qquad a^{(1)}_{n-1,n} = 0, \tag{3.22}$$

*where*

$$B_i(k) := h_{k+i-1}\{d_i(k)\mu_{k+i-1} + 2\lambda_k \xi_i(k)(\sigma'' - \tau') + 2\delta_{i1}\lambda_k[\sigma'(0) - \tau(0)]\}$$
$$(i = 0, 1, 2). \tag{3.23}$$

**Proof.** Substituting $b_{n-1,k}^{(1)} = h_k a_{n-1,k}^{(1)}$ into (3.16), we obtain (3.21) with $\mathscr{L}_2^{(1)} := \mathscr{M}_2^{(1)}(h_k \mathscr{T})$. Formula (3.23) follows by using explicit form of the difference operators $\mathscr{D}$ and $\mathscr{X}$.

Initial conditions (3.22) are obtained by equating the leading coefficients of the polynomials on both sides of (3.14). These coefficients can be deduced from (3.1) and (1.2). □

## 4. Associated polynomials of arbitrary order

The associated polynomials $\{P_k(x; c)\}$ of order $c$ ($c \in \mathbb{N}$) are defined recursively by the formula (1.7).

Let $\hat{P}_k := \pi_k^{-1} P_k$, where $\pi_k$ denotes the leading coefficient of $P_k$ ($k = 0, 1, \ldots$). The *monic* polynomials $\{\hat{P}_k\}$ obey the recurrence relation

$$\hat{P}_{k+1}(x) = (x - \beta_k)\hat{P}_k(x) + \gamma_k \hat{P}_{k-1}(x)$$
$$(k = 0, 1, \ldots;\ \hat{P}_{-1}(x) \equiv 0,\ \hat{P}_0(x) \equiv 1),$$

where $\beta_k$, $\gamma_k$ can be expressed in terms of $\xi_0(k)$, $\xi_1(k)$, $\xi_2(k)$ (cf. (1.2)) and $\pi_k$. Let the corresponding associated polynomials be denoted by $\{\hat{P}_k(x; c)\}$ ($c \in \mathbb{N}$).

### 4.1. Differential equation

Let the first-degree polynomials $\{C_k\}$ and the constants $\{D_k\}$ be defined recursively by

$$C_0 := \tau - \sigma', \tag{4.1}$$

$$C_{k+1} := -C_k + 2\frac{D_k}{\gamma_k}(x - \beta_k) \quad (k \geq 0), \tag{4.2}$$

$$D_{-1} := 0, \qquad D_0 := \tau' - \frac{\sigma''}{2}, \tag{4.3}$$

$$D_{k+1} := -\sigma + \gamma_k \frac{D_{k-1}}{\gamma_{k-1}} + \frac{D_k}{\gamma_k}(x - \beta_k)^2 - C_k(x - \beta_k) \quad (k \geq 0). \tag{4.4}$$

**Lemma 4.1** (Ronveaux [12] and Belmehdi and Ronveaux [3]). *The following differential relations hold*:

$$\mathbb{L}_n^{(c)} \hat{P}_n(x; c) = \kappa_c \mathbb{D} \hat{P}_{n-1}(x; c+1), \quad \kappa_c := 2\frac{\gamma_c}{\gamma_{c-1}} D_{c-1}, \tag{4.5}$$

$$\mathbb{L}_n^{*(c)} \hat{P}_{n-1}(x; c+1) = \kappa_c^* \mathbb{D} \hat{P}_n(x; c), \quad \kappa_c^* := -2\frac{D_c}{\gamma_c}, \tag{4.6}$$

*where*

$$\mathbb{L}_n^{(c)} := \sigma \mathbb{D}^2 + (C_c + \sigma')\mathbb{D} + (\lambda_n - nc\sigma'')\mathbb{I},$$
$$\mathbb{L}_n^{*(c)} := \sigma \mathbb{D}^2 + (\sigma' - C_c)\mathbb{D} + (\lambda_n^* - (n+1)c\sigma'')\mathbb{I}$$

*with* $\lambda_n^* := \lambda_n + \sigma'' - \tau'$.

**Lemma 4.2** (Ronveaux [12] and Belmehdi and Ronveaux [3]). *For $n \in \mathbb{N}$ and $c = 0, 1, \ldots,$ the following fourth-order differential equations hold:*

$$\mathbb{M}_n^{*(c)} P_n(x;c) = 0, \tag{4.7}$$

$$\mathbb{M}_n^{(c+1)} P_{n-1}(x;c+1) = 0, \tag{4.8}$$

*where*

$$\mathbb{M}_n^{*(c)} := \mathbb{N}_n^{*(c)} \mathbb{L}_n^{(c)} - \kappa_c \kappa_c^* \mathbb{D}^2,$$

$$\mathbb{M}_n^{(c+1)} := \mathbb{N}_n^{(c)} \mathbb{L}_n^{*(c)} - \kappa_c \kappa_c^* \mathbb{D}^2.$$

*Here $\mathbb{N}_n^{(c)}$ and $\mathbb{N}_n^{*(c)}$ are the following differential operators:*

$$\mathbb{N}_n^{(c)} := \sigma \mathbb{D}^2 + (C_c + 2\sigma')\mathbb{D} + (\lambda_n + (1 - nc)\sigma'' + C_c')\mathbb{I},$$

$$\mathbb{N}_n^{*(c)} := \sigma \mathbb{D}^2 + (2\sigma' - C_c)\mathbb{D} + (\lambda_n^* + (1 - (n+1)c)\sigma'' - C_c')\mathbb{I}.$$

**Proof.** Obviously, it suffices to show that the Eqs. (4.7) and (4.8) hold in the monic polynomials case.

It can be checked that

$$\mathbb{N}_n^{(c)} \mathbb{D} = \mathbb{D} \mathbb{L}_n^{(c)}, \quad \mathbb{N}_n^{*(c)} \mathbb{D} = \mathbb{D} \mathbb{L}_n^{*(c)} \quad (c = 0, 1, \ldots). \tag{4.9}$$

Using Lemma 4.1 and (4.9), we obtain

$$\mathbb{N}_n^{*(c)} \mathbb{L}_n^{(c)} \hat{P}_n(x;c) = \kappa_c \mathbb{N}_n^{*(c)} \mathbb{D} \hat{P}_{n-1}(x;c+1) = \kappa_c \mathbb{D} \mathbb{L}_n^{*(c)} \hat{P}_{n-1}(x;c+1) = \kappa_c \kappa_c^* \mathbb{D}^2 \hat{P}_n(x;c);$$

hence, Eq. (4.7). Similarly, we have

$$\mathbb{N}_n^{(c)} \mathbb{L}_n^{*(c)} \hat{P}_{n-1}(x;c+1) = \kappa_c^* \mathbb{N}_n^{(c)} \mathbb{D} \hat{P}_n(x;c) = \kappa_c^* \mathbb{D} \mathbb{L}_n^{(c)} \hat{P}_n(x;c) = \kappa_c \kappa_c^* \mathbb{D}^2 \hat{P}_{n-1}(x;c+1),$$

and Eq. (4.8) follows. □

We prove the following.

**Theorem 4.3.** *The differential operator $\mathbb{M}_n^{(c)}$ can be written in the form*

$$\mathbb{M}_n^{(c)} = \mathbb{M}_n^{(1)} + \vartheta_n^{(c)} \mathbb{Q}_n, \tag{4.10}$$

*where*

$$\vartheta_n^{(c)} := (1 - c)[(n + c - 2)\sigma'' + 2\tau'], \tag{4.11}$$

$$\mathbb{Q}_n := 2\sigma \mathbb{D}^2 + 3\sigma' \mathbb{D} - (n^2 - 1)\sigma'' \mathbb{I}. \tag{4.12}$$

**Proof.** Notice that Eqs. (4.7) and (4.8) are equivalent, so that $\mathbb{M}_n^{(c)} = \mathbb{M}_{n-1}^{*(c)}$. Thus, we must have

$$\mathbb{M}_n^{(c+1)} - \mathbb{M}_n^{(c)} = \mathbb{N}_n^{(c)} \mathbb{L}_n^{*(c)} - \mathbb{N}_{n-1}^{*(c)} \mathbb{L}_{n-1}^{(c)}. \tag{4.13}$$

A simple algebra shows that the right-hand side of the above equation can be written as

$$-(n\sigma'' + 2C_c')[2\sigma \mathbb{D}^2 + 3\sigma' \mathbb{D} - (n^2 - 1)\sigma'' \mathbb{I}].$$

It can be deduced from (4.1)–(4.4) that $C'_c = (c-1)\sigma'' + \tau'$ (cf. [12]). Thus, using the notation of (4.11) and (4.12), we obtain

$$\mathbb{M}_n^{(c+1)} - \mathbb{M}_n^{(c)} = \varphi_n^{(c)}\mathbb{Q}_n,$$

where $\varphi_n^{(c)} := -[(n+2c-2)\sigma'' + 2\tau']$, and

$$\mathbb{M}_n^{(c)} - \mathbb{M}_n^{(1)} = \left(\sum_{i=1}^{c-1} \varphi_n^{(i)}\right)\mathbb{Q}_n = \vartheta_n^{(c)}\mathbb{Q}_n. \quad \square$$

**Remark 4.4.** In [18], a program written in MATHEMATICA was used to obtain the coefficients of Eq. (4.7). However, the form (4.10) of the operator $\mathbb{M}_n^{(c)}$ remained unnoticed.

**Remark 4.5.** An alternative proof of the Theorem 4.3 is given in the Appendix.

*4.2. Recurrence relation for the connection coefficients*

We are going to construct a recurrence relation for the Fourier coefficients $a_k[P_{n-1}(\cdot;c)]$, i.e., the connection coefficients $\{a_{n-1,k}^{(c)}\}$ in

$$P_{n-1}(x;c) = \sum_{k=0}^{n-1} a_{n-1,k}^{(c)} P_k(x). \tag{4.14}$$

Let us denote

$$b_{n-1,k}^{(c)} := b_k[P_{n-1}(\cdot;c)] = h_k a_{n-1,k}^{(c)}. \tag{4.15}$$

**Theorem 4.6.** *The coefficients $b_{n-1,k}^{(c)}$ obey the second-order recurrence relation*

$$\mathscr{M}_2^{(c)} b_{n-1,k}^{(c)} = 0 \quad (1 \leqslant k \leqslant n-1), \tag{4.16}$$

*where*

$$\mathscr{M}_2^{(c)} := (\lambda_n - \lambda_k)\mathscr{M}_2^{(1)} + \vartheta_n^{(c)}\mathscr{N}_2, \tag{4.17}$$

*where in turn*

$$\mathscr{M}_2^{(c)} := \mathscr{D}(\mu_k\mathscr{T}) + 2\lambda_k(\sigma' - \tau)(\mathscr{X}),$$

$$\mathscr{N}_2 := \mathscr{D}(\nu_k\mathscr{T}) + \lambda_k(3\sigma' - 2\tau)(\mathscr{X}),$$

*and* $\mu_k := \lambda_n - \lambda_k - \sigma'' + \tau'$, $\nu_k := 2\tau' - 2\lambda_k - (n^2+2)\sigma''$. *Here $\vartheta_n^{(c)}$ is the constant given by* (4.11).

**Proof.** We will show that

$$\mathscr{D} b_k[\mathbb{M}_n^{(c)} f] = \mathscr{M}_2^{(c)} b_k[f]. \tag{4.18}$$

Recall the identity

$$\mathscr{D} b_k[\mathbb{M}_n^{(1)} f] = (\lambda_n - \lambda_k)\mathscr{M}_2^{(1)} b_k[f] \tag{4.19}$$

obtained in the proof of Theorem 3.3.

Writing the operator $\mathbb{Q}_n$ (cf. (4.12)) in the form

$$\mathbb{Q}_n = 2\mathbb{L}_n + \mathbb{D}([3\sigma' - 2\tau]\mathbb{I}) + \nu_k\mathbb{I},$$

we observe that

$$\mathscr{D}b_k[\mathbb{Q}_n f] = \mathscr{N}_2 b_k[f].$$

This together with (4.19), in view of the obvious equation

$$\mathscr{D}b_k[\mathbb{M}_n^{(c)}f] = \mathscr{D}b_k[\mathbb{M}_n^{(1)}f] + \vartheta_n^{(c)}\mathscr{D}b_k[\mathbb{Q}_n f],$$

implies (4.18). By virtue of Theorem 4.3, $f = P_{n-1}(\cdot\,; c)$ satisfies the identity $\mathscr{D}b_k[\mathbb{M}_n^{(c)}f] = 0$ which by (4.18) can be written as $\mathscr{M}_2^{(c)}b_k[f] = 0$. □

Proceeding as in the case of Corollary 3.5, we prove the following.

**Corollary 4.7.** *The connection coefficients $a_{n-1,k}^{(c)}$ satisfy the second-order recurrence relation*

$$\mathscr{L}_2^{(c)}a_{n-1,k}^{(c)} \equiv A_0(k)a_{n-1,k-1}^{(c)} + A_1(k)a_{n-1,k}^{(c)} + A_2(k)a_{n-1,k+1}^{(c)} = 0 \quad (1 \leqslant k \leqslant n-1),$$

*with the initial conditions*

$$a_{n-1,n-1}^{(c)} = \prod_{j=0}^{n-2} \frac{\xi_2(j)}{\xi_2(j+c)}, \qquad a_{n-1,n}^{(c)} = 0,$$

*where*

$$A_i(k) := (\lambda_n - \lambda_k)B_i(k) + \vartheta_n^{(c)}C_i(k) \quad (i = 0, 1, 2),$$

*where in turn*

$$B_i(k) := h_{k+i-1}\{d_i(k)\mu_{k+i-1} + 2\lambda_k\xi_i(k)(\sigma'' - \tau') + 2\delta_{i1}\lambda_k[\sigma'(0) - \tau(0)]\},$$
$$C_i(k) := h_{k+i-1}\{d_i(k)\nu_{k+i-1} + \lambda_k\xi_i(k)(3\sigma'' - 2\tau') + \delta_{i1}\lambda_k[3\sigma'(0) - 2\tau(0)]\}.$$

## 5. Applications

Now, we can consider separately the cases of the Jacobi, Hermite, Laguerre and Bessel polynomials. Substituting in Corollary 4.7 the pertinent expressions for $\sigma$, $\tau$, $\mathscr{X}$, $\mathscr{D}$ and $h_k$ (see Tables 1 and 2), we obtain the results given below. The computer algebra system MAPLE [5] has been quite useful for detailed computations.

We employ the notation of [6] for the classical orthogonal polynomials. The Pochhammer symbol $(a)_m$ has the following meaning:

$$(a)_0 := 1, \qquad (a)_m := a(a+1)\ldots(a+m-1) \quad (m = 1, 2, \ldots).$$

### 5.1. Bessel polynomials

The coefficients $a_{n-1,k}^{(c)}$ in

$$Y_{n-1}^{\alpha}(x; c) = \sum_{k=0}^{n-1} a_{n-1,k}^{(c)} Y_k^{\alpha}(x)$$

Table 1
Data for the Hermite and Bessel polynomials

| | Hermite | Bessel |
|---|---|---|
| $\sigma$ | $1$ | $x^2$ |
| $\tau$ | $-2x$ | $(\alpha+2)x+2$ |
| $\lambda_k$ | $2k$ | $-k(k+\alpha+1)$ |
| $\mathscr{X}$ | $k\mathscr{E}^{-1}+\frac{1}{2}\mathscr{E}$ | $-2\omega_k^{-1}\{k(2k+\alpha+2)\mathscr{E}^{-1}$ $+\alpha(2k+\alpha+1)\mathscr{T}-(k+\alpha+1)(2k+\alpha)\mathscr{E}\}$ |
| $\mathscr{D}$ | $2k\mathscr{E}^{-1}$ | $-2\lambda_k\omega_k^{-1}\{(2k+\alpha+2)\mathscr{E}^{-1}$ $-2(2k+\alpha+1)\mathscr{T}+(2k+\alpha)\mathscr{E}\}$ |
| $h_k$ | $\sqrt{\pi}2^k k!$ | $(-1)^{k+1}\dfrac{k!}{(2k+\alpha+1)(\alpha+1)_k}$ |

Note: $\omega_k := (2k+\alpha)_3$.

Table 2
Data for the Jacobi and Laguerre polynomials

| | Jacobi | Laguerre |
|---|---|---|
| $\sigma$ | $x^2-1$ | $x$ |
| $\tau$ | $(\gamma+1)x+\delta$ | $1+\alpha-x$ |
| $\lambda_k$ | $-k(k+\gamma)$ | $k$ |
| $\mathscr{X}$ | $\omega_k^{-1}\{2(k+\alpha)(k+\beta)(2k+\gamma+1)\mathscr{E}^{-1}$ $-\delta(\gamma-1)(2k+\gamma)\mathscr{T}$ $+2(k+1)(k+\gamma)(2k+\gamma-1)\mathscr{E}\}$ | $-(k+\alpha)\mathscr{E}^{-1}+(2k+\alpha+1)\mathscr{T}$ $-(k+1)\mathscr{E}$ |
| $\mathscr{D}$ | $-2(k+\gamma)\omega_k^{-1}\{(k+\alpha)(k+\beta)$ $\times(2k+\gamma+1)\mathscr{E}^{-1}+\delta k(2k+\gamma)\mathscr{T}$ $-(k)_2(2k+\gamma-1)\mathscr{E}\}$ | $-(k+\alpha)\mathscr{E}^{-1}+k\mathscr{T}$ |
| $h_k$ | $2^\gamma\dfrac{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{(2k+\gamma)k!\Gamma(k+\gamma)}$ | $\dfrac{\Gamma(1+\alpha+k)}{k!}$ |

Note: $\gamma := \alpha+\beta+1$, $\delta := \alpha-\beta$, $\omega_k := (2k+\gamma-1)_3$.

obey the recurrence relation

$$A_0(k)a^{(c)}_{n-1,k-1} + A_1(k)a^{(c)}_{n-1,k} + A_2(k)a^{(c)}_{n-1,k+1} = 0,$$

where

$$\begin{aligned}
A_0(k) &= (k+\alpha)_2(2k+\alpha+2)_2[k^2-(n+2c+\alpha-1)^2](k^2-n^2),\\
A_1(k) &= -k(k+\alpha+1)(2k+\alpha-1)(2k+\alpha+3)\\
&\quad\times\{(k+\alpha+1+n)(k-n)[n(n+\alpha+1)+\alpha+3k(k+\alpha+1)]\\
&\qquad-2(c-1)(n+c+\alpha)[2k(k+\alpha+1)+2n^2+\alpha]\},\\
A_2(k) &= (k)_2(2k+\alpha-1)_2[(k+\alpha+1)^2-n^2][(k+\alpha+1)^2-(n+2c+\alpha-1)^2],
\end{aligned}$$

with the initial conditions

$$a^{(c)}_{n-1,n-1} = \frac{(\alpha+1)_{n-1}(c+\frac{1}{2}\alpha+\frac{1}{2})_{n-1}(c+\frac{1}{2}\alpha+1)_{n-1}}{(\frac{1}{2}\alpha+\frac{1}{2})_{n-1}(\frac{1}{2}\alpha+1)_{n-1}(c+\alpha+1)_{n-1}}, \qquad a^{(c)}_{n-1,n} = 0.$$

The above result seems to be new. It should be compared with the one given in [14], where the special case $c = 1$ is treated; a *fourth-order* recurrence relation for $a^{(1)}_{n-1,k}$ is obtained using MATHEMATICA, under the assumption that both $Y^\alpha_k$ and $Y^\alpha_{n-1}(\cdot;1)$ are monic.

### 5.2. *Laguerre polynomials*

The coefficients $a^{(c)}_{n-1,k}$ in

$$L^\alpha_{n-1}(x;c) = \sum_{k=0}^{n-1} a^{(c)}_{n-1,k} L^\alpha_k(x)$$

obey the recurrence relation

$$(k^2-n^2)a^{(c)}_{n-1,k-1} - [(k-n)(n+3k+1)+2(1-c)(2k+1)]a^{(c)}_{n-1,k} + 2(k-n-2c+2)(k+\alpha+1)a^{(c)}_{n-1,k+1} = 0, \qquad (5.1)$$

with the initial conditions

$$a^{(c)}_{n-1,n-1} = \frac{(n-1)!}{(c+1)_{n-1}}, \qquad a^{(c)}_{n-1,n} = 0. \qquad (5.2)$$

This results agrees with the explicit formula for $a^{(1)}_{n-1,k}$ given in [10]. In [14], a *third-order* recurrence relation for $a^{(1)}_{n-1,k}$ is obtained using MATHEMATICA, under the assumption that both $L^\alpha_k$ and $L^\alpha_{n-1}(\cdot;1)$ are monic.

Notice that in case $\alpha = \frac{1}{2}$, Eq (5.1) implies the existence of a *first-order* recurrence for $a^{(c)}_{n-1,k}$. Indeed, (5.1) can be written in this case as

$$(\mathscr{E}^{-1} - \mathscr{T})\mathscr{Q}a^{(c)}_{n-1,k} = 0,$$

where

$$\mathscr{Q} := [(k+1)^2 - n^2]\mathscr{T} - (2k+3)(k-n-2c+2)\mathscr{E}.$$

It can be seen that the equation $\mathscr{Q}a^{(c)}_{n-1,k} = 0$ holds. Using the first condition of (5.2), we obtain the formula

$$L^{1/2}_{n-1}(x;c) = \frac{(n-1)!}{(c+1)_{n-1}} \sum_{k=0}^{n-1} \frac{(\frac{1}{2}-n)_k(2c)_k}{k!(1-2n)_k} L^{1/2}_{n-1-k}(x).$$

### 5.3. *Hermite polynomials*

The coefficients $a^{(c)}_{n-1,k}$ in

$$H_{n-1}(x;c) = \sum_{k=0}^{n-1} a^{(c)}_{n-1,k} H_k(x)$$

obey the recurrence relation

$$(k^2 - n^2)a^{(c)}_{n-1,k-1} + 4(k)_2(k - n - 2c - 2)a^{(c)}_{n-1,k+1} = 0,$$

with initial conditions

$$a^{(c)}_{n-1,n-1} = 1, \qquad a^{(c)}_{n-1,n} = 0.$$

This together with the symmetry property $H_k(-x; c) = (-1)^k H_k(x; c)$ $(k \geqslant 0;\ c \geqslant 0)$ yields $a^{(c)}_{n-1,n-2k} = 0$ $(k = 1, 2, \ldots, \lfloor n/2 \rfloor)$, and

$$a^{(c)}_{n-1,n-2k-1} = (-2)^k \frac{(c)_k (n-k-1)!}{k!(n-2k-1)!} \quad (k = 0, 1, \ldots, \lfloor \tfrac{1}{2}(n-1) \rfloor).$$

This result was given in [2].

### 5.4. *Jacobi polynomials*

The coefficients $a^{(c)}_{n-1,k}$ in

$$P^{(\alpha,\beta)}_{n-1}(x; c) = \sum_{k=0}^{n-1} a^{(c)}_{n-1,k} P^{(\alpha,\beta)}_k(x)$$

obey the recurrence relation

$$A_0(k)a^{(c)}_{n-1,k-1} + A_1(k)a^{(c)}_{n-1,k} + A_2(k)a^{(c)}_{n-1,k+1} = 0, \tag{5.3}$$

where

$$\begin{aligned}
A_0(k) &= (k + \gamma - 1)_2(2k + \gamma + 1)_2(k^2 - n^2)[k^2 - (n + \gamma + 2c - 2)],\\
A_1(k) &= (\beta - \alpha)(k + \gamma)[(2k + \gamma)^2 - 4]\\
&\quad \times \{(k - n)(k + n + \gamma)[(n + 1)(n + \gamma - 1) + 3k(k + \gamma)]\\
&\qquad - 2(c - 1)(n + \gamma - 1)[2k(k + \gamma) + 2n^2 + \gamma - 1]\},\\
A_2(k) &= -(k + \alpha + 1)(k + \beta + 1)(2k + \gamma - 2)_2\\
&\quad \times [(k + \gamma)^2 - n^2][(k + \gamma)^2 - (n + \gamma + 2c - 2)^2],
\end{aligned}$$

with initial conditions

$$a^{(c)}_{n-1,n-1} = \frac{(n-1)!(\gamma)_{n-1}(c + \frac{1}{2}\gamma)_{n-1}(c + \frac{1}{2}\gamma + \frac{1}{2})_{n-1}}{(\frac{1}{2}\gamma)_{n-1}(\frac{1}{2}\gamma + \frac{1}{2})_{n-1}(c + 1)_{n-1}(c + \gamma)_{n-1}}, \qquad a^{(c)}_{n-1,n} = 0,$$

where $\gamma := \alpha + \beta + 1$. This result is in agreement with the one obtained by the author in [8] for $c = 1$, and in [9] — for arbitrary positive $c \in \mathbb{R}$. For $\alpha = \beta$ the middle term in (5.3) vanishes, hence this equation is in fact of the first order, which yields explicit formulae for $a^{(c)}_{n-1,2k}$ and $a^{(c)}_{n-1,2k-1}$. Defining the *associated Gegenbauer polynomials* by

$$C^{\nu}_k(x; c) := \frac{(2\nu + c)_k}{(c + \nu + \frac{1}{2})_k} P^{(\nu-1/2,\nu-1/2)}_k(x; c) \quad (k \geqslant 0;\ \nu > -\tfrac{1}{2};\ c \geqslant 0),$$

we obtain the formula, already given in [9]:

$$C^{\nu}_{n-1}(x;c) = \frac{(n-1)!(\nu+c)_{n-1}}{(\nu)_n(c+1)_{n-1}} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} g^{(c)}_{n-1,k} C^{\nu}_{n-2k-1}(x),$$

where

$$g^{(c)}_{n-1,k} := \frac{(n+\nu-2k-1)(1-\nu)_k(1-n-\nu)_k(c)_k(2-n-2\nu-2c)_k}{k!(1-n)_k(\nu+c)_k(2-n-\nu-c)_k}$$

$$(k = 0, 1, \ldots, \lfloor \tfrac{1}{2}(n-1) \rfloor).$$

## Appendix. An alternative proof of Theorem 4.3

Let us define $F_k$, $G_k$ and $H_k$ $(k \geqslant 0)$ by

$$F_k := \frac{1}{2}(C'_{n+k} + C'_k) - \sum_{j=k}^{n+k-1} \frac{D_k}{\gamma_k}, \tag{A.1}$$

$$G_k := 4\frac{D_{k-1}D_k}{\gamma_{k-1}} - C_k^2, \tag{A.2}$$

$$H_k := C'_k C_k. \tag{A.3}$$

It has been shown in [3] that the differential equation (4.7) can be written in the form

$$\sum_{i=0}^{4} p_i(x;c)\mathbb{D}^i P_{n-1}(x;c) = 0,$$

where

$$p_4(\cdot\,;c) = \sigma^2, \tag{A.4}$$

$$p_3(\cdot\,;c) = 5\sigma\sigma', \tag{A.5}$$

$$p_2(\cdot\,;c) = 2\sigma(F_c + 2\sigma'') + 4\sigma^2 + G_c, \tag{A.6}$$

$$p_1(\cdot\, c) = 3\sigma'(F_c + \sigma'') - 3H_c, \tag{A.7}$$

$$p_0(\cdot\,;c) = F_c(F_c + \sigma'' - 2C'_c). \tag{A.8}$$

We will show that these formulae can be simplified to the form given in the theorem. We start from an observation made in [12]. By differentiating formulae (4.2) (once) and (4.4) (twice), we obtain

$$C'_{k+1} + C'_k = 2\frac{D_k}{\gamma_k}, \tag{A.9}$$

$$C_k = \left(2\frac{D_k}{\gamma_k} - C'_k\right)(x - \beta_k) - \sigma', \tag{A.10}$$

$$2C'_k = 2\frac{D_k}{\gamma_k} - \sigma'', \tag{A.11}$$

hence — in view of (4.1) — the formulae

$$C'_k = (k-1)\sigma'' + \tau', \tag{A.12}$$

$$\frac{D_k}{\gamma_k} = \frac{1}{2}(2k-1)\sigma'' + \tau' \tag{A.13}$$

and

$$C_k = (k\sigma'' + \tau')(x - \beta_k) - \sigma'. \tag{A.14}$$

Now, we want to express the binom $x - \beta_k$ in terms of $\sigma$, $\tau$ and their derivatives. Polynomial $\hat{P}_1(x) = x - \beta_0$ satisfies (1.4) with $n = 1$, hence $x - \beta_0 = -\tau/\lambda_1 = \tau/\tau'$. Equating the expressions for $C_{k+1}$, implied by (4.2), (A.10) and (A.14), we obtain

$$[(k-1)\sigma'' + \tau'](x - \beta_k) + \sigma' = [(k+1)\sigma'' + \tau'](x - \beta_{k+1}) - \sigma'. \tag{A.15}$$

Thus, we have

$$x - \beta_k = \frac{\tau(\tau' - \sigma'') + \sigma' k[(k-1)\sigma'' + 2\tau']}{[(k-1)\sigma'' + \tau'][k\sigma'' + \tau']} \quad (k \geqslant 0). \tag{A.16}$$

Substituting this into (A.14) and using the notation (1.5), we get

$$C_k = \frac{\tau(\tau' - \sigma'') - 2\sigma'\lambda_k}{(k-1)\sigma'' + \tau'} - \sigma'. \tag{A.17}$$

It is easy now to obtain the formulae

$$F_k = -\tfrac{1}{2}(n-1)[(n+2k-2)\sigma'' + 2\tau'], \tag{A.18}$$

$$H_k = \tau(\tau' - \sigma'') - \sigma'[(k-1)\sigma'' + \tau' + 2\lambda_k]. \tag{A.19}$$

The only lacking expression for $G_k$ may be obtained using the following equation implied by (4.2) and (4.4):

$$G_{k+1} - G_k = -4\sigma\frac{D_k}{\gamma_k};$$

see [3, Eq. (9)]. As $G_0 = -C_0^2 = -(\tau - \sigma')^2$, we have

$$G_k = -(\tau - \sigma')^2 - 2\sigma k[(k-2)\sigma'' + 2\tau'] \quad (k \geqslant 0). \tag{A.20}$$

Eqs (A.4)–(A.8), (A.18)–(A.20) and Lemma 3.1 imply

$$p_4(\cdot\,; c) = p_4(\cdot\,; 1),$$

$$p_3(\cdot\,; c) = p_3(\cdot\,; 1),$$

$$p_2(\cdot\,; c) = p_2(\cdot\,; 1) + 2\sigma\vartheta_n^{(c)},$$

$$p_1(\cdot\,; c) = p_1(\cdot\,; 1) + 3\sigma'\vartheta_n^{(c)},$$

$$p_0(\cdot\,; c) = p_0(\cdot\,; 1) - (n^2 - 1)\sigma''\vartheta_n^{(c)},$$

$\vartheta_n^{(c)}$ being given by (4.11). This — in view of the form (4.12) of the operator $\mathbb{Q}_n$ — completes the proof. □

**Remark A.1.** The formula

$$\gamma_{k-1} = \left\{ \frac{D_k D_{k-1}}{\gamma_k \gamma_{k-1}} \right\}^{-1} (G_k + C_k^2)$$

can be obtained using (A.2). Making use of (A.20), (A.17) and (A.13), we arrive at the formula

$$\gamma_k = \frac{k(k+1)}{\lambda_{k-1}\lambda_{k+1}} \left\{ \left[ (k+1)\frac{2\sigma'\lambda_{k+1} - \tau(\tau' - \sigma'')}{(k+1)\tau + 2\lambda_{k+1}} - \sigma' \right]^2 - (\tau - \sigma')^2 - 4\frac{k+1}{k}\lambda_k \right\},$$

which can be compared to the alternative formulae due to Magnus (see, e.g., [18]) and Yáñez et al. [16].

## Acknowledgements

The author wishes to thank the referees for their valuable comments.

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