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Purely infinite simple Leavitt path algebras

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Abstract

We give necessary and sufficient conditions on a row-finite graph E so that the Leavitt path algebra $L(E)$ is purely infinite simple. This result provides the algebraic analog to the corresponding result for the Cuntz–Krieger C^*-algebra $C^*(E)$ given in [T. Bates, D. Pask, I. Raeburn, W. Szymański, The C^*-algebras of row-finite graphs, New York J. Math. 6 (2000) 307–324].

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An idempotent e in a ring R is called *infinite* if eR is isomorphic as a right R-module to a proper direct summand of itself. R is called *purely infinite* in case every nonzero right ideal of R contains an infinite idempotent. Much recent attention has been paid to the structure of purely infinite simple rings, from both an algebraic (see e.g. [3–5]) as well as an analytic (see e.g. [7,8,11]) point of view. The Leavitt path algebra $L(E)$ of a graph E is investigated in [1]. $L(E)$ is the algebraic counterpart of the Cuntz–Krieger algebra $C^*(E)$; furthermore, the class of algebras of the form $L(E)$ significantly broadens the collection of algebras studied by Leavitt in his seminal papers [9] and [10]. In [1] the authors give necessary and sufficient conditions on E so that $L(E)$ is simple. In the current article we

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provide necessary and sufficient conditions on E so that $L(E)$ is purely infinite simple (Theorem 11).

We recall the definition of the Leavitt path algebra $L(E)$.

Definitions 1. A *(directed) graph* $E = (E^0, E^1, r, s)$ consists of two countable sets E^0, E^1 and functions $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*. Let K be a field. The *path K-algebra over E* is the free associative K-algebra $K[E^0 \cup E^1]$ with relations given by: $v_i v_j = \delta_{ij} v_i$ for every $v_i, v_j \in E^0$, and $e_i = e_i r(e_i) = s(e_i) e_i$ for every $e_i \in E^1$. The *extended graph of E* is the graph $\widehat{E} = (E^0, E^1 \cup (E^1)^*, r', s')$, where $(E^1)^* = \{e_i^* : e_i \in E^1\}$ and the functions r' and s' are defined as: $r'|_{E^1} = r$, $s'|_{E^1} = s$, $r'(e_i^*) = s(e_i)$, and $s'(e_i^*) = r(e_i)$. We call the elements of E^1 (resp., $(E^1)^*$) the *real edges* (resp., the *ghost edges*) of E.

Now suppose that E is *row-finite* (i.e., that $s^{-1}(v)$ is finite for all $v \in E^0$). The *Leavitt path algebra of E with coefficients in K*, denoted by $L_K(E)$ (or $L(E)$ when appropriate), is defined as the path K-algebra over the extended graph $\widehat{E}$, satisfying the so-called *Cuntz–Krieger relations*:

(CK1) $e_i^* e_j = \delta_{ij} r(e_j)$ for every $e_j \in E^1$ and $e_i^* \in (E^1)^*$, and

(CK2) $v_i = \sum_{\{e_j \in E^1 : s(e_j) = v_i\}} e_j e_j^*$ for every $v_i \in E^0$ for which $s^{-1}(v_i) \neq \emptyset$.

Example 2. (i) Let E be the "finite line" graph defined by $E^0 = \{v_1, \ldots, v_n\}$, $E^1 = \{y_1, \ldots, y_{n-1}\}$, $s(y_i) = v_i$, and $r(y_i) = v_{i+1}$ for $i = 1, \ldots, n-1$. Then $L(E) \cong M_n(K)$, via the map $v_i \mapsto e_{ii}$, $y_i \mapsto e_{ii+1}$, and $y_i^* \mapsto e_{i+1i}$ (where e_{ij} denotes the standard (i, j)-matrix unit in $M_n(K)$).

(ii) Let $n \geq 2$. Let E be the "rose with n leaves" graph defined by $E^0 = \{*\}$, $E^1 = \{y_1, \ldots, y_n\}$. Then $L(E) \cong L(1, n)$, the *Leavitt algebra* investigated in [10]. Specifically, $L(E)$ is isomorphic to the free associative K-algebra with generators $\{x_i, y_i : 1 \leq i \leq n\}$ and relations

$$(1)\ x_i y_j = \delta_{ij} \text{ for all } 1 \leq i, j \leq n, \quad \text{and} \quad (2)\ \sum_{i=1}^{n} y_i x_i = 1.$$

Throughout this article all graphs will be assumed to be row-finite. We briefly establish some graph-theoretic notation. For each edge e, $s(e)$ is the *source* of e and $r(e)$ is the *range* of e. A vertex v for which $s^{-1}(v) = \emptyset$ is called a *sink*. A graph E is *finite* if E^0 is a finite set. A *path* μ in a graph E is a sequence of edges $\mu = \mu_1 \ldots \mu_n$ such that $r(\mu_i) = s(\mu_{i+1})$ for $i = 1, \ldots, n-1$. In such a case, $s(\mu) := s(\mu_1)$ is the source of μ and $r(\mu) := r(\mu_n)$ is the range of μ. For vertices we define $r(v) = v = s(v)$. We define a preorder $\leq$ on E^0 given by: $v \leq w$ in the case $w = v$ or there is a path μ such that $s(\mu) = v$ and $r(\mu) = w$. If $s(\mu) = r(\mu)$ and $s(\mu_i) \neq s(\mu_j)$ for every $i \neq j$, then μ is a called a *cycle*. E is *acyclic* if E contains no cycles. The set of paths of length $n > 0$ is denoted by E^n. The set of all paths (and vertices) is $E^* := \bigcup_{n \geq 0} E^n$. It is shown in [1] that $L(E)$ is a $\mathbb{Z}$-graded K-algebra, spanned as a K-vector space by $\{pq^* \mid p, q \text{ are paths in } E\}$. By [1, Lemma 1.6], $L(E)$ is unital if and only if E is finite; otherwise, $L(E)$ is a ring with set of local units consisting of sums of distinct vertices.

If $\alpha \in L(E)$ and $d \in \mathbb{Z}^+$, then we say that α is *representable as an element of degree d in real (resp. ghost) edges* in case when α can be written as a sum of monomials from the aforementioned spanning set of $L(E)$, in such a way that d is the maximum length of a path p (resp. q) which appears in such monomials. We note that an element of $L(E)$ may be representable as an element of different degrees in real (resp. ghost) edges, depending on the particular representation used for α.

Lemma 3. *Let E be a finite acyclic graph. Then $L(E)$ is finite dimensional.*

Proof. Since the graph is row-finite, the given condition on E is equivalent to the condition that E^* is finite. The result now follows from the previous observation that $L(E)$ is spanned as a K-vector space by $\{pq^* \mid p, q \text{ are paths in } E\}$. $\square$

Lemma 3 is precisely the tool we need to establish the following key result.

Proposition 4. *Let E be a graph. Then E is acyclic if and only if $L(E)$ is a union of a chain of finite dimensional subalgebras.*

Proof. Assume first that E is acyclic. If E is finite, then Lemma 3 gives the result. So now suppose E is infinite, and rename the vertices of E^0 as a sequence $\{v_i\}_{i=1}^{\infty}$. We now define a sequence $\{F_i\}_{i=1}^{\infty}$ of subgraphs of E. Let $F_i = (F_i^0, F_i^1, r, s)$ where $F_i^0 := \{v_1, \dots, v_i\} \cup r(s^{-1}(\{v_1, \dots, v_i\}))$, $F_i^1 := s^{-1}(\{v_1, \dots, v_i\})$, and r, s are induced from E. In particular, $F_i \subseteq F_{i+1}$ for all i. For any $i > 0$, $L(F_i)$ is a subalgebra of $L(E)$ as follows. First note that we can construct $\phi : L(F_i) \to L(E)$ a K-algebra homomorphism because the Cuntz–Krieger relations in $L(F_i)$ are consistent with those in $L(E)$, in the following way. Consider v a sink in F_i (which need not be a sink in E), then we do not have CK2 at v in $L(F_i)$. If v is not a sink in F_i, then there exists $e \in F_i^1 = s^{-1}(\{v_1, \dots, v_i\})$ such that $s(e) = v$. But $s(e) \in \{v_1, \dots, v_i\}$ and therefore $v = v_j$ for some j, and then $F_i^1 = s^{-1}(\{v_1, \dots, v_i\})$ ensures that all the edges starting in v are in F_i, so CK2 at v is the same in $L(F_i)$ as in $L(E)$. The other relations offer no difficulty. Now, with a similar construction and argument to that used in [1, Proof of Theorem 3.11] we find $\psi : L(E) \to L(F_i)$ a K-algebra homomorphism such that $\psi\phi = Id|_{L(F_i)}$, so that ϕ is a monomorphism, which we view as the inclusion map. By construction, each vertex in E^0 is in F_i for some i; furthermore, the edge e has $e \in F_j^1$, where $s(e) = v_j$. Thus we conclude that $L(E) = \bigcup_{i=1}^{\infty} L(F_i)$. (We note here that the embedding of graphs $j : F_i \hookrightarrow E$ is a complete graph homomorphism in the sense of [6], so that the conclusion $L(E) = \bigcup_{i=1}^{\infty} L(F_i)$ can also be achieved by invoking [6, Lemma 2.1].)

Since E is acyclic, so is each F_i. Moreover, each F_i is finite since, by the row-finiteness of E, in each step we add only finitely many vertices. Thus, by Lemma 3, $L(F_i)$ is finite dimensional, so that $L(E)$ is indeed a union of a chain of finite dimensional subalgebras.

For the converse, let $p \in E^*$ be a cycle in E. Then $\{p^m\}_{m=1}^{\infty}$ is a linearly independent infinite set, so that p is not contained in any finite dimensional subalgebra of $L(E)$. $\square$

We note that when E is finite and acyclic then $L(E)$ can be shown to be isomorphic to a finite direct sum of full matrix rings over K, and, for any acyclic E, $L(E)$ is a direct limit of subalgebras of this form. The proof follows along the same lines as that given in [8, Corollaries 2.2 and 2.3].

The description of the simple Leavitt path algebras given in [1] will play a key role here, so we briefly review the germane ideas. An edge $e \in E^1$ is an *exit* to the path $\mu = \mu_1 \cdots \mu_n$ if there exists i such that $s(e) = s(\mu_i)$ and $e \neq \mu_i$. A vertex $w \in E^0$ *connects to* $v \in E^0$ if $w \leq v$. A subset $H \subseteq E^0$ is *hereditary* if $w \in H$ and $w \leq v$ imply $v \in H$; H is *saturated* if whenever $s^{-1}(v) \neq \emptyset$ and $\{r(e) : s(e) = v\} \subseteq H$, then $v \in H$. The main result of [1] is the following

Theorem 5 (*[1, Theorem 3.11]*). *Let E be a graph. Then $L(E)$ is simple if and only if:*

(i) *The only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0, and*
(ii) *Every cycle in E has an exit.*

The following proposition will play an important role in the proof of our main result (Theorem 11).

Proposition 6. *Let E be a graph with the property that every cycle has an exit. Then for every nonzero $\alpha \in L(E)$ there exist $a, b \in L(E)$ such that $a\alpha b \in E^0$.*

Proof. Let α be representable by an element having degree d in real edges. If $d = 0$, then by [1, Corollary 3.7] we are done. So suppose $d > 0$. By [1, Lemma 1.5], given a monomial which is not a vertex, either it begins with a real edge or all its edges are ghost edges. Thus we can write

$$\alpha = \sum_{n=1}^{m} e_{i_n} \alpha_{e_{i_n}} + \beta$$

where $m \geq 1$, $e_{i_n}\alpha_{e_{i_n}} \neq 0$ for every n, each $\alpha_{e_{i_n}}$ is representable as an element of degree less than that of α in real edges, and β is a polynomial in only ghost edges (possibly zero). We will present a process by which we will find $\widehat{a}, \widehat{b}$ such that $\widehat{a}\alpha\widehat{b} \neq 0$ and is representable as an element having degree less than d in real edges.

For an arbitrary edge $e_j \in E^1$, we have two cases:

Case 1: $j \in \{i_1, \ldots, i_m\}$. Then $e_j^*\alpha = \alpha_{e_j} + e_j^*\beta$. If this element is nonzero then by choosing $\widehat{a} = e_j^*$ and $\widehat{b}$ a local unit for α we would be done. For later use, we note that if $e_j^*\alpha$ is zero, then $\alpha_{e_j} = -e_j^*\beta$, and therefore $e_j\alpha_{e_j} = -e_j e_j^*\beta$.

Case 2: $j \notin \{i_1, \ldots, i_m\}$. Then $e_j^*\alpha = e_j^*\beta$. If $e_j^*\beta \neq 0$, then with $\widehat{b}$ as before we would have $e_j^*\alpha\widehat{b}$ is a nonzero polynomial which is representable as an element having degree $0 < d$ in real edges, and again we would be done. For later use, we note that if $e_j^*\beta = 0$, then in particular we have $0 = -e_j e_j^*\beta$.

So we may assume that we are in the latter possibilities of both Cases 1 and 2; i.e., we may assume that $e^*\alpha = 0$ for all $e \in E^1$. We show that this situation cannot happen. First, suppose v is a sink in E. Then we may assume $v\beta = 0$, as follows. Multiplying the displayed equation by v on the left gives $v\alpha = v\sum_{n=1}^{m} e_{i_n}\alpha_{e_{i_n}} + v\beta$. Since v is a sink we have $ve_{i_n} = 0$ for all $1 \leq n \leq m$, so that $v\alpha = v\beta$. But if $v\beta \neq 0$ then $\widehat{a} = v$ and $\widehat{b}$ as above would yield a nonzero element in only ghost edges and we would be done as in Case 2.

Now let $S_1 = \{v_j \in E^0 : v_j = s(e_{i_n}) \text{ for some } 1 \leq n \leq m\}$, and let $S_2 = \{v_{k_1}, \ldots, v_{k_t}\}$ where $(\sum_{i=1}^{t} v_{k_i})\beta = \beta$. We note that $w\beta = 0$ for every $w \in E^0 - S_2$. Also, by definition there are no sinks in S_1, and by a previous observation we may assume that there are no sinks in S_2. Let $S = S_1 \cup S_2$. Then in particular we have $(\sum_{v \in S} v)\beta = \beta$.

We now argue that in this situation α must be zero. To this end,

$$\begin{aligned}
\alpha &= \sum_{n=1}^{m} e_{i_n}\alpha_{e_{i_n}} + \beta = \sum_{n=1}^{m} -e_{i_n}e_{i_n}^{*}\beta + \beta \quad \text{(by Case 1)} \\
&= \sum_{n=1}^{m} -e_{i_n}e_{i_n}^{*}\beta - \left(\sum_{\substack{j \notin \{i_1,\dots,i_m\} \\ s(e_j) \in S}} e_j e_j^{*} \right) \beta + \beta \\
&\quad \text{(by Case 2, the newly subtracted terms equal 0)} \\
&= -\left(\sum_{v \in S} v \right) \beta + \beta \quad \text{(no sinks in } S \text{ implies that CK2 applies at each } v \in S) \\
&= -\beta + \beta = 0.
\end{aligned}$$

As we have assumed $\alpha \neq 0$ we have reached the desired contradiction. Thus we are always able to find $\widehat{a}, \widehat{b}$ such that $\widehat{a}\alpha\widehat{b}$ is nonzero, and is representable in degree less than d in real edges. By repeating this process enough times (d at most), we can find $\widehat{a_k}, \dots, \widehat{a_1}, \widehat{b_1}, \dots, \widehat{b_k}$ such that we can represent $\widehat{a_k} \dots \widehat{a_1}\alpha\widehat{b_1} \dots \widehat{b_k} \neq 0$ by an element of degree zero in real edges. Thus [1, Corollary 3.7] applies, and finishes the proof. □

A *closed simple path based at* v_{i_0} is a path $\mu = \mu_1 \dots \mu_n$, with $\mu_j \in E^1$, $n \geq 1$ such that $s(\mu_j) \neq v_{i_0}$ for every $j > 1$ and $s(\mu) = r(\mu) = v_{i_0}$. Denote by $CSP(v_{i_0})$ the set of all such paths. We note that a cycle is a closed simple path based at any of its vertices, but not every closed simple path based at v_{i_0} is a cycle. We define the following subsets of E^0:

$$\begin{aligned}
V_0 &= \{v \in E^0 : CSP(v) = \emptyset\} \\
V_1 &= \{v \in E^0 : |CSP(v)| = 1\} \\
V_2 &= E^0 - (V_0 \cup V_1).
\end{aligned}$$

Lemma 7. *Let E be a graph. If $L(E)$ is simple, then $V_1 = \emptyset$.*

Proof. For any subset $X \subseteq E^0$ we define the following subsets. $H(X)$ is the set of all vertices that can be obtained by one application of the hereditary condition at any of the vertices of X; that is, $H(X) := r(s^{-1}(X))$. Similarly, $S(X)$ is the set of all vertices obtained by applying the saturated condition among elements of X, that is, $S(X) := \{v \in E^0 : \emptyset \neq \{r(e) : s(e) = v\} \subseteq X\}$. We now define $G_0 := X$, and for $n \geq 0$ we define inductively $G_{n+1} := H(G_n) \cup S(G_n) \cup G_n$. It is not difficult to show that the smallest hereditary and saturated subset of E^0 containing X is the set $G(X) := \bigcup_{n \geq 0} G_n$.

Suppose now that $v \in V_1$, so that $CSP(v) = \{p\}$. In this case p is clearly a cycle. By Theorem 5 we can find an edge e which is an exit for p. Let A be the set of all vertices in the cycle. Since p is the only cycle based at v, and e is an exit for p, we conclude that $r(e) \notin A$. Consider then the set $X = \{r(e)\}$, and construct $G(X)$ as described above. Then $G(X)$ is nonempty and, by construction, hereditary and saturated.

Now Theorem 5 implies that $G(X) = E^0$, so we can find $n = \min\{m : A \cap G_m \neq \emptyset\}$. Take $w \in A \cap G_n$. We are going to show that $w \geq r(e)$. First, since $r(e) \notin A$, then $n > 0$ and therefore $w \in H(G_{n-1}) \cup S(G_{n-1}) \cup G_{n-1}$. Here, $w \in G_{n-1}$ cannot happen by the minimality of n. If $w \in S(G_{n-1})$ then $\emptyset \neq \{r(e) : s(e) = w\} \subseteq G_{n-1}$. Since w is in the cycle p, there exists $f \in E^1$ such that $r(f) \in A$ and $s(f) = w$. In that case $r(f) \in A \cup G_{n-1}$ again contradicts the minimality of n. So the only possibility is $w \in H(G_{n-1})$, which means that there exists $e_{i_1} \in E^1$ such that $r(e_{i_1}) = w$ and $s(e_{i_1}) \in G_{n-1}$.

We now repeat the process with the vertex $w' = s(e_{i_1})$. If $w' \in G_{n-2}$ then we would have $w \in G_{n-1}$, again contradicting the minimality of n. If $w' \in S(G_{n-2})$ then, as above, $\{r(e) : s(e) = w'\} \subseteq G_{n-2}$, so in particular would give $w = r(e_{i_1}) \in G_{n-2}$, which is absurd. So therefore $w' \in H(G_{n-2})$ and we can find $e_{i_2} \in E^1$ such that $r(e_{i_2}) = w'$ and $s(e_{i_2}) \in G_{n-2}$.

After n steps we will have found a path $q = e_{i_n} \cdots e_{i_1}$ with $r(q) = w$ and $s(q) = r(e)$. In particular we have $w \geq s(e)$, and therefore there exists a cycle based at w containing the edge e. Since e is not in p we get $|CSP(w)| \geq 2$. Since w is a vertex contained in the cycle p, we then get $|CSP(v)| \geq 2$, contrary to the definition of the set V_1. □

Lemma 8. *Suppose A is a union of finite dimensional subalgebras. Then A is not purely infinite. In fact, A contains no infinite idempotents.*

Proof. It suffices to show the second statement. So just suppose $e = e^2 \in A$ is infinite. Then eA contains a proper direct summand isomorphic to eA, which in turn, by definition and a standard argument, is equivalent to the existence of elements $g, h, x, y \in A$ such that $g^2 = g, h^2 = h, gh = hg = 0, e = g + h, h \neq 0, x \in eAg, y \in gAe$ with $xy = e$ and $yx = g$. But by hypothesis the five elements e, g, h, x, y are contained in a finite dimensional subalgebra B of A, which would yield that B contains an infinite idempotent, and thus contains a non-artinian right ideal, which is impossible. □

Proposition 9. *Let E be a graph. Suppose that $w \in E^0$ has the property that, for every $v \in E^0$, $w \leq v$ implies $v \in V_0$. Then the corner algebra $wL(E)w$ is not purely infinite.*

Proof. Consider the graph $H = (H^0, H^1, r, s)$ defined by $H^0 := \{v : w \leq v\}$, $H^1 := s^{-1}(H^0)$, and r, s induced by E. The only nontrivial part of showing that H is a well defined graph is verifying that $r(s^{-1}(H^0)) \subseteq H^0$. Take $z \in H^0$ and $e \in E^1$ such that $s(e) = z$. But we have $w \leq z$ and thus $w \leq r(e)$ as well, that is, $r(e) \in H^0$.

Using that H is acyclic, along with the same argument as given in Proposition 4, we have that $L(H)$ is a subalgebra of $L(E)$. Thus Proposition 4 applies, which yields that $L(H)$ is the union of finite dimensional subalgebras, and therefore contains no infinite idempotents by Lemma 8. As $wL(H)w$ is a subalgebra of $L(H)$, it too contains no infinite idempotents, and thus is not purely infinite.

We claim that $wL(H)w = wL(E)w$. To see this, given $\alpha = \sum p_i q_i^* \in L(E)$, then $w\alpha w = \sum p_{i_j} q_{i_j}^*$ with $s(p_{i_j}) = w = s(q_{i_j})$ and therefore $p_{i_j}, q_{i_j} \in L(H)$. Thus $wL(E)w$ is not purely infinite as desired. □

We thank P. Ara for indicating the following result, which will provide the direction of proof for our main theorem. A right A-module T is called *directly infinite* in the case T

contains a proper direct summand T' such that $T' \cong T$. (In particular, the idempotent e is infinite precisely when eA is directly infinite.) Recall that a ring A has *local units* if for every finite subset $\{x_1, \ldots, x_n\} \subseteq A$ there exists $e = e^2 \in A$ with $x_i \in eAe$ for every $i = 1, \ldots, n$.

Proposition 10. *Let A be a ring with local units. The following are equivalent:*

(i) *A is purely infinite simple.*
(ii) *A is simple, and for each nonzero finitely generated projective right A-module P, every nonzero submodule C of P contains a direct summand T of P for which T is directly infinite. (In particular, the property 'purely infinite simple' is a Morita invariant of the ring.)*
(iii) *wAw is purely infinite simple for every nonzero idempotent $w \in A$.*
(iv) *A is simple, and there exists a nonzero idempotent w in A for which wAw is purely infinite simple.*
(v) *A is not a division ring, and A has the property that for every pair of nonzero elements α, β in A there exist elements a, b in A such that $a\alpha b = \beta$.*

Proof. (i) $\Leftrightarrow$ (ii). Suppose A is purely infinite simple. Let P be any nonzero finitely generated projective right A-module. Then P is a generator for Mod-A, as follows. Since A generates Mod-A and P is finitely generated we have an integer n such that $P \oplus P' \cong A^n$ as right A-modules. Again using that P is finitely generated, and using that A has local units, we have that P is isomorphic to a direct summand of a right A-module of the form $f_1A \oplus \cdots \oplus f_tA$, where each f_i is idempotent. But this gives $\mathrm{Hom}_A(P, f_1A \oplus \cdots \oplus f_tA) \neq 0$, which in turn gives $0 \neq \mathrm{Hom}_A(P, A^t) \cong (\mathrm{Hom}_A(P, A))^t$, so that $\mathrm{Hom}_A(P, A) \neq 0$. But $\sum\{a \in A \mid a = g(p)$ for some $p \in P$ and some $g \in \mathrm{Hom}_A(P, A)\}$ is then a nonzero two-sided ideal of A, which necessarily equals A as A is simple. Now let $e = e^2 \in A$. Then $e = \sum_{i=1}^{r} g_i(p_i)$ for some $p_i \in P$ and $g_i \in \mathrm{Hom}_A(P, A)$, which gives that $\lambda_e \circ \oplus g_i : P^r \to A \to eA$ is a surjection. Since P generates eA for each idempotent e of A, we conclude that P generates Mod-A.

This observation allows us to argue exactly as in the proof of [5, Lemma 1.4 and Proposition 1.5] that if $e = e^2 \in A$, then there exists a right A-module Q for which $eA \cong P \oplus Q$. Since A is purely infinite, there exists an infinite idempotent $e \in A$. The indicated isomorphism yields that any submodule C of P is isomorphic to a submodule C' of eA, so that by the hypothesis that A is purely infinite we have that C' contains a submodule T' which is directly infinite, and for which T' is a direct summand of eA. But by a standard argument, any direct summand of eA is equal to fA for some idempotent $f \in A$, so that $T' = fA$ for some infinite idempotent f of A. Let T be the preimage of T' in $P \oplus Q$ under the isomorphism. Then T is directly infinite, and since fA is a direct summand of eA we have that T is a direct summand of $P \oplus Q$ which is contained in P, and hence T is a direct summand of P.

By [2, Proposition 3.3], the lattice of two-sided ideals of Morita equivalent rings are isomorphic, so that any ring Morita equivalent to a simple ring is simple. Therefore, since the indicated property is clearly preserved by equivalence functors, we have that "purely infinite simple" is a Morita invariant.

For the converse, let I be a nonzero right ideal of A. We show that I contains an infinite idempotent. Let $0 \neq x \in I$, so that $xA \leq I$. But $x = ex$ for some $e = e^2 \in A$, so

$xA \leq eA$. So by hypothesis, xA contains a nonzero direct summand T of eA, where T is directly infinite. But as noted above we have that $T = fA$ for $f = f^2 \in A$, where f is infinite. Thus $f \in T \leq xA \leq I$ and we are done.

(ii) $\Rightarrow$ (iii). Since we have established the equivalence of (i) and (ii), we may assume A is purely infinite simple. Then the simplicity of A gives that $AwA = A$ for any nonzero idempotent $w \in A$, which yields by [2, Proposition 3.5] that A and wAw are Morita equivalent, so that (iii) follows immediately from (ii).

(iii) $\Rightarrow$ (iv). It is tedious but straightforward to show that if A is any ring with local units, and wAw is a simple (unital) ring for every nonzero idempotent w of A, then A is simple.

(iv) $\Rightarrow$ (i). Since A is simple we get $AwA = A$, so that A and wAw are Morita equivalent by the previously cited [2, Proposition 3.5].

Thus we have established the equivalence of statements (i) through (iv).

(i) $\Rightarrow$ (v). Suppose A is purely infinite simple. Then A is not left artinian, so that A cannot be a division ring. Now choose nonzero $\alpha, \beta \in A$. Then there exists a nonzero idempotent $w \in A$ such that $\alpha, \beta \in wAw$. But wAw is purely infinite simple by (i) $\Leftrightarrow$ (iii), so by [5, Theorem 1.6] there exist $a', b' \in wAw$ such that $a'\alpha b' = w$. But then for $a = a', b = b'\beta$ we have $a\alpha b = \beta$. Conversely, suppose A is not a division ring and that A satisfies the indicated property. Since A is not a division ring and A is a ring with local units, there exists a nonzero idempotent w of A for which wAw is not a division ring. Let $\alpha \in wAw$. Then by hypothesis there exist a', b' in A with $a'\alpha b' = w$. But since $\alpha \in wAw$, by defining $a = wa'w$ and $b = wb'w$ we have $a\alpha b = w$. Thus another application of [5, Theorem 1.6] (noting that w is the identity of wAw) gives the desired conclusion.

(v) $\Rightarrow$ (iv). The indicated multiplicative property yields that any nonzero ideal of A will contain a set of local units for A, so that A is simple. Since A is not a division ring and A has local units there exists a nonzero idempotent w of A such that wAw is not a division ring. Let $\alpha, \beta \in wAw$; in particular, $w\alpha w = \alpha$ and $w\beta w = \beta$. By hypothesis there exists $a, b \in A$ such that $a\alpha b = \beta$. But then $(waw)\alpha(wbw) = w\beta w = \beta$, which yields that wAw is purely infinite simple by [5, Theorem 1.6]. $\square$

We now have all the necessary ingredients in hand to prove the main result of this article.

Theorem 11. *Let E be a graph. Then $L(E)$ is purely infinite simple if and only if E has the following properties.*

(i) *The only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0.*
(ii) *Every cycle in E has an exit.*
(iii) *Every vertex connects to a cycle.*

Proof. First, assume (i), (ii) and (iii) hold. By Theorem 5 we have that $L(E)$ is simple. By Proposition 10 it suffices to show that $L(E)$ is not a division ring, and that for every pair of elements α, β in $L(E)$ there exist elements a, b in $L(E)$ such that $a\alpha b = \beta$. Conditions (ii) and (iii) easily imply that $\mid E^1 \mid > 1$, so that $L(E)$ has zero divisors, and thus is not a division ring.

We now apply Proposition 6 to find $\overline{a}, \overline{b} \in L(E)$ such that $\overline{a}\alpha\overline{b} = w \in E^0$. By condition (iii), w connects to a vertex $v \notin V_0$. Either $w = v$ or there exists a path p such that $r(p) = v$

and $s(p) = w$. By choosing $a' = b' = v$ in the former case, and $a' = p^*, b' = p$ in the latter, we have produced elements $a', b' \in L(E)$ such that $a'wb' = v$.

An application of Lemma 7 yields that $v \in V_2$, so there exist $p, q \in CSP(v)$ with $p \neq q$. For any $m > 0$ let c_m denote the closed path $p^{m-1}q$. Using [1, Lemma 2.2], it is not difficult to show that $c_m^* c_n = \delta_{mn} v$ for every $m, n > 0$.

Now consider any vertex $v_l \in E^0$. Since $L(E)$ is simple, there exist $\{a_i, b_i \in L(E) \mid 1 \leq i \leq t\}$ such that $v_l = \sum_{i=1}^t a_i v b_i$. But by defining $a_l = \sum_{i=1}^t a_i c_i^*$ and $b_l = \sum_{j=1}^t c_j b_j$, we get

$$a_l v b_l = \left(\sum_{i=1}^t a_i c_i^*\right) v \left(\sum_{j=1}^t c_j b_j\right) = \sum_{i=1}^t a_i c_i^* v c_i b_i = v_l.$$

Now let s be a left local unit for β (i.e., $s\beta = \beta$), and write $s = \sum_{v_l \in S} v_l$ for some finite subset of vertices S. By letting $\widetilde{a} = \sum_{v_l \in S} a_l c_l^*$ and $\widetilde{b} = \sum_{v_l \in S} c_l b_l$, we get

$$\widetilde{a} v \widetilde{b} = \sum_{v_l \in S} a_l c_l^* v c_l b_l = \sum_{v_l \in S} v_l = s.$$

Finally, letting $a = \widetilde{a} a' \overline{a}$ and $b = \overline{b} b' \widetilde{b} \beta$, we have that $a\alpha b = \beta$ as desired.

For the converse, suppose that $L(E)$ is purely infinite simple. By Theorem 5 we have (i) and (ii). If (iii) does not hold, then there exists a vertex $w \in E^0$ such that $w \leq v$ implies $v \in V_0$. Applying Proposition 9 we get that $wL(E)w$ is not purely infinite. But then Proposition 10 implies that $L(E)$ is not purely infinite, contrary to hypothesis. □

Example 12. (i) Let E be the graph defined in Example 2(i). Then $L(E) \cong M_n(K)$ which of course is simple, but not purely infinite since no vertex in E^0 connects to a cycle.

(ii) Let $n \geq 2$. Let E be the graph defined in Example 2(ii). Then $L(E) \cong L(1, n)$, the Leavitt algebra. Since $n \geq 2$ we see that all the hypotheses of Theorem 11 are satisfied, so that $L(1, n)$ is purely infinite simple.

(iii) Let E be the graph having $E^0 = \{v, w\}$ and $E^1 = \{e, f, g\}$, where $s(e) = s(f) = v$, $r(e) = r(f) = w$, $s(g) = w, r(g) = v$. Then E satisfies the hypotheses of Theorem 11, so that $L(E)$ is purely infinite simple.

Let $L(1, n)$ denote the Leavitt algebra described in Example 2(ii). We complete this article by providing a realization of the purely infinite simple algebra $M_m(L(1, n))$ as a Leavitt path algebra $L(E)$ for a specific graph E.

Proposition 13. *Let $n \geq 2$ and $m \geq 1$. We define the graph E_n^m by setting $E^0 := \{v_1, \ldots, v_m\}$, $E^1 := \{f_1, \ldots, f_n, e_1, \ldots, e_{m-1}\}$, $r(f_i) = s(f_i) = v_m$ for $1 \leq i \leq n$, $r(e_i) = v_{i+1}$, and $s(e_i) = v_i$ for $1 \leq i \leq m - 1$. Then $L(E_n^m) \cong M_m(L(1, n))$.*

Proof. We define $\Phi : K[E^0 \cup E^1 \cup (E^1)^*] \to M_m(L(1, n))$ on the generators by

$$\Phi(v_i) = e_{ii} \quad \text{for } 1 \leq i \leq m$$

$$\Phi(e_i) = e_{ii+1} \quad \text{and} \quad \Phi(e_i^*) = e_{i+1i} \quad \text{for } 1 \leq i \leq m - 1$$

$$\Phi(f_i) = y_i e_{mm} \quad \text{and} \quad \Phi(f_i^*) = x_i e_{mm} \quad \text{for } 1 \leq i \leq n$$

and extend linearly and multiplicatively to obtain a K-homomorphism. We now verify that Φ factors through the ideal of relations in $L(E_n^m)$.

First, $\Phi(v_i v_j - \delta_{ij} v_i) = e_{ii} e_{jj} - \delta_{ij} e_{ii} = 0$. If we consider the relations $e_i - e_i r(e_i)$ then we have $\Phi(e_i - e_i r(e_i)) = \Phi(e_i - e_i v_{i+1}) = e_{ii+1} - e_{ii+1} e_{i+1i+1} = 0$, and analogously $\Phi(e_i - s(e_i) e_i) = 0$. For the relations $f_i - f_i r(f_i)$ we get $\Phi(f_i - f_i r(f_i)) = \Phi(f_i - f_i v_m) = y_i e_{mm} - y_i e_{mm} e_{mm} = 0$, and similarly $\Phi(f_i - s(f_i) f_i) = 0$. With similar computations it is easy to also see that $\Phi(e_i^* - e_i^* r(e_i^*)) = \Phi(e_i^* - s(e_i^*) e_i^*) = \Phi(f_i^* - f_i^* r(f_i^*)) = \Phi(f_i^* - s(f_i^*) f_i^*) = 0$.

We now check the Cuntz–Krieger relations. First, $\Phi(e_i^* e_j - \delta_{ij} r(e_j)) = \Phi(e_i^* e_j - \delta_{ij} v_{j+1}) = e_{i+1i} e_{jj+1} - \delta_{ij} e_{j+1j+1} = \delta_{ij} e_{i+1j+1} - \delta_{ij} e_{j+1j+1} = 0$. Second, $\Phi(f_i^* f_j - \delta_{ij} r(f_j)) = \Phi(f_i^* f_j - \delta_{ij} v_m) = x_i e_{mm} y_j e_{mm} - \delta_{ij} e_{mm} = 0$, because of the relation (1) in $L(1, n)$. Finally, $\Phi(f_i^* e_j - \delta_{f_i, e_j} r(e_j)) = \Phi(f_i^* e_j - 0 v_{j+1}) = \Phi(f_i^* e_j) = x_i e_{mm} e_{jj+1} = 0$, and similarly $\Phi(e_i^* f_j - \delta_{e_i, f_j} r(f_j)) = 0$.

With CK2 we have two cases. First, for $i < m$, $\Phi(v_i - e_i e_i^*) = e_{ii} - e_{ii+1} e_{i+1i} = 0$. And for v_m we have $\Phi(v_m - \sum_{i=1}^n f_i f_i^*) = e_{mm} - \sum_{i=1}^n y_i e_{mm} x_i e_{mm} = 0$, because of the relation (2) in $L(1, n)$.

This shows that we can factor Φ to obtain a K-homomorphism of algebras $\Phi : L(E_n^m) \to M_m(L(1, n))$. We will see that Φ is onto. Consider any matrix unit e_{ij} and $x_k \in L(1, n)$. If we take the path $p = e_i \ldots e_{n-1} f_k^* e_{n-1}^* \ldots e_j^* \in L(E_n^m)$ then we get $\Phi(p) = e_{ii+1} \ldots e_{n-1n} (x_k e_{nn}) e_{nn-1} \ldots e_{j+1j} = x_k e_{ij}$. Similarly $\Phi(e_i \ldots e_{n-1} f_k e_{n-1}^* \ldots e_j^*) = y_k e_{ij}$. In this way we get that all the generators of $M_m(L(1, n))$ are in $Im(\Phi)$.

Finally, using the same ideas as those presented in [1, Corollary 3.13(i)], we see that E_n^m satisfies the conditions of Theorem 5, which yields the simplicity of $L(E_n^m)$. This implies that Φ is necessarily injective, and therefore an isomorphism. $\square$

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