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Real Even Symmetric Ternary Forms

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Let $S^e_{n,m}$ denote the set of all real symmetric forms of degree $m = 2d$. Let $PS^e_{n,m}$ and $\Sigma S^e_{n,m}$ denote the cones of positive semidefinite (psd) and sum of squares (sos) elements of $S^e_{n,m}$, respectively. For $m = 2$ or 4, these cones coincide. For $m = 6$, they do not, and were analyzed in Even Symmetric Sextics, by M. D. Choi, T. Y. Lam, and B. Reznick (1987, *Math. Z.* **195**, pp. 559–580).

We present an easily checked, necessary and sufficient condition for an even symmetric n-ary octic to be in $PS^e_{n,8}$ and for an even symmetric ternary decic to be in $PS^e_{3,10}$; we also show that there is no corresponding condition for even symmetric ternary forms of degree greater than 10. We proceed to discuss the extremal elements of the cones $PS^e_{3,8}$ and $PS^e_{3,10}$. This leads to the question: how many of these extremal forms have sos representations? We prove that $PS^e_{3,8} = \Sigma S^e_{3,8}$ and demonstrate that $PS^e_{3,10} \setminus \Sigma S^e_{3,10}$ is nonempty, providing many new examples of psd forms which are not sos.

We give a graphic representation of ternary forms which also indicates whether or not an element of $S^e_{3,8}$ or $S^e_{3,10}$ is psd. We interpret elements of $PS^e_{3,m}$ as inequalities; in particular, we give all symmetric polynomial inequalities of degree ≤ 5 satisfied by the sides of a triangle. © 1999 Academic Press

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1. INTRODUCTION

Let p be a real polynomial in n variables and write $\mathbf{x} = (x_1, x_2, \dots, x_n)$. We say that $p(\mathbf{x})$ is a *form of degree m*, or an *n-ary m-ic*, if it is homogeneous of degree m. Let $F_{n,m}$ denote the set of all real n-ary m-ics. Observe that

$$p(\lambda \mathbf{x}) = \lambda^m (p(\mathbf{x})), \qquad \lambda \in \mathbb{R}, \tag{1.1}$$

for any m-ic form p, by homogeneity.

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Recall two familiar properties of forms:

DEFINITION 1.1. A form $p \in F_{n,m}$ is *positive semidefinite* (psd) if $p(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$.

DEFINITION 1.2. A form $p \in F_{n,m}$ is a *sum of squares* (sos) if there exist forms h_j of degree $\frac{m}{2}$, $1 \leq j \leq N$, such that $p = \sum_{j=1}^{N} h_j^2$.

For a fixed m, we associate with an n-ary m-ic p the $N(n, m) = \binom{n+m-1}{n-1}$-tuple of its coefficients. In this way, the vector space $F_{n,m}$ becomes identified with $\mathbb{R}^{N(n,m)}$. After imposing the usual topology, the set of psd forms in $F_{n,m}$ corresponds to a closed convex cone in $\mathbb{R}^{N(n,m)}$. Robinson proved in [12] that the set of sos forms is also a closed convex cone in this space. Following Choi and Lam [3], $P_{n,m}$ and $\Sigma_{n,m}$ denote the cones of psd and sos elements of $F_{n,m}$, respectively.

Clearly, sos forms are psd. Hilbert proved in [8] the following:

THEOREM 1.1. $P_{n,m} = \Sigma_{n,m}$ *if and only if* $(n, m) \in \{(n, 2), (2, m), (3, 4)\}$.

By (1.1), a form can be psd only when m is even. Further, p is psd if and only if $p(\mathbf{x}) \geq 0$ for $\mathbf{x} \in S^{n-1}$, the n-dimensional unit sphere. Let $Z(p)$ denote the set of nontrivial projective real zeros of a form p.

We focus our attention in this paper on even symmetric forms. Let

$$p(x_1, \ldots, x_n) = \sum a(i_1, \ldots, i_n) x_1^{i_1} \cdots x_n^{i_n}.$$

DEFINITION 1.3. The form p is *even* if all nonzero $i_k \in 2\mathbb{N}$; that is, all exponents in all terms are even.

DEFINITION 1.4. The form p is *symmetric* if $a(i_1, \ldots, i_n) = a(\sigma(i_1, \ldots, i_n))$ for all $\sigma \in S_n$, the symmetric group on n letters.

Even symmetric forms are both even and symmetric. A fundamental example of an even symmetric n-ary form is

$$M_{r,n}(\mathbf{x}) = \sum_{i=1}^{n} x_i^r, \qquad r \in 2\mathbb{N}.$$

We will write $M_{r,n}(\mathbf{x}) = M_r(\mathbf{x})$ when the context is clear.

We denote the set of even symmetric n-ary m-ics by $S^e_{n,m}$; $PS^e_{n,m}$ and $\Sigma S^e_{n,m}$ are the cones of psd and sos elements of $S^e_{n,m}$, respectively. The subspace $S^e_{n,m}$ of $F_{n,m}$ is of interest because of its relatively small dimension as a vector space over $\mathbb{R}$. The dimension of $S^e_{n,m}$ is the number of partitions of m into at most n even parts. For fixed m, this is bounded as n goes to infinity and is substantially less than $N(n, m)$. Thus, it is considerably easier to visualize the cones $PS^e_{n,m}$ and $\Sigma S^e_{n,m}$ than $P_{n,m}$ and $\Sigma_{n,m}$.

DEFINITION 1.5. An element c of a closed, convex cone C is called *extremal* if

$$c = c_1 + c_2, \qquad c_i \in C, \qquad \text{implies} \qquad c_i = \lambda_i c, \qquad \lambda_i \geq 0.$$

Any element of a closed, convex cone can be written as a finite, non-negative linear combination of its extremal elements. The following simple observation about extremal psd forms is central to our analysis:

LEMMA 1.1. *If p is extremal in $P_{n,m}$, then $Z(p) \neq \emptyset$.*

Proof. Suppose $p \in P_{n,m}$ and $Z(p) = \emptyset$. Let μ be the minimum value that p takes on S^{n-1}. Clearly, $\mu > 0$ and there exists $\mathbf{y} \in S^{n-1}$ such that $p(\mathbf{y}) = \mu$, by compactness. Let $h(\mathbf{x}) = (M_2(\mathbf{x}))^{m/2}$. Observe that $h(\mathbf{x}) = 1$ for all $\mathbf{x} \in S^{n-1}$. Then $q(\mathbf{x}) = p(\mathbf{x}) - \mu h(\mathbf{x}) \geq 0$ on S^{n-1}, so q is psd and $q(\mathbf{y}) = 0$. If $q \not\equiv 0$, then $p(\mathbf{x}) = q(\mathbf{x}) + \mu h(\mathbf{x})$ is a decomposition of p as a sum of two psd forms, and $Z(p) = \emptyset$, $Z(q) \neq \emptyset$ means that q is not a multiple of p, thus implying that p is not extremal. If $q \equiv 0$, then $p(\mathbf{x}) = \mu(\sum x_i^2)^{m/2}$, which is not extremal by a literal interpretation of the multinomial theorem. ∎

We now introduce two terms that will allow us to state concisely some of our major results.

DEFINITION 1.6. Let G be a set of n-ary forms. We say that $\Omega \subseteq \mathbb{R}^n$ is a *test set* for G if $p \in G$ is psd if and only if $p(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \Omega$.

Any dense subset of $\mathbb{R}^n$ is a test set for $F_{n,m}$; we have already noted in the proof of Lemma 1.1 that S^{n-1} is a test set by homogeneity. Since we are considering only even forms, we may restrict test sets to include only points $\mathbf{x}$ in the first orthant of $\mathbb{R}^n$ without loss of generality.

DEFINITION 1.7. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be in the first orthant of $\mathbb{R}^n$. We say that $\mathbf{x}$ is a *k-point* if there exist distinct positive real numbers $r_1, \ldots, r_k$ such that $x_j \in \{0, r_1, \ldots, r_k\}$, $1 \leq j \leq n$.

We will let $\Omega_{n,k}$ denote the set of k-points in $\mathbb{R}^n$.

The characterization of $PS^e_{n,m}$ and $\Sigma S^e_{n,m}$ within $S^e_{n,m}$ has been completed for $m = 2, 4$, and 6. The cones $PS^e_{n,2}$ and $\Sigma S^e_{n,2}$ coincide and consist of the forms $aM_2(\mathbf{x})$, $a \geq 0$. Likewise, $PS^e_{n,4} = \Sigma S^e_{n,4}$; every element of $PS^e_{n,4}$ can be shown to be a nonnegative linear combination of the two linearly independent sos forms

$$\begin{aligned} q_1 &= nM_4 - M_2^2 = \sum_{i \neq j} (x_i^2 - x_j^2)^2, \\ q_2 &= -M_4 + M_2^2 = 2 \sum_{i \neq j} x_i^2 x_j^2. \end{aligned} \tag{1.2}$$

The case $m = 6$ was studied and completely settled by Choi *et al.* in [5], in which the authors determined necessary and sufficient conditions for an even symmetric sextic p to be psd and to be sos, and showed that, for $n \geq 3$, $\Sigma S^e_{n,6} \subsetneq PS^e_{n,6}$. In particular, they showed that the set of 1-points in $\mathbb{R}^n$ is a test set for $S^e_{n,6}$.

We now turn to the work contained in this paper, which is based on the author's Ph.D. dissertation. In Section 2, we prove that $\Omega_{n,2}$ is a test set for even symmetric n-ary octics. The same set is shown to be a test set for even symmetric ternary decics, but not for even symmetric ternary forms of degree greater than 10, in Section 3. We then examine some specific classes of even symmetric forms. We discuss $S^e_{3,8}$ in Section 4; we parameterize the complete set of extremal forms, and we prove that $PS^e_{3,8} = \Sigma S^e_{3,8}$, a companion result to the portion of Theorem 1.1 regarding ternary quartics. In Section 5 we determine all extremal forms in $PS^e_{3,10}$, and in Section 6 we determine which of these forms have sos representations. In Section 7, we examine inequalities satisfied by the sides of a triangle that arise from our discussion in Sections 4 and 5.

2. TESTS FOR OCTICS

We use some classical results of algebraic geometry to find a surprisingly small test set for even symmetric octics in $n \geq 3$ variables.

DEFINITION 2.1. A *real plane curve* is the set of solutions in $\mathbb{R}^2$ to the polynomial equation $f(x, y) = 0$.

A plane curve f is associated with a dehomogenized ternary form p by taking $f(x, y) = p(x, y, 1)$.

DEFINITION 2.2. A point $\mathbf{x}$ is a *singular point* of f if $f(\mathbf{x}) = \frac{\partial f}{\partial x}(\mathbf{x}) = \frac{\partial f}{\partial y}(\mathbf{x}) = 0$.

Observe that if f is psd and $f(x_1, y_1) = 0$, then (x_1, y_1) is a singular point of f.

DEFINITION 2.3. A plane curve is *irreducible* if f is irreducible over $\mathbb{R}[x, y]$.

The following standard results can be found in, e.g., [14]:

THEOREM 2.1. *If p is an irreducible real plane curve of degree m, then p has at most $\frac{1}{2}(m-1)(m-2)$ singular points.*

LEMMA 2.1. *If a real plane curve $p = g \cdot h$ has a singular point at $\mathbf{x}$, then either $g(\mathbf{x}) = h(\mathbf{x}) = 0$ or one of $\{g, h\}$ has a singular point at $\mathbf{x}$.*

THEOREM 2.2 (A Weak Form of Bezout's Theorem). *Suppose that f and g are real plane curves of degree m and n, respectively, which have more than $m \cdot n$ common points. Then the polynomials f and g have a common nontrivial factor.*

We now examine even symmetric n-ary octics in depth. We shall write $p \in S^e_{n,8}$ as

$$\begin{aligned} p(x_1, \dots, x_n) &:= [\alpha, \beta, \gamma, \delta, \varepsilon] \\ &= \alpha \sum^{n} x_i^8 + \beta \sum^{2\binom{n}{2}} x_i^6 x_j^2 + \gamma \sum^{\binom{n}{2}} x_i^4 x_j^4 \\ &\quad + \delta \sum^{3\binom{n}{3}} x_i^4 x_j^2 x_k^2 + \varepsilon \sum^{\binom{n}{4}} x_i^2 x_j^2 x_k^2 x_l^2. \end{aligned} \tag{2.1}$$

The numbers $n, 2\binom{n}{2}$, etc., on top of the summation symbols indicate the number of terms in the sum. Observe that there is an ε term only for $n \geq 4$. It is convenient also to express elements of $S^e_{n,8}$ in terms of the functions $M_r(\mathbf{x})$ introduced in Section 1:

$$p := (a, b, c, d, e) = aM_8 + bM_6M_2 + cM_4^2 + dM_4M_2^2 + eM_2^4. \tag{2.2}$$

These two bases are related by the equations

$$\alpha = a + b + c + d + e, \qquad \beta = b + 2d + 4e,$$

$$\gamma = 2c + 2d + 6e, \qquad \delta = 2d + 12e, \qquad \varepsilon = 24e.$$

By Newton's theorem on symmetric functions, any even symmetric ternary form is a function of M_2, M_4, and M_6. Since $M_8 = \frac{4}{3}M_6M_2 + \frac{1}{2}M_4^2 - M_4M_2^2 + \frac{1}{6}M_2^4$ when $n = 3$, we may assume $a = 0$ in that case.

Suppose p is psd and $\mathbf{y} = (y_1, \dots, y_n)$ is a nontrivial zero of p in the first orthant of $\mathbb{R}^n$. Let $\overline{M}_r = M_r(\mathbf{y})$. A canonical example of an n-ary octic form which has a singular point at $\mathbf{y}$ is the perfect square

$$t_{n,\mathbf{y}}(\mathbf{x}) = T^2_{n,\mathbf{y}}(\mathbf{x}) = \left(\overline{M}_2^2 \cdot M_4(\mathbf{x}) - \overline{M}_4 \cdot M_2^2(\mathbf{x})\right)^2. \tag{2.3}$$

Let $J(\mathbf{y})$ denote the Jacobian of $\{M_8, M_6M_2, M_4^2, M_4M_2^2, M_2^4\}$ at $\mathbf{y}$; it is a 3×4 matrix for $n = 3$ and an $n \times 5$ matrix for $n \geq 4$. Let $((a), b, c, d, e)$ be a solution to the system

$$J(\mathbf{y}) \begin{pmatrix} (a) \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad n \geq 3. \tag{2.4}$$

We see that the forms $p \in S^e_{n,8}$ which are singular at $\mathbf{y}$ comprise a subspace of $S^e_{n,8}$. Also, if the coefficients of p satisfy (2.4), then $p(\mathbf{y}) = 0$.

If $n = 3$ and rank $J(\mathbf{y}) = 3$, then (2.4) has a one-dimensional solution set. Hence, the subspace of $S^e_{n,8}$ which has a singular point at $\mathbf{y}$ is one-dimensional and is generated by $t_{3,\mathbf{y}}$.

LEMMA 2.2. *Let* $n = 3$. *Then* $\operatorname{rank} J(\mathbf{y}) < 3$ *if and only if* $\mathbf{y}$ *is a k-point*, $k \le 2$.

Proof. We find after a computer-assisted computation that the determinant of the 3×3 submatrix involving the last three columns of $J(\mathbf{y})$ is

$$192\, \overline{M}_4^2\, \overline{M}_2^2\, y_1\, y_2\, y_3 (y_1^2 - y_2^2)(y_1^2 - y_3^2)(y_2^2 - y_3^2).$$

Evidently, rank $J(\mathbf{y}) < 3$ if and only if at least one of $\{y_1, y_2, y_3\}$ is zero or at least two of $\{y_1, y_2, y_3\}$ are equal. ∎

If $n \ge 4$ and rank $J(\mathbf{y}) = 4$, then again (2.4) has a one-dimensional solution set: the subspace of $S^e_{n,8}$ which has $\mathbf{y}$ as a singular point and is generated by $t_{n,\mathbf{y}}$.

LEMMA 2.3. *Let* $n \ge 4$. *Then* $\operatorname{rank} J(\mathbf{y}) < 4$ *if and only if* $\mathbf{y}$ *is a k-point*, $k \le 3$.

Proof. We find after a computer-assisted computation that the determinant of the 4×4 minor of the first four rows and the first four columns of $J(\mathbf{y})$ is

$$1536\, \overline{M}_4^2\, \overline{M}_2^2\, y_1\, y_2\, y_3\, y_4 \prod_{1 \le i < j \le 4} (y_i^2 - y_j^2). \tag{2.5}$$

This is zero if and only if one of $\{y_1, y_2, y_3, y_4\}$ is zero or two of $\{y_1, y_2, y_3, y_4\}$ are equal. Since this holds by symmetry for any subset of four elements from $\{y_1, y_2, \dots, y_n\}$, we conclude that all 4×4 determinants involving the first four columns of $J(\mathbf{y})$ are simultaneously zero if and only if either one of $\{y_i, y_j, y_k, y_l\}$ is zero or at least two of $\{y_i, y_j, y_k, y_l\}$ are equal for all 4-tuples $\{y_i, y_j, y_k, y_l\}$ from $\{y_1, y_2, \dots, y_n\}$. This is equivalent to there being no more than three distinct nonzero values assumed by the coordinates of $\mathbf{y}$. ∎

We have seen thus far that if an even symmetric n-ary octic p has a zero $\mathbf{y}$ which is not sufficiently "simple," then p is a multiple of $t_{n,\mathbf{y}}$. Our next goal is to show that, for $n \ge 4$, often when a form is singular at a 3-point, it is also singular at a 4-point.

The following approach, including Lemma 2.5, has been adapted from an unpublished manuscript on symmetric quartics by Choi *et al.* [6]. Let $p \not\equiv 0$ be a psd even symmetric octic in $n \ge 4$ variables. Assume p has a zero

$\mathbf{y}$ which is a 3-point, say $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, with $y_1 > y_2 > y_3 > 0$. Define the not necessarily symmetric, not necessarily even, psd ternary quartic form

$$H(u, v, w) = p\left(\sqrt{u}, \sqrt{v}, \sqrt{(y_1^2 + y_2^2 + y_3^2)w - u - v}, y_4\sqrt{w}, \ldots, y_n\sqrt{w}\right). \quad (2.6)$$

For example, let $p = (\sum x^4 - 2\sum x^2y^2)^2 = [1, -4, 6, 4, 24] \in S^e_{4,8}$. This form has a zero at $(\sqrt{12}, 2, 1, 1)$. The corresponding H would be

$$(u^2 + v^2 + w^2 + (17w - u - v)^2 - 2(uv + uw + vw + (u + v + w)(17w - u - v)))^2.$$

Let $\{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4, \mathbf{z}_5, \mathbf{z}_6\}$ denote the six points $\{(y_i^2, y_j^2, 1)\}$, $1 \le i \ne j \le 3$, which are zeros of H. Observe that no three of these points are collinear and that each is a singular point of H.

LEMMA 2.4. *The form $H(u, v, w)$ is a perfect square.*

Proof. The plane curve $H(u, v, 1) = 0$ is of degree 4 with at least six singular points. If H were irreducible, then by Theorem 2.1 it could have no more than three singular points. Hence H must factor in one of four ways: (1) four (possibly repeated) linear forms; (2) two (possibly repeated) linear forms and an irreducible quadratic; (3) a linear form and an irreducible cubic; or (4) two (possibly repeated) irreducible quadratics.

We claim H must factor as in (4). Suppose it has factorization (1). We count the number of pairs (i, j) for which $l_i(\mathrm{z}_j) = 0$, where l_i, $i = 1, 2, 3, 4$, is linear. Since no three of $\{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4, \mathbf{z}_5, \mathbf{z}_6\}$ lie in a line, there are at most eight such (i, j). But Lemma 2.1 implies that each of these six zeros must appear in two of the linear terms.

If H has factorization (2), then there are at most four pairs (i, j) such that $l_i(\mathbf{z}_j) = 0$. Hence, by Lemma 2.1, there are at least two of the $\mathbf{z}_j$ which must be singular points of the quadratic, which cannot happen by Theorem 2.1. If H has factorization (3), then the cubic must have $\mathbf{z}_j$ as a singular point whenever $l_i(\mathbf{z}_j) \ne 0$. This implies that the cubic has at least four singular points, again a contradiction of Theorem 2.1.

This means that $H = q \cdot q'$, where q, q' are irreducible quadratics. Since q and q' intersect in at least $6 > 4$ points, they must be multiples of one another by Theorem 2.2. ∎

LEMMA 2.5. *Let $p \not\equiv 0$ be a psd even symmetric octic in $n \ge 4$ variables. Suppose $p(\mathbf{y}) = 0$, where $\mathbf{y}$ is a k-point, $k \ge 3$, which has at least four nonzero coordinates. Then there exists $\mathbf{z} \in \mathbb{R}^n$ such that $p = \lambda t_{n,\mathbf{z}}$, $\lambda > 0$.*

Proof. Let $\mathbf{y}=(y_1,\ldots,y_n)$ with $y_1>y_2>y_3>0$, $y_4\neq 0$, and let $H=q^2$ and let $\{\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3,\mathbf{z}_4,\mathbf{z}_5,\mathbf{z}_6\}$ be as above. If

$$q(u,v,w)=\alpha_1u^2+\alpha_2uv+\alpha_3v^2+\alpha_4uw+\alpha_5vw+\alpha_6w^2,$$

then the six equations $q(\mathbf{z}_j)=0$ yield a linear system which has the unique solution

$$\alpha_1=\alpha_2=\alpha_3,\qquad \alpha_4=\alpha_5=-\alpha_1(y_1^2+y_2^2+y_3^2),$$
$$\alpha_6=\alpha_1(y_1^2y_2^2+y_1^2y_3^2+y_2^2y_3^2).$$

Thus, $(u,v,1)$ is a zero of H if $\{u,v\}$ lies on the nondegenerate ellipse

$$u^2+uv+v^2-(y_1^2+y_2^2+y_3^2)(u+v)+(y_1^2y_2^2+y_1^2y_3^2+y_2^2y_3^2)=0.$$

Hence, $p(\sqrt{u},\sqrt{v},\sqrt{y_1^2+y_2^2+y_3^2-u-v},y_4,y_5,\ldots,y_n)=0$ for all $\{u,v\}$ on this ellipse by (2.6). Choose u and v so that $\{u,v,y_1^2+y_2^2+y_3^2-u-v\}$ are all positive, mutually distinct, and distinct from $y_4^2\neq 0$. We see then that p has a zero $\mathbf{z}$ which is a 4-point, and so $p=\lambda t_{n,\mathbf{z}}$ for some $\lambda>0$ by Lemma 2.3. ∎

We now show that any sum of fourth powers of the coordinates of a point on S^{n-1} can always be formed from a 2-point on S^{n-1}. This will be useful in proving the major result of this section.

LEMMA 2.6. *Suppose* $\mathbf{x}=(x_1,\ldots,x_n)$ *in the first orthant of* $\mathbb{R}^n$ *is such that* $M_2(\mathbf{x})=1$ *and* $M_4(\mathbf{x})=r$. *Then there exists a 2-point* $\mathbf{z}=(a,a,\ldots,a,b)$ *also in the first orthant so that* $M_2(\mathbf{z})=(n-1)a^2+b^2=1$ *and* $M_4(\mathbf{z})=(n-1)a^4+b^4=r$.

Proof. First, observe that when $M_2(\mathbf{x})=1$, we have $\frac{1}{n}\le M_4(\mathbf{x})\le 1$. Let

$$\mathbf{z}_\theta=\left(\frac{\cos\theta}{\sqrt{n-1}},\ldots,\frac{\cos\theta}{\sqrt{n-1}},\sin\theta\right)$$

and let $f(\theta)=M_4(\mathbf{z}_\theta)=\frac{1}{n-1}\cos^4\theta+\sin^4\theta$. Then $M_2(\mathbf{z}_\theta)=1$ for all θ, $f(\frac{\pi}{2})=1$, and $f(\arcsin\frac{1}{\sqrt{n}})=\frac{1}{n}$. Hence, for any r satisfying $\frac{1}{n}\le r\le 1$, there exists $\hat\theta$ between $\arcsin(\frac{1}{\sqrt{n}})$ and $\frac{\pi}{2}$ so that $f(\hat\theta)=r$ and $\mathbf{z}_{\hat\theta}=(a,\ldots,a,b)$, by the intermediate value theorem. ∎

THEOREM 2.3. *The set* $\Omega_{n,2}$ *is a test set for* $S^e_{n,8}$, $n\ge 1$.

Proof. The theorem is trivial for $n=1$ and 2. Let $n=3$. Suppose p is nonnegative at all $\mathbf{x}\in\Omega_{3,2}$, but p is not psd. Let $-\mu$, $\mu>0$, denote the minimum value p takes on the unit sphere S^2, and let $\mathbf{y}=(y_1,y_2,y_3)\in S^2$ be such that $p(\mathbf{y})=-\mu$. Then $q(\mathbf{x})=p(\mathbf{x})+\mu M_2^4(\mathbf{x})$ is psd and $q(\mathbf{y})=0$. By assumption, $\mathbf{y}$ is not a k-point, $k\le 2$; Lemma 2.2 implies that $q=\lambda t_{3,\mathbf{y}}$,

$\lambda > 0$. Thus, $q(\mathbf{x}) = 0$ whenever $x_1^4 + x_2^4 + x_3^4 = y_1^4 + y_2^4 + y_3^4 = \overline{M}_4$. Thus, there exists a 2-point $\mathbf{z} = (a, a, b)$ satisfying $2a^2 + b^2 = 1$ and $2a^4 + b^4 = \overline{M}_4$ by Lemma 2.6. But this means $p(\mathbf{z}) = -\mu < 0$, a contradiction.

Now let $n \geq 4$, and suppose p is nonnegative at all $\mathbf{x} \in \Omega_{n,2}$, as well as at all 3-points with exactly three nonzero coordinates. As above, let $-\mu$, $\mu > 0$, denote the minimum value p takes on the unit sphere S^{n-1}. Then any point $\mathbf{y}$ such that $p(\mathbf{y}) = -\mu$ necessarily satisfies the conditions of Lemma 2.5, and thus $q(\mathbf{x}) = p(\mathbf{x}) + \mu M_2^4(\mathbf{x}) = \lambda t_{n,\mathbf{y}}$, $\lambda > 0$. We then employ Lemma 2.6 again to show that no such forms p can exist.

So, for $n \geq 4$, p is psd if and only if p is nonnegative on $\Omega_{n,2}$ and at all points $(s, u, v, 0, \ldots, 0)$ and their permutations $(s > u > v > 0)$. But $p(s, u, v, 0, \ldots, 0)$ is an even symmetric ternary octic. Hence, being nonnegative on $\{(s, u, v, 0, \ldots, 0)\}$ is equivalent to being nonnegative on $\{(s, u, 0, \ldots, 0)\} \cup \{(s, s, u, 0, \ldots, 0)\}$, a subset of $\Omega_{n,2}$. This proves that $\Omega_{n,2}$ itself is a test set for even symmetric n-ary octics. ∎

Thus, we may determine whether or not an even symmetric n-ary octic is psd by studying a number of binary forms. We conclude this section by examining one of these forms. For $p = [\alpha, \beta, \gamma, \delta, (\varepsilon)] \in S^e_{n,8}$, we define

$$\hat{p}(x, y) = p(x, y, 0, \ldots, 0) = \alpha x^8 + \beta x^6 y^2 + \gamma x^4 y^4 + \beta x^2 y^6 + \alpha y^8. \quad (2.7)$$

The two theorems below describe how the zeros of $\hat{p}$ determine relationships among the coefficients of p.

THEOREM 2.4. *Let* $p = [\alpha, \beta, \gamma, \delta, (\varepsilon)] \in PS^e_{n,8}$, $n \geq 3$, *and suppose that* $\alpha > 0$ *and* $\hat{p}(u, v) = 0$, $u, v \neq 0$.

1. *If* $u \neq v$, *then* $\beta < -4\alpha$ *and* $\gamma = \beta^2/4\alpha + 2\alpha$.
2. *If* $u = v$, *then* $\beta \geq -4\alpha$ *and* $\gamma = -2\alpha - 2\beta$.

Proof. (1) If $\hat{p}(1, \sqrt{u}) = \hat{p}(\sqrt{u}, 1) = 0$, $u \neq 0, 1$, then

$$\begin{aligned} \hat{p}(x, y) &= \alpha u^{-2}(ux^2 - y^2)^2(x^2 - uy^2)^2 \\ &= \alpha(x^4 - (u + u^{-1})x^2y^2 + y^4)^2. \end{aligned} \quad (2.8)$$

Upon expanding (2.8) and comparing coefficients, we find that $\frac{\beta}{\alpha} = -2(u + u^{-1})$ and $\frac{\gamma}{\alpha} = u^2 + u^{-2} + 4 = (\frac{\beta}{2\alpha})^2 + 2$. Since $u + u^{-1} > 2$ when $u \neq 0, 1$, the result follows.

(2) If $\hat{p}(1, 1) = 0$, then

$$\hat{p}(x, y) = \alpha(x^2 - y^2)^2(x^4 + tx^2y^2 + y^4), \qquad t \geq -2, \quad (2.9)$$

since p is psd and $\hat{p}(x, y)$ is symmetric. Expanding (2.9) and comparing coefficients, we find that $\frac{\beta}{\alpha} = t - 2$. Hence, $\beta \geq -4\alpha$. Also note that $\hat{p}(1, 1) = 2\alpha + 2\beta + \gamma$. ∎

THEOREM 2.5. *Suppose* $p = [\alpha, \beta, \gamma, \delta, (\varepsilon)] \in S^e_{n,8}$, $n \geq 3$. *Then* $\hat{p}(x, y)$ *is psd if and only if one of the following sets of conditions holds*:

1. $\alpha = 0, \beta \geq 0, \gamma \geq -2\beta$;
2. $\alpha > 0, \beta < -4\alpha, \gamma \geq \beta^2/4\alpha + 2\alpha$;
3. $\alpha > 0, \beta \geq -4\alpha, \gamma \geq -2\beta - 2\alpha$.

Proof. Since $\hat{p}(1, 0) = \alpha$, $\hat{p}(x, y)$ is psd only if $\alpha \geq 0$.

(1) Suppose that $\hat{p}(x, y)$ is psd and $\hat{p}(1, 0) = \alpha = 0$. If $\beta \neq 0$, then $\hat{p}(x, 1) = \beta x^2(x^4 + \frac{\gamma}{\beta}x^2 + 1)$, which is psd only when $\beta > 0$ and $\gamma \geq -2\beta$. If $\beta = 0$, then $\hat{p}(x, 1) = \gamma x^4$, and so $\gamma \geq 0$.
Suppose the given conditions hold; then $\gamma' = \gamma - 2\beta \geq 0$. We then see that $\hat{p}$ is psd by observing that $\hat{p}(x, 1) = \beta x^2(x^2 - 1)^2 + \gamma' x^4$.

(2) Suppose $\alpha > 0$ and $\beta < -4\alpha$, and let $\gamma' = \gamma - \beta^2/4\alpha - 2\alpha$. Then $\hat{p}(x, 1) = \alpha(x^4 + (\beta/2\alpha)x^2 + 1)^2 + \gamma' x^4$. Since $x^4 + (\beta/2\alpha)x^2 + 1 = 0$ has real solutions in this case, $\hat{p}(x, y)$ is psd if and only if $\gamma' \geq 0$.

(3) Suppose $\alpha > 0$ and $\beta \geq -4\alpha$. Let $\beta' = \beta + 4\alpha$ and let $\gamma' = \gamma + 2\alpha + 2\beta$. Then $\hat{p}(x, 1) = \alpha(x^2 - 1)^4 + \beta'(x^2 - 1)^2 + \gamma' x^4$, which is psd if and only if $\gamma' \geq 0$. ∎

3. A TEST FOR TERNARY DECICS

We now prove a result for $S^e_{3,10}$ which is analogous to Theorem 2.3. Theorem 3.1 below is a special case of Corollary 2.7 in [2].

DEFINITION 3.1. A *transversal zero* of a function f is a point $\mathbf{x} \in \mathbb{R}^n$ such that f is indefinite in every neighborhood of $\mathbf{x}$.

The simplest example of a transversal zero is any zero of a linear form. Let $Z_t(f)$ denote the set of transversal zeros of a polynomial $f \in \mathbb{R}[x_1, x_2, \ldots, x_n]$. We can describe the transversal zeros of a polynomial in terms of the transversal zeros of its irreducible factors.

THEOREM 3.1. *Suppose* $p \not\equiv 0 \in \mathbb{R}[x_1, x_2, \ldots, x_n]$. *Let* $p = \lambda\, p_1^{k_1} \cdots p_s^{k_s}$ *be the factorization of* p *into powers of distinct irreducible polynomials* $p_1, \ldots, p_s$, *with* $\lambda \in \mathbb{R}$. *Then*

$$Z_t(p) = \bigcup_{k_i \text{ odd}} Z_t(p_i). \tag{3.1}$$

We write elements of $S^e_{3,10}$ as

$$\begin{aligned} p(x,y,z) &:= \langle \alpha, \beta, \gamma, \delta, \varepsilon \rangle \\ &= \alpha(x^{10}+y^{10}+z^{10}) \\ &\quad + \beta(x^8y^2+x^2y^8+x^8z^2+x^2z^8+y^8z^2+y^2z^8) \\ &\quad + \gamma(x^6y^4+x^4y^6+x^6z^4+x^4z^6+y^6z^4+y^4z^6) \qquad (3.2) \\ &\quad + \delta(x^6y^2z^2+x^2y^6z^2+x^2y^2z^6) \\ &\quad + \varepsilon(x^4y^4z^2+x^4y^2z^4+x^2y^4z^4) \\ &= \alpha g_1 + \beta g_2 + \gamma g_3 + \delta g_4 + \varepsilon g_5. \end{aligned}$$

LEMMA 3.1. *For $\theta \in (\frac{1}{3}, 1)$, define*

$$\begin{aligned} f_\theta(x,y,z) &= \frac{1}{1-\theta}((x^4+y^4+z^4) - \theta(x^2+y^2+z^2)^2) \\ &= \sum^3 x^4 - \frac{2\theta}{1-\theta}\sum^3 x^2y^2. \end{aligned}$$

Then f_θ is irreducible if and only if $\theta \neq \frac{1}{2}$.

Proof. Since $f_{1/2} = (x+y-z)(x-y+z)(x-y-z)(x+y+z)$, we need only demonstrate that if f_θ is reducible, then $\theta = \frac{1}{2}$.

(1) Suppose f_θ has a linear factor $a\,x + b\,y + c\,z$. Since the coefficients of x^4, y^4, z^4 are 1, we may assume that $c = 1$ and $a, b \neq 0$. This implies that $(1-\theta)f_\theta(x, y, -a\,x - b\,y) = 0$, so

$$x^4 + y^4 + (ax+by)^4 = \theta(x^2+y^2+(ax+by)^2)^2. \qquad (3.3)$$

Expanding (3.3) and taking the quotient of the coefficients of x^3y and xy^3 on both sides of this equation, we discover that $a^2 = b^2 =$ (say) r and $\theta = (1+r^2)/(1+r)^2$. When we then compare coefficients of x^2y^2 on both sides of (3.3), we see that

$$6r^2 = \frac{1+r^2}{(1+r)^2}(2(1+r)^2 + 4r^2).$$

The only real, positive root of this equation is $r = 1$, which means that $\theta = \frac{1}{2}$.

(2) Suppose f_θ has two irreducible quadratic factors q_1 and q_2. We may scale so that the coefficient of x^2 in q_1 and q_2 is 1. Additionally, f_θ is even in z, so either

(a) $q_1(x,y,-z) = q_1(x,y,z)$, in which case both q_1 and q_2 are even in z, or

(b) $q_1(x,y,-z) = q_2(x,y,z)$.

Suppose (b) is true. Let $q_1(x, y, z) = (x^2 + bxy + cy^2 + dz^2) + (ex + gy)z$. If we set $z = 0$, then we find that

$$(x^2 + b\,xy + c\,y^2)^2 = x^4 + y^4 - \left(\frac{2\theta}{1-\theta}\right)x^2y^2.$$

Since the right side of this equation is even, $b = 0$, in which case $c^2 = 1$ and $2c = \frac{-2\theta}{1-\theta}$. This can happen only when $\theta = \frac{1}{2}$, a contradiction of the assumption that q_1 and q_2 are irreducible. Thus, we conclude that q_1 and q_2 are even in z. Similarly, these quadratics must be even in x and y as well. So suppose that

$$f_\theta(x, y, z) = (x^2 + a\,y^2 + b\,z^2)(x^2 + c\,y^2 + d\,z^2).$$

Upon expanding this, we find that $a\,c = b\,d = 1$ and $a + c = b + d = -\frac{2\theta}{1-\theta}$. Together, these two facts imply that $\{a, b, c, d\}$ are all negative. But the positive number $a\,d + b\,c = -\frac{2\theta}{1-\theta}$, too, which is a contradiction. Hence f_θ does not factor in this way. ∎

LEMMA 3.2. *The ternary form* f_θ, $\frac{1}{3} < \theta < 1$, *has a transversal zero.*

Proof. Recall that $f_{1/2}$ factors into four distinct linear terms, each of which has transversal zeros. So suppose $\theta \in (\frac{1}{3}, 1)$, $\theta \neq \frac{1}{2}$. Let $Y_\theta = \{\mathbf{x} \in S^2 : M_4(\mathbf{x}) = \theta\}$. Note that Y_θ is the solution set on S^2 of $f_\theta(\mathbf{x}) = 0$. Since $f_\theta(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}) < 0$ and $f_\theta(1, 0, 0) > 0$ for any θ we are considering, Y_θ is always nonempty by the intermediate value theorem. In addition, Y_θ is closed, since f_θ is continuous and Y_θ is the inverse image of a closed set in the range. When we apply the method of Lagrange multipliers to f_θ subject to the constraint $\mathbf{x} \in S^2$, we find that the only points in the first octant and its boundary where extrema can occur are (up to permutation) $(1, 0, 0)$, $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)$, and $(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$. The corresponding values of θ for which these points are zeros of f_θ are $\theta = 1$, $\theta = \frac{1}{2}$, and $\theta = \frac{1}{3}$, respectively. Thus, any point $\mathbf{y} \in Y_\theta$, $\theta \in (\frac{1}{3}, 1)$, $\theta \neq \frac{1}{2}$, is a transversal zero of f_θ. ∎

THEOREM 3.2. *The set* $\Omega_{3,2}$ *is a test set for* $S^e_{3,10}$.

Proof. Let $\mathbf{y} = (y_1, y_2, y_3)$, $y_1, y_2, y_3 \geq 0$, be a nontrivial zero of p. Let $J(\mathbf{y})$ denote the Jacobian of $\{g_1, g_2, g_3, g_4, g_5\}$ at $\mathbf{y}$; it is a 3×5 matrix. Recall the notation $\overline{M}_r = M_r(\mathbf{y}) = \sum^3 {y_i}^r$.

Define

$$r_{1,\mathbf{y}}(x, y, z) = T^2_{3,\mathbf{y}}(x, y, z)\, M_2(x, y, z),$$

$$r_{2,\mathbf{y}}(x, y, z) = T_{3,\mathbf{y}}\,(\overline{M}_2\overline{M}_4 \cdot M_6(x, y, z) - \overline{M}_6 \cdot M_2(x, y, z)M_4(x, y, z))$$

(recall $T_{3,\mathbf{y}} = \overline{M}_2^2 M_4 - \overline{M}_4 M_2^2$; cf. (2.3)). Observe that $r_{1,\mathbf{y}}$ and $r_{2,\mathbf{y}}$ are linearly independent elements of $S^e_{3,10}$ which are singular at $\mathbf{y}$. If rank $J(\mathbf{y}) = 3$, then the subspace of $S^e_{3,10}$ which is singular at $\mathbf{y}$ is two-dimensional and is spanned by $r_{1,\mathbf{y}}$ and $r_{2,\mathbf{y}}$.

Otherwise, rank $J(\mathbf{y}) < 3$, so each of the $\binom{5}{3}$ 3×3 submatrices of $J(\mathbf{y})$ has determinant zero. We can show with the aid of a computer that the last three columns of $J(\mathbf{y})$ have determinant

$$40\, y_1^3 y_2^3 y_3^3 (y_1^2 - y_2^2)(y_1^2 - y_3^2)(y_2^2 - y_3^2)(y_1^2 + y_2^2 + y_3^2)^2 (y_1^2 y_2^2 + y_1^2 y_3^2 + y_2^2 y_3^2).$$

Thus, rank $J(\mathbf{y}) < 3$ if and only if $\mathbf{y} \in \Omega_{3,2}$.

Suppose $p \in S^e_{3,10}$ is nonnegative on $\mathbf{x} \in \Omega_{3,2}$ but is not psd. As in the proof of Theorem 2.3, let $-\mu$, $\mu > 0$, denote the minimum p takes on the unit sphere S^2, and let $\mathbf{y} = (y_1, y_2, y_3) \in S^2$ be such that $p(\mathbf{y}) = -\mu$. Then $q(\mathbf{x}) = p(\mathbf{x}) + \mu M_2^5(\mathbf{x})$ is psd with $\mathbf{y}$ satisfying $q(\mathbf{y}) = 0$. Since $\mathbf{y}$ is not a k-point, $k \leq 2$, $J(\mathbf{y})$ is of rank 3 and $q(x, y, z) = \lambda_1 r_{1,\mathbf{y}}(x, y, z) + \lambda_2 r_{2,\mathbf{y}}(x, y, z)$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$. Observe that $\frac{1}{3} < \overline{M}_4 < 1$, since the equations $\overline{M}_4 = \frac{1}{3}$ and $\overline{M}_4 = 1$ are satisfied only by 1- or 2-points. Lemma 3.1 implies that, on S^2, $T_{3,\mathbf{y}}$ is irreducible when $\overline{M}_4 \neq \frac{1}{2}$, and Lemma 3.2 implies this term always has a transversal zero. We now apply Theorem 3.1, which implies that if q is psd and divisible by $T_{3,\mathbf{y}}$, then it is also divisible by $T^2_{3,\mathbf{y}}$ (the argument holds for $\overline{M}_4 = \frac{1}{2}$, since we have noted that then each factor of $T_{3,\mathbf{y}}$ has a transversal zero). Here, this means that $\lambda_2 = 0$ and thus

$$p = \lambda_1 M_2 (M_4 - \overline{M}_4 M_2^2)^2 - \mu M_2^5.$$

The proof may now be completed exactly along the lines of the case $n = 3$ of Theorem 2.3. ∎

We have seen that the set $\Omega_{3,2}$ is a test set for even symmetric ternary forms of degree 8 and 10. We now prove that 10 is in fact the highest degree for which this set works.

LEMMA 3.3. *Let* $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ *be defined by*

$$\varphi\colon (x, y, z) \mapsto (x^2 + y^2 + z^2, x^4 + y^4 + z^4, x^6 + y^6 + z^6).$$

Then $\mathbf{y} = (y_1, y_2, y_3)$ *is mapped to the interior of* $\varphi(\mathbb{R}^3)$ *if and only if* $\mathbf{y}$ *has three nonzero coordinates distinct in absolute value.*

REMARK 3.1. This is a special case of a theorem by Ursell in [13].

Proof. The Jacobian of φ at $\mathbf{y}$ has determinant

$$-48 y_1 y_2 y_3 (y_1^2 - y_2^2)(y_1^2 - y_3^2)(y_2^2 - y_3^2).$$

∎

THEOREM 3.3. *If* $2d \geq 12$, *then* $\Omega_{3,2}$ *is not a test set for* $S^e_{3,2d}$.

Proof. It suffices to demonstrate the existence of $h \in S^e_{3,12}$ such that h is nonnegative on $\Omega_{3,2}$ but $h \notin PS^e_{3,12}$, since $M_2^{i-6}h$, $i \in \mathbb{Z}$, $i \geq 7$, is a form in $S^e_{3,2i}$ which also satisfies this condition.

Let $\mathbf{y}$ be in the first octant of $\mathbb{R}^3$. We define an even symmetric ternary dodecic $s_{\mathbf{y}}$ which is singular at $\mathbf{y}$:

$$s_{\mathbf{y}}(x, y, z) = (\overline{M}_2^3 \cdot M_6 - \overline{M}_6 \cdot M_2^3)^2 + (\overline{M}_2^3 \cdot M_4 M_2 - \overline{M}_4 \overline{M}_2 \cdot M_2^3)^2.$$

Clearly, $s_{\mathbf{y}}$ is psd. For $\mathbf{x} \in S^2$, we observe that $s_y(\mathbf{x}) = 0$ if and only if $\sum x_i^4 = \overline{M}_4$ and $\sum x_i^6 = \overline{M}_6$.

Fix $\theta \in (\frac{1}{3}, 1)$. Recall the set Y_θ introduced in Lemma 3.2. As $\mathbf{t}$ ranges over Y_θ, $\zeta_2(\theta) \leq M_6(\mathbf{t}) \leq \zeta_1(\theta)$ for some numbers ζ_1, ζ_2 which depend on θ (note that $\frac{1}{9} < \zeta_2 < \zeta_1 < 1$). Now, select some $\mathbf{v} \in Y_\theta$ such that $\zeta_2(\theta) < M_6(\mathbf{v}) < \zeta_1(\theta)$. Then $\mathbf{v}$ is not a k-point, $k \leq 2$, by Lemma 3.3. Hence, for $\mathbf{z} \in \Omega_{3,2}$,

$$s_{\mathbf{v}}(\mathbf{z}) = (M_6(\mathbf{z}) - M_6(\mathbf{v}))^2 + (M_4(\mathbf{z}) - \theta)^2 \geq \rho > 0,$$

since $M_6(\mathbf{z}) \neq M_6(\mathbf{v})$ when $M_4(\mathbf{z}) = \theta$, and $M_4(\mathbf{z}) - \theta \neq 0$ otherwise. Choose $0 < \lambda < \rho$. Then $s_{\mathbf{v},\lambda} = s_{\mathbf{v}}(x, y, z) - \lambda M_2^6$ is nonnegative on $\Omega_{3,2}$, but is not psd, since $s_{\mathbf{v},\lambda}(\mathbf{v}) = -\lambda < 0$. ∎

4. EVEN SYMMETRIC TERNARY OCTICS

Recall that the set $S^e_{3,8}$ consists of forms

$$\begin{aligned} p(x, y, z) &:= [\alpha, \beta, \gamma, \delta] \\ &= \alpha(x^8 + y^8 + z^8) + \beta(x^6y^2 + x^2y^6 + x^6z^2 + x^2z^6 + y^6z^2 + y^2z^6) \\ &\quad + \gamma(x^4y^4 + x^4z^4 + y^4z^4) + \delta(x^4y^2z^2 + x^2y^4z^2 + x^2y^2z^4). \end{aligned}$$

By Theorem 2.3, we know that $p \in S^e_{3,8}$ is psd if and only if the two binary forms

$$p(x, y, 0) = \alpha x^8 + \beta x^6 y^2 + \gamma x^4 y^4 + \beta x^2 y^6 + \alpha y^8 \tag{4.1}$$

$$\begin{aligned} p(x, x, y) = (2\alpha + 2\beta + \gamma)x^8 + (2\beta + 2\delta)x^6y^2 \\ + (2\gamma + \delta)x^4y^4 + 2\beta\, x^2y^6 + \alpha\, y^8 \end{aligned} \tag{4.2}$$

are psd. We seek to find all extremal elements of $PS^e_{3,8}$, and in doing so we shall prove that $PS^e_{3,8} = \Sigma S^e_{3,8}$. By Lemma 1.1 and homogeneity, it suffices to look for forms with nontrivial zeros of type $(1, z, 0)$ and $(1, 1, z)$.

First, we name some important elements and families of elements of $\Sigma S^e_{3,8}$, and then proceed to demonstrate (with a suitable restriction on the parameters) that they comprise all the extremal forms of $PS^e_{3,8}$. The given representations as sums of squares can be verified easily, and also show that these forms are psd.

$$A(x,y,z) = [0,1,-2,0] = \sum^{6} x^6y^2 - 2\sum^{3} x^4y^4$$
$$= \sum^{3}(x^2-y^2)^2x^2y^2; \tag{4.3}$$

$$B(x,y,z) = [0,0,1,-1] = \sum^{3} x^4y^4 - \sum^{3} x^4y^2z^2$$
$$= \tfrac{1}{2}\sum^{3}(x^2-y^2)^2z^4; \tag{4.4}$$

$$C(x,y,z) = [0,0,0,1] = \sum^{3} x^4y^2z^2; \tag{4.5}$$

$$D_t(x,y,z) = [1, 2t, t^2+2, 2t^2+2t]$$
$$= (x^4+y^4+z^4+t(x^2y^2+x^2z^2+y^2z^2))^2; \tag{4.6}$$

$$E_u(x,y,z) = [1, -(u+1), u^2+2u, -u^2+1]$$
$$= \tfrac{1}{6}\sum^{3}(2x^4-y^4-z^4-(u+1)(x^2y^2+x^2z^2-2y^2z^2))^2. \tag{4.7}$$

In what follows, our interest in the families D_t and E_u will be limited to the ranges $t < -1$ and $u \geq 0$, respectively. Observe that $D_{-1} = E_1$.

The following simple lemmas will be useful here and in Section 5.

LEMMA 4.1. *Suppose f and g are psd forms. If there exist points* $\mathbf{w}$, $\mathbf{w}'$ *such that $f(\mathbf{w}) = 0$, $g(\mathbf{w}) > 0$, $f(\mathbf{w}') > 0$, $g(\mathbf{w}') = 0$, then $h_{\lambda,\mu} = \lambda f + \mu g$ is psd if and only if $\lambda \geq 0$, $\mu \geq 0$.*

LEMMA 4.2. *Let $f(x) = Ax^2 + Bx + C$. Then $f(x) \geq 0$ for $x \geq 0$ if and only if $A \geq 0$, $C \geq 0$, and $B \geq -2\sqrt{AC}$.*

We break our discussion into five parts, based on the different types of zeros a form $p \in S^e_{3,8}$ may have: (a) $(1,0,0)$; (b) $(1,1,0)$; (c) $(1,z,0)$, $z \neq 0,1$; (d) $(1,1,1)$; and (e) $(1,1,z)$, $z \neq 0,1$. The five lemmas which we present now each provide a list of extremal forms in $PS^e_{3,8}$ which share one of these zero types.

LEMMA 4.3. *If $p = [\alpha,\beta,\gamma,\delta] \in PS^e_{3,8}$ and $p(1,0,0) = 0$, then p is a nonnegative linear combination of A, B, and C.*

Proof. By assumption, $\alpha = p(1,0,0) = 0$. Thus, the subspace of forms in $S^e_{3,8}$ with a zero at $(1,0,0)$ is three-dimensional and the sos forms A, B, and C comprise a basis for it.

Let $\eta = 2\beta + \gamma$ and $\zeta = 2\beta + \gamma + \delta = \eta + \delta$. Then $p = \beta A + \eta B + \zeta C$. By Theorem 2.5, if p is psd, then $\beta \geq 0$, $\eta \geq 0$. Additionally, $p(1,1,1) = 3\zeta \geq 0$. ∎

In the rest of the paper, we assume that $p(1,0,0) > 0$, and we will scale so that $\alpha = 1$ when convenient.

LEMMA 4.4. *If $p = [1, \beta, \gamma, \delta] \in PS^e_{3,8}$ and $p(1,1,0) = 0$, then p is a nonnegative linear combination of A, C, D_{-2}, and E_0.*

Proof. Let $p \in PS^e_{3,8}$. By Theorem 2.4, $p = [1, \beta, -2\beta - 2, \delta]$, $\beta \geq -4$. Let $\hat{\delta} = \max(1, -\beta)$. We claim that $\delta \geq \hat{\delta}$. First, observe that

$$p(1,1,z) = z^8 + 2\beta z^6 + (-4\beta + \delta - 4)z^4 + (2\beta + 2\delta)z^2.$$

For this to be psd, the last coefficient must be nonnegative. Thus, $\delta \geq -\beta$. Also, $p(1,1,1) = 3\delta - 3$, so $\delta \geq 1$, as well. We now have two cases to consider.

(1) $-4 \leq \beta \leq -1$: Here, $\hat{\delta} = -\beta$, so

$$\begin{aligned} p &= [1, \beta, -2\beta - 2, -\beta] \\ &= \lambda[1, -4, 6, 4] + (1-\lambda)[1, -1, 0, 1] + (\delta - \hat{\delta})[0,0,0,1] \\ &= \lambda D_{-2} + (1-\lambda)E_0 + (\delta - \hat{\delta})C, \end{aligned}$$

where $0 < \lambda = \frac{-1-\beta}{3} < 1$. Hence p is a nonnegative linear combination of the given forms.

(2) $\beta > -1$: Here, $\hat{\delta} = 1$. Write $\beta = -1 + \lambda, \lambda > 0$, so $p = [1, -1 + \lambda, -2\lambda, 1] = E_0 + \lambda A + (\delta - \hat{\delta})C$. Again, we have a nonnegative linear combination of the given forms. ∎

LEMMA 4.5. *If $p = [1, \beta, \gamma, \delta] \in PS^e_{3,8}$ and $p(1, \sqrt{u}, 0) = 0$, $u \neq 0, 1$, then p is a nonnegative linear combination of C and D_t, where $t = -(u + u^{-1}) < -2$.*

Proof. Let $p \in PS^e_{3,8}$. By Theorem 2.4, we know that $p = [1, \beta, \beta^2/4 + 2, \delta]$, where $\beta = -2(u + u^{-1}) < -4$. Let $t = \beta/2$. Recall that $p(x,y,0) = (x^4 + tx^2y^2 + y^4)^2$, and so by symmetry,

$$\begin{aligned} p(x,y,z) &= (x^4 + y^4 + z^4 + t(x^2y^2 + x^2z^2 + y^2z^2))^2 \\ &\quad + \delta' x^2y^2z^2(x^2 + y^2 + z^2) \\ &= D_t + \delta' C, \qquad \text{where } \delta' = \delta - (2t^2 + 2t). \end{aligned}$$

We claim that $\delta' \geq 0$. Observe that

$$p(1,1,z) = (z^4 + 2tz^2 + t + 2)^2 + \delta'(z^4 + 2z^2).$$

Let z_t denote the positive real solution of $z^2 + 2tz + t + 2 = 0$ for $t < -2$; then $p(1, 1, \sqrt{z_t}) = \delta'(z_t^2 + 2z_t) \geq 0$ implies $\delta' \geq 0$, and so p is a nonnegative linear combination of D_t and C. ∎

LEMMA 4.6. *If $p = [\alpha, \beta, \gamma, \delta] \in PS^e_{3,8}$ and $p(1, 1, 1) = 0$, then p is a nonnegative linear combination of A, B, and a finite number of members of the family E_u.*

Proof. Let $p \in PS^e_{3,8}$. We have $p(1, 1, 1) = 3(\alpha + 2\beta + \gamma + \delta) = 0$. A basis for the solution space of this homogeneous equation in $\{\alpha, \beta, \gamma, \delta\}$ is given by the coefficients of E_0, A, and B. Thus, any $p = [1, \beta, \gamma, \delta] \in S^e_{3,8}$ satisfying $p(1, 1, 1) = 0$ can be expressed as $p_{\lambda,\mu} = E_0 + \lambda A + \mu B = [1, \lambda - 1, -2\lambda + \mu, -\mu + 1]$. We now determine the conditions on λ and μ such that p is psd. Since $p(1, 1, 0) = \mu$, we must have $\mu \geq 0$. If $\lambda \geq 0$, then $p_{\lambda,\mu}$ is a nonnegative linear combination of sos forms, and so is psd. Thus, fix $\lambda < 0$ and consider

$$p_{\lambda,\mu}(1, 1, z) = (z^2 - 1)^2(z^4 + 2\lambda z^2 + \mu).$$

This is psd if and only if $\mu \geq \lambda^2$ by Lemma 4.2. Additionally, $p_{\lambda,\lambda^2} = E_{-\lambda}$ for $\lambda < 0$, which is sos, and so in this case p is a nonnegative linear combination of $E_{-\lambda}$ and B. ∎

LEMMA 4.7. *Suppose $p = [\alpha, \beta, \gamma, \delta] \in PS^e_{3,8}$ and $p(1, 1, \sqrt{u}) = 0$, $u \neq 0, 1$. Define $\hat{u} = -(u^2 + 2)/(2u + 1)$. Then p is a nonnegative linear combination of $D_{\hat{u}}$ and E_u.*

Proof. When the system of equations in $\{\gamma, \delta\}$

$$\begin{aligned} 0 = p(1, 1, \sqrt{u}) &= \alpha u^4 + 2\beta u^3 + (2\gamma + \delta)u^2 \\ &\quad + (2\beta + 2\delta)u + (2\alpha + 2\beta + \gamma), \\ 0 = \frac{\partial p}{\partial x}(1, 1, \sqrt{u}) &= 2\beta u^3 + (4\gamma + 2\delta)u^2 \\ &\quad + (6\beta + 6\delta)u + (8\alpha + 8\beta + 4\gamma) \end{aligned}$$

is solved, we have a two-dimensional subspace of solutions given in terms of $\{\alpha, \beta\}$. We now show that this subspace is spanned by the specified members of the families $D_{\_}$ and $E_{\_}$.

First, we note that both $D_{\hat{u}}$ and E_u possess $(1, 1, \sqrt{u})$ as a zero. A lengthy calculation (which we omit) demonstrates that

$$D_{\hat{u}}(1, 1, z) = (2u + 1)^{-2}(z^2 - u)^2((2u + 1)z^2 + u - 4)^2. \tag{4.8}$$

One may check directly that $(1, 1, \sqrt{u})$ is a zero of E_u by substituting into the sum-of-squares representation given in (4.7).

We now show that $D_{\hat{u}}$ and E_u are linearly independent by determining that their zero sets are not identical. The zero set of E_u is $\{(1, 1, 1), (1, 1, \sqrt{u})\}$. For $0 < u < 4$, $u \neq 1$ (which corresponds to $-2 < \hat{u} < -1$), we see from (4.8) above that the zero set of $D_{\hat{u}}$ is $\{(1, 1, \sqrt{u}), (1, 1, \sqrt{\frac{4-u}{2u+1}})\}$. For $u \geq 4$ (which corresponds to $\hat{u} \leq -2$), the zero set of $D_{\hat{u}}$ is $\{(1, 1, \sqrt{u}), (1, \overline{u}, 0)\}$, where

$$\overline{u} = \frac{u^2 + 2 + (u+2)\sqrt{u(u-4)}}{2(2u+1)},$$

a real solution of $D_{u^*}(1, y, 0) = (y^4 + u^* y^2 + 1)^2 = 0$ (the other solution is $\overline{u}^{-1}$).

Finally, define $F_{\lambda,\mu,u} = \lambda D_{\hat{u}} + \mu E_u$. To prove that $F_{\lambda,\mu,u}$ is psd if and only if λ and μ are both nonnegative, we consider two cases.

1. $0 < u < 4$, $u \neq 1$. Let $\mathbf{w} = (1, 1, \sqrt{\frac{4-u}{2u+1}})$, $\mathbf{w}' = (1, 1, 1)$. Apply Lemma 4.1.

2. $u \geq 4$. Let $\mathbf{w} = (1, \overline{u}, 0)$, where $\overline{u}$ is as defined above, $\mathbf{w}' = (1, 1, 1)$. Apply Lemma 4.1. ∎

THEOREM 4.1. $PS^e_{3,8} = \Sigma S^e_{3,8}$.

Proof. In Lemmas 4.3, 4.4, 4.5, 4.6, and 4.7, we found all extremal psd forms for each of the five zero types. Since all of these have sos representations, the result follows. ∎

5. EVEN SYMMETRIC TERNARY DECICS

Recall the five-tuple notation $p(x, y, z) := \langle \alpha, \beta, \gamma, \delta, \varepsilon \rangle$ for elements of $S^e_{3,10}$ introduced in Section 3. By Theorem 3.2, $p \in PS^e_{3,10}$ if and only if the two binary forms

$$\begin{aligned} p_1(x, y) &= p(x, y, 0) \\ &= \alpha x^{10} + \beta x^8 y^2 + \gamma x^6 y^4 + \gamma x^4 y^6 + \beta x^2 y^8 + \alpha y^{10}, \qquad (5.1) \end{aligned}$$

$$\begin{aligned} p_2(x, y) &= p(x, x, y) \\ &= (2\alpha + 2\beta + 2\gamma) x^{10} + (2\beta + 2\delta + \varepsilon) x^8 y^2 \\ &\quad + (2\gamma + 2\varepsilon) x^6 y^4 + (2\gamma + \delta) x^4 y^6 + 2\beta x^2 y^8 + \alpha y^{10} \qquad (5.2) \end{aligned}$$

are psd. As before, we examine forms which have a nontrivial zero.

We first prove results for $p \in S^e_{3,10}$ analogous to Theorems 2.4 and 2.5.

THEOREM 5.1. *Let* $p = \langle \alpha, \beta, \gamma, \delta, \varepsilon \rangle \in PS^e_{3,10}$, *and suppose that* $\alpha > 0$ *and* $p_1(u, v) = 0$, $u, v \neq 0$.

1. *If* $u \neq v$, *then* $\beta < -3\alpha$ *and* $\gamma = \beta^2/4\alpha + \frac{1}{2}\beta + \frac{5}{4}\alpha$.
2. *If* $u = v$, *then* $\beta \geq -3\alpha$ *and* $\gamma = -\alpha - \beta$.

Proof. (1) If $p_1(1, \sqrt{u}) = p_1(\sqrt{u}, 1) = 0$, $u \neq 0, 1$, then

$$\begin{aligned} p_1(x, y) &= \alpha u^{-2}(ux^2 - y^2)^2(x^2 - uy^2)^2(x^2 + y^2) \\ &= \alpha(x^4 - (u + u^{-1})x^2y^2 + y^4)^2(x^2 + y^2). \end{aligned} \tag{5.3}$$

Upon expanding (5.3), and comparing coefficients with (5.1), we find that $\frac{\beta}{\alpha} = -2(u + u^{-1}) + 1$ and $\frac{\gamma}{\alpha} = u^2 - 2u + 4 - 2u^{-1} + u^{-2} = (\frac{\beta}{2\alpha})^2 + \frac{\beta}{2\alpha} + \frac{5}{4}$. Since $u + u^{-1} > 2$ when $u \neq 0, 1$, the result follows.

(2) If $p_1(1, 1) = 0$, then

$$p_1(x, y) = \alpha(x^2 - y^2)^2(x^6 + tx^4y^2 + tx^2y^4 + y^6), \qquad t \geq -1, \tag{5.4}$$

since $p_1(x, y)$ is symmetric. Expanding (5.4) and comparing coefficients, we find that $\frac{\beta}{\alpha} = t - 2$. Hence, $\beta \geq -3\alpha$. The result follows upon noting that $p_1(1, 1) = 2\alpha + 2\beta + 2\gamma$. ∎

THEOREM 5.2. *Let* $p = \langle \alpha, \beta, \gamma, \delta, \varepsilon \rangle \in S^e_{3,10}$. *Then* $p_1(x, y) \geq 0$ *if and only if one of the following is true*:

1. $\alpha = 0$, $\beta \geq 0$, $\gamma \geq -\beta$.
2. $\alpha \geq 0$, $\beta < -3\alpha$, $\gamma \geq \beta^2/4\alpha + \frac{1}{2}\beta + \frac{5}{4}\alpha$.
3. $\alpha \geq 0$, $\beta \geq -3\alpha$, $\gamma \geq -\alpha - \beta$.

Proof. Since $p(x, 0, 0) = \alpha x^{10} = 0$ if and only if $\alpha = 0$, we may assume $xy \neq 0$. Let $u = \frac{x}{y}$. Then $p_1(x, y) = x^5y^5(\alpha(u^5 + u^{-5}) + \beta(u^3 + u^{-3}) + \gamma(u + u^{-1}))$. Let $v^2 = (u + u^{-1})^2 - 4$; then p_1 is psd if and only if

$$\alpha v^4 + (\beta + 3\alpha)v^2 + (\alpha + \beta + \gamma) \geq 0$$

for $v^2 \geq 0$. We now apply Lemma 4.2. If $\alpha = 0$, then we must have $\beta \geq 0$, $\beta + \gamma \geq 0$. When $\alpha > 0$, then $\beta + 3\alpha \geq 0$ implies that $\alpha + \beta + \gamma \geq 0$; if $\beta + 3\alpha < 0$, then $\alpha(\alpha + \beta + \gamma) \geq \frac{1}{4}(\beta + 3\alpha)^2$, which can be rewritten $\gamma \geq \beta^2/4\alpha + \frac{1}{2}\beta + \frac{5}{4}\alpha$. ∎

We now discuss $p_2(x, y)$. Suppose

$$p_2(1, \sqrt{u}) = \frac{\partial p_2}{\partial y}(1, \sqrt{u}) = 0.$$

We may express these two equations as the linear system $AD = B$, where

$$A = \begin{bmatrix} u^3 + 2u & 2u^2 + u \\ 3u^2 + 2 & 4u + 1 \end{bmatrix}, \qquad D = \begin{bmatrix} \delta \\ \varepsilon \end{bmatrix},$$

and

$$B = \begin{bmatrix} -(\alpha(u^5+2)+\beta(2u^4+2u+2)+\gamma(2u^3+2u^2+2)) \\ -(\alpha(5u^4)+\beta(8u^3+2)+\gamma(6u^2+4u)) \end{bmatrix}.$$

The determinant of A is $-2u^2(u-1)^2$, so the system has a unique solution when $u \neq 0, 1$. This solution is linear in α, β, and γ; the expressions for $\delta_{\alpha,\beta,\gamma,u}$ and $\varepsilon_{\alpha,\beta,\gamma,u}$ are

$$\begin{aligned} -\big[&\alpha(3u^5+5u^4+5u^3+5u^2+5u+1) \\ &+\beta(4u^4+7u^3+7u^2+5u+1)+\gamma(2u^3+4u^2+5u+1)\big] \\ &\cdot [u^2(u+2)]^{-1}, \end{aligned} \tag{5.5}$$

$$\begin{aligned} \big[&\alpha(u^6+u^5+5u^4+5u^3+5u^2+2u+2) \\ &+\beta(u^5+u^4+7u^3+5u^2+2u+2)+\gamma(-u^3+3u^2+2u+2)\big] \\ &\cdot [u^2(u+2)]^{-1}, \end{aligned} \tag{5.6}$$

respectively.

We now present a number of families of forms in $PS^e_{3,10}$. For the families $Q_{u,v}$ and $R_{\beta,u}$, we give only α, β, and γ; the expressions for δ and ε can be derived from (5.5) and (5.6). We will first show that these forms are psd, and then will proceed to prove that these forms comprise the entire set of extremal forms in $PS^e_{3,10}$.

LEMMA 5.1. *The following forms are in* $PS^e_{3,10}$:

1. $F = \langle 0,0,1,-2,0\rangle = \sum^3 (x^2-y^2)^2 z^6$;
2. $G = \langle 0,0,0,1,-1\rangle = \sum^3 x^2y^2z^2(x^2-y^2)^2$;
3. $H = \langle 0,0,0,0,1\rangle = \sum^3 x^4y^4z^2$;
4. $J_u = \langle 0,1,u^2-1,-2(u+1)^2,4u+2\rangle$ $= \sum^3 x^2(y^2-z^2)^2(y^2+z^2-(u+1)x^2)^2,\ u \geq 0$;
5. $K_u = \langle 0,(5u+1)^2,(u-1)^2(u^2-12u-1),-(2u^4+72u^3+68u^2+120u+26),24u^4+4u^3+114u^2+24u+50\rangle,\ 0 \leq u \leq 7$;
6. $L_u = \langle 0,u+2,-u-2,-(4u^2+5u+3),u^3+u^2+8u+2\rangle,\ u \geq 7$;
7. $N_u = \langle 1,-u-1,u,(u+1)^2,-u(u+2)\rangle,\ 0 \leq u \leq 2$;
8. $Q_{\beta,u} = \langle 4u^2(u+2), 4u^2(u+2)\beta, u^2(u+2)(\beta^2+2\beta+5), \delta_{\beta,u}, \varepsilon_{\beta,u}\rangle$, *where* $\delta_{\beta,u}$ *and* $\varepsilon_{\beta,u}$ *are as given in* (5.5) *and* (5.6), *and* $r(u) = (u-\sqrt{u^2+u-2})^2-u-1 \leq \beta \leq -3,\ u \geq 2$.

9. $R_{u,v} = \langle 2(f_{u,v})^2, f_{u,v}\beta_{u,v}, \gamma_{u,v}, 2\delta_{u,v}, 2\varepsilon_{u,v}\rangle$, *where*

$$f_{u,v} = (5v+2)u + (v+4),$$

$$\beta_{u,v} = (-4-10v)u^2 + (5v^2+2v)u + (v^2-10v-20),$$

$$\begin{aligned}\gamma_{u,v} = {} & (v^3+8v^2+16v)u^4 - (14v^3+26v^2+40v-8)u^3 \\ & + (24v^3+80v^2+100v+8)u^2 - (10v^3-6v^2-24v-8)u \\ & - (v^3-4v^2-44v-48),\end{aligned}$$

$\delta_{u,v}$ *and* $\varepsilon_{u,v}$ *are as given in* (5.5) *and* (5.6) *and* $0 \le u < 7$, $u \ne 1$, $v \ge \max\{0, \frac{4u-10}{7-u}\}$.

REMARK 5.1. Note that $36J_1 = K_1$, $9K_0 = \frac{1}{144}K_7 = L_7$, $72N_1 = Q_{1,0}$, and $64N_2 = Q_{-3,2}$.

Proof of Lemma 5.1. Since we have presented sums of squares representations for F, G, H, and J_u, we already know these are psd.

We apply Theorem 5.2 to show that the other forms are nonnegative at all points of type $\{(x, y, 0)\}$. We can easily determine that we must verify the first set of conditions stated in that result hold for K_u and L_u, the second set of conditions hold for $Q_{\beta,u}$, and the third set of conditions hold for N_u and $R_{u,v}$. We observe that for K_u, $\beta + \gamma = u^2(u-7)^2$ and for L_u, $\beta + \gamma = 0$. We can verify that for $Q_{\beta,u}$, $\gamma = \beta^2/4\alpha + \frac{1}{2}\beta + \frac{5}{4}\alpha$. For N_u, $\alpha + \beta + \gamma = 0$, and for $R_{u,v}$, $\alpha + \beta + \gamma = u^2v(uv + 4u - 7v - 10)^2 > 0$ when $v \ge 0$.

Next, we verify that these forms are nonnegative at all points of type $\{(x, x, y)\}$. We calculate

$$K_u(x,x,y) = 2x^2(u\,x^2 - y^2)^2((u-7)x^2 + (5u+1)y^2)^2;$$

$$L_u(x,x,y) = x^2y^2(u\,x^2 - y^2)^2((u-7)x^2 + (2u+4)y^2);$$

$$N_u(x,x,y) = y^2(x-y)^2(x+y)^2(u\,x^2 - y^2)^2;$$

$$\begin{aligned}R_{u,v}(x,x,y) = {} & 2(v\,x^2 + y^2)(u\,x^2 - y^2)^2 \\ & \times ((u\,v + 4u - 7v - 10)x^2 + (5u\,v + 2u + v + 4)y^2)^2.\end{aligned}$$

Since all of these binary forms are psd on the ranges of u (and v) given in the statement of the lemma, it follows by Theorem 3.2 that K_u, L_u, N_u, and $R_{u,v}$ are psd. Lastly, $Q_{\beta,u}(1,1,z) = (z^2 - u)^2 S_{z,u}(\beta)$, where

$$\begin{aligned}S_{z,u}(\beta) = {} & (-(5u+1)z^2 + (2u+4))\beta^2 \\ & + (8u^2z^4 + (4u^3 - 28u^2 - 30u - 6)z^2 + (12u+24))\beta \\ & + (4u^3 + 8u^2)z^6 + 8u^3z^4 \\ & + (4u^4 - 20u^3 - 20u^2 - 45u - 9)z^2 + (18u+36).\end{aligned}$$

A calculation shows that

$$S_{z,u}(-3) = 4u^2z^2((u+2)z^4 + (2u^2 - 2u - 12)z^2 + (u-4)^2).$$

We see this is psd by Lemma 4.2 for all values $u \geq 2$, since this is the range of u for which $2u^2 - 2u - 12 \geq -2\sqrt{(u+2)(u-4)^2}$. Also,

$$S_{z,u}(r(u)) = 4u^2(u+2)(z^2-1)^2(z^2 + 2(u - \sqrt{u^2+u-2})^2) \geq 0$$

for $u \geq 2$.

Thus, we have that $S_{z,u}(-3) \geq 0$ and $S_{z,u}(r(u)) \geq 0$ for all $z \geq 0$ when $u \geq 2$. Also, for $z^2 > \frac{2u+4}{5u+1}$, $S_{z,u}(\beta) \geq 0$ for $r(u) \leq \beta \leq -3$, since then it is a quadratic in β with negative leading coefficient. In particular, this means that $Q_{\beta,u}(1,1,z) \geq 0$ for $z \geq 1$ on the stated ranges of u and β.

Recall that we may write any even symmetric ternary form in terms of M_2, M_4, and M_6 by Newton's theorem on symmetric functions. When $n = 3$,

$$M_{10} = \tfrac{5}{6}M_6M_4 + \tfrac{5}{6}M_6M_2^2 - -\tfrac{5}{6}M_4M_2^3 + \tfrac{1}{6}M_2^5,$$

$$M_8M_2 = \tfrac{4}{3}M_6M_2^2 + \tfrac{1}{2}M_4^2M_2 - M_4M_2^3 + \tfrac{1}{6}M_2^5.$$

Thus we may write elements of $S^e_{3,10}$ as

$$p(M_2, M_4, M_6) = a\,M_6M_4 + b\,M_6M_2^2 + c\,M_4^2M_2 + d\,M_4M_2^3 + e\,M_2^5. \tag{5.7}$$

The relationship between (5.7) and $\langle \alpha, \beta, \gamma, \delta, \varepsilon \rangle$ is

$$\begin{gathered} \alpha = a + b + c + d + e, \qquad \beta = 2b + c + 3d + 5e, \\ \gamma = a + b + 2c + 4d + 10e, \\ \delta = b + 6d + 20e, \qquad \varepsilon = 2c + 6d + 30e. \end{gathered} \tag{5.8}$$

Let $V = \{(x,y,z) \in S^2 \mid x \geq y \geq z \geq 0\}$. The boundary of V contains all points (up to permutation) of the type $(u, v, 0)$ and (u, u, v), and that the interior of the region comprises all other possible 3-tuples of nonnegative numbers (again, up to permutation). We may parameterize the boundary of V as $\partial V = C_1 \cup C_2 \cup C_3$, where $C_1 = \{(\cos t, \sin t, 0), 0 \leq t \leq \pi/4\}$, $C_2 = \{(\frac{1}{\sqrt{2}}\cos t, \frac{1}{\sqrt{2}}\cos t, \sin t), 0 \leq t \leq \pi/4\}$, and $C_3 = \{(\frac{1}{\sqrt{2}}\sin t, \frac{1}{\sqrt{2}}\sin t, \cos t), 0 \leq t \leq \pi/4\}$.

By Theorems 2.3 and 3.2 and the homogeneity of forms, an even symmetric ternary octic or decic is psd if and only if $p(1, M_4, M_6) \geq 0$, where M_4 and M_6 take the values induced by the above parameterizations.

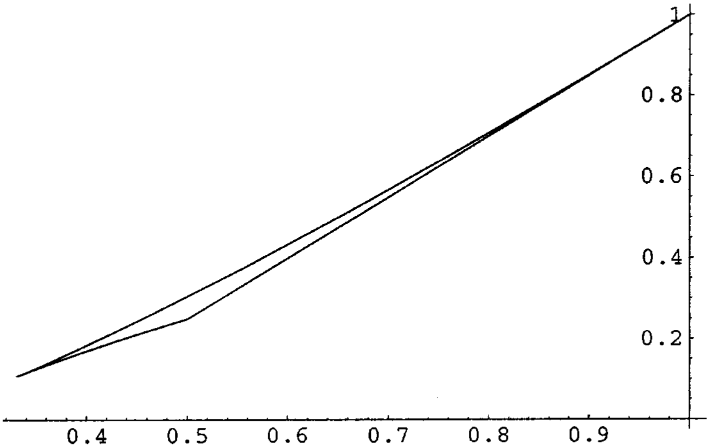

FIG. 1. A parametric representation of 3-tuples: M_4 vs. M_6.

We wish to plot M_4 vs. M_6 for these parameterizations. However, when we do this with Mathematica, a graph which is visually inconvenient results (see Fig. 1). We get a better drawing when we plot M_4 vs. $M_4 - \frac{3}{4}M_6$ (see Fig. 2).

The upper vertex in Fig. 2 corresponds to the point (1,1,0), the lower left vertex to $(1, 1, 1)$, and the lower right vertex to $(1, 0, 0)$. The portion of this "shark's-fin"-shaped region connecting $(1, 0, 0)$ to $(1, 1, 0)$, which corresponds to the plot of C_1 above, is a line; the other two curves, which do not individually, but combined, correspond to C_2 and C_3, are concave. The interior of this plotted region corresponds to the interior of the region

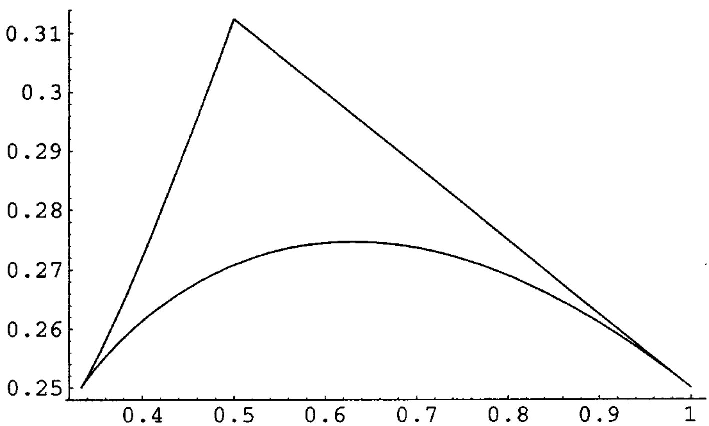

FIG. 2. A better representation of 3-tuples: M_4 vs. $M_4 - \frac{3}{4}M_6$.

of S^2 such that $x > y > z > 0$. We let $X = M_4$ and $Y = M_4 - \frac{3}{4}M_6$, and define $q(X, Y) = p(1, M_4, M_6)$. Under this substitution, (5.7) becomes

$$q(X, Y) = -\tfrac{4}{3}a\, X\, Y - \tfrac{4}{3}b\, Y + (\tfrac{4}{3}a + c)X^2 + (\tfrac{4}{3}b + d)X + e, \qquad (5.9)$$

and then the preceding discussion establishes the equivalence we exploit:

The form $p \in S^e_{3,10}$ *is psd if and only if its corresponding* q *is nonnegative on* $C_1 \cup C_2 \cup C_3$.

Note that the curve $q(X, Y) = 0$ is in general a hyperbola. When we plot for any even symmetric ternary form p its associated curve $q(X, Y) = 0$, the condition that p is psd is equivalent to asserting that $q(X, Y) \geq 0$ on the "shark's-fin" region.

We can now complete the proof that the forms $Q_{\beta,u}$ are psd. Recall that all which remains for us to show is that $Q_{\beta,u}(1, 1, \sqrt{z}) \geq 0$ for $0 < z < 1$.

Let $(a_1, b_1, c_1, d_1, e_1)$ be the coefficients of $Q_{\beta,u}$ written in terms of M_2, M_4, and M_6, computed from (5.8). In particular,

$$a_1 = -\tfrac{1}{6}(u+2)^2\big((2u+1)\beta^2 + (4u^3 + 20u^2 + 12u + 6)\beta + (4u^4 + 12u^3 - 4u^2 + 18u + 9)\big).$$

The associated curve $q = 0$ for $Q_{\beta,u}$ is a hyperbola unless $a_1 = 0$. This occurs only when

$$u = -2, \qquad \beta = -2u^2 - 8u - 3, \qquad \beta = \frac{-2u^2 + 2u - 3}{2u + 1} = m(u).$$

The first solution can be excluded since we are considering only $u > 2$. We can also eliminate the second solution, since $-2u^2 - 8u - 3 < r(u)$ when $u > 2$. Lastly, we can ignore the third solution, since we will show in the next section that the forms $Q_{m(u),u}$ are perfect squares and hence are psd. Thus, for $r(u) < \beta < -3$, $\beta \neq m(u)$, the curve $q = 0$ is a hyperbola.

Since we know that $Q_{\beta,u}(1, \sqrt{y}, 0) \geq 0$ for $y \geq 0$ and $Q_{\beta,u}(1, 1, \sqrt{z}) \geq 0$ for $z \geq 1$ for all $\{\beta, u\}$ on their given ranges, the curve $q = 0$ touches, but does not cross, the lower and right sides of our "shark's-fin" region. Since each branch of a hyperbola is convex, the hyperbola cannot cross the left side of this region, either. This means that $q < 0$ has no intersection with the region, and thus the forms are psd. ∎

We note that the limits placed on β for $Q_{\beta,u}$ as expressed in this result are meaningful. When $\beta > -3$, $Q_{\beta,u}$ does not have a zero of type $(1, z, 0)$, $z \neq 1$. Also,

$$Q_{r(u)+t,u}(1, 1, 1) = -3(u-1)^3 t\big(-4u\sqrt{u^2 + u - 2} + t\big) < 0$$

whenever $t < 0$, $u > 2$, so $Q_{\beta,u}$ is not psd when $\beta < r(u)$.

We now show that forms in $PS^e_{3,10}$ cannot be extremal unless their zero sets are “large” enough. The first two lemmas allow us to perturb the coefficients of $p \in PS^e_{3,10}$ with a zero at $(1, \sqrt{v}, 0)$, $v \geq 0$, by a multiple of $\langle 0, 0, 0, 1, 0\rangle$ or $\langle 0, 0, 0, 1, -1\rangle$ ($p_1(x, y) \equiv 0$ for these forms) and shows that, if p is extremal and has a zero of type $(1, \sqrt{v}, 0)$ or $(1, 1, 1)$, then there exists some $u > 0$ such that $p_2(1, \sqrt{u}) = 0$. The third result allows us to conclude that an extremal form in $PS^e_{3,10}$ cannot have its zero set consist only of $(1, 1, \sqrt{u})$.

Lemma 5.2. *Let $p = \langle \alpha, \beta, \gamma, \delta, \varepsilon\rangle \in PS^e_{3,10}$ be such that $p(1, \sqrt{y}, 0) = 0$ for some $y \geq 0$. Define $p^* = \langle \alpha, \beta, \gamma, 0, \varepsilon\rangle$ and define δ_0 by*

$$\delta_0 = \inf_{u>0} \frac{p^*(1, 1, u)}{u^6 + 2u^2}.$$

Then $\langle \alpha, \beta, \gamma, -\delta_0, \varepsilon\rangle \in PS^e_{3,10}$ and has a zero at $(1, 1, u_0)$, where u_0 is the value of u for which δ_0 is realized. Further, there is no value $\delta^* < -\delta_0$ so that $\langle \alpha, \beta, \gamma, \delta^*, \varepsilon\rangle$ is psd.

Proof. If $p_2(1, u)$ is psd, then by (5.2),

$$\delta \geq \frac{-(\alpha u^{10} + 2\beta u^8 + 2\gamma u^6 + (2\gamma + 2\varepsilon)u^4 + (2\beta + \varepsilon)u^2 + 2\alpha + 2\beta + 2\gamma)}{u^6 + 2u^2}.$$

The numerator of the expression on the right side of this inequality is $p^*(1, 1, u)$. The minimum value of δ so that $p(1, 1, u)$ remains positive semidefinite occurs at $\delta = -\delta_0$ defined above. ∎

Lemma 5.3. *Let $p = \langle \alpha, \beta, \gamma, \delta, \varepsilon\rangle \in PS^e_{3,10}$ be such that $p(1, 1, 1) = 0$. Define $\overline{p} = \langle \alpha, \beta, \gamma, 0, -\alpha - 2\beta - 2\gamma\rangle$ and define δ_1 by*

$$\delta_1 = \inf_{u>0} \frac{\overline{p}(1, 1, u)}{u^2}.$$

Then $\langle \alpha, \beta, \gamma, -\delta_1, \varepsilon - \delta + \delta_1\rangle \in PS^e_{3,10}$ and has a zero at $(1, 1, u_0)$, where u_0 is the value of u for which δ_1 is realized. Further, there is no value $\delta_2 < -\delta_1$ so that $\langle \alpha, \beta, \gamma, \delta_2, \varepsilon - \delta - \delta_2\rangle$ is psd.

Proof. If

$$p_2(1, 1) = \frac{\partial p_2}{\partial y}(1, 1) = 0,$$

then we may write

$$p_2(1, z) = (z^2 - 1)^2\big(\alpha z^6 + (2\alpha + 2\beta)z^4 + (3\alpha + 4\beta + 3\gamma + \delta)z^2 + (2\alpha + 2\beta + 2\gamma)\big),$$

since $\varepsilon = -\alpha - 2\beta - 2\gamma - \delta$. Thus, if p is psd, then

$$\delta \geq \frac{-(\alpha u^6 + (2\alpha + 2\beta)u^4 + (3\alpha + 4\beta + 2\gamma)u^2 + (2\alpha + 2\beta + 2\gamma))}{u^2}.$$

The numerator of the right side of this inequality is $\overline{p}(1, 1, u)$. The minimum value of δ so that $p(1, 1, u)$ remains positive semidefinite occurs at $\delta = -\delta_1$ defined above. Note that we are subtracting $(\delta - \delta_1)G$ from p. ∎

LEMMA 5.4. *Let p be extremal in $PS^e_{3,10}$, and suppose $p_1(x, y)$ is definite. Then $p_2(x, y)$ has two zeros, say $(1, \sqrt{u})$ and $(1, \sqrt{w})$, $u, w \neq 0$.*

Proof. Suppose that $p \in PS^e_{3,10}$ with a zero at $(1, 1, \sqrt{u})$, $u \neq 0$; then $p_2(x, y) = (u x^2 - y^2)^2 \overline{p}_2(x, y)$, where $\overline{p}_2(x, y)$ is definite. Observe that the two sets

$$\begin{aligned} \chi_1 &= \{(\alpha, \beta, \gamma) \mid p_1(x, y) \text{ is psd}\}, \\ \chi_2 &= \{(\alpha, \beta, \gamma) \mid \overline{p}_2(x, y) \text{ is psd}\} \end{aligned} \tag{5.10}$$

are closed convex cones in $\mathbb{R}^3$ (recall that δ and ε are determined here), so $\chi_1 \cap \chi_2$ is also a closed convex cone. The interior points of $\chi_1 \cap \chi_2$ correspond to forms for which both $p_1(x, y)$ and $\overline{p}_2(x, y)$ are definite. Thus, p cannot be extremal in $\chi_1 \cap \chi_2$, and hence also in $PS^e_{3,10}$, unless it has another zero. If p_1 is definite, then $\overline{p}_2$ has a zero, say at $(1, \sqrt{w})$. ∎

We now show an application of Lemma 5.3. Let $p \in PS^e_{3,10}$ be such that $p(1, 1, 1) = 0$. Then Lemma 5.3 implies that we may perturb p by subtracting a multiple of G until it has a root $(1, 1, t)$. If $t \neq 1$, then δ and ε are as in (5.5) and (5.6). If $t = 1$, then we have two additional conditions on the coefficients:

$$8\alpha + 8\beta + 4\gamma + \delta = 0, \qquad 10\alpha + 8\beta + 2\gamma + \delta = 0. \tag{5.11}$$

This implies $\alpha = \gamma$, which in turn means that $\delta = -12\alpha - 8\beta$ and $\varepsilon = 9\alpha + 6\beta$. Thus, $p = \langle \alpha, \beta, \alpha, -12\alpha - 8\beta, 9\alpha + 6\beta \rangle$. If $\alpha = 0$, then p is a multiple of $J_1 = \langle 0, 1, 0, -8, 6 \rangle$. If $\alpha > 0$, then p is a multiple of $N_1 = \langle 1, -2, 1, 4, -3 \rangle$. These two forms span the subspace of $S^e_{3,10}$ which possesses a fourth order zero at $(1, 1, 1)$. By Lemma 4.1 with $\mathbf{w} = (1, 0, 0)$, $\mathbf{w}' = (1, 1, 0)$, we see that only nonnegative linear combinations of J_1 and N_1 are psd.

We now proceed as in Section 4, separating our discussion into five parts based on the zero types $(1, 0, 0)$, $(1, 1, 0)$, $(1, 1, 1)$, $(1, z, 0)$, $z \neq 0, 1$, and $(1, 1, z)$, $z \neq 0, 1$.

LEMMA 5.5. *If $p \in PS^e_{3,10}$ such that $p(1, 0, 0) = 0$, then p is a nonnegative linear combination of forms F, G, and H, and several members of the families J_u, $K_{u'}$, and $L_{u''}$, where the u, u', u'' are in the ranges of u given in the statement of Lemma 5.1.*

Proof. Note that $\alpha = 0$, so we are examining forms $p = \langle 0, \beta, \gamma, \delta, \varepsilon \rangle$. Recall that by Theorem 5.2, $p_1(x, y)$ is psd if and only if $\beta \geq 0, \gamma \geq -\beta$. Also, $p_2(x, y)$ becomes

$$p_2(x, y) = (2\beta + 2\gamma)x^{10} + (2\beta + 2\delta + \varepsilon)x^8y^2 + (2\gamma + 2\varepsilon)x^6y^4 + (2\gamma + \delta)x^4y^6 + 2\beta x^2y^8. \tag{5.12}$$

We consider three cases.

(1) $\beta = 0$. A basis for the subspace of forms $\langle 0, 0, \gamma, \delta, \varepsilon \rangle$ is given by the forms F, G, and H. Thus, all elements $p \in S^e_{3,10}$ satisfying $\alpha = \beta = 0$ may be written as

$$p = \lambda_1 A + \lambda_2 B + \lambda_3 C = \langle 0, 0, \lambda_1, -2\lambda_1 + \lambda_2, -\lambda_2 + \lambda_3 \rangle, \qquad \lambda_i \in \mathbb{R}.$$

In this case, $p_2(1, z) = \lambda_2 z^6 + 2(\lambda_1 - \lambda_2 + \lambda_3)z^4 + (-4\lambda_1 + \lambda_2 + \lambda_3)z^2 + 2\lambda_1$. If p is psd, then

1. $p_2(1, 0) = 2\lambda_1 \geq 0$;
2. $p_2(1, z) \geq 0$ for large z implies $\lambda_2 \geq 0$;
3. $p_2(1, 1) = 3\lambda_3 \geq 0$.

Thus all forms $p \in PS^e_{3,10}$ such that $\alpha = \beta = 0$ may be expressed as a linear combination with nonnegative coefficients of the sos forms F, G, and H.

(2) $\beta > 0, \gamma = -\beta$. Here, $(1, 1, 0) \in Z(p)$. We may assume without loss of generality that $\beta = 1$, so that $p = \langle 0, 1, -1, \delta, \varepsilon \rangle$. Hence

$$p_2(1, z) = z^2(2z^6 + (\delta - 2)z^4 + (2\varepsilon - 2)z^2 + (2\delta + \varepsilon + 2)). \tag{5.13}$$

We break into two subcases.

(2a) $2\delta + \varepsilon + 2 = 0$. In this case $p = \langle 0, 1, -1, \delta, -2\delta - 2 \rangle$, and $p_2(1, z)$ simplifies to $z^4(2z^4 + (\delta - 2)z^2 - 4\delta - 6)$. Then p is psd if and only if $-26 \leq \delta \leq -2$ by Lemma 4.2. It follows that p is a convex combination of $\langle 0, 1, -1, -2, 2 \rangle = J_0$ and $\langle 0, 1, -1, -26, 50 \rangle = K_0$.

(2b) $2\delta + \varepsilon + 2 > 0$, $\delta \leq -26$. If p is psd, then the second factor of (5.13) is either definite or it has a zero $(1, 1, \sqrt{u})$. If it is definite, then by Lemma 5.2, we may perturb δ until p has a double zero at some $(1, 1, \sqrt{u})$, in which case δ and ε are as in (5.5) and (5.6) if $u \neq 1$. Letting $\{\alpha, \beta, \gamma\} = \{0, 1, -1\}$, we obtain the coefficients of the forms L_u after multiplying through by $u + 2$. By our work above, we know that in this case $u \geq 7$.

(3) $\beta > 0, \gamma > -\beta$. Let $p = \langle 0, \beta, \gamma, \delta, \varepsilon \rangle$. If (5.12) is definite, then we may apply Lemma 5.2 and assume p has a double zero, say at $(1, 1, \sqrt{u})$, and thus δ and ε are as in (5.5) and (5.6) if $u \neq 1$. For $0 < u < 7$, $u \neq 1$, the forms J_u and K_u form a basis for the subspace. Let $g_{\lambda,\mu,u} = \lambda J_u + \mu K_u$,

$0 < u < 7$, $u \neq 1$. Apply Lemma 4.1 with $\mathbf{w} = (1, 1, 1)$, $\mathbf{w}' = (1, 1, \sqrt{\frac{7-u}{5u+1}})$ to show that p must be a nonnegative linear combination of J_u and K_u. Recall that $36J_1 = K_1$ has a fourth order zero at $(1, 1, 1)$.

For $u \geq 7$, we already have seen that the form L_u has a zero at $(1, 1, \sqrt{u})$. The forms J_u and L_u are a basis for this subspace. Let $h_{\lambda,\mu,u} = \lambda J_u + \mu L_u$, $u \geq 7$, and apply Lemma 4.1 with $\mathbf{w} = (1, 1, 1)$, $\mathbf{w}' = (1, 1, 0)$. ∎

LEMMA 5.6. *If $p \in PS^e_{3,10}$ and $p(1, 1, 0) = 0$, then p is a nonnegative linear combination of forms F and G, and several members of the families L_u, $N_{u'}$, $Q_{-3,u'''}$, and R_{u'',u^*}, where $u^* = \max\{(4u'' - 10)/(7 - u'')\}$ and u, u', u'', u''' are contained in the ranges as stated in Lemma* 5.1.

Proof. Here, $\alpha + \beta + \gamma = 0$, i.e., $p = \langle \alpha, \beta, -2\alpha - 2\beta, \delta, \varepsilon \rangle$, and

$$p_2(1, z) = z^2(\alpha z^8 + 2\beta z^6 + (-2\alpha - 2\beta + \delta)z^4 + (-2\alpha - 2\beta + 2\varepsilon)z^2 + (-2\alpha + \varepsilon)). \tag{5.14}$$

For a given α and β, by Lemma 5.2 we may perturb δ until (5.14) has a double zero at some $z = \sqrt{u}$. As before, we then have the expressions for δ and ε as in (5.5) and (5.6) when $u \neq 1$. We may write p as a linear combination of:

1. N_u and $R_{u,0}$ for $0 < u \leq 2$, $u \neq 1$;
2. $Q_{-3,u}$ and R_{u,u^*} for $2 < u < 7$;
3. $Q_{-3,u}$ and L_u for $u \geq 7$.

(1) When $0 < u \leq 2$, $u \neq 1$, let $\mathbf{w} = (1, 1, 1)$, $\mathbf{w}' = (1, 1, \sqrt{\frac{5-2u}{u+2}})$, and then apply Lemma 4.1 to $j_{\lambda,\mu,u} = \lambda N_u + \mu R_{u,0}$ to conclude that such forms are psd if and only if $\lambda \geq 0, \mu \geq 0$. Recall that $72N_1 = R_{1,0}$ has a fourth order zero at $(1, 1, 1)$.

(2a) When $2 < u \leq \frac{5}{2}$, let $k_{\lambda,\mu,u} = \lambda Q_{-3,u} + \mu R_{u,0}$. Now $\mu \geq 0$, since otherwise $\beta < -3\alpha$, and $\lambda \geq 0$, since $Q_{-3,u}(1, 1, \sqrt{\frac{5-2u}{u+2}}) \neq 0$.

(2b) When $\frac{5}{2} < u < 7$, let $m_{\lambda,\mu,u} = \lambda Q_{-3,u} + \mu R_{u,u^*}$. Again, $\mu \geq 0$, since otherwise $\beta < -3\alpha$. To see that we must have $\lambda \geq 0$, observe that R_{u,u^*} has a fourth order zero at (1,1,0), while $Q_{-3,u}$ does not.

(3) Suppose $u \geq 7$, and assume that $p_2(1, z)$ has factorization

$$\alpha z^2(z^2 - u)^2(z^4 + s\,z^2 + r). \tag{5.15}$$

Expand (5.15) and compare coefficients with $p_2(1, z)$; we find that $s = \frac{2\beta}{\alpha} + 2u$. Let $\beta = -3\alpha + t$, $t \geq 0$. Then $s = 2u - 6 + t$, and $r = \frac{1}{2}((32 - 7t) + (-16 + t)u + 2u^2)$. Substituting this into (5.15), we find that $p = Q_{-3,u} + t\,L_u$, $t \geq 0$. ∎

LEMMA 5.7. *If* $p \in PS^e_{3,10}$ *and* $p(1,1,1) = 0$, *then* p *is a nonnegative linear combination of forms* F *and* G, *and several members of the families* J_u, $N_{u'}$, *and* $Q_{r(u''),u''}$, *where* u, u', u'' *are contained in the ranges as stated in Lemma* 5.1.

REMARK 5.2. While the forms $R_{1,v}$ have zero at $(1,1,1)$, we note that $R_{1,v} = 72(v+1)^2 N_1 + 36v(v+1)^2 J_1$, so these forms are not extremal.

Proof of Lemma 5.7. Here, $p(1,1,1) = 3\alpha + 6\beta + 6\gamma + 3\delta + 3\varepsilon = 0$. A basis for the solution space of this homogeneous equation in $\{\alpha, \beta, \gamma, \delta, \varepsilon\}$ is given by the coefficients of the forms N_0, J_0, F, and G. Thus any $p = \langle 1, \beta, \gamma, \delta, \varepsilon\rangle \in S^e_{3,10}$ with this zero can be written as

$$\begin{aligned} p &= N_0 + \lambda_1 J_0 + \lambda_2 F + \lambda_3 G \\ &= \langle 1, \lambda_1 - 1, -\lambda_1 + \lambda_2, -2\lambda_1 - 2\lambda_2 + \lambda_3 + 1, 2\lambda_1 - \lambda_3\rangle. \end{aligned}$$

We are interested in finding the set of $\{\lambda_1, \lambda_2, \lambda_3\}$ such that p is psd. Since all four elements of the basis are psd, p is psd if all three of the λ_i's are nonnegative. Also, since $p(1,1,0) = 2\lambda_2$, if p is psd, then $\lambda_2 \geq 0$. Observe that

$$p_1(1,z) = (z^2+1)((z^8+1) + (\lambda_1 - 2)(z^6 + z^2) + (-2\lambda_1 + \lambda_2 + 2)z^4),$$
$$p_2(1,z) = (z^2-1)^2(z^6 + 2\lambda_1 z^4 + \lambda_3 z^2 + 2\lambda_2).$$

By Theorem 5.2, $p_1(1,z)$ is psd if and only if one of the following sets of conditions holds:

1. $\lambda_1 \geq -2$, $\lambda_2 \geq 0$;
2. $\lambda_1 \leq -2$, $\lambda_2 \geq \frac{1}{4}(\lambda_1 + 2)^2$.

Also, we can show that $p_2(1,z)$ is psd if and only if one of the following sets of conditions holds:

1. $\lambda_2 = 0$, $\lambda_1 \geq 0$, $\lambda_3 \geq 0$;
2. $\lambda_2 = 0$, $\lambda_1 < 0$, $\lambda_3 \geq \lambda_1^2$;
3. $\lambda_2 > 0$, $\lambda_1 < 0$, $\lambda_3 \geq -4\lambda_1 z_0 - 3z_0^2$, where z_0 is the unique positive root of $z^3 + 2\lambda_1 z^2 - \lambda_2 = 0$.

So, we consider fixed $\lambda_1 < 0$ and determine the minimum $\{\lambda_2, \lambda_3\}$ so that both $p_1(1,z)$ and $p_2(1,z)$ are psd. We consider two cases.

(1) $-2 \leq \lambda_1 < 0$. By the discussion above, the minimum λ_2 so that $p_1(1,z)$ is psd is $\lambda_2 = 0$ and the minimum λ_3 so that $p_2(1,z)$ is psd is $\lambda_3 = \lambda_1^2$. If we let $\lambda_1 = -u$, so that $\lambda_2 = s$, $s \geq 0$, and $\lambda_3 = u^2 + t$, $t \geq 0$, then $p = J_u + sF + tG$, $0 \leq u \leq 2$.

(2) $\lambda_1 < -2$. By the discussion above, the minimum value of λ_2 so that $p_1(1, z)$ is psd is $\lambda_2 = \frac{1}{4}(\lambda_1 + 2)^2 > 0$. Hence the minimum value of λ_3 so that $p_2(1, z)$ is psd is given by $\lambda_3 = -4\lambda_1 z_0 - 3z_0^2$, where z_0 is defined as above. Let $u = z_0$. When the λ's assume these minimum values, we see that

$$p_2(1, z) = (z^2 - 1)^2(z^2 - u)^2(z^2 + 2(\lambda_1 + u)). \tag{5.16}$$

Let $v = 2(\lambda_1 + u)$. Upon solving this for λ_1, substituting and expanding in (5.16), we find that $p = J_u + \frac{1}{2}vD_u$. We now find v in terms of u. We know that $\gamma = \frac{1}{4}\beta^2 + \frac{1}{2}\beta + \frac{5}{4}$, and so by setting the two expressions we have for γ equal to one another, we find that $v^2 + (-8u^2 - 4u - 8)v + (4u^2 - 16u + 16) = 0$. The solutions to this quadratic are $v = 2(u \pm \sqrt{u^2 + u - 2})^2$. A check reveals that $\beta = -u + \frac{1}{2}v - 1 \geq -3$ for all $u \geq 0$ when we take the positive square root of this solution, so we are interested only in the negative square root. Denote this latter solution by v_0. Then we have the forms

$$\begin{aligned} p = \big\langle 1, -u + \tfrac{1}{2}v_0 - 1, u + \tfrac{1}{2}v_0(u^2 - 1), \\ (v_0 + 1)(u + 1)^2, v_0(2u + 1) - u^2 - 2u\big\rangle, \end{aligned}$$

where $u > 2$. After observing that $-u + \frac{1}{2}v_0 - 1 = r(u)$, a lengthy calculation (which we omit) verifies that these forms are $Q_{r(u),u}$. ∎

LEMMA 5.8. *If* $p \in PS^e_{3,10}$ *and* $p(1, \sqrt{v}, 0) = 0, v \neq 0, 1$, *then* p *is a nonnegative linear combination of* G, H, *and several members of the family* $Q_{\beta,u}$, $r(u) \leq \beta < -3$, $u \geq 2$.

Proof. Suppose $p \in PS^e_{3,10}$ satisfies $p(1, \sqrt{z_0}, 0) = 0$ for some $z_0 > 0$, $z_0 \neq 1$. Then $p = \langle 1, \beta, \frac{1}{4}\beta^2 + \frac{1}{2}\beta + \frac{5}{4}, \delta, \varepsilon\rangle$, $\beta < -3$. Since $p_2(x, y)$ is psd, we may perturb δ by Lemma 5.2 until p has a double zero at some point $(1, 1, \sqrt{u})$, with δ and ε as in (5.5) and (5.6) when $u \neq 1$. Then, substituting $\gamma = \frac{1}{4}\beta^2 + \frac{1}{2}\beta + \frac{5}{4}$ into (5.5) and (5.6) and multiplying through by $4u^2(u + 2)$, we have the coefficients of the family $Q_{\beta,u}$, forms which we know are psd when $r(u) \leq \beta < -3$, $u \geq 2$, where $r(u)$ is as defined above. If these forms are not extremal, then necessarily there are other forms X_u satisfying $Q_{\beta,u} \geq X_u \geq 0$ and $Z(Q_{\beta,u}) \subseteq Z(X_u)$. But we know that for a given α, β, and u, the coefficients δ and ε are uniquely determined. Hence the $Q_{\beta,u}$ are the only psd forms with the zeros $(1, 1, \sqrt{u})$ and $(1, \sqrt{y}, 0)$, where y satisfies $\beta = 1 - 2(y + y^{-1})$. ∎

LEMMA 5.9. *If* $p \in PS^e_{3,10}$ *and* $p(1, 1, \sqrt{u}) = 0$, $u \neq 0, 1$, *then* p *is a nonnegative linear combination of several members of the families* 4 *through* 9 *listed in Lemma* 5.1.

Proof. Thus far, for a given u, we have found forms with zeros at $(1, 1, \sqrt{u})$ and also at one or more of the points $(1, 0, 0)$, $(1, 1, 0)$, $(1, 1, 1)$, and $(1, \sqrt{y}, 0)$. Suppose that p is a psd form which has a zero at $(1, 1, \sqrt{u})$, $u \neq 0, 1$, but is positive at all of the other points listed above. By Lemma 5.4, p is not extremal unless it has another zero. Here, we may assume this zero is of type $(1, 1, \sqrt{w})$, $w \neq 0, 1$. Then $p_2(x, y)$ factors as

$$\begin{gathered} p_2(x, y) = (u\,x^2 - y^2)^2(w\,x^2 - y^2)^2(v\,x^2 + y^2), \\ u, v, w > 0, \;\; u, w \neq 1. \end{gathered} \tag{5.17}$$

When we expand (5.17) and compare it to $p_2(x, y)$, we have two expressions for ε,

$$\varepsilon = -\tfrac{1}{2}u^2vw^2 - u^2w - uw^2 + \tfrac{1}{2}u^2v + \tfrac{1}{2}vw^2 + 2uvw - u + \tfrac{1}{2}v - w + 1,$$

$$\begin{aligned} \varepsilon = 2u^2vw^2 + u^2w^2 - 2u^2vw - 2uvw^2 - 2u^2 + 4uv \\ - 8uw + 4vw - 2w^2 + 6u - 3v + 6w - 4. \end{aligned}$$

The relation obtained by setting these two expressions equal to one another is

$$(u-1)(w-1)(5uvw+uv+2uw+vw+4u-7v+4w-10)=0. \tag{5.18}$$

Since we assume $u, w \neq 1$, we are interested in the third factor equalling zero, or

$$w = \frac{(7-u)v + (10-4u)}{(5u+1)v + (2u+4)}. \tag{5.19}$$

Since u, v, w are all positive, we must have $0 < u < 7$. Further, $w > 0$ implies $v > \frac{10-4u}{7-u}$; since $v > 0$, this applies only when $\frac{5}{2} < u < 7$. So when the factorization in (5.17) occurs, we must have $0 < u < 7$, $v > \max\{0, \frac{4u-10}{7-u}\}$. When we substitute (5.19) into the expressions for $\{\beta, \gamma, \delta, \varepsilon\}$ which arise, we obtain the coefficients of the forms $R_{u,v}$, which we know are psd; automatically, then, $p_1(x, y)$ is psd in this case. These forms are extremal by the same argument presented at the end of the previous lemma. ∎

THEOREM 5.3. *The forms listed in Lemma* 5.1 *comprise the set of extremal forms in* $PS^e_{3,10}$.

Proof. In Lemmas 5.5, 5.6, 5.7, 5.8, and 5.9, we found all extremal forms for each of the five zero types. ∎

We conclude this section with a summary of the zero sets (up to permutation) of each of the forms discussed here.

$F:\{(1,0,0),(1,1,1)\}$;

$G:\{(1,z,0), z\geq 0, (1,1,1)\}$;

$H:\{(1,z,0), z\geq 0\}$;

$J_u:\{(1,0,0),(1,1,1),(1,1,\sqrt{u})\}$, where $u\geq 0$;

$K_u:\{(1,0,0),(1,1,\sqrt{u}),(1,1,\sqrt{\frac{7-u}{5u+1}})\}$, where $0\leq u\leq 7$;

$L_u:\{(1,0,0),(1,1,\sqrt{u})\}$, where $u\geq 7$;

$N_u:\{(1,1,0),(1,1,1),(1,1,\sqrt{u})\}$, where $0\leq u\leq 2$;

$Q_{\beta,u}:\{(1,\sqrt{y},0),(1,1,\sqrt{u})\}$, where y satisfies $\beta = 1-2(y+y^{-1})$ and $u\geq 2$; $Q_{r(u),u}$ also has the zero $(1,1,1)$;

$R_{u,v}:\{(1,1,\sqrt{u}),(1,1,\sqrt{w})\}$, where $0\leq u\leq 7$, $v\geq \max\{0,\frac{4u-10}{7-u}\}$, and $w=\frac{(7v+10)-(v+4)u}{(v+4)+(5v+2)u}$.

6. REPRESENTATION OF EVEN SYMMETRIC TERNARY DECICS AS SUMS OF SQUARES

In this section we show that, like $S^e_{3,6}$ and unlike $S^e_{3,8}$, not all of the extremal elements of $PS^e_{3,10}$ are sos. There are two results in the literature which will be particularly useful in proving various forms are not sos:

THEOREM 6.1 (Theorem 3.5 and Corollary 4.7 of [4]). *If p is a psd ternary form of degree m and $|Z(p)| > m^2/4$, then either p is not sos or p is divisible by the square of an indefinite form. In particular, $|Z(p)| = \infty$ if and only if p is divisible by the square of an indefinite form.*

THEOREM 6.2 (Theorem 4.1 of [5]). *Suppose p is an even sos form. Then we may write $p=\sum_k\sum_j q_{jk}^2$, where each form q_{jk}^2 is even, and for each k, $\sum_j q_{jk}^2$ is symmetric. In particular, $q_{jk}(x)=\sum c_j x^\alpha$, where the sum is taken over α's in one congruence class* mod 2 *componentwise.*

We first consider psd forms p such that $|Z(p)|=\infty$; by Theorem 6.1, such forms are divisible by the square of some indefinite form. We will use some elementary algebraic techniques to determine which irreducible indefinite forms may divide a psd even symmetric ternary decic.

DEFINITION 6.1. Let p be an n-ary m-ic form. We say that $p(\mathbf{x})$ is *invariant* under a group G of $n\times n$ real matrices if for all $\sigma\in G$, $p(\sigma(\mathbf{x}))=p(\mathbf{x})$.

If p is a symmetric form, then p is invariant under the group S_n of $n\times n$ permutation matrices. If p is an even form, then p is invariant under the group V_n of $n\times n$ diagonal matrices with entries ± 1. Thus, even symmetric n-ary m-ics are invariant under the group $G = S_n\oplus V_n$.

LEMMA 6.1. *Let G be a group of n by n real matrices. Suppose p is an n-ary m-ic which is invariant under G. If $h(\mathbf{x}) \mid p(\mathbf{x})$, then $h(\sigma(\mathbf{x})) \mid p(\mathbf{x})$ for all $\sigma \in G$.*

Proof. We may write $p(\mathbf{x}) = h(\mathbf{x}) \cdot f(\mathbf{x})$. Then $p(\mathbf{x}) = p(\sigma(\mathbf{x})) = h(\sigma(\mathbf{x})) \cdot f(\sigma(\mathbf{x}))$. ∎

LEMMA 6.2. *An n-ary m-ic p is invariant under a finite group G if and only if*

$$p(\mathbf{x}) = \frac{1}{|G|} \sum_{\sigma \in G} p(\sigma(\mathbf{x})).$$

Proof. One direction is clear: if p is invariant, then $p(\mathbf{x}) = p(\sigma(\mathbf{x}))$ for all $\sigma \in G$, and so of course $p(\mathbf{x}) = \frac{1}{|G|} \sum_{\sigma \in G} p(\sigma(\mathbf{x}))$.

To prove the other direction, suppose $p(\mathbf{x}) = \frac{1}{|G|} \sum_{\sigma \in G} p(\sigma(\mathbf{x}))$. Pick arbitrary $\sigma_1 \in G$. Then

$$p(\sigma_1(\mathbf{x})) = \frac{1}{|G|} \sum_{\sigma \in G} p(\sigma\sigma_1(\mathbf{x})) = p(\mathbf{x}),$$

since $\{\sigma\sigma_1\}$ ranges over G as $\{\sigma\}$ ranges over G. Hence p is invariant. ∎

We now present two lemmas which count the projectively distinct elements in the orbit of a 3-tuple (a, b, c) when acted upon by two groups mentioned above. We will apply Lemma 6.3 to even terms in proposed irreducible, indefinite factors of psd even symmetric ternary decics p, and apply Lemma 6.4 to terms in such factors that have at least one odd exponent. Recall that (a, b, c) and $(-a, -b, -c)$ are equivalent projectively.

LEMMA 6.3. *The number of projectively distinct elements in the orbit of a 3-tuple (a, b, c) when acted upon by S_3 is:*

1. 1 *if* $a = b = c$;
2. 3 *if* $(a, b, c) = (t, t, u)$, $0 \neq t \neq u$, *perhaps after a permutation*;
3. 3 *if* $(a, b, c) = (t, -t, 0)$, $t \neq 0$, *perhaps after a permutation*;
4. 6 *otherwise*.

Proof. The orbit of (a, b, c) under S_3 is a subset of

$$\{(a, b, c), (a, c, b), (b, a, c), (b, c, a), (c, a, b), (c, b, a)\}.$$

We leave the analysis of the circumstances under which $(a, b, c) = \lambda(a, c, b)$, $\lambda = \pm 1$, etc. to the reader. ∎

LEMMA 6.4. *The number of projectively distinct elements in the orbit of a nonzero 3-tuple* (a, b, c) *when acted upon by* $S_3 \oplus V_3$ *is*:

1. 3 *if* $b = c = 0$, $a \neq 0$, *etc.*;
2. 4 *if* $|a| = |b| = |c|$, $a \neq 0$;
3. 6 *if* $|a| = 0$, $|b| = |c| \neq 0$, *etc.*;
4. 12 *if* $|a| \neq |b| = |c|$, *but* $abc \neq 0$, *etc.*;
5. 12 *if* $|a|$, $|b|$, $|c|$ *are distinct*, *but* $abc = 0$;
6. 24 *if* $|a|$, $|b|$, $|c|$ *are distinct*, *but* $abc \neq 0$.

The etc.'s refer to permutations of $\{a, b, c\}$.

Proof. The orbit of (a, b, c) under $S_3 \oplus V_3$ is a subset of

$$\big\{(\pm a, \pm b, \pm c), (\pm a, \pm c, \pm b), (\pm b, \pm a, \pm c),$$
$$(\pm b, \pm c, \pm a), (\pm c, \pm a, \pm b), (\pm c, \pm b, \pm a)\big\}.$$

Again, we leave to the reader the analysis of the circumstances under which $(a, b, c) = \lambda(a, c, b)$, $\lambda = \pm 1$, etc. ∎

LEMMA 6.5. *Let* $p \in PS^e_{3,10}$. *Suppose* q *is an irreducible indefinite form of degree* $d \geq 2$ *such that* $q^2 \mid p$. *Then* q *has no terms with odd exponents. In fact*, q *is even and symmetric*.

Proof. All terms of q except powers of xyz can be grouped in threes, related under a cyclic permutation of $\{x, y, z\}$. Note that $p \in PS^e_{3,10}$ will always have fewer than three irreducible factors of a given degree (since $6d > 10$). Thus, the coefficients of the even triples must be equal by Lemma 6.3 and the coefficients of the odd triples must be zero by Lemma 6.4. Odd powers of xyz occur only if q is cubic, but xyz is not irreducible. ∎

THEOREM 6.3. *Let* $p \in PS^e_{3,10}$. *Suppose that* q *is an irreducible indefinite form such that* $q^2 \mid p$. *Then* q *is one of*

$$x, y, z, x \pm y \pm z, \tag{6.1}$$

$$a(x^4 + y^4 + z^4) + b(x^2y^2 + x^2z^2 + y^2z^2), \qquad b < -a, \; b \neq -2a. \tag{6.2}$$

Proof. Lemma 6.5 proves that there are no irreducible cubic or quintic factors, since they are composed completely of odd terms. The only even symmetric quadratic form is $x^2 + y^2 + z^2$, which is not indefinite. By Lemmas 6.5 and 3.1, the only possible irreducible, indefinite quartic factors are those in (6.2) (recall that $x^4 + y^4 + z^4 - 2(x^2y^2 + x^2z^2 + y^2z^2)$ factors). This leaves the linear case to consider. Suppose $q = a\,x + b\,y + c\,z$ is a factor of p. By Lemma 6.1, we know that $ax + by - cz$, $bx - ay + cz$, etc. are also factors of p. By degree considerations, we are only interested in 3-tuples

(a, b, c) whose orbit when acted upon by $S_3 \oplus V_3$ is 5 or less. This occurs only in cases 1 and 2 in Lemma 6.4 above. We conclude that the only linear factors p may have are those listed in (6.1). ∎

Thus, the only symmetric indefinite forms which may divide $p \in PS^e_{3,10}$ are xyz and $a(x^4+y^4+z^4)+b(x^2y^2+x^2z^2+y^2z^2)$, $b < -a$. We can now identify all elements in $PS^e_{3,10}$ which have these factors.

If $p = (xyz)^2 \cdot f$, then necessarily f is a psd even symmetric ternary quartic. Recall that the only extremal elements of $PS^e_{3,4}$ are those given in (1.2). Thus the extremal members of $PS^e_{3,10}$ which arise here are $(xyz)^2 \cdot q_1 = G$ and $(xyz)^2 \cdot q_2 = H$.

The only forms in $S^e_{3,10}$ which have $a(x^4+y^4+z^4)+b(x^2y^2+x^2z^2+y^2z^2)$, $b < -a$, as a factor are multiples of

$$\begin{aligned} T_b &:= (x^2+y^2+z^2)(x^4+y^4+z^4+b(x^2y^2+x^2z^2+y^2z^2))^2 \\ &= \langle 1, 2b+1, b^2+2b+2, 2b^2+6b, 5b^2+4b+2\rangle, \qquad b < -1. \end{aligned} \tag{6.3}$$

Let u be such that $b = -(u^2+2)/(2u+1)$. Then $T_b = (1/4u^2(u+2))Q_{m(u),u}$, where $m(u) = (-2u^2+2u-3)/(2u+1)$, $u \geq 4$ (i.e., $b \leq -2$), and $T_b = (1/72(2u+1))R_{u,2}$, $0 < u < 4$, $u \neq 1$ (i.e., $-2 < b < -1$). Thus, the forms T_b, $b < -1$, are extremal by our work in Section 5. We note that T_{-1} is not extremal, since $T_{-1} = N_1 + J_1$.

We now restate Theorem 6.1 in terms of the above discussion:

THEOREM 6.4. *If $p \in PS^e_{3,10}$ is such that $|Z(p)|$ is greater than or equal to $10^2/4 = 25$, and p is not one of G, H, or T_b, $b < -1$, then p is not sos.*

We present three applications of this result.

LEMMA 6.6. *The forms K_u, $u \neq 1$, are not sos.*

Proof. We saw in Section 5 that these forms have zeros at $(1,0,0)$, $(1,1,\sqrt{u})$, and $(1,1,\sqrt{\frac{7-u}{5u+1}})$. After accounting for the images under $S_3 \oplus V_3$, we see that this adds up to $3 + 2\cdot 12 = 27$ zeros when $u \neq 1$. ∎

Recall that $K_1 = 36J_1$ is sos.

LEMMA 6.7. *The forms $Q_{r(u),u}$, $u > 2$, are not sos.*

Proof. The forms $Q_{r(u),u}$ have zeros at $(1,1,1)$, $(1,1,\sqrt{u})$, and $(1,\sqrt{y},0)$, where y is as at the end of Section 5. This means that any $Q_{r(u),u}$ has $4 + 2\cdot 12 = 28$ zeros. ∎

We will show that $Q_{-3,2} = 64N_2$ is not sos below.

LEMMA 6.8. *The forms $R_{u,0}$, $0 < u < \frac{5}{2}$, $u \neq 1$, are not sos.*

Proof. The forms $R_{u,0}$ have zeros at $(1, 1, 0)$, $(1, 1, \sqrt{u})$, and $(1, 1, \sqrt{\frac{5-2u}{u+2}})$; these add up to $6 + 2 \cdot 12 = 30$ zeros. ∎

Note that $R_{0,0} = R_{0,5/2}$; we shall demonstrate it is not sos below. Additionally, we have noted previously that $R_{1,0} = 72N_1$ is sos.

REMARK 6.1. The form $p(x, y, z) = \sum^3 x^6 - \sum^6 x^4y^2 + 3x^2y^2z^2$, given by Robinson in [12], is the only other symmetric ternary form known to satisfy the hypotheses of Theorem 6.1.

We present a result on binary forms which will help us show that one family in $PS^e_{3,10}$ cannot be expressed as a sum of squares.

LEMMA 6.9. *Let $f(u, v)$ and $g(u, v)$ be real forms of degree $2n$. Suppose f is the square of a product of real linear forms, and $f(u, v) \geq g(u, v) \geq 0$ for all $(u, v) \in \mathbb{R}^2$. Then g is a scalar multiple of f. In particular, if $g(u_0, v_0) = 0$ but $f(u_0, v_0) \neq 0$, then $g = 0$.*

Proof. Let $f(u, v) = \prod(a_iu - b_iv)^{2r_i}$, $a_i, b_i \in \mathbb{R}$. Then $f(b_i, a_i) = 0$ for $i = 1, \ldots, n$. In particular, $f(b_1, a_1) \geq g(b_1, a_1) \geq 0$ implies $g(b_1, a_1) = 0$. If g is not the zero polynomial, then $(a_1u - b_1v)^2$ divides g, since g is psd. Repeating this process n times yields $f = \prod_{i=1}^n (a_iu - b_iv)^2 \mid g$. The fact that f and g have the same degree proves the last statement of the lemma. ∎

LEMMA 6.10. *The forms N_u, $u \neq 1$, are not sos.*

Proof. Choi and Lam use a "term-inspection" method in [3] to prove certain symmetric forms are not sos. We adapt this method (specifically, we adapt the proof of Proposition 2.7 in that paper) in order to prove this lemma. Suppose $N_u = \sum r_i^2$, where the r_i are of degree 5. Write $r_i = xf_i + g_i$, where both f_i and g_i are even in x. Then $N_u = \sum r_i^2 = \sum(x^2f_i^2 + 2xf_ig_i + g_i^2)$. Now, $N_u(1, 0, 0) \neq 0$ implies that $\sum f_i^2(1, 0, 0) \neq 0$, since x^{10} can arise only from $x^2f_i^2$. Since N_u is even, $2\sum xf_ig_i = 0$. Thus, $N_u \geq x^2f_i^2$ for each i.

Recall that the forms N_u have zeros at $(1, 1, 0)$, $(1, 1, 1)$, and $(1, 1, \sqrt{u})$, so we know that

$$N_u(x, \pm x, z) = z^2(z^2 - x^2)^2(z^2 - ux^2)^2 \geq x^2f_i^2(x, \pm x, z).$$

Since the left side of this inequality does not have x^2 as a factor, $f_i(x, \pm x, z)$ must equal zero by Lemma 6.9, so $(x^2 - y^2) \mid f_i(x, y, z)$. Similarly, by looking at $N_u(x, y, \pm x)$, we see $(x^2 - z^2) \mid f_i(x, y, z)$. Hence $f_i = \alpha_i(x^2 - y^2)(x^2 - z^2)$, $\alpha_i \in \mathbb{R}$. Finally, consider $N_u(x, y, y)$. Then

$$x^2(x^2 - y^2)^2(x^2 - uy^2)^2 \geq \alpha_i x^2(x^2 - y^2)^4.$$

When $u \neq 1$, Lemma 6.9 implies that $\alpha_i = 0$. Thus, $f_i = 0$ and $\sum f_i^2(1, 0, 0) = 0$, a contradiction. ∎

We note that N_1 is sos; $N_1 = \sum^3 (x(x^2 - y^2)(x^2 - z^2))^2$.

The following lemma limits the types of terms which can appear in an sos representation of forms in $PS^e_{3,10}$.

LEMMA 6.11. *If p is an extremal element of $PS^e_{3,10}$ such that $p(1, 1, \sqrt{u}) = 0$, $u > 0, u \neq 1$, then no nonzero term of the form $h(x, y, z) = (\hat{a}x^3yz + \hat{b}xy^3z + \hat{c}xyz^3)^2$ may appear in an sos representation of p.*

Proof. The system of equations

$$\{h(1, 1, \sqrt{u}) = 0, h(1, \sqrt{u}, 1) = 0, h(\sqrt{u}, 1, 1) = 0\}$$

has the unique solution $\hat{a} = \hat{b} = \hat{c} = 0$ when $u \neq 1$. ∎

Thus, by this lemma and Theorem 6.2, if $p \in \Sigma S^e_{3,10}$, then we may write

$$p = \sum_{j=1}^{3} \sum_{i} \big(a_{i,j}x^5 + b_{i,j}x^3y^2 + c_{i,j}xy^4 + d_{i,j}x^3z^2 + e_{i,j}xz^4 + f_{i,j}xy^2z^2\big)^2 \tag{6.4}$$

As j runs from 1 to 3, we assume that the variables in the summation are cyclically permuted. We will often suppress subscripts for convenience.

We now summarize relations which arise when the terms in (6.4) have various zeros. Recall that if $p = \sum h_k^2$ and $\mathbf{x} \in Z(p)$, then $\mathbf{x} \in Z(h_k)$ for all k.

$$h(1, 1, 0) = 0 \implies a + b + c = 0; \tag{6.5}$$

$$h(1, 0, 1) = 0 \implies a + d + f = 0; \tag{6.6}$$

$$\frac{\partial h}{\partial x}\Big|_{(x,y,z)=(1,1,0)} = 0 \implies 5a + 3b + c = 0; \tag{6.7}$$

$$\frac{\partial h}{\partial x}\Big|_{(x,y,z)=(1,0,1)} = 0 \implies 5a + 3d + e = 0; \tag{6.8}$$

$$h(1, \sqrt{v}, 0) = 0 \implies a + bv + cv^2 = 0; \tag{6.9}$$

$$h(\sqrt{v}, 1, 0) = 0 \implies av^2 + bv + c = 0; \tag{6.10}$$

$$h(1, 0, \sqrt{v}) = 0 \implies a + dv + ev^2 = 0; \tag{6.11}$$

$$h(\sqrt{v}, 0, 1) = 0 \implies av^2 + dv + e = 0. \tag{6.12}$$

$$h(1, 1, \sqrt{z}) = 0 \implies (a + b + c) + (d + f)z + ez^2 = 0; \tag{6.13}$$

$$h(1, \sqrt{z}, 1) = 0 \implies (a + d + e) + (b + f)z + cz^2 = 0; \tag{6.14}$$

$$h(\sqrt{z}, 1, 1) = 0 \implies (c + e + f) + (b + d)z + az^2 = 0; \tag{6.15}$$

LEMMA 6.12. *The forms L_u are not sos.*

Proof. Suppose that L_u is sos. Recall that L_u is defined only for $u \geq 7$. Since $L_u(1, 0, 0) = 0$, we know $a = 0$. Then by (6.5) and (6.6), $b = -c$ and $d = -e$. Relations (6.13) and (6.14) imply that neither b nor d is zero. Thus, the relations (6.13) through (6.15) can be represented as

$$\begin{bmatrix} 0 & u^2 - u & u \\ u^2 - u & 0 & u \\ 1 - u & 1 - u & 1 \end{bmatrix} \begin{bmatrix} b \\ d \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{6.16}$$

The coefficient matrix has zero determinant only when $u = 0, 1$. ∎

LEMMA 6.13. *The forms $Q_{\beta,u}$ are sos only if $\beta = m(u) = (-2u^2 + 2u - 3)/(2u + 1)$, $u \geq 4$.*

Proof. Recall we showed in Lemma 6.7 that the forms $Q_{r(u),u}$, $u > 2$, are not sos. We divide the remaining members of this family into two parts.

(1) $Q_{-3,u}$. Note that $Q_{-3,2} = 64N_2$, which we have already seen is not sos. Also, $Q_{-3,4}$ is a multiple of $R_{4,2}$, so we know that $Q_{-3,4}$ is sos ($m(4) = -3$). Suppose that $Q_{-3,u}$ is sos. Using the relations (6.5), (6.6), and (6.13) through (6.15), we find that $b = d = -\frac{1}{3}a(u + 2)$, $c = e = \frac{1}{3}a(u - 1)$, and $f = -au^2 + 2a$. Comparing coefficients of x^8y^2 between (6.4) and $Q_{-3,u}$ (after scaling leading coefficients appropriately), we find that $2(\frac{1}{3}(u + 2)) + (\frac{1}{3}(u - 1))^2 = -3$, or $u = 4$.

(2) $Q_{\beta,u}$, $r(u) < \beta < -3$. Recall that $m(u) < -3$ only when $u > 4$. Recall $Q_{\beta,u}$ has zeros at $(1, 1, \sqrt{u})$ and $(1, \sqrt{y}, 0)$, where y is as given at the end of Section 5. If $Q_{\beta,u}$ is sos, then relations (6.9) through (6.12) imply that $a = c = e$. Substituting this into (6.13), (6.14), and (6.15), we obtain the system

$$\begin{bmatrix} u^2 + 2 & 1 & u & u \\ u^2 + 2 & u & 1 & u \\ u^2 + 2 & u & u & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ d \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{6.17}$$

The coefficient matrix has rank 3, so there is a one-dimensional family of solutions. It is easily checked that this family is

$$\{a, b, d, f\} = \lambda\{2u + 1, -(u^2 + 2), -(u^2 + 2), -(u^2 + 2)\}, \qquad \lambda \in \mathbb{R}. \tag{6.18}$$

As above, by comparing coefficients of x^8y^2 between (6.4) and $Q_{\beta,u}$, we find that these forms are sos only when

$$\beta = 2\left(-\frac{u^2+2}{2u+1}\right)+1 = \frac{-2u^2+2u-3}{2u+1}.$$

Recall that we saw in (6.3) that these are the sos forms T_b, $b \le -2$, where $b = -(u^2+2)/(2u+1)$. ∎

LEMMA 6.14. *The forms $R_{u,v}$, $v \ne 2$, are not sos.*

Proof. Recall we proved in Lemma 6.8 that the forms $R_{u,0}$, $0 < u < \frac{5}{2}$, $u \ne 1$, are not sos. We divide the remaining members of this family into two parts.

(1) R_{u,u^*}, $\frac{5}{2} \le u \le 7$, where $u^* = \frac{4u-10}{7-u}$. Suppose R_{u,u^*} is sos. Since R_{u,u^*} has a fourth order zero at (1,1,0), we may assume that the relations (6.5) through (6.8) and (6.13) through (6.15) apply. Using the first four of these relations, we find that $b = d = -2a$ and $c = e = a$. By (6.15), then, we have $f = -au^2 + au + a$. By comparing coefficients of x^8y^2 between (6.4) and R_{u,u^*}, we find that $-3 = (u^2-5u-5)/(7-u)$, or $u = 4$, so $u^* = 2$.

(2) $R_{u,v}, 0 < u < 7$, $u \ne 1$, $v > \max\{0, \frac{4u-10}{7-u}\}$. Recall that these forms have zeros at both $(1,1,\sqrt{u})$ and $(1,1,\sqrt{w})$, where $w = \frac{(7v+10)-(v+4)u}{(v+4)+(5v+2)u}$, $u, w \ne 1$. Hence the three relations (6.13) through (6.15), which are quadratic in z, hold for both $(1,1,\sqrt{u})$ and $(1,1,\sqrt{w})$. This means that

$$\begin{aligned} h(1,1,\sqrt{z}) &= e(z-u)(z-w),\\ h(1,\sqrt{z},1) &= c(z-u)(z-w), \qquad (6.19)\\ h(\sqrt{z},1,1) &= a(z-u)(z-w). \end{aligned}$$

Expanding and comparing the two expressions for each of $h(1,1,\sqrt{z})$, $h(1,\sqrt{z},1)$, and $h(\sqrt{z},1,1)$, we find two ways to express the sum $b+d+f$:

$$b+d+f = -\tfrac{1}{2}(a+c+e)(u+w) = (a+c+e)(uw-2). \qquad (6.20)$$

Thus, these forms are sos only when $(a+c+e)(2uw-4+u+w) = 0$. Since u and w are both positive, $a+c+e = 0$ if and only if $b+d+f = 0$ by (6.20). But $a+b+c+d+e+f = 0$ would imply a zero at $(1,1,1)$, which the forms under discussion do not have. Thus, $2uw-4+u+w = 0$, or $w = \frac{4-u}{2u+1}$. A calculation shows this occurs only when $v = 2$, $0 \le u \le 4$, $u \ne 1$. Recall that $R_{u,2}$ is a multiple of the sos form $T_{-(u^2+2)/(2u+1)}$. ∎

THEOREM 6.5. *The four forms F, G, H, N_1 and the two families of forms J_u and T_b are the only extremal elements of $PS^e_{3,10}$ which are also elements of $\Sigma S^e_{3,10}$.*

Proof. The sums of squares representations of F, G, H, N_1, and J_u, along with Lemmas 6.6, 6.7, 6.8, 6.10, 6.12, 6.13, and 6.14, prove the theorem. ∎

We thus have provided many new examples of psd forms which do not have representations as a sum of squares.

7. SYMMETRIC TERNARY FORMS AND TRIANGLE INEQUALITIES

We conclude this paper by discussing the relationship between even symmetric ternary forms and triangle inequalities. It is natural to think of a psd form as an inequality; extremal psd forms p are "best possible" inequalities in the sense that there is no form q such that $p(\mathbf{x}) \geq q(\mathbf{x}) \geq 0$ other than multiples of p.

Let p be a psd even symmetric ternary form, and define

$$a = y^2 + z^2, \qquad b = x^2 + z^2, \qquad c = x^2 + y^2, \tag{7.1}$$

so that

$$x^2 = \tfrac{1}{2}(b + c - a), \qquad y^2 = \tfrac{1}{2}(a + c - b), \qquad z^2 = \tfrac{1}{2}(a + b - c). \tag{7.2}$$

For any real $\{x, y, z\}$, the quantities $\{a, b, c\}$ may be viewed as the lengths of the sides of a (possibly degenerate) triangle. Applying this transformation, we find that there is a one-to-one correspondence between psd even symmetric ternary forms of degree m and inequalities of degree $\frac{m}{2}$ which are satisfied by a triangle with sides of length $\{a, b, c\}$. Recall that in Sections 4 and 5 we found all extremal psd even symmetric ternary octics and decics. Hence, after the substitution (7.2) into these forms, we obtain all symmetric triangle inequalities of degree 4 and 5.

A number of papers on triangle inequalities written in the 1970s, e.g., [1, 10, 11], pay particular attention to what are called *special* inequalities, that is, inequalities where equality holds for equilateral triangles. We see by (7.1) that these must correspond to psd even symmetric ternary forms with a zero at $(1, 1, 1)$. We now present the special triangle inequalities which arise from the results in Sections 4 and 5.

1. *Special Quartic Inequalities*

In $PS^e_{3,8}$, the extremal forms with a zero at $(1, 1, 1)$ are A, B, and E_u. Up to a scalar multiple, A, B, and E_u become, respectively,

$$-\sum^{3} a^4 + 2\sum^{6} a^3 b - 2\sum^{3} a^2 b^2 - \sum^{3} a^2 b\, c \geq 0,$$

$$\sum^{3} a^4 - \sum^{6} a^3 b + \sum^{3} a^2 b\, c \geq 0,$$

$$(u+1)^2 \sum^{3} a^4 - (u+3)(u+1) \sum^{6} a^3 b + 4\,(u+2) \sum^{3} a^2 b^2 + (u+3)(u-1) \sum^{3} a^2 b\, c \geq 0.$$

By Lemma 4.6, any special inequality of degree 4 must be a nonnegative linear combination of these inequalities. The inequalities corresponding to A, B, and E_0 were presented in [11, p. 198], but the expression for A (which Rigby calls B) is incorrect; the inequality given there is the sum of those given for B and E_0 (Rigby calls these C and A, respectively).

2. *Special Quintic Inequalities*

Recall that in $PS^e_{3,10}$, the extremal forms with a zero at $(1,1,1)$ are F, G, J_u, N_u, and $Q_{r(u),u}$. Up to a scalar multiple, F, G, J_u, N_u, and $Q_{r(u),u}$ become, respectively,

$$\sum^{3} a^5 - \sum^{6} a^4 b + \sum^{3} a^3 b\, c \geq 0,$$

$$-\sum^{3} a^5 + 2\sum^{6} a^4 b - \sum^{6} a^3 b^2 - 3\sum^{3} a^3 b\, c + 2\sum^{3} a^2 b^2 c \geq 0,$$

$$(u+1)^2 \sum^{3} a^5 - (u+3)(u+1) \sum^{6} a^4 b + 2\,(u+1) \sum^{6} a^3 b^2 + (u+3)^2 \sum^{3} a^3 b\, c - 4\,(u+2) \sum^{3} a^2 b^2 c \geq 0,$$

$$-(u+1)^2 \sum^{3} a^5 + 2(u+2)(u+1) \sum^{6} a^4 b + (-u^2-4u+1) \sum^{6} a^3 b^2 - 3(u+3)^2 \sum^{3} a^3 b\, c + 2(u^2+8u+9) \sum^{3} a^2 b^2 c \geq 0,$$

$$\begin{aligned}
&(u+1)^2(4u^2+2u-5-4u\sqrt{u^2+u-2}) \sum^{3} a^5 \\
&\quad + 2(u+1)(-2u^3-7u^2+8+2u(u+3)\sqrt{u^2+u-2}) \sum^{6} a^4 b \\
&\quad + (8u^3+11u^2-8u-7-8u(u-1)\sqrt{u^2+u-2}) \sum^{6} a^3 b^2 \\
&\quad + (u+3)^2(4u^2+2u-7-4u\sqrt{u^2+u-2}) \sum^{3} a^3 b\, c \\
&\quad + (-16u^3-38u^2+16u+50+16u(u+2)\sqrt{u^2+u-2}) \sum^{3} a^2 b^2 c \geq 0.
\end{aligned}$$

By Lemma 5.7, any special inequality of degree 5 must be a nonnegative linear combination of these inequalities. There is discussion in the literature on quartic and sextic triangle inequalities, but we have found no previous work on quintic triangle inequalities.

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