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Radiation in (2 + 1)-dimensions

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ABSTRACT

In this paper we discuss the radiation equation of state $p = \rho/2$ in (2 + 1)-dimensions. In (3 + 1)-dimensions the equation of state $p = \rho/3$ may be used to describe either actual electromagnetic radiation (photons) or a gas of massless particles in a thermodynamic equilibrium (for example neutrinos). In this work it is shown that in the framework of (2 + 1)-dimensional Maxwell electrodynamics the radiation law $p = \rho/2$ takes place only for plane waves, i.e. for $E = B$. Instead of the linear Maxwell electrodynamics, to derive the (2 + 1)-radiation law for more general cases with $E \neq B$, one has to use a conformally invariant electrodynamics, which is a (2 + 1)-nonlinear electrodynamics with a trace free energy–momentum tensor, and to perform a volumetric spatial average of the corresponding Maxwell stress–energy tensor with its electric and magnetic components at a given instant of time t .

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1. Introduction

It is well known that radiation or black body radiation (as a superposition of plane waves of different frequencies) from the point of view of a perfect fluid obeys the equation of state $p = \rho/3$. Additionally, there are massless particles which in the standard framework may be treated in terms of a fluid with energy density ρ and isotropic pressure p , which satisfies the same equation of state. This law of radiation has been established in the theory of gases, in particular, by means of the virial theorem [1,2]. The virial theorem to describe radiation of electromagnetically interacting ultra-relativistic particles has been used in Ref. [1] (Ch. 5), where it is pointed out that one arrives at the 1/3-radiation law since the Maxwell energy–momentum tensor is characterized by the vanishing of its trace. This property plays a crucial role in establishing the correspondence “gas–particle–field”. In (3 + 1)-dimensions within nonlinear electrodynamics there is no way to establish the quoted radiation law by averaging the corresponding energy–momentum tensor with non-vanishing trace.

To obtain in (2 + 1)-spacetime the radiation law $p = \rho/2$, which can be established from the corresponding formal gas thermodynamics, from the view point of electrodynamics, one has to construct an energy–momentum tensor by means of the electrodynamics we proposed previously [3]. If one were averaging the

(2 + 1)-(linear) Maxwell energy–momentum tensor $T_{\mu\nu}$ one never should obtain the relation $p = \rho/2$ for ultra-relativistic particles interacting via this electrodynamics. This fact compelled us to search for an equivalent to the (3 + 1)-radiation formulation.

The main goal of this paper is just to establish a parallelism between radiation equation of states $p = \rho/3$ and $p = \rho/2$ in (3 + 1)- and (2 + 1)-gravities respectively.

From Cosmology, described by the Friedmann–Robertson–Walker (FRW) model,

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \right), \quad (1)$$

where $k = -1, 0, 1$ and $a(t)$ is the scale factor, one can easily obtain the 1/3-state equation for radiation: In this model all distances are proportional to the scale factor $a(t)$, thus the volume V scales as $a(t)^3$, and the wavelength λ of electromagnetic waves is proportional to $a(t)$, therefore the corresponding radiation energy density of one photon in a volume V is given by $\rho_{\text{photon}} = \frac{h\nu}{V} = \frac{hc}{\lambda V}$, this means that this energy density behaves like $\rho_{\text{photon}} = \rho_0 a(t)^{-4}$. On the other hand, using the energy conservation equation of a perfect fluid,

$$\dot{\rho} + 3\frac{\dot{a}}{a}(p + \rho) = 0, \quad (2)$$

one derives the expression of the pressure in terms of the density, which occurs to be just

$$p = \frac{1}{3}\rho, \quad (3)$$

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and with the help of the Friedmann equation

$$3\left(\frac{\dot{a}}{a}\right)^2 + \frac{3k}{a^2} = \kappa_4 \rho, \quad (4)$$

where $\kappa_4 = 8\pi G$, we may find, for example, that for a flat FRW model the scale factor is given by $a(t) = a_0 t^{1/2}$, where a_0 is a constant.

In (2 + 1)-FRW cosmology

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 \right), \quad (5)$$

one may accomplish a similar treatment for (2 + 1)-radiation. In this case the radiation energy density $\rho_{\text{photon}} = \frac{h\nu}{S} = \frac{hc}{\lambda S}$ amounts to $\rho_{\text{photon}} = \rho_0 a(t)^{-3}$. Hence by using the energy conservation equation of a perfect fluid,

$$\dot{\rho} + 2\frac{\dot{a}}{a}(p + \rho) = 0, \quad (6)$$

one obtains that the pressure fulfills

$$p = \frac{\rho}{2}, \quad (7)$$

and with the help of the 3-dimensional Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \kappa_3 \rho, \quad (8)$$

we may find, for example, that for a flat FRW model the scale factor is given by $a(t) = a_0 t^{2/3}$, where a_0 is a constant.

In Section 2 we briefly recall the averaging of the energy-momentum tensor approach to radiation in (3 + 1)-dimensions. In Section 3 the averaging approach is applied to the (2 + 1)-Maxwellian tensor and also to the (2 + 1)-nonlinear electrodynamics tensor singled out by the vanishing of its trace: in the Maxwell case one arrives at the stiff matter state equation $p = \rho$, which is far from being the 1/2-radiation law, while for the energy-momentum tensor of a conformally invariant electrodynamics, which is a (2 + 1)-nonlinear electrodynamics with vanishing trace, one obtains the radiation law $p = \rho/2$. The Cornish–Frankel cosmological radiative solution is commented and some concluding remarks are added.

2. Radiation in (3 + 1)-dimensions

Many years ago, in 1930, Tolman and Ehrenfest [4] analyzed the problem of the black body radiation in the framework of thermodynamic equilibrium of a matter distribution described by a perfect fluid energy-momentum tensor, using a static spherically symmetric spacetime. These authors established the black body radiation state equation (3) by treating this phenomenon from Maxwell electrodynamics via an averaging procedure of the components of the Maxwell energy-momentum tensor.

For the sake of reference, we repeat this procedure in details: the electromagnetic field is specified at any point by the Maxwell tensor $F_{\alpha\beta}$, and the Maxwell energy-momentum tensor is given by

$$T_{\alpha\beta} = -F_{\alpha\gamma} F_{\beta}{}^{\gamma} + \frac{1}{4} g_{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta}, \quad (9)$$

where $F_{\gamma\delta} F^{\gamma\delta} = 2(B^2 - E^2)$ is an invariant of the electromagnetic field.

It is well known that electromagnetic fields are not compatible with highly symmetric spacetimes, such as isotropic and homogeneous FRW spacetimes, due to inherent anisotropic nature of

Maxwell field sources. This electromagnetic field can be included as a source of such spacetimes through an averaging procedure, yielding then to an effective perfect fluid source with an isotropic pressure. In order to achieve the isotropy of the fields one has to require that the electric and magnetic fields components do not possess preferred directions thus the mean values fulfill the following relations [4]

$$\overline{E}_i = \overline{B}_i = \overline{E_i B_i} = 0, \quad (10)$$

and

$$\overline{E_i E_j} = -\frac{1}{3} E^2 g_{ij}, \quad \overline{B_i B_j} = -\frac{1}{3} B^2 g_{ij}, \quad (11)$$

where the bar over physical quantities stands for volumetric spatial average of the corresponding quantities at a given instant of time t .

Then, considering the metric (+1, -1, -1, -1), one has that $\overline{E_i^2} = E^2/3$, and $\overline{B_i^2} = B^2/3$. Using the above average relations, one obtains by averaging the energy-momentum tensor (9), the following perfect fluid configuration

$$T^{00} = \rho_{em}, \quad T^{11} = T^{22} = T^{33} = p_{em}, \quad (12)$$

where

$$p_{em} = \frac{1}{3} \rho_{em} = \frac{1}{6} (E^2 + B^2), \quad (13)$$

hence the radiation state equation (3) takes place.

Note that the state equation (13) is not only valid for a field of plane waves, for which $E = B$. In the deduction of Eq. (13) we did not say anything about the properties of the electromagnetic field, so the strengths of the electric and magnetic fields may take any value. It may be in particular a chaotic magnetic field (for which $E = 0$, $B \neq 0$) or a random magnetic field [5].

Consequently, “radiation” may be used to describe either actual electromagnetic radiation (for massless photons and neutrinos this state equation is exactly valid), or massive particles moving at relative velocities sufficiently close to the speed of light for which the state equation (3) takes place asymptotically. In this case due to that the velocities of the gas particles approach that of light their rest energy becomes negligible compared to their total energy. Thus by neglecting their rest masses fluid behaves like electromagnetic radiation.

Although radiation is a perfect fluid and thus has an energy-momentum tensor given by $T_{\mu\nu}^{PF} = (p + \rho)u_\mu u_\nu - pg_{\mu\nu}$ (with trace $T^{PF} = \rho - 3p$), we also know that $T^{\mu\nu}$ can be expressed in terms of the field strength (9). The trace of this is given by $T^\mu{}_\mu = \frac{1}{4\pi} [F^{\mu\nu} F_{\mu\nu} - \frac{1}{4}(4)F^{\lambda\sigma} F_{\lambda\sigma}] = 0$. But this must also equal $T^{PF} = \rho - 3p$, so the equation of state is (3).

From here we deduce that the term “radiation”, or more exactly the state equation (3), is used in a more general sense. Effectively, a such interpretation of “radiation” state equation can be introduced into the study of cosmological models in the early universe, where matter should be identified with a primordial plasma. This is equivalent to put the squared electric field $E^2 = 0$ in Eq. (13), neglecting bulk viscosity terms in the electric conductivity of the primordial plasma [6].

Also one can study a universe filled with a non-interacting chaotic or random magnetic field and radiation.

3. Radiation in (2 + 1)-dimensions

Now we shall treat the (2 + 1)-dimensional case. Usually one considers the same Maxwell electromagnetic energy-momentum

tensor (9) to describe electromagnetic phenomena in $(2 + 1)$ -dimensions. In these dimensions the Maxwell tensor $F_{\alpha\beta}$ has only three independent components, two for the vector electric field E_i and one for the magnetic field B , which now is a pseudo-scalar field, in contrast to 4-dimensional Maxwell field. Nevertheless, the use of the Maxwell electrodynamics in different dimensions deserves some attention. From Eq. (9) we obtain that in an $(N + 1)$ -dimensional spacetime the energy–momentum tensor trace is given by

$$T_{N+1} = T_{\alpha\beta} g^{\alpha\beta} = \left(-1 + \frac{N+1}{4}\right) F_{\gamma\delta} F^{\gamma\delta}. \quad (14)$$

It becomes clear that in $(3 + 1)$ -dimensions the electromagnetic tensor has a vanishing trace; this property, being an invariant one, singles out the Maxwell theory as the only trace free linear theory in $(3 + 1)$ -dimensions. Additionally, the propagation velocity of electromagnetic waves coincides with the velocity of propagation of the gravitational waves. Moreover, the eigenvalue problem, as it should be, presents its own features depending on the dimensionality. On the other hand, from Eq. (14) we see that $T_{2+1} = -1/4 F_{\gamma\delta} F^{\gamma\delta}$, hence in $(2 + 1)$ -dimensions the Maxwell energy–momentum tensor possesses a non-vanishing trace. As we shall see below, this implies that, in the framework of Maxwell electromagnetism, the equation of state $p = \rho/2$ takes place only for plane waves.

Notice that any comparison of the velocity of the electromagnetic waves with gravitational waves in $(2 + 1)$ -dimensions is empty, since there are no (vacuum) gravitational waves in these dimensions.

In what follows, we shall extend the well-established averaging procedure of the $(3 + 1)$ -theory to the $(2 + 1)$ -case for linear and conformally invariant nonlinear electrodynamics.

3.1. Maxwell electrodynamics; stiff matter and dust

Assuming (29) in $(2 + 1)$ -gravity that electrodynamics is described by Maxwell theory with the energy–momentum tensor (9), the averaging procedure yields

$$\overline{E_i} = \overline{E_i B} = 0, \quad (15)$$

and

$$\overline{E_i E_j} = -\frac{1}{2} E^2 g_{ij}, \quad \overline{B^2} = B^2 \quad (16)$$

($\overline{E_i^2} = E^2/2$, for the metric $(+1, -1, -1)$). Hence the averaging of the $(2 + 1)$ -Maxwell electromagnetic energy–momentum tensor yields

$$T_{2+1}^{00} = \rho_{2+1}^{em}, \quad T_{2+1}^{11} = T_{2+1}^{22} = p_{2+1}^{em}, \quad (17)$$

where

$$\rho_{2+1}^{em} = \frac{1}{2} (E^2 + B^2), \quad p_{2+1}^{em} = \frac{1}{2} B^2. \quad (18)$$

Consequently, from Eqs. (18), if $E = 0$ one concludes that a stiff matter state equation arises:

$$p_{2+1}^{em} = \rho_{2+1}^{em} = B^2/2, \quad (19)$$

which can be called plasma, in correspondence with the terminology used in $(3 + 1)$ -dimensions. In this case, by using the 3-dimensional Friedmann equation (8) with $k = 0$, we have that for a flat FRW model the energy density takes the form $\rho(t) = \rho_0 a^{-4}$, i.e. $B \sim a^{-2}$ and the scale factor is given by $a(t) = a_0 t^{1/2}$, where a_0 is a constant.

Next, if $B = 0$ the matter distribution can be viewed as dust:

$$\rho_{2+1}^{em} = E^2/2, \quad p_{2+1}^{em} = 0. \quad (20)$$

In this case, by using Eq. (8) with $k = 0$, we have that for a flat FRW model the energy density takes the form $\rho(t) = \rho_0 a^{-2}$, i.e. $E \sim a^{-1}$ and the scale factor is given by $a(t) = a_0 t$, where a_0 is a constant.

Notice that in general, the Maxwell electromagnetic tensor has only three independent components, two for the vector electric field $\vec{E} = (E_1, E_2)$ and one for the magnetic field B . Thus, in $(2 + 1)$ -dimensions only the electric component is inherently anisotropic. If $E_1 = E_2 = 0$ the magnetic component behaves like a perfect fluid with an equation of state of the stiff matter. Thus, in $(2 + 1)$ -dimensions, the equation of state (19) with $\rho(t) = \rho_0 a^{-4}$, $B \sim a^{-2}$ and $a(t) = a_0 t^{1/2}$ provide the general solution for flat FRW cosmologies sourced by a magnetic field.

From Eq. (18) the standard state equation $p = \rho/2$ is obtained only if one considers a sum of plane waves, i.e. incoherent isotropic black body radiation, where we have vacuum transverse electromagnetic waves with $E = B$. Thus, as well as we have in $(3 + 1)$ -dimensions for incoherent isotropic black body radiation, in $(2 + 1)$ -dimensions the radiation equation of state (7) is also fulfilled for plane waves. However, in $(3 + 1)$ -dimensions we have that the radiation equation of state (3) is also fulfilled for $B \neq E = 0$, $E \neq B = 0$; and $E \neq B$ with non-vanishing magnetic and electric fields, thus the whole parallelism with the 4-dimensional case does not extend to Maxwell electromagnetic fields satisfying the relation $E \neq B$ in 3-dimensional gravity.

3.2. $(2 + 1)$ -FRW cosmologies with a mixture of non-interacting electric and magnetic fields

In the framework of FRW spacetimes the 3-dimensional Maxwell electromagnetic field may be interpreted as a cosmological configuration with a mixture of two barotropic perfect fluids: a matter component

$$\rho_1(t) = \frac{B(t)^2}{2} \quad (21)$$

with a stiff equation of state ($p_1 = \rho_1$) representing the magnetic field, and a fluid

$$\rho_2(t) = \frac{E(t)^2}{2} \quad (22)$$

with a dust equation of state ($p_2 = 0$) representing the electric field. In this case both components ρ_1 and ρ_2 satisfy the conservation equation

$$\dot{\rho}_1 + \dot{\rho}_2 + 2\frac{\dot{a}}{a}(\rho_1 + \rho_2 + p_1 + p_2) = 0, \quad (23)$$

implying that the sum of two fluids is conserved.

In order to find solutions, formally we can consider scenarios where the electric field does not interact with the magnetic field, and scenarios where the electric and magnetic fields interact with each other.

Let us now consider the Maxwell equations for the studied gravitational configuration. For the metric (5) we may write

$$F = E_1 \theta^{(1)} \wedge \theta^{(0)} + E_2 \theta^{(2)} \wedge \theta^{(0)} + B \theta^{(1)} \wedge \theta^{(2)}, \quad (24)$$

where we have introduced the proper orthonormal basis $\theta^{(0)} = dt$, $\theta^{(1)} = a(t)/\sqrt{1 - kr^2} dr$ and $\theta^{(2)} = a(t) r d\theta$. Thus, the Maxwell tensor, in the coordinate basis, takes the form

$$F^{\mu\nu} = \begin{pmatrix} 0 & \frac{(1-kr^2)E_1}{a\sqrt{1-kr^2}} & \frac{E_2}{ra} \\ -\frac{(1-kr^2)E_1}{a\sqrt{1-kr^2}} & 0 & \frac{(1-kr^2)B}{ra^2\sqrt{1-kr^2}} \\ -\frac{E_2}{ra} & -\frac{(1-kr^2)B}{ra^2\sqrt{1-kr^2}} & 0 \end{pmatrix},$$

and the Maxwell equations $F^{\mu\nu}{}_{;v} = j^\mu$ and $F_{\alpha\nu;\mu} + F_{\mu\alpha;\nu} + F_{\nu\mu;\alpha} = 0$ are respectively given by (the Greek indices run from 0 to 2)

$$F^{\mu\nu}{}_{;v} = \begin{pmatrix} \frac{\sqrt{1-kr^2}}{ra} E_1 \\ -\frac{\sqrt{1-kr^2}}{a^2} (a\dot{E}_1 + E_1\dot{a}) \\ -\frac{1}{ra^2} (a\dot{E}_2 + E_2\dot{a}) \end{pmatrix} = j^\mu, \quad (25)$$

$$E_2 + \frac{r}{\sqrt{1-kr^2}} (a\dot{B} + 2B\dot{a}) = 0. \quad (26)$$

Let us first consider the case of a vanishing electric field. We obtain from Eq. (25) that $j^\mu = 0$, while Eq. (26) implies that $a\dot{B} + 2B\dot{a} = 0$, which is consistent with the homogeneity and isotropy of the FRW metric. Hence, the magnetic field is given by $B(t) = B_0/a^2(t)$, in agreement with what we have stated above, in the previous section, for the (2+1)-FRW magnetic solution.

We consider next the inclusion of the electric field into the study. It becomes clear that its vector character breaks the isotropy and homogeneity symmetries of the FRW spacetimes. For non-vanishing electric and magnetic fields, the inhomogeneous character of Eq. (26) requires first that

$$\dot{B} + 2B\frac{\dot{a}}{a} = \frac{1}{2} \frac{d}{dt} (B^2) + 2\frac{\dot{a}}{a} B^2 = 0, \quad (27)$$

and second that the electric component $E_2 = 0$. Consequently in Eq. (25) $F^{2\nu}{}_{;v} = 0 = j^2$. Now, by taking into account Eq. (21) and that

$$\dot{\rho}_1 + 2\frac{\dot{a}}{a}(\rho_1 + p_1) = \frac{1}{2} \frac{d}{dt} (B^2) + 2\frac{\dot{a}}{a} B^2 = 0, \quad (28)$$

from Eqs. (22) and (23) we obtain that

$$\dot{\rho}_2 + 2\frac{\dot{a}}{a}(\rho_2 + p_2) = \frac{1}{2} \frac{d}{dt} (E^2) + \frac{\dot{a}}{a} E^2 = 0. \quad (29)$$

These two equations indicate us that the electric and magnetic fields are conserved separately, and hence the conservation equation (23) is fulfilled.

Notice that due to that $E_2 = 0$ the RHS of Eq. (29) implies that

$$a\dot{E}_1 + E_1\dot{a} = 0, \quad (30)$$

then in Eq. (25) we have that $F^{1\nu}{}_{;v} = 0 = j^1$. Thus, the radial electric field takes the form $E_1 = E_0/a$. Since, for time dependent electric and magnetic fields, the Maxwell equations impose a restriction only on the non-radial electric component E_2 , which must vanish, then the Lorentz invariant is given by $F/2 = E^2 - B^2 = E_1^2 - B^2$. We can have pure electric field for $B^2 = 0$ or pure magnetic field for $E_1^2 = 0$, as well as a mixture of both fields.

The presence of the radial electric component E_1 clearly breaks the symmetries of the FRW spacetime. In order to fulfill them we conclude that an average procedure must be applied for the radial electric component E_1 , and consequently in Eq. (25) $F^{0\nu}{}_{;v} = 0 = j^0$.

However, it must be remarked that this scenario requires that $\overline{E_2^2} = 0$, therefore it is not related to the requirements of the spatial averaging procedure defined before in Eqs. (15) and (16), since this one requires that $\overline{E_1^2} = \overline{E_2^2} = E^2/2$, and $E^2 = \overline{E_1^2} + \overline{E_2^2}$, for a non-vanishing electric field. In this case the Maxwell equations are trivially fulfilled since for a such spatial averaging procedure we

have that $\overline{E_1} = \overline{E_2} = 0$, as we can see from Eq. (15). In what follows we shall discuss FRW solutions fulfilling the average procedure defined in Eqs. (15) and (16).

From Eqs. (28) and (29) we conclude that for non-interacting electric and magnetic fields we have that

$$B(t) = \frac{B_0}{a^2(t)}, \quad (31)$$

$$E(t) = \frac{E_0}{a(t)}, \quad (32)$$

respectively. The Friedmann equation (8) in this case takes the following form:

$$2\left(\frac{\dot{a}}{a}\right)^2 = \kappa_3 \left(\frac{B_0^2}{a^4} + \frac{E_0^2 - 2k/\kappa_3}{a^2} \right). \quad (33)$$

It becomes clear, for example, that for a flat FRW cosmology at early times the magnetic component dominates over the electric field, while for late times the electric field dominates over the magnetic field. Note that from Eq. (33) we have that the general form of the scale factor is given by

$$a^2(t) = \frac{\kappa_3(E_0^2 - 2k/\kappa_3)}{2} (C+t)^2 - \frac{B_0^2}{(E_0^2 - 2k/\kappa_3)}, \quad (34)$$

where C is a constant of integration.

For $B \neq 0$ and $E \neq 0$ the invariant $F/2$ is given by

$$B^2 - E^2 = \frac{B_0^2}{a^4} - \frac{E_0^2}{a^2}, \quad (35)$$

then if $0 < a \leq B_0/E_0$, $B^2 \geq E^2$, while if $B_0/E_0 \leq a < \infty$, $B^2 \leq E^2$.

It is useful to remark that, in the case of 3-dimensional static Einstein-Maxwell spacetimes, there exist the (2+1)-analog of the magnetic Reissner-Nordström spacetime, and separately the electric Reissner-Nordström analog [7]. It is noteworthy that the (2+1)-magnetic Reissner-Nordström analog is not a black hole in contrast with the (2+1)-electric Reissner-Nordström analog, where a black hole is present [7].

In order to close this subsection, we want to make some comments on the possibility of considering (2+1)-FRW cosmologies with a mixture of interacting electric and magnetic fields. In principle one can introduce more general scenarios where the magnetic and electric fields do not conserve separately and are coupled to each other. One coupling mechanism can be formally introduced into the Friedmann equations by defining a homogeneous interacting term $Q(t)$ in the following form [8]:

$$\dot{\rho}_1 + 2\frac{\dot{a}}{a}(\rho_1 + p_1) = Q(t), \quad (36)$$

$$\dot{\rho}_2 + 2\frac{\dot{a}}{a}(\rho_2 + p_2) = -Q(t), \quad (37)$$

In this case $Q > 0$ is interpreted as a transfer of energy from fluid ρ_2 to fluid ρ_1 , while for $Q < 0$, we should have an energy transfer from fluid ρ_1 to fluid ρ_2 . With the help of Eqs. (21) and (22), and by taking into account that $p_1 = \rho_1$ and $p_2 = 0$, Eqs. (36) and (37) may be rewritten in the form

$$\frac{1}{2} \frac{d}{dt} (B^2) + 2\frac{\dot{a}}{a} B^2 = Q(t), \quad (38)$$

$$\frac{1}{2} \frac{d}{dt} (E^2) + \frac{\dot{a}}{a} E^2 = -Q(t). \quad (39)$$

The interpretation of these equations is direct: for the case $Q > 0$ we have a transfer of energy from the electric field E to the magnetic field B , while if $Q < 0$, we should have an energy transfer from the magnetic to the electric fields. Notice that Eqs. (38)

and (39) imply that the whole conservation equation (23) is satisfied.

Nevertheless, in this case the inhomogeneous character of Eq. (26) requires that $a\dot{B} + 2B\dot{a} = 0$, implying that $Q(t) = 0$, so we are not allowed to consider such interacting scenarios for time-dependent electric and magnetic fields in the framework of homogeneous and isotropic cosmologies. However, it must be noticed that, in principle, such an interaction between electric and magnetic fields can be appropriately introduced in the framework of more general metrics than FRW ones, such as for example inhomogeneous circularly symmetric spacetimes depending, as well as the electric and magnetic fields, on the time and radial coordinates. For these interacting models the interacting term must be considered in the form $Q = Q(t, r)$. This work is currently in progress.

3.3. 3-dimensional conformally invariant electrodynamics

In $(3+1)$ -dimensions the equation of state $p = \rho/3$ is a direct consequence of the conformal invariant character of the Maxwell equations. Effectively, it can be shown that Maxwell equations in four dimensions are invariant under conformal transformation $\tilde{g}_{\alpha\beta} = \Omega^2 g_{\alpha\beta}$ and $\tilde{F}_{\mu\nu} = F_{\mu\nu}$ [9]. The conformal invariance of these equations is encoded by the traceless condition $T = T_{\mu\nu} g^{\mu\nu} = 0$ of the energy–momentum tensor (9) in $(3+1)$ -dimensions. Thus, the Maxwell field in four dimensions has conformal symmetry. This result is true regardless of whether spacetime is flat or curved. In spacetime dimensions with $N \neq 3$ this is not true anymore because the Maxwell energy–momentum tensor possesses a non-vanishing trace [9].

Fortunately, we can take advantage of this conformal symmetry by using an extension of the Maxwell action that possesses the conformal invariance in an arbitrary dimension. The Maxwell action in $(N+1)$ -dimensions may be written as [10]

$$S_M = \alpha \int \sqrt{-g} (F_{\mu\nu} F^{\mu\nu})^{\frac{N+1}{4}} d^{N+1}x, \quad (40)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. It is simple to see that under a conformal transformation acting on the metric and the electromagnetic fields as $\tilde{g}_{\alpha\beta} = \Omega^2 g_{\alpha\beta}$ and $A_\mu \rightarrow A_\mu$, this action remains unchanged [10]. The energy–momentum tensor associated to the action (40) is given by

$$T_{\alpha\beta} = 4\alpha (F_{\gamma\delta} F^{\gamma\delta})^{\frac{N+1}{4}} \times \left[-\frac{N+1}{4} F_{\alpha\gamma} F_{\beta\gamma} (F_{\gamma\delta} F^{\gamma\delta})^{-1} + \frac{1}{4} g_{\alpha\beta} \right]. \quad (41)$$

It can be shown that the traceless condition for energy–momentum tensor (41) is fulfilled.

The Maxwell action (40) in $(2+1)$ -dimensions takes the form

$$S_M = \alpha \int \sqrt{-g} (F_{\mu\nu} F^{\mu\nu})^{3/4} d^3x, \quad (42)$$

hence the energy–momentum tensor associated to the action (42) is given by

$$T_{\alpha\beta} = 3\alpha (F_{\gamma\delta} F^{\gamma\delta})^{-1/4} \left[-F_{\alpha\gamma} F_{\beta\gamma} + \frac{1}{3} g_{\alpha\beta} (F_{\gamma\delta} F^{\gamma\delta}) \right]. \quad (43)$$

It is interesting to note that this $(2+1)$ -electrodynamics is a nonlinear electrodynamics. Such 3-dimensional nonlinear electrodynamics was discussed before in the literature [3,11–13]. In general, one can construct a $(2+1)$ -Einstein theory coupled with nonlinear electrodynamics starting from the action

$$S_{NL} = \int \sqrt{-g} L(F) dx^3, \quad (44)$$

where the electromagnetic Lagrangian $L(F)$ depends upon a single invariant

$$F = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \frac{1}{2} (B^2 - E^2). \quad (45)$$

Physically one requires the Lagrangian to coincide with the linear Maxwell $L(F) = -F/4\pi$ at small values of the electromagnetic fields. The energy–momentum tensor associated to action (44) is given by

$$T_{\mu\nu} = g_{\mu\nu} L(F) - F_{\mu\gamma} F_{\nu}{}^\gamma L_{,F}, \quad (46)$$

where $L_{,F}$ denotes the derivative of $L(F)$ with respect to F . The trace of this tensor is given by

$$T = 3L(F) - 4FL_{,F}, \quad (47)$$

therefore, by requiring T to vanish, we establish the existence of the unique $(2+1)$ -nonlinear electrodynamics, with vanishing energy–momentum trace, given by the action (42). This nonlinear electrodynamics was considered first for obtaining a $(2+1)$ -dimensional static black hole with Coulomb-like field [3].

Thus, the conformally invariant $(2+1)$ -electrodynamics (42) is a particular case of 3-dimensional nonlinear electrodynamics described by the action (44). The same can be said about any higher dimension as well. Indeed, any $(N+1)$ -electrodynamics described by the action (40) is a particular case of $(N+1)$ -nonlinear electrodynamics theories, characterized by having a traceless energy–momentum tensor, and hence by being conformally invariant.

The averaging procedure, applied to the electric component E_i and the magnetic field B , yields relations (15) and (16), namely, for the metric $(+1, -1, -1)$: $\bar{E}_i = \bar{E}_i \bar{B} = 0$, $\bar{E}_i^2 = E^2/2$, $\bar{B}^2 = B^2$.

Consequently the average of the energy–momentum tensor of the nonlinear electrodynamics under consideration gives rise to the relations

$$T_{2+1}^{00} = \rho_{2+1}^{nonIE}, \quad T_{2+1}^{11} = T_{2+1}^{22} = p_{2+1}^{nonIE}, \quad (48)$$

$$p^{nonIE} = \frac{1}{2} \rho^{nonIE} = \frac{\alpha(E^2 + 2B^2)}{2|4F^{\gamma\delta} F_{\gamma\delta}|^{1/4}} \quad (49)$$

as one should expect. Note that Eqs. (45) and (49) imply that in this specific electrodynamics we may consider only cases with $E \neq B$ in order to have finite energy density and pressure in Eq. (49).

Therefore, we conclude that if one considers a 3-dimensional perfect fluid with the radiation equation of state $p = \rho/2$, this must be done in the framework of the nonlinear conformally invariant electromagnetic theory described by the action (42) and energy–momentum tensor (43).

Cornish and Frankel [14] derived, among others, a cosmological solution, using the $(2+1)$ -FRW metric, which fulfills the law (7), referring to it as radiation-dominated FRW universe, see Eqs. (4.1)–(4.7) of the quoted work. On the light of the present results, the Cornish–Frankel solution (see also Ref. [15]) has to be associated, from the point of view of electrodynamics, to plane waves in the framework of the linear Maxwell electrodynamics, and for more general cases with $E \neq B$ to the nonlinear conformally invariant electrodynamics exhibited above.

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