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Monotone Properties of Certain Classes of Solutions of Second-Order Difference Equations

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Abstract—The authors consider the difference equations

$$\Delta(a_n \Delta x_n) = q_n x_{n+1} \tag{$*$}$$

and

$$\Delta(a_n \Delta x_n) = q_n f(x_{n+1}), \tag{$**$}$$

where $a_n > 0$, $q_n > 0$, and $f : \mathbb{R} \to \mathbb{R}$ is continuous with $uf(u) > 0$ for $u \neq 0$. They obtain necessary and sufficient conditions for the asymptotic behavior of certain types of nonoscillatory solutions of $(*)$ and sufficient conditions for the asymptotic behavior of certain types of nonoscillatory solutions of $(**)$. Sufficient conditions for the existence of these types of nonoscillatory solutions are also presented. Some examples illustrating the results and suggestions for further research are included.

Keywords—Difference equations, Linear, Nonlinear, Asymptotic behavior, Nonoscillatory solutions, Monotone solutions.

1. INTRODUCTION

In this paper, we study the asymptotic behavior of certain solutions of the second-order difference equations

$$\Delta(a_n \Delta x_n) = q_n x_{n+1} \tag{E$_1$}$$

and

$$\Delta(a_n \Delta x_n) = q_n f(x_{n+1}), \tag{E$_2$}$$

where $\{a_n\}$ and $\{q_n\}$ are real sequences, $a_n > 0$ and $q_n > 0$, for all $n \geq 0$, and $f : \mathbb{R} \to \mathbb{R}$ is continuous with $uf(u) > 0$ for $u \neq 0$. By a solution of (E_1) or (E_2), we mean a real sequence $\{x_n\}$ that satisfies the equation and is not eventually identically zero. Such a solution is said to be *nonoscillatory* if it is eventually positive or eventually negative, and it is said to be oscillatory otherwise.

From results of Cheng, Li and Patula [1] and Thandapani, Graef and Spikes [2], it is known that any nontrivial solution $\{x_n\}$ of (E_1) or (E_2) is nonoscillatory and belongs to one of the two

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Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

classes

$$\mathcal{A} = \{\{x_n\} : \text{ there exists an integer } N \geq 0 \text{ such that } x_n \Delta x_n > 0 \text{ for } n \geq N\},$$
$$\mathcal{B} = \{\{x_n\} : \text{ there exists an integer } N \geq 0 \text{ such that } x_n \Delta x_n < 0 \text{ for } n \geq N\}.$$

The purpose of this paper is to study the asymptotic behavior of the solutions that belong to class $\mathcal{B}$. In Section 2, we consider the linear equation (E_1) and obtain necessary and sufficient conditions which ensure that $\lim_{n\to\infty} x_n = 0$ or $\lim_{n\to\infty} a_n \Delta x_n = 0$. These results extend previously known ones such as those found in [1]. In addition, we obtain some asymptotic estimates for the solutions in $\mathcal{B}$. Using fixed-point techniques, in Section 3 we obtain conditions that guarantee the existence of solutions of the nonlinear equation (E_2) belonging to class $\mathcal{B}$. Section 4 contains some results giving sufficient conditions, similar to those in Section 2, for the asymptotic behavior of the class $\mathcal{B}$ solutions of (E_2). The paper also includes some examples and some suggestions for future research. Results on the asymptotic behavior of solutions of (E_1) and (E_2) that belong to class $\mathcal{A}$ can be found in [1] and [2], respectively. Related results and additional references can be found in [3–6].

2. BEHAVIOR OF CLASS $\mathcal{B}$ SOLUTIONS OF (E_1)

Let

$$S_1 = \sum_{n=0}^{\infty} \frac{1}{a_n} \sum_{s=0}^{n} q_s \quad \text{and} \quad S_2 = \sum_{n=0}^{\infty} q_n \sum_{s=0}^{n} \frac{1}{a_s}.$$

It is known that equation (E_1) always has solutions in both class $\mathcal{A}$ and class $\mathcal{B}$ [1]. Clearly, every solution in class $\mathcal{B}$ is bounded. We recall the following result, which will be applied in the sequel.

THEOREM 1. [1, Theorem 4]. *Every solution of (E_1) is bounded if and only if $S_1 < \infty$.*

We now investigate the convergence of those solutions of (E_1) that belong to the class $\mathcal{B}$.

THEOREM 2.

(a) *Every solution of (E_1) in class $\mathcal{B}$ tends to zero if and only if $S_2 = \infty$.*
(b) *Every solution of (E_1) in class $\mathcal{B}$ tends to a nonzero limit if and only if $S_1 = \infty$ and $S_2 < \infty$.*

PROOF. Part (a) was proved in [1, Theorem 6]. To prove (b), assume $S_1 = \infty$, $S_2 < \infty$, and suppose that (E_1) has a class $\mathcal{B}$ solution $\{x_n\}$ such that $\lim_{n\to\infty} x_n = 0$. By Part (a), there exists at least one solution $\{y_n\}$ in $\mathcal{B}$ such that $\lim_{n\to\infty} y_n \neq 0$. Since $\{x_n\}$ and $\{y_n\}$ are two linearly independent solutions of (E_1) and are bounded for $n \geq 0$, all solutions of (E_1) are bounded for $n \geq 0$. This contradicts Theorem 1.

Now, assume that for every solution $\{x_n\}$ of (E_1) in $\mathcal{B}$, we have $\lim_{n\to\infty} x_n \neq 0$. The assertion follows from the fact that if $S_1 < \infty$, then there always exists a solution $\{z_n\}$ of (E_1) in $\mathcal{B}$ such that $\lim_{n\to\infty} z_n = 0$. This completes the proof of the theorem. ∎

EXAMPLE 1. The equation

$$\Delta\left(\frac{3}{2^n}\Delta x_n\right) = \frac{1}{2^n} x_{n+1}$$

satisfies the hypotheses of Theorem 2(a) and has the class $\mathcal{B}$ solution $\{x_n\} = \{(2/3)^{n+1}\} \to 0$ as $n \to \infty$.

EXAMPLE 2. The equation

$$\Delta((n+2)\Delta x_n) = \frac{1}{(n+1)(n+3)} x_{n+1}$$

satisfies the hypotheses of Theorem 2(b) and has the solution $\{x_n\} = \{(n+2)/n+1\} \to 1$ belonging to the class $\mathcal{B}$. Define

$$\mathcal{B}_0 = \left\{x_n \in \mathcal{B} : \lim_{n\to\infty} x_n = 0\right\},$$
$$\mathcal{B}_L = \left\{x_n \in \mathcal{B} : \lim_{n\to\infty} x_n \neq 0\right\}.$$

As an immediate consequence of Theorem 2, we have the following corollary.

COROLLARY 3. *For equation (E_1), the sets $\mathcal{B}_0$ and $\mathcal{B}_L$ are both nonempty if and only if $S_1 < \infty$ and $S_2 < \infty$.*

REMARK. To summarize Theorem 2 and Corollary 3, we have

$$S_2 = \infty \text{ if and only if } \mathcal{B} = \mathcal{B}_0,\ \mathcal{B}_L = \emptyset,$$
$$S_1 = \infty \text{ and } S_2 < \infty \text{ if and only if } \mathcal{B} = \mathcal{B}_L,\ \mathcal{B}_0 = \emptyset,$$
$$S_1 < \infty \text{ and } S_2 < \infty \text{ if and only if } \mathcal{B}_0 \neq \emptyset,\ \mathcal{B}_L \neq \emptyset.$$

A similar result concerning the asymptotic behavior of $\{a_n \Delta x_n\}$ can be obtained by noticing that $\{z_n\} = \{a_n \Delta x_n\}$ is a solution of the equation

$$\Delta\left(\frac{1}{q_n}\Delta z_n\right) = \frac{1}{a_{n+1}} z_{n+1}. \tag{E_1'}$$

Applying Theorem 2 to equation (E_1'), we obtain the following theorem.

THEOREM 4. *Let $\{x_n\}$ be a class $\mathcal{B}$ solution of* (E_1). *Then,*

(a) $\lim_{n\to\infty} a_n \Delta x_n = 0$ *if and only if* $S_1 = \infty$,
(b) $\lim_{n\to\infty} a_n \Delta x_n \neq 0$ *if and only if* $S_1 < \infty$ *and* $S_2 = \infty$.

EXAMPLE 3. Consider the equation

$$\Delta(2^n \Delta x_n) = \frac{2^{n+1}}{3} x_{n+1}.$$

This equation has the solution $\{x_n\} = \{3^{-n}\}$ which satisfies the conditions of Theorem 4(a).

EXAMPLE 4. The equation

$$\Delta((n+1)(n+3)\Delta x_n) = \frac{1}{n+3} x_{n+1}$$

satisfies the hypotheses of Theorem 4(b) and has the solution $\{x_n\} = \{1/(n+1)\}$ belonging to the class $\mathcal{B}$ and satisfying $\{a_n \Delta x_n\} = \{-(n+3)/n+2\} \to -1 \neq 0$.

From Theorems 2 and 4, we can relate the asymptotic behavior of a class $\mathcal{B}$ solution $\{x_n\}$ of (E_1) with the behavior of $\{a_n \Delta x_n\}$.

COROLLARY 5. *Let $\{x_n\}$ be a class $\mathcal{B}$ solution of (E_1). Then,*

(a) $\lim_{n\to\infty} x_n = \lim_{n\to\infty} a_n \Delta x_n = 0$ *if and only if* $S_1 = S_2 = \infty$,
(b) $0 = \lim_{n\to\infty} x_n \neq \lim_{n\to\infty} a_n \Delta x_n$ *if and only if* $S_1 < \infty$ *and* $S_2 = \infty$,
(c) $\lim_{n\to\infty} x_n \neq \lim_{n\to\infty} a_n \Delta x_n = 0$ *if and only if* $S_1 = \infty$ *and* $S_2 < \infty$.

If we apply Theorem 1 to equation (E_1'), we obtain that for every solution $\{x_n\}$ of (E_1), the sequence $\{a_n \Delta x_n\}$ is bounded if and only if $S_2 < \infty$. This is exactly the content of Lemma 6 in [1].

Next, we give asymptotic estimates for the solutions of (E_1) in $\mathcal{B}$.

COROLLARY 6. *Let $\{x_n\}$ be a class $\mathcal{B}$ solution of (E_1).*

(a) *If $S_1 < \infty$ and $S_2 = \infty$, then $\{x_n\}$ is asymptotically equivalent to $\sum_{s=n}^{\infty}(1/a_s)$, i.e.,*

$$\lim_{n\to\infty} \frac{x_n}{\sum_{s=n}^{\infty}(1/a_s)}$$

exists, is finite, and is different from zero.

(b) *If $S_1 = \infty$ and $S_2 < \infty$, then $\{x_n - \ell\}$ is asymptotically equivalent to*

$$\sum_{s=n}^{\infty} q_s \sum_{r=n}^{s} \frac{1}{a_r},$$

where $\ell = \lim_{n\to\infty} x_n \neq 0$.

(c) *If $S_1 < \infty$, $S_2 < \infty$ and $\lim_{n\to\infty} x_n = 0$, then $\{x_n\}$ is asymptotically equivalent to*

$$\sum_{s=n}^{\infty} \frac{1}{a_s}.$$

PROOF. Part (a) follows from Theorem 5 and L'Hôpital's rule. A similar proof holds for Parts (b) and (c). ∎

3. EXISTENCE OF CLASS $\mathcal{B}$ SOLUTIONS OF (E_2)

In this section, we use a fixed-point theorem to prove the existence of solutions of the nonlinear equation (E_2) belonging to the classes $\mathcal{B}_0$ and $\mathcal{B}_L$ under the assumption that both the sums

$$S_a = \sum_{n=0}^{\infty} \frac{1}{a_n} \quad \text{and} \quad S_q = \sum_{n=0}^{\infty} q_n$$

are finite. Of special interest here is the fact that no growth conditions are needed on the nonlinear function f.

THEOREM 7. *Let $S_a < \infty$ and $S_q < \infty$. Then, equation (E_2) has at least one solution in the class $\mathcal{B}_0$ and at least one solution in the class $\mathcal{B}_L$.*

PROOF. First, we prove the existence of a positive decreasing solution of (E_2) that approaches a nonzero limit as $n \to \infty$. Let $M = \max\{|f(u)| : 1 \le u \le 2\}$ and choose n_0 large enough so that

$$M\left[\sum_{n=n_0}^{\infty} q_n \sum_{s=n_0}^{n} \frac{1}{a_s} + \left(\sum_{n=n_0}^{\infty} \frac{1}{a_n}\right)\left(\sum_{n=n_0}^{\infty} q_n\right)\right] < \frac{1}{2} \tag{1}$$

and

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n} \le \frac{1}{2}. \tag{2}$$

Let B_{n_0} denote the Banach space of all real sequences $X = \{x_n\}$, $n \ge n_0$, with the supremum norm

$$\|X\| = \sup_{n \ge n_0} |x_n|,$$

and let

$$S = \{X \in B_{n_0} : 1 \le x_n \le 2, n \ge n_0\}.$$

Clearly, S is a bounded, convex, and closed subset of B_{n_0}. We define an operator $T : S \to B_{n_0}$ by

$$Tx_n = \frac{3}{2} + \sum_{s=n}^{\infty} \frac{1}{a_s} - \sum_{s=n_0}^{n-1} q_s \left(\sum_{t=n_0}^{s} \frac{1}{a_t}\right) f(x_{s+1}) - \left(\sum_{s=n_0}^{n-1} \frac{1}{a_s}\right)\left(\sum_{s=n}^{\infty} q_s f(x_{s+1})\right), \qquad n \ge n_0. \tag{3}$$

Next, we show that T satisfies the hypotheses of Schauder's fixed-point theorem.

(a) T maps S into itself. In fact, if $X \in S$, then from (1) and (3), we have

$$Tx_n \geq \frac{3}{2} - M\left[\sum_{n=n_0}^{\infty} q_n \sum_{t=n_0}^{n} \frac{1}{a_t} + \left(\sum_{n=n_0}^{\infty} \frac{1}{a_n}\right)\left(\sum_{n=n_0}^{\infty} q_n\right)\right] \geq \frac{3}{2} - \frac{1}{2} = 1, \tag{4}$$

and from (2), we have

$$Tx_n \leq \frac{3}{2} + \sum_{s=n_0}^{\infty} \frac{1}{a_s} \leq \frac{3}{2} + \frac{1}{2} = 2.$$

Therefore, $T(S) \subset S$.

(b) T is continuous. Let $X = \{x_n\} \in S$, let $\varepsilon > 0$ be given, and choose n_1 large enough so that

$$\max\left\{M \sum_{n=n_1}^{\infty} q_n,\ 2M \sum_{n=n_1}^{\infty} q_n \left(\sum_{t=n_0}^{n} \frac{1}{a_t}\right)\right\} < \varepsilon, \tag{5}$$

for $n \geq n_1$. For each i, let $Y^i = \{y_n^i\}$ be a sequence in S such that $\lim_{i\to\infty} \|Y^i - X\| = 0$. Then, for $n \geq n_1$, we have

$$\begin{aligned} |Ty_n^i - Tx_n| &\leq \sum_{s=n_0}^{\infty} q_s \left(\sum_{t=n_0}^{s} \frac{1}{a_t}\right) \left|f\left(y_{s+1}^i\right) - f(x_{s+1})\right| \\ &\quad + \left(\sum_{s=n_0}^{n-1} \frac{1}{a_s}\right)\left(\sum_{s=n}^{\infty} q_s \left|f\left(y_{s+1}^i\right) - f(x_{s+1})\right|\right) \\ &\leq \sum_{s=n_0}^{n_1-1} q_s \left(\sum_{t=n_0}^{s} \frac{1}{a_t}\right)\left|f\left(y_{s+1}^i\right) - f(x_{s+1})\right| + 2M \sum_{s=n_1}^{\infty} q_s \sum_{t=n_0}^{s} \frac{1}{a_t} + M \sum_{s=n_1}^{\infty} q_s. \end{aligned}$$

From (5) and the continuity of f, it follows that $\lim_{i\to\infty} |Ty_n^i - Tx_n| = 0$, so T is continuous.

(c) $T(S)$ is relatively compact. As proved by Cheng and Patula [4, Theorem 3.3], it suffices to show that $T(S)$ is uniformly Cauchy, so let $X \in S$ and $m > n \geq n_0$. Then,

$$|Tx_n - Tx_m| \leq \sum_{s=m}^{\infty} \frac{1}{a_s} + M \sum_{s=m}^{\infty} q_s \left(\sum_{t=n_0}^{s} \frac{1}{a_t}\right) + M \sum_{s=n_0}^{\infty} \frac{1}{a_s}\left(\sum_{s=m}^{\infty} q_s\right).$$

From the hypotheses, it is clear that for a given $\varepsilon > 0$, there exists an integer $n_1 \geq n_0$ such that for all $m > n \geq n_1$, $|Tx_n - Tx_m| < \varepsilon$. Thus, $T(S)$ is uniformly Cauchy and, hence, $T(S)$ is relatively compact.

Applying Schauder's fixed-point theorem [7], there exists $X \in S$ such that $TX = X$. That is,

$$x_n = \frac{3}{2} + \sum_{s=n}^{\infty} \frac{1}{a_s} - \sum_{s=n_0}^{n-1} q_s \left(\sum_{t=n_0}^{s} \frac{1}{a_t}\right) f(x_{s+1}) - \left(\sum_{s=n_0}^{n-1} \frac{1}{a_s}\right)\left(\sum_{s=n}^{\infty} q_s f(x_{s+1})\right).$$

It is easy to see that $\{x_n\}$ is a solution of (E_2). Since

$$\Delta x_n = -\frac{1}{a_n}\left(1 + \sum_{s=n}^{\infty} q_s f(x_{s+1})\right) < 0,$$

and $1 \leq x_n \leq 2$, we see that $\{x_n\}$ is an eventually positive decreasing solution of (E_2) with $\lim_{n\to\infty} x_n = \ell \neq 0$. Hence, $\mathcal{B}_L \neq \emptyset$.

Next, we prove the existence of an eventually positive decreasing solution of (E_2) that tends to zero as $n \to \infty$. Let $M = \max\{|f(u)| : 0 \le u \le 1\}$ and choose n_0 such that

$$M \sum_{n=n_0}^{\infty} \frac{1}{a_n} \left(\sum_{s=n}^{\infty} q_s \right) \le 1.$$

Let B_{n_0} be the Banach space defined above, let $S = \{X \in B_{n_0} : 0 \le x_n \le 1, n \ge n_0\}$, and define the operator T by

$$Tx_n = \sum_{s=n}^{\infty} \frac{1}{a_s} \sum_{t=s}^{\infty} q_t f(x_{t+1}), \qquad n \ge n_0.$$

Using an argument similar to the one above, we can show that the operator T satisfies the assumptions of Schauder's fixed-point theorem. Therefore, there exists an $X \in S$ such that $TX = X$, i.e.,

$$x_n = \sum_{s=n}^{\infty} \frac{1}{a_s} \sum_{t=s}^{\infty} q_t f(x_{t+1}).$$

Since

$$\Delta x_n = -\frac{1}{a_n} \left(\sum_{s=n}^{\infty} q_s f(x_{s+1}) \right) < 0,$$

and $x_n \to 0$ as $n \to \infty$, we see that $\{x_n\} \in \mathcal{B}_0 \neq \emptyset$. This completes the proof of the theorem. ∎

In conclusion, we note that our results in this section can be extended to nonlinear difference equations of the form

$$\Delta(a_n \psi(x_n) \Delta x_n) = h(n, x_{n+1}),$$

where $\psi : \mathbb{R} \to \mathbb{R}$ is a positive continuous function, $h : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ is continuous with $uh(n,u) > 0$ for $u \neq 0$. By requiring $|h(n,u)| \le q_n|u|$, the results in Section 2 can be extended to this equation as well.

4. FURTHER RESULTS ON THE CLASS $\mathcal{B}$ SOLUTIONS OF (E_2)

It is reasonable to ask whether it is possible to give conditions under which solutions of (E_2) satisfy $\lim_{n\to\infty} x_n = 0$ or $\lim_{n\to\infty} a_n \Delta x_n = 0$ as we were able to do for equation (E_1) in Section 2. In this section, we present a couple of results in this direction.

THEOREM 8. *If $S_a = \infty$, then any class $\mathcal{B}$ solution $\{x_n\}$ of (E_2) satisfies $\lim_{n\to\infty} a_n \Delta x_n = 0$.*

PROOF. Suppose that $\{x_n\}$ is a class $\mathcal{B}$ solution of (E_2), say $x_n > 0$ for $n \ge n_1 \ge 0$. Then, $a_n \Delta x_n < 0$ and increasing. If $a_n \Delta x_n \not\to 0$, there exists $K > 0$ such that $a_n \Delta x_n \le -K < 0$ for $n \ge n_1$. Summing, we have

$$x_{n+1} \le x_{n_1} - K \sum_{s=n_1}^{n} \frac{1}{a_s} \to -\infty,$$

as $n \to \infty$, which is a contradiction. ∎

REMARK. Under the assumption $S_a = \infty$, it is not difficult to see that $S_q < \infty$ is a necessary condition for a class $\mathcal{B}$ solution of (E_2) to converge to a nonzero limit as $n \to \infty$.

Clearly, $S_a = \infty$ implies $S_1 = \infty$. It would be interesting to know if the conclusion of Theorem 8 holds under this weaker hypothesis (see, Theorem 4(a)). Also, the question of whether the conditions $S_1 < \infty$ and $S_2 = \infty$ are enough to ensure that class $\mathcal{B}$ solutions of (E_2) satisfy $\lim_{n\to\infty} a_n \Delta x_n \neq 0$ remains open.

THEOREM 9. *If $S_2 = \infty$, then any class $\mathcal{B}$ solution $\{x_n\}$ of (E_2) satisfies $\lim_{n\to\infty} x_n = 0$.*

PROOF. Suppose that $\{x_n\}$ is a class $\mathcal{B}$ solution of (E_2), say $x_n > 0$ for $n \geq N \geq 0$, and $\lim_{n\to\infty} x_n = L > 0$. Let $M = \min\{f(u) : L \leq u \leq x_{n_1}\}$. Summing equation ($E_2$), we have

$$a_n \Delta x_n - a_N \Delta x_N = \sum_{s=N}^{n-1} q_s f(x_{s+1}) \geq M \sum_{s=N}^{n-1} q_s.$$

It follows that

$$M \sum_{s=n}^{\infty} q_s \leq -a_n \Delta x_n,$$

so

$$M \sum_{s=N}^{n} \frac{1}{a_s} \sum_{t=s}^{\infty} q_t \leq -\sum_{s=N}^{n} \Delta x_s = x_N - x_{n+1}. \tag{6}$$

Since $S_2 = \infty$, a summation by parts shows that the left-hand side of (6) tends to ∞ as $n \to \infty$, and this contradicts $x_n > 0$ for $n \geq N \geq 0$. ∎

It would be interesting to know if $S_2 = \infty$ is a necessary condition for class $\mathcal{B}$ solutions of (E_2) to converge to zero so that we would have the complete counterpart of Theorem 2(a).

We conclude this paper with one more suggestion for further research. Are the conditions $S_2 < \infty$ and $S_a = \infty$ together enough to ensure that class $\mathcal{B}$ solutions of (E_2) converge to a nonzero limit?

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