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HAMILTONIAN CIRCUITS IN PRISMS OVER CERTAIN SIMPLE 3-POLYTOPES

P.R. GOODEY
Department of Mathematics, Royal Holloway College, Englefield Green, Surrey, United Kingdom

M. ROSENFELD
Department of Mathematics, Ben Gurion University of the Negev, Beer Sheba, Israel

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In this paper it is shown that the prisms over cyclically 4-connect simple 3-polytopes admit Hamiltonian circuits. It is also shown that if P is a simple 3-polytope all of whose faces are polygons with six sides or less than the prism over P admits a Hamiltonian circuit.

1. Introduction

It is well-known (see for example [1, 5, 9]) that there are simple 3-polytopes which do not admit Hamiltonian circuits. But it is suggested in [4] that, for $d \geq 4$, every simple d-polytope has a Hamiltonian circuit. This conjecture is still unresolved even in the case of prisms over simple 3-polytopes. It is shown in [7] that, for any 3-polytope P, the k-fold prism ($k \geq 2$) over P has a Hamiltonian circuit. The authors also prove that if the faces of the simple 3-polytope P are 4-colourable then the prism over P has a Hamiltonian circuit.

Since the completion of this work, the four colour theorem has, in fact, been established and therefore, as mentioned above, all simple 4-dimensional prisms admit Hamiltonian circuits. This is a stronger result than the ones proved in this article, though the latter may still retain some significance because of the techniques employed, in fact, we obtain these results using Tutte's important theory of bridges in planar graphs. Though it seems that this theory is very fundamental in the study of planar graphs, only one application of it can be found in the literature, Tutte's proof that 4-connected planar graphs admit Hamiltonian circuits. We show that if P is a simple cyclically 4-connected 3-polytope then the prism over P admits a Hamiltonian circuit. For such a polytope P, each face is an n-gon with $n \geq 4$. The second result is concerned with simple 3-polytopes all of whose faces are n-gons with $n \leq 6$. It seems likely that such polytopes admit Hamiltonian circuits. Here we show that the prisms over these polytopes have Hamiltonian circuits.

2. Definitions and notation

Throughout the work we shall use the notation of [4]. Let G be a graph and H a subgraph of G. An *H-avoiding* path in G is a path $P(a_0, a_n)$ in which no edges or vertices belong to H except possibly the end vertices a_0 and a_n. Two edges E_1, E_2 not belonging to H are said to be *bridge equivalent* with respect to H when an H-avoiding path $P(E_1, E_2)$ exists, beginning in E_1 and ending in E_2. This relation is an equivalent relation (Ore [6, Chapter 2]). All edges which are bridge equivalent to an edge E, form a *bridge* of H in G. We denote such a bridge by $B(E)$. The bridge is *degenerate* if $B(E) = E$. The vertices of a bridge B that also belong to H are the *vertices of attachment* of B in H. We denote their number by $w(B)$. The subgraph spanned in G by the vertices of a bridge B, that are not attachment vertices of B in H, is the *nucleus of B*. A bridge is *simple* if its nucleus is a single vertex. The following important result due to Tutte [10] (see also Ore [6, p. 68]) is one of the main tools in our proofs.

Theorem 2.1. *Let G be a planar graph. If E_1 and E_2 are distinct edges of G lying in the same face, then there is a circuit T of G having the following properties:*

(i) $E_1, E_2 \in T$;
(ii) *if B is a bridge of T in G then $w(B) \leq 3$; and*
(iii) *if B is a bridge of T in G which meets either of the two faces containing E_1, then $w(B) \leq 2$.*

A circuit T in G that has the above properties will be called a *Tutte circuit*. We shall also use the following result of Steinitz [8].

Theorem 2 *The graph G is the graph of a 3-polytope if and only if G is planar and 3-connected.*

In fact we shall not usually distinguish between the polytope and its graph.

The prism over a graph G, is a graph $P(G)$, formed by taking two disjoint copies of G and adding the *vertical edges*, that is the edges connecting a vertex of G to the same vertex in the second copy of G.

3. Graphs of type S

An *S-circuit* is a graph consisting of a circuit $C = \{a_1, a_2, \ldots, a_n\}$ together with a family of disjoint paths $\{P_i\}$ such that each P_i has exactly one end vertex on C and no other vertex of P_i belongs to C (Fig. 1). An S-circuit is *even* if C is an even circuit. A connected graph G is of *type S* if its vertices are of degree not greater than 3, it has a family of disjoint even S-circuits and a family of disjoint paths, such that the two end vertices of each path belong to two distinct circuits of the S-circuits, and no other vertex of the path belongs to an S-circuit.

Lemma 3.1. *If G is of type S then the prism over G admits a Hamiltonian circuit*

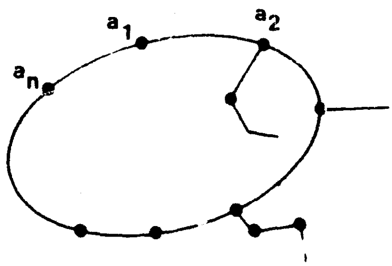

Fig. 1. An S circuit

Proof. Let $C=\{c_1, c_2, \ldots, c_{2k}\}$ be an even circuit. Let $C'=\{c'_1, c'_2, \ldots, c'_{2k}\}$ be another copy of C. In the prism over C, $H=\{c_1, c'_1, c'_2, c_2, \ldots, c'_{2k}, c_{2k}\}$ is a Hamiltonian circuit that uses all vertical edges. If $P_i=\{a_1, a_2, \ldots, a_k\}$ is a path, and $a_1=c_j$, then in the prism over the S-circuit $C\cup P_i$, $\{H\backslash(c_j, c'_j)\}\cup\{c_j, a_2, \ldots, a_k, a'_k, \ldots, a'_2, c'_j\}$ is a Hamiltonian circuit. Continuing in this fashion we see that if G is an even S-circuit then $P(G)$ admits a Hamiltonian circuit that uses all vertical edges at the vertices of degree 2 of C. If G is of type S, we first remove all paths that connect distinct S-circuits of G, obtaining a family of disjoint even S-circuits. In the prism over each one of them we construct a Hamiltonian circuit of the type described above. For every path $P=\{b_1, b_2, \ldots, b_k\}$ that has been removed, we have $b_1\in S$, $b_k\in S'$ for two of the even S-circuits with $S\cap S'=\emptyset$, and P has no other vertices in common with any of the S-circuits. We remove the vertical edges (b_1, b'_1) and (b_k, b'_k) from the Hamiltonian circuits in the prisms over S and S', introduce the paths $(b_1, b_2, \ldots, b_k)$ and $(b'_1, b'_2, \ldots, b'_k)$ and obtain a Hamiltonian circuit in the prism over $S\cup S'\cup P$. Obviously, the same process can be repeated for all other paths thus obtaining a Hamiltonian circuit in $P(G)$.

4. Prisms over cyclically 4-connected 3-polytopes

A simple 3-polytope is called *cyclically k-connected* if its graph cannot be broken into two separate parts, each containing a circuit, by the removal of fewer than k edges (Grünbaum [4, p. 365]).

Lemma 4.1. *If G is a cyclically 4-connected polytope and T is a Tutte circuit in G then all bridges of T in G are degenerate or simple.*

Proof. Since the removal of the attachment vertices of a bridge B would disconnect G unless the nucleus of B is empty, there are no bridges with $w(B)=1$. If $w(B)=2$, since G is 3-connected its nucleus must be empty. If $w(B)=3$, let E_1, E_2, E_3 be the three edges of B that contain the three attachment vertices. The removal of these edges from G breaks G into two parts, one containing the circuit T and the other, the nucleus of B. Since G is cubic, the nucleus of B must consist of a single vertex. Since every bridge of T has at most three vertices of attachment, the lemma is proved.

Theorem 4.2. *The prism over any cyclically 4-connected, 3-polytope G admits a Hamiltonian circuit.*

Proof. By Lemma 3.1, it suffices to show that G admits a spanning subgraph of type S. Let T be a Tutte circuit for G. By Lemma 4.1, all bridges of T in G are degenerate or simple. Assume first that T is even. Since every vertex of $G \setminus T$ is the nucleus of some bridge of T, by removing two of the edges incident with it we obtain in G a spanning subgraph which is an even S-circuit. Thus we may assume that T is odd.

Consider now all non-degenerate bridges of T (there must be such bridges since the number of vertices of G is even). Each non-degenerate bridge is simple. The three attachment vertices of such a bridge determine six arcs on T. Obviously, at least one of the arcs is of even length. Assume that B_0, with nucleus v_0 and attachment vertices $\{a_0, b_0, c_0\}$ determines an even arc of maximal length among all bridges of T in G (Fig. 2). Without loss of generality, we may assume that the arc $A_0 = (a_0 - b_0 - c_0)$ (Fig. 2) is the longest even arc. If B is another simple bridge of T with nucleus x_0, that has an attachment vertex on A_0, we delete from G the two other edges incident with x_0. If G does not contain any additional non-degenerate bridges, it is easily seen that G has a spanning even S-circuit based on the circuit $A_0' = (a_0, v_0, c_0 - b_0 - a_0)$. We denote the even S-circuit based on A_0' with the edges (x_i, y), $y \in A_0$, by S_0. Consider now all other non-degenerate bridges of T in G. Obviously, each such bridge must have its three vertices of attachment on $T \setminus A_0$. Let B_1, with nucleus v_1 and attachment vertices $\{a_1, b_1, c_1\}$ be a bridge that determines a longest even arc among these bridges. Because of the maximality of A_0, $A_1 = (a_1 - b_1 - c_1)$ must be an even arc (Fig. 2). The circuit $A_1' = (a_1 - b_1 - c_1 - v_1)$ is even and disjoint from S_0. Again, for every vertex x' of $G \setminus (T \cup S_0)$, that has a vertex $y \in A_1'$ such that (x', y) is an edge in G, we remove from G the two other edges incident with x' to form an S-circuit S_1. If there are additional vertices in G, each one of them is the nucleus of a simple bridge. The

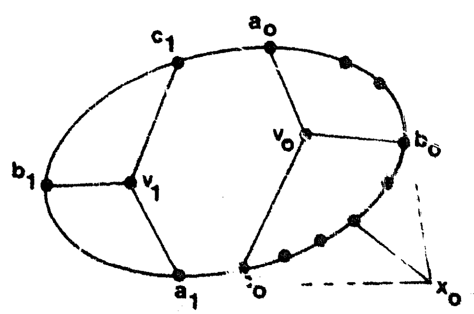

Fig. 2

three vertices of attachment of such a bridge must belong to a connected component of $T\setminus(A_0\cup A_1)$, and the arc determined by them on this component must be even. Thus, the process described above can be continued until all vertices of G are accounted for. The even S-circuits $\{S_i\}$ are disjoint. These S-circuits together with the paths $T\setminus\bigcup A_i$ span G. Therefore G has a spanning subgraph of type S, as required.

5. Prisms over small polytopes

A simple 3-polytope will be called a *small polytope* if none of its faces have more than six sides. We shall use the following result of Ewald [2].

Theorem 5.1. *If G is the graph of a small polytope and J is a longest circuit in G then J contains a vertex of every face of G.*

Lemma 5.2. *If G is the graph of a small polytope which has a triangular face then G admits a circuit of even length which contains a vertex of every face of G.*

Proof. Assume that v_1, v_2, v_3 are the vertices of a triangular face F of G. For $i = 1, 2, 3$ let w_i be the vertex of G adjacent to v_i which does not belong to F. Let J be a longest circuit in G, then by Theorem 5.1, we may assume that $P_1 = (w_1, v_1, v_2, v_3, w_3)$ is a subpath of J. If J is of even length there is nothing to prove. Otherwise it is clear that replacing P_1 by (w_1, v_1, v_3, w_3) will give us the required circuit.

Lemma 5.3. *If G is the graph of a small polytope which has a pentagonal face then G admits an even circuit which contains a vertex of every face of G.*

Proof. Let J be a longest circuit in G, which we may assume to be of odd length. Let F be a pentagonal face of G with vertices v_1, v_2, v_3, v_4, v_5 and let w_i be the vertex of G adjacent to v_i with $w_i \notin F$ for $1 \leq i \leq 5$. Also let F_i be the face of G with $F_i \cap F = (v_i, v_{i+1})$ for $1 \leq i \leq 5$, where $v_6 = v_1$ (Fig. 3).

If J intersects F in a single component then this component must have length 3 or 4 because of the maximality of J. Assume that the component has length 4 and that $(w_1, v_1, v_2, v_3, v_4, v_5, w_5)$ is a subpath of J. Note that $(v_2, v_3) \neq J \cap F_2$ and $(v_3, v_4) \neq J \cap F_3$ since J is maximal. Thus if we replace the path $(v_1, v_2, \ldots, v_5)$ by the edge (v_1, v_5) in J we get a circuit of even length which contains a vertex of every face, since F_2, F_3 have vertices other than v_2, v_3, v_4 on J. Similarly, we can deal with the case there the length of the component is 3.

So we now assume that J intersects F in two components and that one of them is (v_1, v_2). Let the other component be J_1, which is of length either 1 or 2. Now if J intersects F_1 in a single component J_2 then J_2 is of length at least 3 and by the

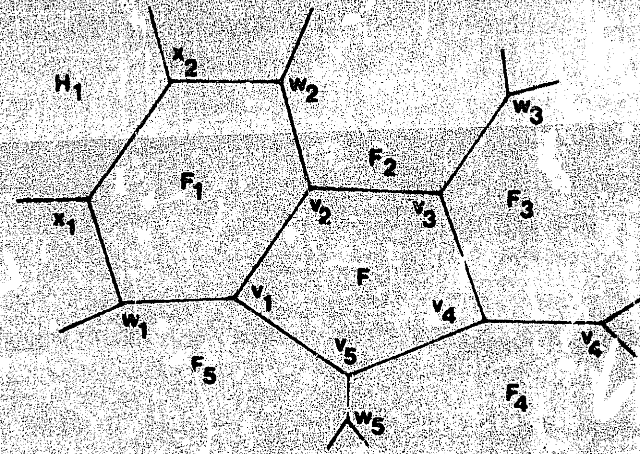

Fig. 3

above argument we can assume that F_1 is either a quadrilateral or a hexagon. Thus if we replace J_2 by its complementary arc in F_1 and J_1 by its complementary arc in F we get an even circuit which, because of the maximality of J contains a vertex of every face. So we must assume that J intersects F_1 in two components.

Thus F_1 must be a hexagon with vertices $w_1, v_1, v_2, w_2, x_2, x_1$ and the two components of J in F_1 are $P_1 = (w_1, v_1, v_2, w_2)$ and (x_1, x_2). Let H_1 be the face of G such that $H_1 \cap F_1 = (x_1, x_2)$, note that $H_1 \neq F$ (Fig. 3). If J intersects H_1 in a single component J_3 then H_1 must be either a quadrilateral or a hexagon. We replace J_3 by its complementary arc in H_1, P_1 by (w_1, x_1, x_2, w_2) and J_1 by its complementary arc in F to obtain a circuit of even length which contains a vertex of every face. Otherwise J must intersect H_1 in two components and then we can construct a face H_2 just as we did H_1, and H_2 will be distinct from F, F_1, H_1. Again we can construct an even circuit or else find yet another face H_3. Since the number of faces is finite, this process must terminate and when it does it is clear that we obtain the required circuit.

Theorem 5.4. *The prism over any small polytope admits a Hamiltonian circuit.*

Proof. If the polytope has a triangular or pentagonal face then by Lemmas 5.2 and 5.3 it has an even circuit which contains a vertex of every face. Thus there is an even S-circuit which spans the graph of the polytope and so the result follows from Lemma 3.1.

If the polytope has only quadrilateral or hexagonal faces then we know by [3] that it admits a Hamiltonian circuit. Clearly then the prism over the polytope will also have a Hamiltonian circuit.

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