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Decomposition of Regular Matroids

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It is proved that every regular matroid may be constructed by piecing together graphic and cographic matroids and copies of a certain 10-element matroid.

1. INTRODUCTION

We shall assume familiarity with matroid theory; for an introduction, see Welsh [16].

Let M_1, M_2 be binary matroids with element sets S_1, S_2, respectively, where S_1, S_2 may intersect. We define a new binary matroid $M_1 \Delta M_2$ to be the matroid with element set $S_1 \Delta S_2$ and with cycles all subsets of $S_1 \Delta S_2$ of the form $C_1 \Delta C_2$, where C_i is a cycle of M_i $(i = 1, 2)$. (For sets S_1, S_2, $S_1 \Delta S_2$ denotes $(S_1 - S_2) \cup (S_2 - S_1)$. A *cycle* of a binary matroid is a subset of the elements expressible as a disjoint union of circuits. It is easy to see that if C, C' are cycles then $C \Delta C'$ is a cycle.)

We are only concerned with three special cases of this operation, as follows.

(i) When $S_1 \cap S_2 = \varnothing$ and $|S_1|, |S_2| < |S_1 \Delta S_2|$ (that is, $S_1, S_2 \neq \varnothing$) $M_1 \Delta M_2$ is a *1-sum* of M_1 and M_2.

(ii) When $|S_1 \cap S_2| = 1$, and $S_1 \cap S_2 = \{z\}$, say, and z is not a loop or coloop of M_1 or M_2, and $|S_1|, |S_2| < |S_1 \Delta S_2|$ (that is, $|S_1|, |S_2| \geqslant 3$), $M_1 \Delta M_2$ is a *2-sum* of M_1 and M_2.

(iii) When $|S_1 \cap S_2| = 3$, and $S_1 \cap S_2 = Z$, say, and Z is a circuit of M_1 and M_2, and Z includes no cocircuit of either M_1 or M_2, and $|S_1|$, $|S_2| < |S_1 \Delta S_2|$ (that is, $|S_1|, |S_2| \geqslant 7$), $M_1 \Delta M_2$ is a *3-sum* of M_1 and M_2.

In each case M_1 and M_2 are called the *parts* of the sum.

It is helpful to visualize these in terms of graphs. Let G be a connected graph, and let $Y \subseteq V(G)$ be a minimal cut-set (that is, the deletion of Y from G produces a disconnected graph, but no proper subset of Y has this

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property). Choose non-empty subsets T_1, T_2 of $V(G)$, such that (T_1, Y, T_2) is a partition of $V(G)$, and no edge joins a vertex of T_1 to a vertex of T_2. Add a set Z of new edges, one joining every pair of distinct vertices in Y. Let G_1, G_2 be subgraphs so that $V(G_i) = Y \cup T_i$ $(i = 1, 2)$, $E(G_1) \cup E(G_2) = E(G) \cup Z$, and $E(G_1) \cap E(G_2) = Z$. Then if $|Y| = k$ and $1 \leqslant k \leqslant 3$, the polygon matroid $\mathscr{M}(G)$ of G is the k-sum of $\mathscr{M}(G_1)$ and $\mathscr{M}(G_2)$, provided that these both have fewer elements than $\mathscr{M}(G)$.

A matroid is *regular* if it is binary and has no minor isomorphic to the Fano matroid F_7 or its dual F_7^*. (M^* will always denote the dual of M.) Tutte proved that a matroid is regular if and only if it can be represented by the columns of a totally unimodular matrix [11], but we do not assume this result. The object of this paper is to show that every regular matroid may be obtained by means of 1-, 2-, and 3-sums, starting from graphic and cographic matroids and copies of a certain 10-element matroid which we call R_{10}. (A matroid is *graphic* if it is the polygon matroid of some graph.) Brylawski [4] proved the converse, that taking the 1-, 2-, or 3-sum of two regular matroids produces a regular matroid.

This theorem has several applications in combinatorics. Edmonds used it in an algorithm to recognize totally unimodular matrices (combining it with the work of Bixby and Cunningham, and Cunningham and Edmonds); combined with a theorem of Wagner it reduces Tutte's "tangential 2-block" conjecture to a conjecture about graphs; combined with Brylawski's theorem above it implies Tutte's characterization of regular matroids; combined with the Kuratowski–Wagner theorem it implies Tutte's characterization of graphic matroids (however, the proof of our result uses this theorem); the method of proof may be adapted to give a "short" proof of Tutte's characterization of graphic matroids, and a very short proof of a matroid max-flow min-cut theorem of the author; and there are some applications to "multicommodity flows" in matroids. Some of these will appear in subsequent papers [8–10].

The operations of taking 1-, 2-, and 3-sums, even in the extended sense of the next section, are not new. A 1-sum is simply a "direct sum" or "disjoint union", and 2-sums have also been investigated (for example, [2, 3]); 3-sums are rather less well-known, but are a special case of Brylawski's "modular flat" construction [4], and also occur in [5]. Our theorem too is not the first of its kind. There is a famous decomposition theorem in graph theory, due to Wagner [15], which is closely analogous to ours. Put into matroid language, it asserts that all (and only) graphic matroids without minors isomorphic to $\mathscr{M}(K_5)$ may be obtained by means of 1-, 2-, and 3-sums, starting from polygon matroids of planar graphs and copies of two special graphic matroids. (Actually, in Wagner's paper, it is only necessary to use one "special" graph, because he uses a different definition of 3-sum. But the adaptation is straightforward.) The close resemblance between Wagner's theorem and ours may help to make least the form of ours more plausible.

The proof of the theorem is in three steps.

(A) We define the (10-element, regular) matroid R_{10}, and show that every regular matroid may be obtained by 1- and 2-sums from regular matroids without R_{10} minors and copies of R_{10}. To do this we show that every regular matroid with an R_{10} minor is 2-separable, in Tutte's sense, except R_{10} itself. This step of the proof is essentially easy. (However, some of the lemmas are stated in greater generality than is needed here, because they have other applications.)

(B) We define the (12-element, regular) matroid R_{12}, and show that every regular matroid may be obtained by 1-, 2-, and 3-sums from regular matroids without R_{12} minors. To do this we partition the elements of R_{12} into two sets A, B of size 6, and show that every regular matroid with an R_{12} minor has a 3-separation (X, Y) with $A \subseteq X$, $B \subseteq Y$. This requires an analysis of the minimal matroids with an R_{12} minor, in which the connectivity between A and B is larger than it is in R_{12}.

(C) We show that every 3-connected regular matroid which is neither graphic nor cographic has an R_{10} or R_{12} minor. This is lengthy, but the matroid theory involved is simple—the difficulties are purely graph-theoretic.

Let us fix some notation and terminology. $E(M)$ is the set of elements of a matroid M, and if $E(M) = S$, M is said to be *on* S. For $Z \subseteq E(M)$, $M \times Z$ is the *restriction* of M to Z, that is, the matroid on Z with circuits those circuits of M included in Z. For convenience, $M \backslash Z$ denotes $M \times (E(M) - Z)$, and M/Z denotes $(M^* \backslash Z)^*$. $M \backslash Z$ and M/Z are the results of *deleting* and *contracting* Z, respectively. We abbreviate $M \backslash \{x\}$ by $M \backslash x$, etc. r_M is the rank function of M, and $r_M(E(M))$ is abbreviated $r(M)$. We shall also omit the subscript M when there is no risk of confusion. A *loop* is an element x such that $\{x\}$ is a circuit. Distinct x, y are *parallel* if $\{x, y\}$ is a circuit, and x and y are then both said to be *parallel elements*. The prefix "co-" dualizes a term: thus, coloop, cocircuit. Distinct x, y are *in series* if they are coparallel. A *parallel class* of M is a maximal subset $X \subseteq E(M)$ which contains no loops, so that every two distinct members of X are parallel. The parallel classes of M therefore form a partition of the non-loops of M. A *series class* is defined similarly. $X \subseteq E(M)$ is said to *span* $Y \subseteq E(M)$ if Y is a subset of the closure of X. $M \cong N$ denotes that M is isomorphic to N.

2. Sums and Separability

There is obviously a connection of some sort between being a k-sum (for $k = 1$, 2, or 3) and being k-separable in Tutte's sense. In this section we make the connection precise.

Before we do so, however, we shall extend the definition of 1- and 2-sums to non-binary matroids. Let M_1, M_2 be matroids on sets S_1, S_2, respectively. If $S_1 \cap S_2 = \emptyset$, and yet S_1, $S_2 \neq \emptyset$, we define the *1-sum* $M_1 \Delta M_2$ of M_1 and M_2 to be the matroid on $S_1 \cup S_2$ which has as circuits those sets which are circuits of M_1 or circuits of M_2. Clearly this extends the definition of a 1-sum for binary matroids.

Secondly, suppose that M_1, M_2 are matroids on S_1, S_2, and that $S_1 \cap S_2 = \{z\}$. Suppose also that z is not a loop or coloop of M_1 or M_2, and $|S_1|$, $|S_2| \geqslant 3$. Then the *2-sum* $M_1 \Delta M_2$ of M_1, M_2 is the matroid on $S_1 \Delta S_2$ with circuits the following sets:

(i) all sets $X_1 \subseteq S_1 - \{z\}$ which are circuits of M_1,
(ii) all sets $X_2 \subseteq S_2 - \{z\}$ which are circuits of M_2,
(iii) all sets $X_1 \cup X_2$, where $X_i \subseteq S_i - \{z\}$ and $X_i \cup \{z\}$ is a circuit of M_i $(i = 1, 2)$.

It is easy to see that this extends the definition of a 2-sum for binary matroids.

However, there appears to be no simple way to extend the definition of a 3-sum to non-binary matroids. There would, in any case, be no use for such a definition in this paper. (Brylawski's method [4] seems the least complicated.)

If (X_1, X_2) is a partition of S, we say that (X_1, X_2) is a *k-separation* of M if $|X_1|$, $|X_2| \geqslant k$ and

$$r(X_1) + r(X_2) \leqslant r(M) + k - 1.$$

It is easy to see that no matroid has a k-separation for $k < 1$. A k-separation is *exact* if equality holds in the displayed inequality. (Thus every 1-separation is exact.) M is *k-separable* if it has a k-separation. M is *k-connected* if it has no k'-separation for any $k' < k$.

The connection between 1-sums and 1-separation is particularly simple, as follows.

(2.1) *If (X_1, X_2) is a 1-separation of M then M is the 1-sum of $M \times X_1$ and $M \times X_2$; and conversely, if M is the 1-sum of M_1 and M_2 then $(E(M_1)$, $E(M_2))$ is a 1-separation of M, and M_1, M_2 are isomorphic to (indeed, are) minors of M.*

The proof is trivial. A similar result holds for 2-sums, but the proof takes several steps.

(2.2) *Suppose that M is the 2-sum of M_1 and M_2. Then $(E(M_1) - E(M_2)$, $E(M_2) - E(M_1))$ is an exact 2-separation of M.*

Proof. Put $E(M_1) \cap E(M_2) = \{z\}$, and $X_i = E(M_i) - \{z\}$ $(i = 1, 2)$. z is not a coloop of M_1, and so there is a set $B_1 \subseteq X_1$ independent in M_1, with $|B_1| = r(M_1) = r_M(X_1)$. Also, z is not a loop of M_2, and so there is a set $B_2 \subseteq X_2 \cup \{z\}$ with $z \in B_2$, independent in M_2, and with $|B_2| = r_M(X_2)$. From the definition of the circuits of $M_1 \Delta M_2$, we see that $(B_1 \cup B_2) - \{z\}$ includes no circuits of M, and so

$$r(M) \geqslant r_M(X_1) + r_M(X_2) - 1.$$

Moreover, $|X_1|, |X_2| \geqslant 2$, since $|E(M_i)| \geqslant 3$ $(i = 1, 2)$ and so (X_1, X_2) is a 2-separation. It is not a 1-separation of M, because (since z is neither a loop nor coloop of M_1 or M_2) M has a circuit intersecting both X_1 and X_2. Thus it is an exact 2-separation, as required.

(2.3) *Let (X_1, X_2) be a 2-separation of M. If C_1, C_2 are circuits of M and both intersect both X_1 and X_2, then $C_1 \cap X_1 \not\subset C_2 \cap X_1$.*

Proof. Suppose that $C_1 \cap X_1 \subset C_2 \cap X_1$. Choose $x_1 \in C_1 \cap X_1$, $x_2 \in (C_2 - C_1) \cap X_1$. Now $C_2 \cap X_1$ is independent in M, and so there is an independent set $B_1 \subseteq X_1$ with $C_2 \cap X_1 \subseteq B_1$, and with $|B_1| = r(X_1)$. Choose an independent set $B_2 \subseteq X_2$ with $|B_2| = r(X_2)$. Then B_2 spans X_2 in M, and so $(B_1 \cup B_2) - \{x_1, x_2\}$ spans x_1, x_2 and hence $E(M)$. Thus $r(M) \leqslant |B_1| + |B_2| - 2 = r(X_1) + r(X_2) - 2$, contrary to the hypothesis that (X_1, X_2) is a 2-separation.

(2.4) *Let (X_1, X_2) be a 2-separation of M, and let $Y_i \subseteq X_i$ $(i = 1, 2)$ be non-empty. Suppose that for $i = 1, 2$, there is a circuit C_i with $C_i \cap X_1 \neq \varnothing$, $C_i \cap X_2 \neq \varnothing$, $C_i \cap X_i = Y_i$. Then $Y_1 \cup Y_2$ is a circuit of M.*

Proof. Choose C_1, C_2 with these properties so that $C_1 \cup C_2$ is minimal. We prove that $C_1 = C_2$. First, suppose that $(C_1 \cup C_2) \cap X_1$ is not independent, and let C be a circuit with $C \subseteq (C_1 \cup C_2) \cap X_1$. Now $C \not\subseteq C_1$, and so there exists $x \in (C_2 - C_1) \cap C$. Choose $y \in C_2 \cap X_2$. There is a circuit $C_2' \subseteq (C_2 \cup C) - \{x\}$ with $y \in C_2'$. Then $C_2' \cap X_2 \subseteq C_2 \cap X_2$ and so $C_2' \cap X_1 \neq \varnothing$; but $y \in C_2'$ and so $C_2' \cap X_2 \neq \varnothing$. By (2.3), $C_2' \cap X_2 = C_2 \cap X_2 = Y_2$, and yet $C_1 \cup C_2' \subseteq (C_1 \cup C_2) - \{x\}$. This contradicts the minimality of $C_1 \cup C_2$. Thus $(C_1 \cup C_2) \cap X_1$ is independent, and similarly so is $(C_1 \cup C_2) \cap X_2$. For $i = 1, 2$, choose $B_i \subseteq X_i$, independent, with $(C_1 \cup C_2) \cap X_i \subseteq B_i$ and with $|B_i| = r(X_i)$. Now if $C_1 \cap X_2 = Y_2$ the result is true. We assume therefore that $C_1 \cap X_2 \neq Y_2$, and so by (2.3), $C_1 \cap X_2 \not\subseteq Y_2$. Choose $x_2 \in (C_1 \cap X_2) - Y_2$, and, similarly, choose $x_1 \in (C_2 \cap X_1) - Y_1$. Then $x_i \in B_i$ $(i = 1, 2)$, and $B_1 \cup B_2 - \{x_1, x_2\}$ spans x_1, x_2 and hence $E(M)$. Thus

$$r(M) \leqslant |B_1| + |B_2| - 2 = r(X_1) + r(X_2) - 2,$$

contrary to the hypothesis that (X_1, X_2) is a 2-separation.

(2.5) *Let* (X_1, X_2) *be an exact* 2*-separation of* M. *Then there are matroids* M_1, M_2 *on* $X_1 \cup \{z\}$, $X_2 \cup \{z\}$, *respectively* (*where* z *is a new element*), *so that* M *is the* 2*-sum of* M_1 *and* M_2.

Proof. Define the matroid M_1 on $X_1 \cup \{z\}$ to have as circuits those circuits of M which do not intersect X_2, together with all sets $Y \cup \{z\}$, where $\varnothing \neq Y \subseteq X_1$ is such that there is a circuit C of M with $C \cap X_1 = Y$, $C \cap X_2 \neq \varnothing$. The result follows easily from (2.4).

We summarize the results in the following.

(2.6) *If* (X_1, X_2) *is an exact* 2*-separation of* M *then there are matroids* M_1, M_2 *on* $X_1 \cup \{z\}$, $X_2 \cup \{z\}$, *respectively* (*where* z *is a new element*), *such that* M *is the* 2*-sum of* M_1 *and* M_2. *Conversely, if* M *is the* 2*-sum of* M_1 *and* M_2 *then* $(E(M_1) - E(M_2), E(M_2) - E(M_1))$ *is an exact* 2*-separation of* M, *and* M_1, M_2 *are isomorphic to minors of* M.

Proof. Because of (2.2) and (2.5), it only remains to show that if M is the 2-sum of M_1 and M_2 then M_1, M_2 are isomorphic to minors of M. Suppose then that $E(M_1) \cap E(M_2) = \{z\}$, and put $X_i = E(M_i) - \{z\}$ $(i = 1, 2)$. Then (X_1, X_2) is an exact 2-separation of M by (2.2) and so there is a circuit C with $C \cap X_1 \neq \varnothing$, $C \cap X_2 \neq \varnothing$. Choose $y \in C \cap X_2$, and put $M'_1 = M \times (X_1 \cup C)/((C \cap X_2) - \{y\})$. Then M' is isomorphic to M_1, by (2.4). Similarly M_2 is isomorphic to a minor of M, as required.

For $k = 3$ the situation is a little more complicated. First, we are confined to binary matroids. Even there, being expressible as a 3-sum is not the same as having an exact 3-separation; indeed, we shall see that it is equivalent to having an exact 3-separation (X_1, X_2) such that $|X_1|, |X_2| \geqslant 4$. This is a technicality, but there is a second and more awkward difference—the parts of a 3-sum are not always isomorphic to minors of the whole. The most we can say is that the parts are isomorphic to minors of the whole, provided that the whole is 3-connected, and even this is non-trivial to prove; we defer the proof to Section 4.

We begin with a lemma.

(2.7) *Suppose that* M_1, M_2 *are binary matroids on* S_1, S_2, *respectively, and* M *is the* 3*-sum of* M_1 *and* M_2. *If* $Y_1 \cup Y_2$ *is a circuit of* M, *where* $\varnothing \neq Y_i \subseteq S_i - (S_1 \cap S_2)$ $(i = 1, 2)$, *then there exists* $z \in S_1 \cap S_2$ *such that for* $i = 1, 2$, $Y_i \cup \{z\}$ *is a circuit of* M_i.

Proof. Put $Z = S_1 \cap S_2$. Choose cycles C_1, C_2 of M_1, M_2, respectively, so that $Y_1 \cup Y_2 = C_1 \mathbin{\Delta} C_2$. Then $C_1 \cap Z = C_2 \cap Z$, and moreover Z is a circuit of both M_1 and M_2; and so by replacing C_1, C_2 by $C_1 \mathbin{\Delta} Z$, $C_2 \mathbin{\Delta} Z$ if necessary, we may assume that $|C_1 \cap Z| \leqslant 1$. Now $C_1 \not\subseteq Y_1$, because $Y_2 \neq \varnothing$ and $Y_1 \cup Y_2$ is a circuit of M. Thus $C_1 \cap Z \neq \varnothing$. Put $C_1 \cap Z = \{z\}$;

then $C_2 \cap Z = \{z\}$. Now every circuit of M_i included in C_i contains z, because $Y_1 \cup Y_2$ is a circuit of M; yet C_i is a cycle of M_i, and so $C_i = Y_i \cup \{z\}$ is a circuit of M_i $(i = 1, 2)$, as required.

When $A_1, ..., A_k$ are subsets of S and $Z \subseteq S$ is expressible as $\Delta(A_i : i \in I)$ for some $I \subseteq \{1, ..., k\}$, we say that $A_1, ..., A_k$ *surround* Z. When M is a binary matroid on S and $Z \subseteq S$, $\rho(Z)$ denotes the maximum number of circuits of M included in Z, with the property that no one of them is surrounded by the others. $\rho^*(Z)$ denotes the same quantity for M^*. (It is easy to see that $\rho(Z)$ is also the maximum number of cycles of M included in Z with the property that no one is surrounded by the others.)

It is well known that

(2.8) *If M is binary and $Z \subseteq S$, $r_M(Z) + \rho(Z) = |Z|$.*

(2.9) *If (X_1, X_2) is an exact 3-separation of a binary matroid M, with $|X_1|, |X_2| \geqslant 4$, then there are binary matroids M_1, M_2 on $X_1 \cup Z$, $X_2 \cup Z$, respectively (where Z contains three new elements), such that M is the 3-sum of M_1 and M_2. Conversely if M is the 3-sum of M_1 and M_2 then $(E(M_1) - E(M_2), E(M_2) - E(M_1))$ is an exact 3-separation of M, and $|E(M_1) - E(M_2)|, |E(M_2) - E(M_1)| \geqslant 4$.*

Proof. Suppose that (X_1, X_2) is an exact 3-separation of the binary matroid M, and $|X_1|, |X_2| \geqslant 4$. Let $\mathscr{C}_i$ be the set of circuits of M included in X_i $(i = 1, 2)$. Now

$$\rho(X_1) + \rho(X_2) = \rho(X_1 \cup X_2) - 2,$$

and so there are two circuits C_1, C_2 of M so that $\mathscr{C}_1, \mathscr{C}_2$ and C_1, C_2 together surround all cycles of M. Take three new elements z_1, z_2, z_3 and put $Z = \{z_1, z_2, z_3\}$. For $i = 1$, 2, let M_i be the binary matroid on $X_i \cup Z$ with cycles the sets surrounded by $\mathscr{C}_i$ together with $(C_1 \cap X_i) \cup \{z_1\}$, $(C_2 \cap X_i) \cup \{z_2\}$ and Z. We claim that M is the 3-sum of M_1 and M_2. For certainly $M = M_1 \Delta M_2$, and for $i = 1$, 2, $(C_1 \cap X_i) \cup \{z_1\}$, $(C_2 \cap X_i) \cup \{z_2\}$ and Z are cycles of M_i, and so Z includes no cocircuits of M_i. It remains to check that Z is a circuit of M_i, and that M_i has fewer elements than M $(i = 1, 2)$. The second assertion is true because $|X_1|, |X_2| \geqslant 4$ and $|Z| = 3$. If the first is false, then one of $\{z_1\}$, $\{z_2\}$, $\{z_1, z_2\}$ is a cycle of M_1 or M_2 (M_1, say), and so one of $C_1 \cap X_1$, $C_2 \cap X_1$, $(C_1 \Delta C_2) \cap X_1$ is surrounded by $\mathscr{C}_1$. But if Y is a cycle of M and $Y \cap X_1$ is surrounded by $\mathscr{C}_1$, then $Y \cap X_2$ is surrounded by $\mathscr{C}_2$, because $Y \cap X_2 \subseteq X_2$ and $Y \cap X_2 = Y \Delta (Y \cap X_1)$; and so Y is surrounded by $\mathscr{C}_1 \cup \mathscr{C}_2$. Thus one of C_1, C_2, $C_1 \Delta C_2$ is surrounded by $\mathscr{C}_1 \cup \mathscr{C}_2$, contrary to

$$\rho(X_1) + \rho(X_2) = \rho(X_1 \cup X_2) - 2.$$

Conversely, suppose that M is the 3-sum of M_1 and M_2. Put $E(M_1)\cap E(M_2)=Z=\{z_1,z_2,z_3\}$, and $E(M_i)-Z=X_i$ $(i=1,2)$. Certainly $|X_1|$, $|X_2|\geqslant 4$, because $|E(M_i)|<|E(M)|$ $(i=1,2)$. We must prove that

$$\rho(X_1)+\rho(X_2)=\rho(X_1\cup X_2)-2.$$

Let $\mathscr{C}_i$ be the set of circuits of M included in X_i $(i=1,2)$. Now Z includes no cocircuits of M_i, and so for each $z_j\in Z$ there is a circuit C_j^i of M_i with $C_j^i\cap Z=\{z_j\}$. Put $C_j=C_j^1\,\Delta\,C_j^2$ so that C_j is a cycle of M $(j=1,2,3)$. Now by (2.7), if C is any circuit of M such that $C\cap X_1$, $C\cap X_2\neq\varnothing$, then $(C\cap X_i)\cup\{z_j\}$ is a circuit of M_i $(i=1,2)$, for some j; and then $C\,\Delta\,C_j$ is surrounded by $\mathscr{C}_1\cup\mathscr{C}_2$. Thus $\mathscr{C}_1\cup\mathscr{C}_2$ together with C_1, C_2 and C_3 surround all circuits of M. Moreover, $(C_1\,\Delta\,C_2\,\Delta\,C_3)\cap X_1=C_2^1\,\Delta\,C_2^1\,\Delta\,C_3^1\,\Delta\,Z$ and so it is surrounded by $\mathscr{C}_1$; and similarly $(C_1\,\Delta\,C_2\,\Delta\,C_3)\cap X_2$ is surrounded by $\mathscr{C}_2$. Thus $C_1\,\Delta\,C_2\,\Delta\,C_3$ is surrounded by $\mathscr{C}_1\cup\mathscr{C}_2$. It follows that

$$\rho(X_1)+\rho(X_2)\geqslant\rho(X_1\cup X_2)-2.$$

Suppose that strict inequality holds here; then $\mathscr{C}_1\cup\mathscr{C}_2$ surrounds one of C_1, C_2, $C_1\,\Delta\,C_2$. Thus $\mathscr{C}_1$ surround one of $C_1\cap X_1$, $C_2\cap X_1$, $(C_1\,\Delta\,C_2)\cap X_1$, and so one of these sets is a cycle of M_1. But $(C_1\cap X_1)\cup\{z_1\}$, $(C_2\cap X_1)\cup\{z_2\}$, $((C_1\,\Delta\,C_2)\cap X_1)\cup\{z_1,z_2\}$ are cycles of M_1, and so one of $\{z_1\}$, $\{z_2\}$, $\{z_1,z_2\}$ is a cycle of M_1. This is contrary to the hypothesis that M is the 3-sum of M_1 and M_2, and completes the proof.

(2.10) *Let M be a matroid. Then*

(a) *the following are equivalent*:

(i) *M is 1-separable.*

(ii) *M is expressible as a 1-sum.*

(b) *The following are equivalent*:

(i) *M is 1- or 2-separable.*

(ii) *M is expressible as a 1- or 2-sum.*

(c) *If M is binary, the following are equivalent*:

(i) *M is 1- or 2-separable, or has a 3-separation (X_1,X_2) with $|X_1|$, $|X_2|\geqslant 4$.*

(ii) *M is expressible as a 1-, 2-, or 3-sum.*

Proof. That (ii) implies (i) follows from the previous results. To see that (i) implies (ii), observe that if M has a k-separation then it has an exact k'-separation for some $k'\leqslant k$.

3. Arcs and Fundamentals

If M is a matroid on S and $Z \subseteq S$, a *Z-arc* is a minimal non-empty subset $A \subseteq S - Z$ such that there is a circuit C with $C - Z = A$ and $C \cap Z \neq \varnothing$. A *Z-fundamental* (*for* A) is a circuit C such that $C - Z = A$, where A is a Z-arc.

We observe that all Z-arcs are non-empty and independent in M; and that no Z-arc is a proper subset of another.

(3.1) *If C is a circuit with $C \cap Z \neq \varnothing$, then $C - Z$ is expressible as a union of Z-arcs.*

Proof. We proceed by induction on $|C - Z|$. If $C \subseteq Z$ the result is trivial. We assume that $C - Z \neq \varnothing$. But $C \cap Z \neq \varnothing$, and so $C - Z$ includes a Z-arc A. Choose a Z-fundamental F for A. Choose $x \in A$. Now for each $y \in C - (A \cup Z)$ there is a circuit $C_y \subseteq (C \cup F) - \{x\}$ with $y \in C_y$, and $C_y \cap Z \neq \varnothing$, because $C_y \nsubseteq C - \{x\}$. By induction $C_y - Z$ is a union of Z-arcs; but

$$C - Z = A \cup \bigcup (C_y - Z\colon y \in C - (A \cup Z))$$

and the result follows.

(3.2) *If M is a matroid on S and $x, y \in S$ are distinct but are contained in the same circuits of M, then either x, y are in series or they are both coloops.*

The proof is elementary.

(3.3) *If A is a Z-arc and $x, y \in A$ are distinct, then x, y are in series in $M \times (A \cup Z)$.*

Proof. Because A is a Z-arc, there is no circuit of $M \times (Z \cup A)$ containing just one of x, y. The result follows from (3.2). (x, y cannot be coloops because they are contained in a circuit.)

When A is a Z-arc and $P \subseteq Z$, we say $A \to P$ if there is a Z-fundamental for A included in $A \cup P$. For its negation we write $A \nrightarrow P$.

(3.4) *Suppose that A is a Z-arc, $P \subseteq Z$, $A \nrightarrow P$, and $x_0 \in A$. If C is a circuit with $x_0 \in C$ and $C \cap Z \subseteq P$, then there is a Z-arc A' such that $A' \subseteq (A \cup C) - \{x_0\}$ and such that $A' \nrightarrow P$.*

Proof. If possible, choose a counterexample in which $C - (A \cup Z)$ is minimal. Certainly $C \nsubseteq A \cup Z$, because A is a Z-arc and $A \nrightarrow P$. Choose $y \in C - (A \cup Z)$. Let F be a Z-fundamental for A, and choose a circuit C_1

with $y \in C_1$ and $C_1 \subseteq (C \cup F) - \{x_0\}$. If there is a Z-arc $A' \subseteq C_1 - Z$ with $y \in A'$ then $A' \to P$, for otherwise the theorem is true; and if there is no such Z-arc, then by (3.1) $C_1 \cap Z = \emptyset$. Thus in either case there is a circuit $C_2 \subseteq C \cup A \cup P$ with $x_0 \notin C_2$ and $y \in C_2$. Choose a circuit $C_3 \subseteq (C \cup C_2) - \{y\}$ with $x_0 \in C_3$. Then $C_3 \cap Z \subseteq P$, and $C_3 - (A \cup Z) \subseteq (C - (A \cup Z)) - \{y\}$. This contradicts the minimality of $C - (A \cup Z)$, as required.

(3.5) *If $P \subseteq Z$ and A_1, A_2 are Z-arcs with $A_1 \cap A_2 \neq \emptyset$ and $A_2 \not\to P$, then for every $x \in A_1$ there is a Z-arc A with $x \in A \subseteq A_1 \cup A_2$ such that $A \not\to P$.*

Proof. Keeping A_1 fixed, choose the Z-arc A_2 with $A_1 \cap A_2 \neq \emptyset$ and $A_2 \not\to P$, such that $A_1 \cup A_2$ is minimal. By setting $A = A_2$ and $x_0 \in A_2 - A_1$, we see from (3.4) that every circuit included in $A_1 \cup A_2 \cup P$ is included in $A_1 \cup P$.

Now if $x \in A_1 \cap A_2$ the result is true, and we assume that $x \in A_1 - A_2$. Choose $y \in A_1 \cap A_2$. Let F_1, F_2 be Z-fundamentals for A_1, A_2, respectively. Choose a circuit C with $x \in C$ and $C \subseteq (F_1 \cup F_2) - \{y\}$. Then by (3.1) we can choose a circuit C_1 with $x \in C_1 - Z \subseteq C - Z$ so that either C_1 is a Z-fundamental or $C_1 \cap Z = \emptyset$. But $y \notin C_1$ and so $C_1 - Z \not\subseteq A_1$, because A_1 is a Z-arc; thus $(C_1 - Z) \cap (A_2 - A_1) \neq \emptyset$. From the result of the previous paragraph, $C_1 - Z$ is not a circuit, and so is a Z-arc and C_1 is a Z-fundamental; and moreover, $C_1 - Z \not\to P$, for the same reason. The result follows.

(3.6) *If A_1, A_2 are Z-arcs with $A_1 \cap A_2 \neq \emptyset$, and $P_1, P_2 \subseteq Z$ are such that $A_1 \not\to P_1$, $A_2 \not\to P_2$, then there is a Z-arc $A \subseteq A_1 \cup A_2$ such that $A \not\to P_1$, P_2.*

Proof. If possible, choose a counterexample with $A_1 \cup A_2$ minimal. Then certainly we have $A_1 \to P_2$, $A_2 \to P_1$. Let F_2 be a Z-fundamental for A_2 with $F_2 \subseteq A_2 \cup P_1$. Choose $x_0 \in A_1 \cap A_2$. Then by (3.4), taking $C = F_2$, we see that there is a Z-arc A_1' with $A_1' \subseteq (A_1 \cup A_2) - \{x_0\}$ such that $A_1' \not\to P_2$. Similarly, there is a Z-arc A_2' with $A_2' \subseteq (A_1 \cup A_2) - \{x_0\}$ such that $A_2' \not\to P_1$. Now $A_1' \not\subseteq A_1$; choose $y \in A_1' \cap (A_2 - A_1)$. In particular, $A_1' \cap A_2 \neq \emptyset$, and similarly $A_2' \cap A_1 \neq \emptyset$. By minimality of $A_1 \cup A_2$ we have $A_1 \cup A_2' = A_1 \cup A_2$, and hence $y \in A_2$. Thus $A_1' \cap A_2' \neq \emptyset$; but

$$A_1' \cup A_2' \subseteq (A_1 \cup A_2) - \{x_0\},$$

contrary to the minimality of $A_1 \cup A_2$.

(3.7) *Let M be a connected matroid on S, and let $Z \subseteq S$ be non-empty. Let (X_1, X_2) be a partition of S, so that no Z-arc intersects both X_1 and X_2,* *and such that for* $i=1, 2$, *if* $A \subseteq X_i$ *is a Z-arc then* X_i *includes a Z-fundamental for A. Then*

$$r(X_1)+r(X_2)-r(S)=r(X_1 \cap Z)+r(X_2 \cap Z)-r(Z).$$

Proof. We proceed by induction on $|S-Z|$. If this is zero the result is trivial, and we assume not. Choose $X \subseteq S$ with $Z \subseteq X$ and $X \neq S$, maximal such that Z intersects every component of $M \times X$. Then $X \neq \varnothing$ and so there is a circuit C intersecting both X and $S-X$, and so by maximality of X, $S-X \subseteq C$. Thus $S-X$ is independent. Moreover, if $x, y \in S-X$ are distinct, then a circuit of M containing x must intersect X and so must contain y; and so by (3.2), x and y are in series in M. There is a circuit intersecting $S-X$ and Z, and so by (3.1) there is a Z-arc A including $S-X$. Now $A \subseteq X_1$ or X_2 (X_1, say), and so $S-X \subseteq X_1$. Let F be a Z-fundamental for A with $F \subseteq X_1$. Then $S-X \subseteq F$.

By induction, applied to the components of $M \times X$,

$$r(X_1 \cap X)+r(X_2 \cap X)-r(X)=r(X_1 \cap Z)+r(X_2 \cap Z)-r(Z).$$

But $r(X_1)=r(X_1 \cap X)+|S-X|-1$ since the elements of $S-X$ are in series in M and $F \subseteq X_1$, and

$$r(S)=r(X)+|S-X|-1$$

for the same reason. The result follows.

(3.8) *Let M be a matroid on S, let* $Z \subseteq S$, *and let* (P_1, P_2) *be a partition of Z. Then either there is a Z-arc A such that* $A \nrightarrow P_1$, $A \nrightarrow P_2$, *or there is a partition* (X_1, X_2) *of S such that* $X_i \cap Z = P_i$ $(i=1, 2)$ *and*

$$r(X_1)+r(X_2)-r(S)=r(P_1)+r(P_2)-r(Z).$$

Proof. If $Z=\varnothing$ we may take $X_1=S$, $X_2=\varnothing$. Assume then that $Z \neq \varnothing$. If M is not connected we may treat its components separately, for if the theorem is true for each component it is true for M. Assume then that M is connected. Define

$$X_1=P_1 \cup \bigcup (A \subseteq S-X\text{: } A \text{ is a } Z\text{-arc and } A \nrightarrow P_2).$$

Put $X_2=S-X_1$. We claim that no Z-arc intersects both X_1 and X_2. For if A is a Z-arc and $A \cap X_1 \neq \varnothing$, then by (3.5), $A \subseteq X_1$. Moreover, if A is a Z-arc and $A \subseteq X_2$ then $A \to P_2$ by definition of X_1. There are now two possibilities:

(i) there is a Z-arc $A_1 \subseteq X_1$ such that $A_1 \nrightarrow P_1$. Pick a Z-arc A_2 with $A_1 \cap A_2 \neq \varnothing$ and $A_2 \nrightarrow P_2$; and then from (3.6) there is a Z-arc A such that $A \nrightarrow P_1$, $A \nrightarrow P_2$.

(ii) there is no such Z-arc $A_1 \subseteq X_1$. Then $A_1 \to P_1$ for every Z-arc A_1 included in X_1, and so the hypotheses of (3.7) are satisfied. The result follows.

4. Parts of a 3-Sum

Our first application of the results of the previous section is to prove the following theorem, postponed from Section 2.

(4.1) *If M is binary and is the 3-sum of M_1 and M_2, and M is 3-connected, then M_1, M_2 are isomorphic to minors of M.*

The main step in its proof is the following lemma.

(4.2) *Let Z be an independent cocircuit of a binary matroid M, with $|Z| = 3$. Then either there is a partition (R_1, R_2, R_3) of $E(M) - Z$ such that for each i,*

$$r(R_i) + r(E(M) - R_i) \leqslant r(M) + 1$$

or M has a minor M' such that:

(i) $|E(M')| = 6$,
(ii) *$Z \subseteq E(M')$ and Z is a cocircuit of M',*
(iii) *each $z \in Z$ is in a 2-element cocircuit of M'.*

Proof. We may assume that M is connected, for if the theorem is true for the component of M containing Z then it is true for M. Let $Z = \{z_1, z_2, z_3\}$ and put $P_i = Z - \{z_i\}$ $(i = 1, 2, 3)$. For every circuit C of M, $C \cap Z = \varnothing$ or $= P_i$ for some i, because Z is a cocircuit and M is binary. In particular, for every Z-arc A, $A \to P_i$ for some i. In fact $A \to P_i$ for a unique i, for if $A \to P_1$, $A \to P_2$, say, then $A \cup P_1$, $A \cup P_2$ are circuits, and so $P_1 \mathbin{\Delta} P_2 = P_3$ is a cycle of M (since M is binary), which is impossible since Z is independent. Define $v(A) = i$, where $A \to P_i$.

Suppose first that there are two Z-arcs A_1, A_2 with $v(A_1) \neq v(A_2)$ and $A_1 \cap A_2 \neq \varnothing$. Choose such a pair A_1, A_2 with $A_1 \cup A_2$ minimal. Let $A_1 \to P_1$, $A_2 \to P_2$, say. Put

$$\mathscr{C} = \{\varnothing, A_1 \cup P_1, A_2 \cup P_2, (A_1 \mathbin{\Delta} A_2) \cup P_3\}.$$

Then every member of $\mathscr{C}$ is a cycle of M. We claim that these are the only cycles included in $Z \cup A_1 \cup A_2$. For suppose that C_1 is another. Then for some $C' \in \mathscr{C}$, $C_1 \mathbin{\Delta} C' \subseteq A_1 \cup A_2$; and $C_1 \mathbin{\Delta} C' \notin \mathscr{C}$, because $C_1 \mathbin{\Delta} C' \neq \varnothing$ (since $C_1 \neq C'$). $C_1 \mathbin{\Delta} C'$ includes a circuit C, say; and then $C \subseteq A_1 \cup A_2$, and so $C \notin \mathscr{C}$. Now $C \not\subseteq A_2$, and so $C \cap (A_1 - A_2) \neq \varnothing$; choose $x_0 \in C \cap (A_1 - A_2)$. By (3.4) there is a Z-arc $A \subseteq (A_1 \cup A_2) - \{x_0\}$ such that $A \not\rightarrow P_2$ (since $A_1 \not\rightarrow P_2$). But $A \cap A_2 \neq \varnothing$, since $A \not\subseteq A_1 - A_2$, and yet

$$A \cup A_2 \subseteq (A_1 \cup A_2) - \{x_0\}.$$

This contradicts the minimality of $A_1 \cup A_2$. Thus the only cycles of M included in $A_1 \cup A_2 \cup Z$ are the members of $\mathscr{C}$. Choose $x_1 \in A_1 - A_2$, $x_2 \in A_2 - A_1$, $x_3 \in A_1 \cap A_2$, and put $X = \{x_1, x_2, x_3\}$. Put

$$M' = M \times (A_1 \cup A_2 \cup Z)/((A_1 \cup A_2) - X).$$

Then M' satisfies the theorem.

We may therefore assume that for any two Z-arcs A_1, A_2, if $v(A_1) \neq v(A_2)$ then $A_1 \cap A_2 = \varnothing$. Put

$$R_i = \bigcup (A \subseteq E(M) - Z\colon A \text{ is a } Z\text{-arc and } A \rightarrow P_i) \qquad (i = 1, 2, 3).$$

By (3.1), every element in $E(M) - Z$ is in a Z-arc, since M is connected, and so (R_1, R_2, R_3) is a partition of $E(M) - Z$.

Take a new element e and let M_1 be the binary matroid on $E(M) \cup \{e\}$ such that $M_1 \backslash e = M$ and such that $\{e, z_1, z_2\}$ is a circuit of M_1. Put $Z_1 = Z \cup \{e\}$, and put $X_1 = R_1 \cup \{e\}$, $X_2 = R_2 \cup R_3 \cup Z$. Then (X_1, X_2) is a partition of $E(M_1)$. Now let A be a Z_1-arc of M_1. Let F be a Z_1-fundamental for A (of M). Then either F or $F \Delta \{e, z_1, z_2\}$ is a circuit of M, and so A is a Z-arc of M. Hence $A \subseteq R_i$ for some i, and so A does not intersect both X_1 and X_2. If $A \subseteq R_2 \cup R_3$ then $A \cup P_2$ or $A \cup P_3$ is a Z_1-fundamental for A (of M_1), and if $A \subseteq R_1$ then $A \cup \{e\}$ is a Z_1-fundamental.

Thus by (3.7),

$$\begin{aligned} r_{M_1}(X_1) + r_{M_1}(X_2) - r(M_1) &= r_{M_1}(\{e\}) + r_{M_1}(Z) - r_{M_1}(Z_1) \\ &= 1 + 3 - 3 \\ &= 1. \end{aligned}$$

But $r_{M_1}(X_1) \geqslant r_M(R_1)$, $r_{M_1}(X_2) = r_M(E(M) - R_1)$, and $r(M_1) = r(M)$; thus $r(R_1) + r(E(M) - R_1) \leqslant r(M) + 1$.

Inequalities for R_2, R_3 are proved similarly. This completes the proof.

(4.3) *Suppose that M is the 3-sum of binary matroids M_1 and M_2, and that M is 3-connected. If (Y_1, Y_2) is a 2-separation of M_1 then for some i, $Y_i = \{x, z\}$, where $x \in E(M_1) - E(M_2)$, $z \in E(M_1) \cap E(M_2)$, and x, z are parallel in M_1.*

Proof. Put $Z = E(M_1) \cap E(M_2)$ and $X_i = E(M_i) - Z$ $(i = 1, 2)$, as usual. Now $Z \subseteq Y_1 \cup Y_2$ and $|Z| = 3$; we assume that $|Y_1 \cap Z| \geqslant 2$. If $Z \subseteq Y_1$ it is easy to see that $((Y_1 - Z) \cup X_2, Y_2)$ is a 2-separation of M, which is impossible. We deduce that $Y_1 \cap Z = Z - \{z\}$, say, and that $z \in Y_2$; and furthermore that $(Y_1 \cup \{z\}, Y_2 - \{z\})$ is not a 2-separation of M_1, for the same reason. But $r_{M_1}(Y_1 \cup \{z\}) = r_{M_1}(Y_1)$ since Z is a circuit of M_1, and so $|Y_2 - \{z\}| \leqslant 1$. Equality holds, since $|Y_2| \geqslant 2$; put $Y_2 - \{z\} = \{x\}$. Now Y_2 includes a circuit or cocircuit of M_1, because if not then $r_{M_1}(Y_2) = |Y_2| = 2$, and $r_{M_1}(Y_1) = r_{M_1}(Y_1 \cup Y_2)$, contrary to $r_{M_1}(Y_1) + r_{M_1}(Y_2) \leqslant r(M_1) + 1$. Thus $\{x, z\}$ includes a circuit or cocircuit of M_1. Now Z is a circuit of M_1 and so z is not a loop or coloop of M_1. And x is not a loop or coloop of M_1 because M is 3-connected and so x is not a loop or coloop of M. Thus $\{x, z\}$ is a circuit or cocircuit of M_1. It is not a cocircuit because $|\{x, z\} \cap Z| = 1$ and Z is a circuit. Hence x, z are parallel in M_1, as required.

(4.4) *If M_1 and M_2 are binary matroids on S_1, S_2, respectively, and Z_1, Z_2 are disjoint subsets of $S_1 - S_2$, then*

$$(M_1 \Delta M_2)\backslash Z_1/Z_2 = (M_1\backslash Z_1/Z_2) \Delta M_2.$$

Proof. C is a cycle of $M\backslash Z_1/Z_2$ if and only if $C \cup X$ is a cycle of M for some $X \subseteq Z_2$. The result follows.

Proof of (4.1). Suppose then that M is the 3-sum of M_1 and M_2, and M is 3-connected. Put $E(M_i) = S_i$ $(i = 1, 2)$, and $S_1 \cap S_2 = Z$. Then Z is an independent cocircuit of M_1^*, and so we may apply (4.2) to it. Suppose that (R_1, R_2, R_3) is a partition of $E(M_1^*) - Z$ such that for each i,

$$r_{M_1^*}(R_i) + r_{M_1^*}(S_1 - R_i) \leqslant r(M_1^*) + 1.$$

Now $|E(M_1^*)| \geqslant 7$ and so $|R_i| \geqslant 2$ for some i; $|R_1| \geqslant 2$, say. Then $(R_1, S_1 - R_1)$ is a 2-separation of M_1^* and hence of M_1, but it does not satisfy (4.3), a contradiction.

We deduce from (4.2) that M_1 has a six-element minor M_1', in which Z is a circuit and each $z \in Z$ is parallel to some other element of M_1'. Choose F_1, F_2 so that $M_1'\backslash F_1/F_2$. Then

$$\begin{aligned} M\backslash F_1/F_2 &= (M_1 \Delta M_2)\backslash F_1/F_2 \\ &= (M_1\backslash F_1/F_2) \Delta M_2 \qquad \text{by (4.4)} \\ &= M_1' \Delta M_2 \\ &\cong M_2, \end{aligned}$$

and so M has a minor isomorphic to M_2, and similarly for M_1, as required.

The reader may be puzzled by the fact that although M is regular if and only if M^* is, the decomposition of M provided by our main theorem is not invariant under duality; for if M is the 3-sum of M_1 and M_2 then M^* need not (indeed, cannot) be the 3-sum of M_1^* and M_2^*. It can be shown that if $M = M_1 \Delta M_2$ then $M^* = M_1^* \Delta M_2^*$, and one may wonder why we do not define a 3-sum to be $M_1 \Delta M_2$ when $|S_1 \cap S_2| = 2$, for then it would be invariant under duality. Indeed, if M is the 3-sum of M_1 and M_2, and $z \in E(M_1) \cap E(M_2)$, then $M = (M_1 \backslash z) \Delta (M_2 \backslash z)$, and the third common element of the matroids M_1, M_2 would appear to be irrelevant. The reason is that although our theorem could still be true (a fortiori) with this broader sense of 3-sum, the converse would become false; under this new definition one could produce the Fano matroid by a 3-sum, starting with, say, the polygon matroids of K_4 and the graph obtained from K_3 by replacing two edges by parallel pairs of edges.

There is an alternative definition of 3-sum which is invariant under duality and which would make our theorem and its converse both true. When M is a binary matroid, we say that distinct $e_1, e_2 \in E(M)$ are *adjacent* in M if M has no minor M' isomorphic to $\mathscr{M}(K_4)$ in which e_1, e_2 both appear and in which they correspond to non-adjacent edges of K_4. Clearly e_1, e_2 are adjacent in M if and only if they are adjacent in M^*. Then when M_1, M_2 have element sets S_1, S_2 and $S_1 \cap S_2 = \{e_1, e_2\}$, we could say that $M_1 \Delta M_2$ is a 3-sum of M_1 and M_2 if e_1, e_2 are adjacent in both M_1 and M_2. This definition has all the desirable qualities we mentioned, and some more; however, it is rather complicated, and we prefer the version given. In passing, it can be shown that if M is regular and $e_1, e_2 \in E(M)$ are distinct, then e_1, e_2 are adjacent if and only if the matroid M' is regular, where M' has one extra element e_3, $M' \backslash e_3 = M$, and $\{e_1, e_2, e_3\}$ is a cycle of M'.

5. Kelmans' Theorem

Kelmans [7] recently proved a result concerning the structure of 3-connected graphs. We need a matroid generalization of it, which we shall prove in this section.

We say that M is a *subdivision* of N if $E(M)$ can be partitioned into non-empty sets $P_1, \ldots, P_k$, where $k = |E(N)|$, so that every pair of elements in the same P_i are in series, and so that if $y_i \in P_i$ $(1 \leqslant i \leqslant k)$ then

$$M/(E(M) - \{y_1, \ldots, y_k\}) \cong N.$$

We say that M is *cyclically 3-connected* if it is a subdivision of a 3-connected matroid. It is an easy exercise to prove the following.

(5.1) *If M is connected, the following are equivalent*:

(i) *M is cyclically 3-connected,*

(ii) *for every partition* (X_1, X_2) *of* $E(M)$ *with* $r(X_1) + r(X_2) \leqslant r(M) + 1$, *one of* X_1, X_2 *is independent,*

(iii) *for every such partition* (X_1, X_2) *either all the members of* X_1 *or all the members of* X_2 *are pairwise in series.*

When $Z \subseteq E(M)$ and $A \subseteq E(M) - Z$ is a Z-arc such that for every series class P of $M \times Z$, $A \not\rightarrow P$, we say A is an *adjoinable* Z-arc. The reason for the name is the following theorem.

(5.2) *Let* $Z \subseteq E(M)$ *be such that* $M \times Z$ *is cyclically* 3-*connected, and let* A *be an adjoinable* Z-*arc. Then* $M \times (Z \cup A)$ *is cyclically* 3-*connected.*

Proof. Let (X_1, X_2) be a partition of $Z \cup A$ such that

$$r(X_1) + r(X_2) \leqslant r(Z \cup A) + 1.$$

Suppose that both X_1 and X_2 include circuits of $M \times (Z \cup A)$. Now it is easy to see that

$$r(X_1 \cap Z) + r(X_2 \cap Z) - r(Z) \leqslant r(X_1) + r(X_2) - r(X_1 \cup X_2)$$

(this also follows from (8.1)), and so

$$r(X_1 \cap Z) + r(X_2 \cap Z) \leqslant r(Z) + 1.$$

Since $M \times Z$ is cyclically 3-connected, the elements in one of $X_1 \cap Z$, $X_2 \cap Z$ ($X_1 \cap Z$, say) are in series in $M \times Z$ by (5.1). Thus $X_1 \cap Z \subseteq P$ for some series class P of $M \times Z$. Let C be a circuit of M included in X_1. Then $C \subseteq A \cup P$. C is not a Z-fundamental, because $A \not\rightarrow P$, and so $C \subseteq P$. But P is a series class of $M \times Z$, and $M \times Z$ is connected, and so $P = C = Z$. This contradicts $A \not\rightarrow P$.

Thus for every such partition (X_1, X_2), one of X_1, X_2 is independent, and so by (5.1), M is cyclically 3-connected, as required.

It follows from (5.2) that if $Z_1 \subseteq Z_2 \subseteq \cdots \subseteq Z_r = E(M)$, and $M \times Z_1$ is cyclically 3-connected, and for each i $(1 \leqslant i \leqslant r - 1)$ $Z_{i+1} - Z_i$ is an adjoinable Z_i-arc, then M is cyclically 3-connected. We would like a converse to this. The most appealing one would be that $\varnothing \neq Z \subseteq E(M)$ and M, $M \times Z$ are both cyclically 3-connected then we can find such a sequence $Z_1, \ldots, Z_r$ with $Z = Z_1$ and $Z_r = E(M)$. But this unfortunately is not true. For example, take $M = \mathscr{M}(K_5)$, and let Z be the set of edges in a K_4 subgraph; then there is no adjoinable Z-arc. For graphs, there are at least two ways to avoid this difficulty, both at the cost of some extra complication. Tutte [12] proved that a sequence $Z_1, \ldots, Z_r$ with prescribed Z_1, Z_r could always be found with the property that each $Z_{i+1} - Z_i$ is either an adjoinable Z_i-arc or a slightly less simple structure. This theorem of Tutte can be generalized to all cyclically 3-connected matroids (indeed, an earlier version of this paper proved and used such a generalization) but it becomes cumbersome. The following generalization of Kelmans' theorem is both easier to prove and for our purposes easier to apply.

(5.3) *Let $\emptyset \neq Z \subseteq E(M)$, and let M and $M \times Z$ both be cyclically 3-connected. Suppose that $M \times Z$ has at least two series classes. Then for some $r \geqslant 1$ there is a sequence $Z_1 \subseteq \cdots \subseteq Z_r = E(M)$ such that for each i $(1 \leqslant i \leqslant r-1)$, $Z_{i+1} - Z_i$ is an adjoinable Z_i-arc, and such that $M \times Z$ and $M \times Z_1$ are both subdivisions of the same 3-connected matroid.*

To prove this we use the following.

(5.4) *Let M be 3-connected, and let $Z \subset E(M)$ be such that $M \times Z$ is connected and $M \times Z$ has at least two series classes. Choose $Z_1 \subseteq E(M)$ with $|Z_1|$ maximum such that $M \times Z_1$ is a subdivision of $M \times Z$. Then there is an adjoinable Z_1-arc.*

Proof. Let $M \times Z$ have k series classes. Then $k \geqslant 2$. But no connected matroid has exactly two series classes, and so $k \geqslant 3$. M has at least $k+1$ series classes, since $Z \neq E(M)$. But $M \times Z_1$ is a subdivision of $M \times Z$, and so has k series classes; thus $Z_1 \neq E(M)$. Let the series classes of $M \times Z_1$ be $P_1, \ldots, P_k$. Let A_1 be any Z_1-arc (this must exist since M is connected and $Z_1 \neq E(M)$). We may assume that A_1 is not adjoinable, and so $A_1 \to P_1$, say. Suppose for a contradiction that $|P_1| = 1$, and $P_1 = \{p\}$, say. Then $A_1 \cup \{p\}$ is a circuit of M. Choose $a \in A_1$; and then a, p are parallel in $(M \times (Z_1 \cup A))/(A - \{a\}) = M_1$, say, and

$$\begin{aligned} M_1 \backslash a &= (M \times (Z_1 \cup A)) \backslash \{a\} / (A - \{a\}) \\ &= (M \times (Z_1 \cup A)) \backslash \{a\} \backslash (A - \{a\}) \end{aligned}$$

since all elements in $A - \{a\}$ are coloops of $(M \times (Z_1 \cup A)) \backslash \{a\}$. Thus

$$M_1 \backslash a = M \times Z_1,$$

and so $M_1 \backslash p \cong M \times Z_1$. Hence $M \times ((Z_1 - \{p\}) \cup A)$ is a subdivision of $M \times Z_1$. From the maximality of $|Z_1|$ we have $|A| = 1$ and $A = \{a\}$; but then $\{a, p\}$ is a circuit of M, which is impossible since M is 3-connected and $|E(M)| \geqslant k+1 \geqslant 4$.

It follows that $|P_1| \geqslant 2$. Now P_1 is a series class of $M \times Z_1$, and so

$$r(P_1) + r(Z_1 - P_1) \leqslant r(Z_1) + 1.$$

However, M is 3-connected, and so there is no partition (X_1, X_2) of $E(M)$ such that $X_1 \cap Z_1 = P_1$ and such that

$$r(X_1) + r(X_2) \leqslant r(M) + 1,$$

for if so then $|X_1| \geqslant |P_1| \geqslant 2$ and $|X_2| \geqslant k-1 \geqslant 2$ and so (X_1, X_2) is a 2-separation.

From (3.8) there is a Z_1-arc A such that $A \nrightarrow P_1$, $A \nrightarrow (Z_1 - P_1)$. Then A is adjoinable, as required.

Proof of (5.3). By contracting all elements except one from every series class of M, we see that it suffices to prove (5.3) when M is 3-connected. The proof in that case is by induction on $|E(M) - Z|$. The result is trivial if this number is zero; assume then that $Z \neq E(M)$. By (5.4) there exists $Z' \subseteq E(M)$ such that $M \times Z'$ is a subdivision of $M \times Z$, and such that there is an adjoinable Z'-arc A, say. By (5.2), $M \times (Z' \cup A)$ is cyclically 3-connected, and $|Z' \cup A| > |Z|$, and so by induction there is a sequence $Z_2 \subseteq \cdots \subseteq Z_r = E(M)$ such that for $2 \leqslant i \leqslant r-1$ each $Z_{i+1} - Z_i$ is an adjoinable Z_i-arc, and such that $M \times Z_2$ and $M \times (Z' \cup A)$ are subdivisions of the same 3-connected matroid N, say. The series classes of $M \times Z_2$ and of $M \times (Z' \cup A)$ are in 1–1 correspondence with the elements of N and so with each other in an obvious way. Let A' be the series class of $M \times Z_2$ corresponding to A. Put $Z_1 = Z_2 - A'$; and then A' is an adjoinable Z_1-arc, and $M \times Z_1$, $M \times Z$ are both subdivisions of the same 3-connected matroid, and so the sequence $Z_1 \subseteq \cdots \subseteq Z_r$ satisfies the theorem.

Actually, in our applications (5.4) is more convenient to use than (5.3)—we have proved (5.3) merely to show the connection with Kelmans' theorem, which is (5.3) restricted to graphic matroids.

6. A "Wheels and Whirls" Theorem

For $n \geqslant 2$, the *wheel* W_n is the binary matroid on $\{1,..., 2n\}$ which has cycles all sets surrounded by $\{2n-1, 2n, 1\}$ and the sets $\{i, i+1, i+2\}$ $(1 \leqslant i \leqslant 2n-3,\ i \text{ odd})$. It is the polygon matroid of the graph consisting of a circuit with n vertices and one extra vertex joined (by one edge) to every vertex of the circuit.

We observe that $\{2, 4, 6,..., 2n\}$ is a circuit of W_n. The *whirl* $\mathscr{W}_n$ is the matroid on the same set, in which the same sets are independent, except that $\{2, 4, 6,..., 2n\}$ is independent in $\mathscr{W}_n$.

These matroids may be familiar to the reader, because Tutte [13] proved that if M is a matroid which is not 2-separable and M is not isomorphic to a wheel or whirl, then for some $x \in E(M)$, one of $M\backslash x$, M/x is not 2-separable. However, there is no connection between that theorem and the result we are concerned with here, as far as I can see.

First we prove the following lemma.

(6.1) *Let M be a connected matroid on S, and let $z_1,\ldots, z_{2n} \in S$ be distinct, where $n \geqslant 2$. Let $Z_i = \{z_i, z_{i+1}, z_{i+2}\}$ $(1 \leqslant i \leqslant 2n)$, reading subscripts modulo $2n$. Suppose that for i odd, Z_i is a circuit, and for i even, Z_i is a cocircuit. Then $S = \{z_1,\ldots, z_{2n}\}$ and M is isomorphic to W_n or $\mathscr{W}_n$.*

Proof. Put $P = \{z_i : i \text{ even}\}$ and $Q = \{z_i : i \text{ odd}\}$. For i odd, Z_i is a circuit and $Z_i - Q = \{z_{i+1}\}$; thus Q spans P, and $r(P \cup Q) = r(Q) \leqslant n$. Similarly, if r^* denotes r_{M^*}, $r^*(P \cup Q) \leqslant n$. But for any subset X of a matroid, $r(X) + r^*(X) \geqslant |X|$, with equality only if X is a separator; yet M is assumed to be connected. It follows that $S = P \cup Q$, and that $r(M) = n$.

Let $\mathscr{C}$ contain all sets $X \subseteq S$ such that $|X \cap Q| = 2$ $(X \cap Q = \{z_{j_1}, z_{j_2}\}$, say, where $j_2 > j_1)$ and such that $X \cap P$ is either $P \cap \{z_i : j_1 < i < j_2\}$ or $P - \{z_i : j_1 < i < j_2\}$. It is easy to see that if some nonempty $X \subseteq S$ is such that $|X \cap Z_i| \neq 1$ for $1 \leqslant i \leqslant 2n$, i even, then X includes either P or a member of $\mathscr{C}$; and so in particular, every circuit of M includes either P or a member of $\mathscr{C}$.

On the other hand, every member of $\mathscr{C}$ is a circuit. To show this, it suffices (by symmetry) to show that

$$\{z_1, z_{2r+1}\} \cup \{z_2, z_4,\ldots, r_{2r}\}$$

is a circuit $(r = 1,\ldots, n-1)$, and we prove this by induction on r. It is true by hypothesis if $r = 1$. If $r > 1$, then by induction,

$$\{z_1, z_{2r-1}\} \cup \{z_2, z_4,\ldots, z_{2r-2}\}$$

is a circuit, and so is $\{z_{2r-1}, z_{2r}, z_{2r+1}\}$; and so

$$\{z_1, z_{2r+1}\} \cup \{z_2, z_4,\ldots, z_{2r}\}$$

includes a circuit. But this circuit must include P or a member of $\mathscr{C}$, and so the inclusion is an equality, as required. Thus all members of $\mathscr{C}$ are circuits of M, and every other circuit includes P. If P is a circuit then $M \cong W_n$. If not, then P spans S, because $r(M) = n$, and every set $P \cup \{z\}$ $(z \in Q)$ is a circuit. Then $M \cong \mathscr{W}_n$, as required.

The main result of this section is the following.

(6.2) *Let M be a matroid on S, and let $x, y \in S$ be distinct, and let N be another matroid. Suppose that $M\backslash x/y \cong N$, and that N is connected, and has no loops, coloops, or series or parallel elements. Then one of the following is true:*

(i) *x is a loop, coloop, or parallel element of M;*
(ii) *y is a loop, coloop, or series element of M;*

(iii) *for some* $x', y' \in S$, $M\backslash x'/y' \cong N$ *but* x' *is not a loop, coloop, or parallel element of* M/y';

(iv) *for some* $x', y' \in S$, $M\backslash x'/y' \cong N$ *but* y' *is not a loop, coloop, or series element of* $M\backslash x'$;

(v) *there exists* $z \in S - \{x, y\}$ *so that* $\{x, y, z\}$ *is both a circuit and a cocircuit of* M;

(vi) *for some* $n \geqslant 2$, $N \cong W_n$ *or* $\mathscr{W}_n$ *and* $M \cong W_{n+1}$ *or* $\mathscr{W}_{n+1}$, *respectively.*

Proof. We assume that (i)–(iv) are all false. It follows that M has no loops, coloops, or series or parallel elements. For suppose that C is a circuit of M with $|C| \leqslant 2$. C does not include a circuit of $M\backslash x/y$ because N has no circuits of size $\leqslant 2$, and so $x \in C$. Thus x is a loop or parallel element of M and (i) is true, contrary to our assumption. Similarly, M has no coloops or series elements. It follows that M is connected (because $M\backslash x/y$ is connected, and $\{x, y\}$ includes no circuits or cocircuits).

We define inductively a sequence $x_1, y_1, x_2, y_2, \ldots$ of elements of M with the following properties:

(i) for all $i \geqslant 1$, $M\backslash x_i/y_i \cong N$ and $M/y_i\backslash x_{i+1} \cong N$,

(ii) for $i \geqslant 1$, $\{x_i, y_i, x_{i+1}\}$ is a circuit, and $\{y_i, x_{i+1}, y_{i+1}\}$ is a cocircuit.

To do so, we define $x_1 = x$ and $y_1 = y$. Inductively, having defined $x_1, y_1, \ldots, x_i, y_i$, we have that $M\backslash x_i/y_i \cong N$ and so by the falsity of (iii), x_i is a loop, coloop, or parallel element of M/y_i. M has no loops, coloops, or parallel elements, and so for some element $x_{i+1} \neq x_i, y_i$, $\{x_i\, y_i, x_{i+1}\}$ is a circuit of M. We see that $M\backslash x_i/y_i \cong M/y_i\backslash x_{i+1}$, because x_i, x_{i+1} are parallel in M/y_i; and so $M/y_i\backslash x_{i+1} \cong N$. Now by the falsity of (iv), y_i is a series element of $M\backslash x_{i+1}$, and there is an element $y_{i+1} \neq y_i, x_{i+1}$ so that $\{y_i, x_{i+1}, y_{i+1}\}$ is a cocircuit of M. And again, $M\backslash x_{i+1}/y_{i+1} \cong M/y_i\backslash x_{i+1}$, because y_i, y_{i+1} are in series in $M\backslash x_{i+1}$, and so $M\backslash x_{i+1}/y_{i+1} \cong N$. This completes the inductive definition.

For $i \geqslant 1$, define $z_{2i-1} = x_i$, $z_{2i} = y_i$. Define $Z_i = \{z_i, z_{i+1}, z_{i+2}\}$. Then for i odd, Z_i is a circuit, and for i even it is a cocircuit. Since M is finite, there is repetition in the sequence $z_1, z_2, \ldots$. Choose $i < j$ with j minimum so that $z_i = z_j$. Then $j \geqslant i + 3$, since z_i, z_{i+1}, z_{i+2} are distinct by the construction.

Suppose first that $j = i + 3$. If $i = 1$ then $\{z_1, z_2, z_3\} = Z_1$ is both a circuit and a cocircuit of M, and so (v) is true.

Secondly, suppose that $j = i + 3$ and $i = 2$. By the falsity of (iv), y_1 is a series element of $M\backslash x_1$, and so there is a cocircuit $\{z_1, z_2, z\}$ for some $z \in S - \{z_1, z_2\}$. But Z_3 is a circuit and $Z_3 = \{z_3, z_4, z_2\}$, and so $z \in \{z_3, z_4\}$. Thus if Y denotes $\{z_1, z_2, z_3, z_4\}$, we have $r(Y) = 2$ and $r^*(Y) = 2$, and so Y is a separator. Hence $Y = S$, which is impossible because $|S| = |E(N)| + 2 \geqslant 6$ (since N has no loops, coloops, or series or parallel elements).

Thirdly, suppose that $j=i+3$ and $i\geqslant 3$. One of Z_{i-2}, Z_{j-2} is a circuit and one is a cocircuit, and yet they have intersection $\{z_i\}$, by the minimality of j. This is impossible, and so we may assume that $j\geqslant i+4$.

Suppose that $j-i$ is odd. Then one of Z_i, Z_{j-2} is a circuit and one is a cocircuit, but their intersection is $\{z_i\}$, a contradiction. Thus $j-i$ is even.

Suppose that $i\geqslant 2$. Then one of Z_{i-1}, Z_{j-2} is a circuit and one is a cocircuit, and we obtain a contradiction as before.

Hence $i=1$, and for some $n\geqslant 2$, $x_1=x_{n+1}$ and $x_1, y_1, \ldots, x_n, y_n$ are all distinct. To apply Lemma (6.1), it suffices to prove that $\{y_n, x_1, y_1\}$ is a cocircuit.

Suppose now that $n=2$. Then $\{x_1, y_1, x_2\}$ and $\{x_2, y_2, x_1\}$ are circuits, and $\{y_1, x_2, y_2\}$ is a cocircuit, and $\{y_2, x_1, y_3\}$ is a cocircuit. Therefore $|\{y_2, x_1, y_3\}\cap\{x_1, y_1, x_2\}|\neq 1$, and so $y_3=y_1$ or x_2. In either case, $r^*(\{x_1, y_1, x_2, y_2\})=2$ and as before $Y=S$, a contradiction.

Thus $n\geqslant 3$. Then $\{y_n, x_1, y_{n+1}\}$ is a cocircuit, but $\{x_1, y_1, x_2\}$ and $\{x_2, y_2, x_3\}$ are circuits, and so $y_{n+1}=y_1$ and $\{y_n, x_1, y_1\}$ is a cocircuit. Thus the hypotheses of (6.1) are satisfied, and we conclude that M (and hence N) is isomorphic to a wheel or a whirl. Then (vi) holds.

7. Splitters

Let $\mathscr{F}$ be a class of matroids, closed under minors and under isomorphism. $N\in\mathscr{F}$ is said to be a *splitter* for $\mathscr{F}$ if every $M\in\mathscr{F}$ with a minor isomorphic to N is 1- or 2-separable unless $M\cong N$.

Such objects occur in graph theory, but only in relatively out-of-the-way places. The class of graphic matroids, for example, has no splitters. In a non-graphic context, however, they appear to be more central. We shall see that the class of regular matroids does have a splitter, the matroid we call R_{10}.

(7.1) *Suppose that $N\in\mathscr{F}$, and that for every $M\in\mathscr{F}$, if $M/x\cong N$ then x is a loop, coloop, or series element of M. Then for every $M\in\mathscr{F}$, if M has a minor isomorphic to N, then there exists $Z\subseteq E(M)$ such that $M\times Z$ is a subdivision of N.*

Proof. M has a minor isomorphic to N. Choose disjoint subsets X, $Y\subseteq E(M)$ such that $M\backslash X/Y\cong N$, with Y minimal. For $y\in Y$, let M_y be $(M\backslash X)/(Y-\{y\})$. Then $M_y/y\cong N$, and so by hypothesis y is a loop, coloop or series element of M_y. If y is a loop or coloop then $M_y/y=M_y\backslash y$, and so $M_y\backslash(X\cup\{y\})/(Y-\{y\})\cong N$, contrary to the minimality of Y. Thus each $y\in Y$ is a series element of M_y. Let the elements of $M\backslash X/Y$ be $z_1, \ldots, z_k$. Let L_i be the subset of $\{z_1, \ldots, z_k\}\cup Y$ containing z_i and those $y\in Y$ for which in

M_y, y is in series with z_i but with no z_j for $j < i$. Then $(L_1,..., L_k)$ is a partition of $E(M\backslash X)$ and $M\backslash X$ is a subdivision of N, as required.

For example, we have the following, which will be used later in the paper.

(7.2) *If M is regular and has an $\mathscr{M}(K_{3,3})$ minor, then there exists $Z \subseteq E(M)$ such that $M \times Z$ is a subdivision of $\mathscr{M}(K_{3,3})$.*

Proof. We merely verify the hypotheses of (7.1), taking $\mathscr{F}$ to be the class of regular matroids. A mechanical way to do so is to take a representation of $\mathscr{M}^*(K_{3,3})$ over $GF(2)$, and examine the matroids obtained from this by adding one new vector in all possible ways. Such a matroid M will have an element e so that $M\backslash e \cong \mathscr{M}^*(K_{3,3})$, and all such matroids which are binary will be constructed (except the one in which e is a coloop). Their duals are the matroids which need to be checked. For details, see the Appendix.

(7.3) *Suppose that $N \in \mathscr{F}$, and is non-null and connected, and that the following statements are true:*

(i) *all circuits and cocircuits of N have at least three elements;*

(i) *for every $M \in \mathscr{F}$, if $M\backslash x \cong N$ then x is a loop, coloop, or parallel element of M;*

(iii) *for every $M \in \mathscr{F}$, if $M/x \cong N$ then x is a loop, coloop, or series element of M;*

(iv) *if $N \cong W_n$ or $\mathscr{W}_n$ for some $n \geqslant 2$, then $W_{n+1} \notin \mathscr{F}$ or $\mathscr{W}_{n+1} \notin \mathscr{F}$, respectively.*

Then N is a splitter for $\mathscr{F}$.

The proof is in two steps.

Step 1. *If $M \in \mathscr{F}$ and $Z \subseteq E(M)$, and $M \times Z$ is a subdivision of N, then for every Z-arc A there is a series class P of $M \times Z$ such that $A \to P$.*

Proof. If possible, take a counterexample M, Z, A with $|E(M)|$ minimum. Then obviously $E(M) = Z \cup A$, and $|A| = 1$. (For all the elements in A are in series, and so if $|A| > 1$ we may produce a smaller counterexample by contracting.) Let $A = \{a\}$, say. Let the series classes of $M\backslash a$ be $P_1,..., P_k$, where $k = |E(N)|$, so that $(P_1,..., P_k)$ is a partition of Z.

Now if $|P_i| = 1$ for all i, then $M\backslash a \cong N$, and yet a is not a loop, coloop, or parallel element of M, contrary to hypothesis. Thus for some i, $|P_i| > 1$. We assume $|P_i| > 1$. Choose $x_1, x_1' \in P_1$. Now $(M/x_1) \times (Z - \{x_1\})$ is a subdivision of N, and A is a $(Z - \{x_1\})$-arc of M/x_1 (since $A \cup \{x_1\}$ includes no circuits of M); and so by induction there is a circuit of M/x_1 containing a and included in $\{a\} \cup P_{j_1}$, for some j_1. This is therefore not a circuit of M, and so there is a circuit C_1 of M with x_1, $a \in C_1$ and $C_1 - \{x_1, a\} \subseteq P_{j_1}$, for some $j_1 \neq 1$. Similarly there is a circuit C'_1 with x'_1, $a \in C'_1$ and $C'_1 - \{x'_1, a\} \subseteq P_{j'_1}$ for some $j'_1 \neq 1$. But $(C_1 \cup C'_1) - \{a\}$ includes a circuit C, and $C \subseteq Z$ and so is a union of at least three P_i's (because N has no circuits with $\leqslant 2$ elements). Thus $|P_1| = 2$, and $P_{j_1} \subseteq C_1$. Suppose now that for some $i > 1$, $|P_i| \geqslant 2$, $|P_2| \geqslant 2$, say. Then we have similarly $|P_2| = 2$. Choose $x_2 \in P_2$; then there is a circuit C_2 with x_2, $a \in C_2$ and $C_2 - \{x_2, a\} \subseteq P_{j_2}$ for some $j_2 \neq 2$. Certainly $C_1 \neq C_2$ (because even if $j_1 = 2$ and $j_2 = 1$, $C_1 \neq C_2$ since $P_{j_1} \subseteq C_1$), and so there is a circuit $C \subseteq (C_1 \cup C_2) - \{a\}$. But then C is a union of at least three P_i's, and yet $C_1 \cup C_2$ only includes two P_i's, a contradiction.

This proves that $|P_1| = 2$, $|P_i| = 1$ $(2 \leqslant i \leqslant k)$. Thus $M\backslash a/x_1 \cong N$, and we may apply (6.2). We discover from (6.2) that one of the following holds.

(i) a is a loop, coloop, or parallel element of M. This is impossible since $\{a\}$ is a Z-arc of M and $\{a\} \nrightarrow P$ for any series class P of $M \times Z$.

(ii) x_1 is a loop, coloop, or series element of M. But $x_1 \in C_1$, and so it is not a loop or coloop, and if it is in series with y, say, then $y \in C_1$. But $y \neq a$, and so $y \in Z$, and so $y \in P_1$ since P_1 is a series class of $M\backslash a$. This is impossible since $C_1 \cap P_1 = \{x_1\}$.

(iii) For some $x', y' \in E(M)$, $M\backslash x'/y' \cong N$ but x' is not a loop, coloop, or parallel element of M/y'. This is impossible by hypothesis (ii) of (7.3).

(iv) For some $x', y' \in E(M)$, $M\backslash x'/y' \cong N$ but y' is not a loop, coloop, or series element of $M\backslash x'$. This is impossible by hypothesis (iii) of (7.3).

(v) There exists $z \neq a$, x_1 such that $\{a, x_1, z\}$ is both a circuit and a cocircuit of M. Thus $z \in Z$, and z, x_1 are in series in $M\backslash a$; and so $z \in P_1$. This is impossible because $\{a\} \nrightarrow P_1$.

(vi) For some $n \geqslant 2$, $N \cong W_n$ or $\mathscr{W}_n$, and $M \cong W_{n+1}$ or $\mathscr{W}_{n+1}$, respectively. This contradicts hypothesis (iv) of (7.3), and completes the proof of Step 1.

Step 2. *If $M \in \mathscr{F}$ is* 3*-connected and has a minor isomorphic to N, then $M \cong N$, that is, N is a splitter for $\mathscr{F}$.*

Proof. $M \in \mathscr{F}$ has a minor isomorphic to N, and so by (7.1) there exists $Z \subseteq E(M)$ such that $M \times Z$ is a subdivision of N. Suppose for a contradiction that we can choose Z thus with $Z \neq E(M)$. Then by (5.4) we can choose Z thus so that in addition there is an adjoinable Z-arc A. But this contradicts Step 1. So $Z = E(M)$, and M is a subdivision of N. But M is 3-connected, and $|E(M)| \geqslant |E(N)| \geqslant 4$, and so M has no pairs of elements in series. It follows that $M \cong N$, as required.

For the result of this paper we need only two applications of (7.3) (more will appear in subsequent papers, however).

The matroid R_{10} was, as far as I know, first introduced by Bixby [1]. (He calls matroids isomorphic to it matroids of *Type R*.) We define R_{10} to be the linear independence matroid of the ten 5-tuples over $GF(2)$ with three 1's and two 0's. It may be verified that another, less symmetric but more convenient, representation of R_{10} is the following: take the nine 6-tuples over $GF(2)$ with two 1's and four 0's which represent $\mathscr{M}(K_{3,3})$, and add the 6-tuple of all 1's.

It may be verified that R_{10} is regular, and is isomorphic to its dual (although not self-dual). Its automorphism group is doubly transitive. $R_{10}\backslash e \cong \mathscr{M}(K_{3,3})$ and $R_{10}/e \cong \mathscr{M}^*(K_{3,3})$ for any element e. A totally unimodular representation of R_{10} is given in Fig. 1.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 \end{bmatrix}$$

R_{10}

FIGURE 1

(7.4) *R_{10} is a splitter for the class of regular matroids.*

Proof. It is necessary only to check that R_{10} satisfies the hypotheses of (7.3). To verify this, take a 5-dimensional representation of R_{10} over $GF(2)$, and examine the result of adding any other vector to this representation. It will be found that either the vector added is the zero vector, or is one already in the representation, or the enlarged binary matroid produced will not be regular. For details, see the Appendix. This verifies (7.3)(ii), and (7.3)(iii) follows from this, because R_{10} is isomorphic to its dual.

(7.5) *$\mathscr{M}(K_5)$ is a splitter for the class of regular matroids without minors isomorphic to $\mathscr{M}(K_{3,3})$.*

Proof. Again, we verify that the hypotheses of (7.3) hold; with the difference that we must check (7.3)(ii) and (7.2)(iii) separately. (See the Appendix.)

There is another application of (7.3) which is not needed for this paper, but which we state here for convenient reference in subsequent papers.

(7.6) *F_7 is a splitter for the class of binary matroids without F_7^* minors.*

The proof is again case-checking. (See the Appendix.)

8. Connectivity

If $A, B \subseteq S$ are disjoint and M is a matroid on S, we define $k_M(A, B)$ to be the minimum over all partitions (X, Y) of S with $A \subseteq X$, $B \subseteq Y$, of

$$r(X) + r(Y) - r(M).$$

$k_M(A, B)$ should be thought of as the "connectivity" between A and B in M. It is easy to see that

$$0 \leqslant k_M(A, B) \leqslant |A|, |B|$$

and that $k_M(A, B) = k_{M^*}(A, B)$.

(8.1) *If (X, Y) is a partition of $E(M)$, and $M' = M\backslash Z_1/Z_2$, where $Z_1 \cap Z_2 = \varnothing$ and $Z_1 \cup Z_2 = Z$, say, then*

$$k_{M'}(X - Z, Y - Z) \leqslant k_M(X, Y);$$

that is,

$$r_{M'}(X - Z) + r_{M'}(Y - Z) - r(M') \leqslant r_M(X) + r_M(Y) - r(M).$$

Proof. Using duality, it suffices to prove the result when $Z_1 = \{z\}$, say and $Z_2 = \varnothing$. By symmetry we may assume that $z \in X$. We must prove that

$$r_M(X - \{z\}) + r_M(Y) - r(M') \leqslant r_M(X) + r_M(Y) - r(M).$$

But $r_M(X - \{z\}) \leqslant r_M(X)$, and if equality occurs, then z is not a coloop of M and so $r(M') = r(M)$.

(8.2) *If $A, B \subseteq E(M)$ are disjoint, and M' is obtained from M by deleting and contracting elements from $E(M) - (A \cup B)$, then*

$$k_{M'}(A, B) \leqslant k_M(A, B).$$

Proof. Choose a partition (X, Y) of $E(M)$ with $A \subseteq X$, $B \subseteq Y$, so that $k_M(X, Y) = k_M(A, B)$. If $Z = E(M) - E(M')$, we have from (8.1) that $k_{M'}(X - Z, Y - Z) \leqslant k_M(X, Y)$, and so $k_{M'}(A, B) \leqslant k_M(A, B)$, as required.

For the remainder of this section, we shall assume that M is a matroid on a set S, and that N is a minor of M; that (A, B) is a partition of $E(N)$, and $k_N(A, B) = k$; that $k_M(A, B) > k$; and that for every $x \in S - (A \cup B)$, and for $M' = M\backslash x$, M/x, either N is not a minor of M' or $k_{M'}(A, B) = k$. We shall analyse the structure of M with respect to N. (We should perhaps stress that N is a minor of M; no isomorphism is involved.)

(8.3) *For each* $x \in S - (A \cup B)$ *either* $k_{M\backslash x}(A, B) > k$ *or* $k_{M/x}(A, B) > k$.

Proof. Suppose that $k_{M\backslash x}(A, B) \leqslant k$. Then there exists a partition (X_1, Y_1) of $S - \{x\}$ with $A \subseteq X_1$, $B \subseteq Y_1$, and

$$r_M(X_1) + r_M(Y_1) - r(M\backslash x) \leqslant k < r_M(X_1 \cup \{x\}) + r_M(Y_1) - r(M).$$

Thus $r_M(X_1) \neq r_M(X_1 \cup \{x\})$, so x is not a loop, and $r(M\backslash x) = r(M)$, so x is not a coloop. We have

$$r_M(X_1) + r_M(Y_1) - r(M) \leqslant k.$$

Now suppose that $k_{M/x}(A, B) \leqslant k$. Then there exist X_2, $Y_2 \subseteq S$ such that $X_2 \cup Y_2 = S$, $X_2 \cap Y_2 = \{x\}$, $A \subseteq X_2$, $B \subseteq Y_2$, and

$$r_M(X_2) - 1 + r_M(Y_2) - 1 - (r(M) - 1) \leqslant k;$$

that is, $r_M(X_2) + r_M(Y_2) - r(M) \leqslant k + 1$. Adding, we obtain

$$r_M(X_1) + r_M(X_2) + r_M(Y_1) + r_M(Y_2) - 2r(M) \leqslant 2k + 1,$$

and hence, using submodularity of the rank function,

$$\begin{aligned} r_M(X_1 \cup X_2) + r_M(Y_1 \cap Y_2) + r_M(X_1 \cap X_2) \\ + r_M(Y_1 \cup Y_2) - 2r(M) \leqslant 2k + 1. \end{aligned}$$

Now every element of S is in precisely two of X_1, X_2, Y_1, Y_2, and so $(X_1 \cup X_2,\ Y_1 \cap Y_2)$ and $(X_1 \cap X_2,\ Y_1 \cup Y_2)$ are both partitions of S; and $A \subseteq X_1 \cup X_2$, $X_1 \cap X_2$, and $B \subseteq Y_1 \cap Y_2$, $Y_1 \cup Y_2$. Thus

$$r_M(X_1 \cup X_2) + r_M(Y_1 \cap Y_2) - r(M) \geqslant k + 1$$

and

$$r_M(X_1 \cap X_2) + r_M(Y_1 \cup Y_2) - r(M) \geqslant k + 1.$$

Adding, we obtain a contradiction, as required.

(8.4) *For each* $x \in S - (A \cup B)$, *either* N *is not a minor of* $M\backslash x$ *or* N *is not a minor of* M/x.

Proof. This follows from (8.3) and our hypothesis about M, N.

Thus N may be expressed as $M\backslash P/Q$, where (P, Q) is a partition of $S - (A \cup B)$, in a unique way.

(8.5) *There is no circuit* C *with* $C \subseteq P \cup Q$, $|C - Q| \leqslant 1$, *and there is no* *cocircuit D with $D \subseteq P \cup Q$, $|D - P| \leqslant 1$. In particular, Q is independent, and $A \cup B \cup Q$ spans P.*

Proof. Suppose that C is a circuit with $C \subseteq Q$. Choose $q \in C$; then q is a loop of $M \backslash P/(Q - \{q\})$, and so $M \backslash P/(Q - \{q\}) \backslash \{q\} = N$, contrary to (8.4). Secondly, suppose that $C \subseteq P \cup Q$ is a circuit with $C \cap P = \{p\}$. Then p is a loop of $M \backslash (P - \{p\})/Q$, and so M/p has N as a minor, contrary to (8.4). The second assertion of the theorem follows by duality.

For $z \in P \cup Q$, let M_z be $M \backslash z$ if $z \in P$, and be M/z if $z \in Q$. Now N is a minor of M_z, and so by hypothesis $k_{M_z}(A, B) = k$. Thus for each $z \in P \cup Q$ there is a partition (X_z, Y_z) of $S - z$ such that $A \subseteq X_z$, $B \subseteq Y_z$, and

$$r_{M_z}(X_z) + r_{M_z}(Y_z) = r(M_z) + k.$$

(8.6) *For distinct z, $z' \in P \cup Q$, either $z \in X_{z'}$ or $z' \in X_z$ but not both. In particular, for each $z \in P \cup Q$, (X_z, Y_z) is uniquely defined.*

Proof. The second statement follows from the first immediately. Suppose the first is false; then by exchanging A and B if necessary, we may assume that $z \in X_{z'}$, $z' \in X_z$. By duality we may assume that $z \in P$. Now $(X_z \cup X_{z'}, Y_z \cap Y_{z'})$ is a partition of S, and so

$$r_M(X_z \cup X_{z'}) + r_M(Y_z \cap Y_{z'}) - r(M) \geqslant k + 1 \qquad (*)$$

since $k_M(A, B) > k$.

There are now two cases, $z' \in P$, and $z' \in Q$. Suppose first that $z' \in P$. Then $M_z = M \backslash z$ and $M_{z'} = M \backslash z'$. By (8.5), z, z' are not coloops of M, and so $r(M_z) = r(M_{z'}) = r(M)$. Thus

$$r_M(X_z) + r_M(Y_z) - r(M) \leqslant k$$

and

$$r_M(X_{z'}) + r_M(Y_{z'}) - r(M) \leqslant k.$$

Adding, and using submodularity and (*), we obtain

$$r_M(X_z \cap X_{z'}) + r_M(Y_z \cup Y_{z'}) - r(M) \leqslant k - 1.$$

Let M' be $M \backslash \{z, z'\}$. By (8.5), $r(M') = r(M)$, and since M' has N as a minor, and since $(X_z \cap X_{z'}, Y_z \cup Y_{z'})$ is a partition of the elements of M', we have a contradiction to (8.2).

Now suppose that $z' \in Q$. Then $M_z = M \backslash z$ and $M_{z'} = M/z'$. By (8.5), z is not a coloop of M, and z' is not a loop, and so $r(M_z) = r(M)$, $r(M_{z'}) = r(M) - 1$. Thus

$$r_M(X_z) + r_M(Y_z) - r(M) \leqslant k,$$

and

$$r_M(X_{z'} \cup \{z'\}) - 1 + r_M(Y_{z'} \cup \{z'\}) - 1 - (r(M) - 1) \leqslant k,$$

that is,

$$r_M(X_{z'} \cup \{z'\}) + r_M(Y_{z'} \cup \{z'\}) - r(M) \leqslant k + 1.$$

Adding, and using submodularity and (*), we obtain

$$r_M((X_z \cap X_{z'}) \cup \{z'\}) + r_M((Y_z \cup Y_{z'}) \cup \{z'\}) - r(M) \leqslant k.$$

Let M' be $M \backslash z/z'$. By (8.5), $r(M') = r(M) - 1$, and since M' has N as a minor, since $(X_z \cap X_{z'}, Y_z \cup Y_{z'})$ is a partition of the elements of M', and since $r_{M'}(Z) = r_M(Z \cup \{z'\}) - 1$ for $Z \subseteq S - \{z, z'\}$, we have a contradiction to (8.2).

(8.7) *For distinct* $z, z' \in P \cup Q$, *if* $z \in X_{z'}$ *then* $X_z \subseteq X_{z'}$. *In particular,* $P \cup Q$ *may be ordered as* $z_1, \ldots, z_n$, *so that for each* i,

$$X_{z_i} = A \cup \{z_1, \ldots, z_{i-1}\},$$
$$Y_{z_i} = \{z_{i+1}, \ldots, z_n\} \cup B.$$

Proof. The second statement follows from the first and (8.6), because we would have $z \in X_{z'}$ if and only if $|X_z| < |X_{z'}|$. To prove the first, we may assume from duality that $z \in P$. There are two cases, $z' \in P$ and $z' \in Q$.

Suppose first that $z' \in P$. As in (8.6), we have

$$r_M(X_z) + r_M(Y_z) - r(M) \leqslant k$$

and

$$r_M(X_{z'}) + r_M(Y_{z'}) - r(M) \leqslant k.$$

Now $(X_z \cup X_{z'}, Y_z \cap Y_{z'})$ is a partition of $E(M \backslash z')$, and so

$$r_M(X_z \cup X_{z'}) + r_M(Y_z \cap Y_{z'}) - r(M) \geqslant k.$$

Using these three inequalities and submodularity, we obtain

$$r_M(X_z \cap X_{z'}) + r_M(Y_z \cup Y_{z'}) - r(M) \leqslant k.$$

But $(X_z \cap X_{z'}, Y_z \cup Y_{z'})$ is a partition of $E(M \backslash z)$, and so, by the uniqueness of (X_z, Y_z) asserted in (8.6), we have $X_z \cap X_{z'} = X_z$, that is, $X_z \subseteq X_{z'}$, as required.

Now suppose that $z' \in Q$. Then as in (8.6) we have

$$r_M(X_z) + r_M(Y_z) - r(M) \leqslant k$$

and

$$r_M(X_{z'} \cup \{z'\}) + r_M(Y_{z'} \cup \{z'\}) - r(M) \leqslant k + 1.$$

But $(X_z \cap X_{z'}, Y_z \cup Y_{z'})$ is a partition of the elements of $M\backslash z$, and so

$$r_M(X_z \cap X_{z'}) + r_M(Y_z \cup Y_{z'}) - r(M) \geqslant k.$$

Using these three inequalities and submodularity, we obtain

$$r_M((X_z \cup X_{z'}) \cup \{z'\}) + r_M((Y_z \cap Y_{z'}) \cup \{z'\}) - r(M) \leqslant k + 1,$$

that is,

$$r_{M_{z'}}(X_z \cup X_{z'}) + r_{M_{z'}}(Y_z \cap Y_{z'}) - r(M_{z'}) \leqslant k.$$

Thus, by the uniqueness of $(X_{z'}, Y_{z'})$, we have $X_z \cup X_{z'} = X_{z'}$, that is, $X_z \subseteq X_{z'}$, as required.

(8.8) *$z_1, \ldots, z_n$ are alternately members of P and members of Q.*

Proof. Suppose this is false; then by duality, we may assume that z_i, $z_{i+1} \in P$. By (8.7),

$$r_M(A \cup \{z_1, \ldots, z_{i-1}\}) + r_M(\{z_{i+1}, \ldots, z_n\} \cup B) - r(M) \leqslant k$$

and

$$r_M(A \cup \{z_1, \ldots, z_i\}) + r_M(\{z_{i+2}, \ldots, z_n\} \cup B) - r(M) \leqslant k.$$

Now

$$r_M(A \cup \{z_1, \ldots, z_i\}) + r_M(\{z_{i+1}, \ldots, z_n\} \cup B) - r(M) > k$$

since $k_M(A, B) > k$, and so

$$r_M(A \cup \{z_1, \ldots, z_{i-1}\}) + r_M(\{z_{i+2}, \ldots, z_n\} \cup B) - r(M) < k.$$

Let M' be $M\backslash\{z_i, z_{i+1}\}$. Then as before, $r(M') = r(M)$; and so $k_{M'}(A, B) < k$, contrary to (8.2).

(8.9) *For all $i > 1$, if $z_i \in P$ there is a circuit C of M with $z_{i-1}, z_i \in C$, and $C - \{z_{i-1}, z_i\} \subseteq (Q \cap \{z_j : j > i\}) \cup B$. If $z_i \in Q$ there is a cocircuit D with $z_{i-1}, z_i \in D$ and $D - \{z_{i-1}, z_i\} \subseteq (P \cap \{z_j : j > i\}) \cup B$.*

A similar result holds for all $i < n$ with A and B exchanged.

Proof. By duality we may assume that $z_i \in P$, so that $z_{i-1} \in Q$, by (8.7). Let M_0 be M/z_{i-1}. By (8.7),

$$r_{M_0}(A \cup \{z_j: 1 \leqslant j \leqslant i-2\}) + r_{M_0}(\{z_j: i \leqslant j \leqslant n\} \cup B) - r(M_0) \leqslant k.$$

Let M' be $M_0 \backslash (P - \{z_i\})$, and then by (8.1)

$$r_{M'}(A \cup (Q \cap \{z_j: 1 \leqslant j \leqslant i-2\})) \\ + r_{M'}((Q \cap \{z_j: i+1 \leqslant j \leqslant n\}) \cup \{z_i\} \cup B) - r(M') \leqslant k.$$

Let M'' be $M' \backslash z_i$. By (8.2), since N is a minor of M'', we have

$$r_{M''}(A \cup (Q \cap \{z_j: 1 \leqslant j \leqslant i-2\})) \\ + r_{M''}((Q \cap \{z_j: i+1 \leqslant j \leqslant n\}) \cup B) - r(M'') \geqslant k.$$

The first terms in these inequalities are equal; and by (8.5), $r(M'') = r(M')$; thus

$$r_{M'}((Q \cap \{z_j: i+1 \leqslant j \leqslant n\}) \cup \{z_i\} \cup B) \\ \leqslant r_{M''}((Q \cap \{z_j: i+1 \leqslant j \leqslant n\}) \cup B),$$

that is, there is a circuit C of M' and hence of M_0 with $z_i \in C$, and $C - \{z_i\} \subseteq (Q \cap \{z_j: i+1 \leqslant j \leqslant n\}) \cup B$. Now C is not a circuit of M, because by (8.7),

$$r_M(A \cup \{z_j: 1 \leqslant j \leqslant i-1\}) + r_M\{(z_j: i+1 \leqslant j \leqslant n\} \cup B) - r(M) \leqslant k$$

and yet

$$r_M(A \cup \{z_j: 1 \leqslant j \leqslant i-1\}) + r_M(\{z_j: i \leqslant j \leqslant n\} \cup B) - r(M) > k$$

since $k_M(A, B) > k$, and thus $\{z_j: i+1 \leqslant j \leqslant n\} \cup B$ does not span z_i. Hence $C \cup \{z_{i-1}\}$ is a circuit of M. This completes the proof.

(8.10) *If $z_1 \in P$, there is no circuit C with $z_1 \in C \subseteq P \cup Q \cup B$. If $z_1 \in Q$, there is no cocircuit D with $z_1 \in D \subseteq P \cup Q \cup B$. Similar results hold for z_n and A.*

Proof. If $z_1 \in P$, then by (8.7),

$$r_M(A) + r_M(\{z_2, \dots, z_n\} \cup B) - r(M) \leqslant k,$$

but

$$r_M(A) + r_M(\{z_1, \dots, z_n\} \cup B) - r(M) > k$$

since $k_M(A, B) > k$. Thus there is no circuit C with $z_1 \in C \subseteq P \cup Q \cup B$. The other results follow by duality and symmetry.

9. SEPARABILITY BECAUSE OF R_{12}

Now we use the results of the previous section to prove that every regular matroid with an R_{12} minor is 3-separable. In an attempt to clarify the proof, the relevant properties of R_{12} have been abstracted, in the following theorem. (R_{12} itself is defined later in this section.)

(9.1) *Let $\mathscr{F}$ be a class of matroids, closed under minors and under isomorphism. Let $N \in \mathscr{F}$, and let (A, B) be a partition of $E(N)$, with $k_N(A, B) = k$. Suppose that N, $\mathscr{F}$ have the following properties*:

(i) *for each $x \in A$ there is a circuit C and a cocircuit D of N containing x, with $C, D \subseteq A$,*

(ii) *for each $M \in \mathscr{F}$, if $M \backslash x = N$ and x is not a coloop of M, there is a circuit C of M with $x \in C$ and $C - \{x\}$ included in one of A, B,*

(iii) *for each $M \in \mathscr{F}$, if $M/y = N$ and y is not a loop of M, there is a cocircuit D of M with $y \in D$ and $D - \{y\}$ included in one of A, B,*

(iv) *for each $M \in \mathscr{F}$, if $M\backslash x/y = N$, suppose that there is a cocircuit D of M with $\{x, y\} \subset D \subseteq B \cup \{x, y\}$; then either there is a circuit C of M with $x \in C \subseteq B \cup \{x, y\}$ or x is parallel to an element of A in M/y,*

(v) *for each $M \in \mathscr{F}$, if $M\backslash x/y = N$, suppose that there is a circuit C of M with $\{x, y\} \subset C \subseteq B \cup \{x, y\}$; then either there is a cocircuit D of M with $y \in D \subseteq B \cup \{x, y\}$, or y is in series with an element of A in $M\backslash x$.*

Then $k_M(A, B) = k$ for each $M \in \mathscr{F}$ with N as a minor.

(*Remark.* Statements (ii) and (iii) form a dual pair, as do (iv) and (v), and (i) is invariant under duality. Thus if N, $\mathscr{F}$ have these properties, then so do N^*, $\mathscr{F}^* = \{M^*: M \in \mathscr{F}\}$. However, there is no symmetry between A and B.)

Proof. We proceed by induction on $|E(M)|$. The result is clear if $|E(M)| = |E(N)|$, because then $M = N$, and so we assume that $|E(M)| > |E(N)|$. We assume for a contradiction that $k_M(A, B) > k$. Thus, by induction, M and N have the properties discussed in Section 8, and so (8.4)–(8.10) are true for them. We assume by duality that $z_1 \in P$, where P, Q, $z_1, \dots, z_n$ are defined as before.

(1) *z_1 is parallel in $M\backslash(P - \{z_1\})/Q$ to some element $a \in A$.*

Put $M_1 = M\backslash(P - \{z_1\})/Q$. Then $M_1\backslash z_1 = N$, and z_1 is not a coloop of M_1, by (8.5), and so by (ii) there is a circuit C_0 of M_1 with $z_1 \in C_0$ and $C_0 - \{z_1\}$ included in one of A, B. If $C_0 - \{z_1\} \subseteq B$, then there is a circuit C'

of M with $z_1 \in C'$ and $C' - \{z_1\} \subseteq Q \cup B$, contrary to (8.10). Thus $C_0 - \{z_1\} \subseteq A$, and there is a circuit C' of M with $z_1 \in C'$, $C' - \{z_1\} \subseteq Q \cup A$. By (8.10) applied to z_n and A, we deduce $n > 1$. By (8.9) with A, B exchanged, we see that there is a circuit C of M with $z_1, z_2 \in C$, $C - \{z_1, z_2\} \subseteq A$. By (8.9) again, there is a cocircuit D' of M with z_1, $z_2 \in D'$, $D' - \{z_1, z_2\} \subseteq P \cup B$. Let M_2 be $M\backslash(P - \{z_1\})/(Q - \{z_2\})$. Now $D' \cap (Q - \{z_2\}) = \varnothing$, and so $D' - (P - \{z_1\})$ is a union of cocircuits of M_2. Choose a cocircuit D of M_2 with $z_1 \in D \subseteq D' - (P - \{z_1\})$. C is a circuit of $M\backslash(P - \{z_1\})$, and D is a cocircuit of this, and so $|C \cap D| \neq 1$; thus $z_2 \in D$. Now $D \cap B \neq \varnothing$, because by (8.5), $P \cup \{z_2\}$ includes no cocircuits of M, and so $D \neq \{z_1, z_2\}$. But $M_2\backslash z_1/z_2 = N$, and so by (iv), either there is a circuit C'' of M_2 with $z_1 \in C'' \subseteq B \cup \{z_1, z_2\}$ or z_1 is parallel to an element of A in M_2/z_2. The first alternative implies that there is a circuit C_1 of M with $z_1 \in C_1 \subseteq B \cup Q \cup P$, contrary to (8.10); and the second is the desired result.

We observe that N is isomorphic to $N' = M\backslash((P - \{z_1\}) \cup \{a\})/Q$, because z_1 and a are parallel in $M\backslash(P - \{z_1\})/Q$, and an isomorphism is given by the map ϕ: $A \cup B \to ((A - \{a\}) \cup \{z_1\}) \cup B$, defined by

$$\phi(x) = x \qquad (x \neq a),$$
$$\phi(a) = z_1.$$

We observe that under this isomorphism, $\phi(A) = (A - \{a\}) \cup \{z_1\}$ and $\phi(B) = B$. Now N' is a minor of $M\backslash a$, and so by induction, $k_{M\backslash a}((A - \{a\}) \cup \{z_1\}, B) = k$; that is, there is a partition X, Y of $S - \{a\}$ with $(A - \{a\}) \cup \{z_1\} \subseteq X$, $B \subseteq Y$, and

$$r_M(X) + r_M(Y) - r(M\backslash a) = k.$$

a is not a coloop of M, because it is not a coloop of N by (i), and so $r(M\backslash a) = r(M)$. Thus

$$r_M(X) + r_M(Y) - r(M) = k.$$

However, $r_M(X \cup \{a\}) + r_M(Y) - r(M) > k$ since $k_M(A, B) > k$, and so $r_M(X \cup \{a\}) > r_M(X)$. In particular, there is no circuit C of M with $a \in C \subseteq A$. However, by (i), there is a circuit C of M with $a \in C$, $C - Q \subseteq A$, $C \cap P = \varnothing$. Thus $C \cap Q \neq \varnothing$; choose $q \in C \cap Q$. Then $q = z_i$ for some i. Now $i \neq 1$, because $z_1 \in P$ by hypothesis; and so by (8.9) there is a cocircuit D of M with $z_i \in D$, $D - \{z_i\} \subseteq P \cup B$. But then $|C \cap D| = 1$, which is impossible. This completes the proof of (9.1).

R_{12} is defined to be the linear independence matroid of the columns of the matrix of Fig. 2, which has entries over $GF(2)$. It will be seen that the matrix obtained by deleting the first six columns is symmetric, and so R_{12} is isomorphic to its dual. (However, it is not self-dual). R_{12} has just two circuits of cardinality 3, and they are disjoint. We define B to be the union of these two circuits, and A to be the set of the remaining six elements. Thus (A, B) is a partition of $E(R_{12})$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

R_{12}

FIGURE 2

(9.2) *If M is regular, and M has a minor isomorphic to R_{12}, then M has an exact 3-separation (X_1, X_2) with $|X_1|, |X_2| \geqslant 4$ (indeed, $\geqslant 6$).*

Proof. We take $\mathscr{F}$ to be the class of regular matroids, and N to be R_{12}; we take (A, B) to be the partition of $E(R_{12})$ defined as above, and $k = 2$. It suffices to verify that the hypotheses of (9.1) hold. Each of these is an assertion about the members of $\mathscr{F}$ with at most two more elements than N (that is, at most 14 elements) and so checking each is a finite problem, which in principal could be left to the reader. However, the following observations may considerably reduce the amount of work the reader needs to do.

Checking hypothesis (i) is straightforward, and (iii) follows from (ii) because R_{12} is isomorphic to its dual, and the isomorphism exchanges A and B. It remains to check (ii), (iv), and (v). First it should be verified that there are only two non-isomorphic matroids M in $\mathscr{F}$ such that for some element e, $M \backslash e \cong R_{12}$ and e is not a loop, coloop, or parallel element of M. Then simply checking these two matroids verifies (ii). To verify (iv) we observe that M/y must be one of our two matroids (in a counterexample) and so we just check the ways of reinserting y. To verify (v), we observe that $M \backslash x$ must be the dual of one of our two matroids, and proceed similarly. For full details, see the Appendix.

10. GRAFTS

Now we begin the third part of the proof—that every regular matroid which is not 2-separable is either graphic or cographic, or has a minor isomorphic to R_{10} or R_{12}. The proof of this is essentially graph theory, in the sense that although we need a mild application of (5.4), all the difficulties are graph-theoretic.

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A graph is *simple* if it has no loops or multiple edges. $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. When $u, v \in V(G)$ are distinct, u, v are said to be 3-*connected* if there are three paths linking u and v, vertex-disjoint except for u and v. G is 3-*connected* if $|V(G)| \geqslant 4$ and every pair of vertices is 3-connected; or equivalently, if $|V(G)| \geqslant 4$ and the result of deleting any two vertices is connected.

The following is due to Tutte [13].

(10.1) *If G is connected and $|E(G)| \geqslant 4$, then $\mathscr{M}(G)$ is 3-connected if and only if G is simple and 3-connected.*

If $e \in E(G)$, $G \backslash e$ and G/e are the graphs obtained from G by deleting and contracting e, respectively (in the sense of graph theory). Then for disjoint subsets F_1, F_2 of $E(G)$, we define $G \backslash F_1/F_2$ in the natural way.

Let G be a graph, and let $T \subseteq V(G)$. The pair (G, T) is called a *graft*. With every graft (G, T) we can associate a binary matroid on the set $E(G) \cup \{\Omega\}$, where Ω is a new element. To define $\mathscr{M}(G, T)$, we associate with each $e \in E(G) \cup \{\Omega\}$ a vector $v(e)$ over $GF(2)$, and then we say $X \subseteq E(G) \cup \{\Omega\}$ is a cycle of $\mathscr{M}(G, T)$ if and only if $\sum_{e \in X} v(e) = 0$. The vectors $v(e)$ are defined as follows.

Let V be a $|V(G)|$-dimensional vector space over $GF(2)$, and let $\{b_v : v \in V(G)\}$ be a basis of V. Then for $e \in E(G) \cup \{\Omega\}$,

$$v(e) = 0 \text{ if } e \in E(G) \text{ and is a loop,}$$

$$v(e) = b_u + b_v \text{ if } e \in E(G) \text{ and has distinct ends } u, v,$$

$$v(\Omega) = \sum_{v \in T} b_v .$$

It is easy to see that an equivalent definition of $\mathscr{M}(G, T)$ is as follows. A *T-join* of G is a subset X of $E(G)$ such that X includes no circuits of G and such that the vertices incident with an odd number of edges in X are precisely the vertices in T. Then $\mathscr{M}(G, T)$ is a matroid on $E(G) \cup \{\Omega\}$, where Ω is a new element, in which for $C \subseteq E(G) \cup \{\Omega\}$, C is a circuit of $\mathscr{M}(G, T)$ if and only if and only if either $\Omega \notin C$ and C is the edge-set of a circuit of G, or $\Omega \in C$ and $C - \{\Omega\}$ is a T-join.

We observe that $\mathscr{M}(G, T) \backslash \Omega = \mathscr{M}(G)$; that if $|T|$ is odd then Ω is a coloop of $\mathscr{M}(G, T)$ (because there are no T-joins); and that if $|T| = 0$ or 2 then $\mathscr{M}(G, T)$ is graphic. It follows that if $\mathscr{M}(G, T)$ is not graphic then $|T|$ is even and $|T| \geqslant 4$. (This is not a sufficient condition for $\mathscr{M}(G, T)$ to be non-graphic. For example, it may be shown that if G is outerplanar then $\mathscr{M}(G, T)$ is graphic for all $T \subseteq V(G)$.)

We observe also that if M is binary and $M \backslash e$ is graphic, and $M \backslash e \cong \mathscr{M}(G)$, say (where $V(G) \neq \varnothing$), then $M \cong \mathscr{M}(G, T)$ for some $T \subseteq V(G)$. To see this, observe that every representation of $M\backslash e$ over $GF(2)$ may be extended to a representation of M (unless e is a coloop, when the result is clear), and so in particular the representation $e \to v(e)$ ($e \in E(G)$) (defined as above) may be extended, using a vector v, say, to a representation of M. Choose $T \subseteq V(G)$ so that $v = v(\Omega)$ (again, defined as above), and then $M \cong \mathscr{M}(G, T)$.

Some important examples of grafts are given in Fig. 3. In the figure, the circles contain the vertices in T in each case. (The last graft has been labelled so that the element labelled i of $\mathscr{M}(G, T)$ corresponds to the ith column of the representation in Section 9.)

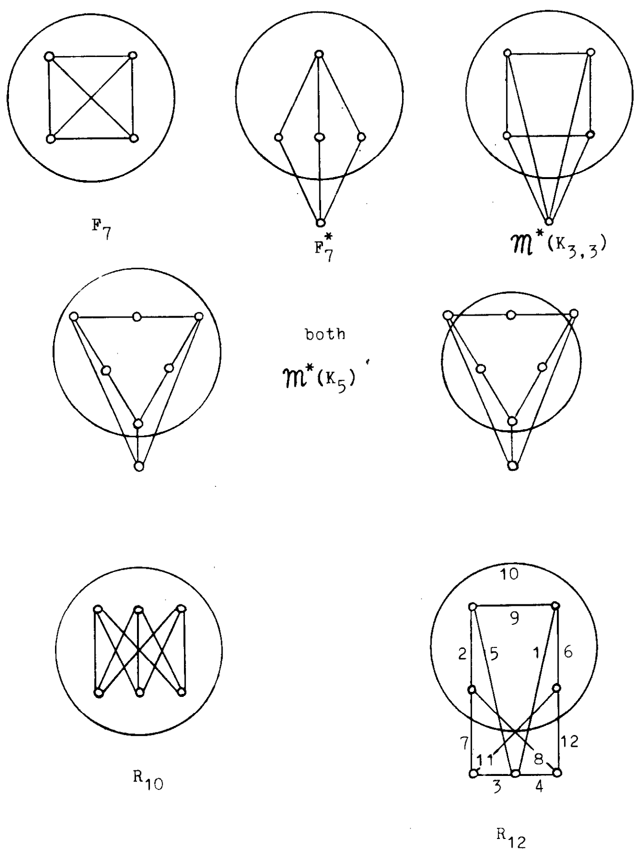

FIGURE 3

When $T \subseteq V(G)$ and e is an edge of G with ends u, v, say, let T/e be a subset of $V(G/e)$ defined as follows (here w is the vertex of G/e obtained by identifying u, v under contraction of e).

$$T/e = T \quad \text{if} \quad u, v \notin T \text{ or if } u = v,$$
$$T/e = (T - \{u\}) \cup \{w\} \quad \text{if} \quad u \in T, v \notin T,$$
$$T/e = (T - \{v\}) \cup \{w\} \quad \text{if} \quad u \notin T, v \in T,$$
$$T/e = T - \{u, v\} \quad \text{if} \quad u, v \in T \text{ and } u \neq v.$$

We observe that if $|T|$ is even then so is $|T/e|$, and indeed $|T/e| = |T|$ unless both $u, v \in T$. When $F \subseteq E(G)$ and $F = \{e_1, ..., e_k\}$, T/F is defined to be

$$(\cdots ((T/e_1)/e_2) \cdots /e_k).$$

This definition is seen to be independent of the ordering of the members of F. Then it is easy to check the following.

(10.2) *If (G, T) is a graft and F_1, $F_2 \subseteq E(G)$ are disjoint, then*

$$\mathscr{M}(G, T)\backslash F_1/F_2 = \mathscr{M}(G\backslash F_1/F_2, T/F_2).$$

11. Some Graph Theory Lemmas

When G is simple and $e \in E(G)$, $G \circ e$ denotes the graph obtained as follows: let the ends of e be u, v; delete e; if u is cubic in G, contract one edge incident with it; and similarly for v.

(11.1) *If G is simple and 3-connected and $|V(G)| \geqslant 5$, and $e \in E(G)$, then one of G/e, $G \circ e$ is 3-connected.*

Proof. The result is true when $|V(G)| = 5$, as may easily be verified by checking cases. We therefore assume that $|V(G)| \geqslant 6$, so that $|V(G \circ e)| \geqslant 4$.

Let the ends of e be u, v. Assume that $G \circ e$ is not 3-connected. Then $G \circ e$ has a two-vertex cut-set. There are therefore vertices x_1, x_2 of G, so that $\{x_1, x_2\}$ is a cut-set of $G\backslash e$ (but not of G, and so every path linking u to v in G uses one of x_1, x_2, except for the path using e), and such that the neighbour set of u in G is not $\{x_1, x_2, v\}$, and the neighbour set of v is not $\{x_1, x_2, u\}$.

Assume that G/e is not 3-connected, and so there is a vertex y and two non-empty sets X_1, X_2 such that $(X_1, X_2, \{y\}, \{u, v\})$ is a partition of $V(G)$, and such that no vertex in X_1 is adjacent to any vertex in X_2.

Choose $z_1 \in X_1$. G is 3-connected, and so there are three paths linking z_1 with u, v, y, respectively, vertex-disjoint except for z_1. Thus there is a path linking u and v within $X_1 \cup \{u, v\}$, not using e, and so one of x_1, x_2 (x_1, say) is in X_1. Similarly $x_2 \in X_2$, and so $y \neq x_1$, x_2.

Let U_1, V_1 be the sets of vertices in $X_1 \cup \{y\}$ adjacent to u, v, respectively. Let G_1 be the restriction of G to $X_1 \cup \{y\}$. Then every path of G_1 linking U_1 to V_1 pásses through x_1; and so there is a partition $(U, V, \{x_1\})$ of $X_1 \cup \{y\}$ with $U_1 \subseteq U \cup \{x_1\}$, $V_1 \subseteq V \cup \{x_1\}$, and with no vertex in U adjacent in G_1 to any in V. Now $y \in U \cup V$; we assume $y \in V$ without loss of generality. Then no vertex in U is adjacent in G to any vertex of G not in U except u, x_1. But G is 3-connected, and so $U = \emptyset$. Thus $U_1 = \emptyset$ or $\{x_1\}$; but $U_1 \neq \emptyset$ since $\{v, y\}$ is not a cut-set of G, and so $U_1 = \{x_1\}$. Thus x_1 is the only vertex in X_1 adjacent to u. Similarly, for one of u, v, x_2 is the only vertex in X_2 adjacent to it.

Now $|V(G)| \geqslant 6$, and so one of $|X_1|$, $|X_2| \neq 1$. We assume $|X_1| \geqslant 2$ without loss of generality. Choose $z_1 \in X_1 - \{x_1\}$, and choose three paths of G linking z_1 to x_1, v, y, respectively, vertex-disjoint except for z_1. We deduce that there is a path of G linking v to y avoiding x_1 and contained within $X_1 \cup \{v, y\}$. Hence u is not adjacent to y (for this would give us a u–v path avoiding x_1, x_2, and e). Now we recall that the neighbour set of u is not $\{x_1, x_2, v\}$, and so some $z_2 \in X_2 - \{x_2\}$ is adjacent to u. Thus x_2 is the only neighbour of v in X_2 (because it is clearly not the only neighbour of u), and $|X_2| \geqslant 2$. But then as before there is a path linking u to y avoiding x_2 and contained within $X_2 \cup \{u, y\}$; this, composed with the path within $X_1 \cup \{v, y\}$ linking v and y and avoiding x_1 gives us a u–v path avoiding x_1, x_2, and e, a contradiction.

G is said to be a *subdivision* of a graph H if G may be obtained from H by (repeatedly) picking an edge and replacing it by two adjacent edges in series and a vertex of valency 2. It is easy to see that if G is connected, and $\mathcal{M}(G)$ has at least four series classes, then $\mathcal{M}(G)$ is cyclically 3-connected if and only if G is a subdivision of a simple 3-connected graph.

A series class of $\mathcal{M}(G)$ which is independent in $\mathcal{M}(G)$ is called a *line* of a graph G. However, when G is a subdivision of a 3-connected graph, a line is the edge-set of a path subgraph of G, and it will be convenient to use the word "line" to refer to such a path as well. We shall use expressions such as "the line L passes through vertex v" and "vertex v is an end of L" without further explanation.

G is said to be a *minor* of a graph H if $G = H \backslash F_1 / F_2$ for disjoint subsets F_1, F_2 of $E(H)$. (We observe that this definition does not permit the removal of isolated vertices, and so if G has at least k components then so does every minor G. This is for technical convenience only.) It is easy to see that if G has a $K_{3,3}$ minor then it has a subgraph which is a subdivision of $K_{3,3}$.

(11.2) *Let G be a 3-connected graph with a $K_{3,3}$ minor, and let v_1, v_2, v_3 be distinct vertices of G, pairwise adjacent. Then G has the graph of Fig. 4 as a minor.*

The proof is in three steps, as follows.

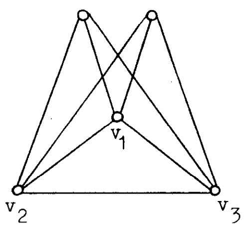

FIGURE 4

(11.3) *If* $|V(G)| = 6$ *then* (11.2) *is true.*

Proof. Check cases.

(11.4) *If G is a subdivision of* $K_{3,3}$ *and* v_1, v_2, $v_3 \in V(G)$ *are distinct, and do not all lie on the same line of G, then either* $G \cong K_{3,3}$, *or there is an edge e, not joining two of* v_1, v_2, v_3, *such that G/e is a subdivision of* $K_{3,3}$ *and* v_1, v_2, v_3 *do not all lie on the same line of G/e.*

Proof. We assume $G \not\cong K_{3,3}$, so that G has a vertex v of valency 2. Let e_1, e_2 be the edges incident with v, and let their respective other ends be u_1, u_2. Now if $v \neq v_1$, v_2, v_3 we may take $e = e_1$ and (11.4) holds. We assume therefore that $v = v_3$. Now u_1, u_2, v all lie on the same line of G and so $\{u_1, u_2\} \neq \{v_1, v_2\}$. We assume without loss of generality that $u_1 \neq v_1, v_2$. We may assume that (11.4) is not satisfied taking $e = e_1$, and so u_1 is an end of the line of G through v_1 and u_1, v_1, v_2 are all on the same line of G. Thus $u_2 \neq v_1$, v_2, and so similarly, u_2, v_1, v_2 are all on the same line of G. This is impossible.

Proof of (11.2). We use induction on $|E(G)|$. Evidently we may assume that G is simple. Let H be a subgraph of G which is a subdivision of $K_{3,3}$, with as few edges as possible. Suppose that $v_1 \notin V(H)$. Let e be an edge of G incident with v_1 but not with v_2 or v_3 (this exists since G is 3-connected). Now G/e, $G \circ e$ both have $K_{3,3}$ minors, and v_1, v_2, v_3 is a triangle of G/e; thus we may assume that G/e is not 3-connected. Hence by (11.1) $G \circ e$ is 3-connected, and we may assume that v_1, v_2, v_3 is not a triangle of $G \circ e$. Thus v_1 is cubic in G.

Now G/e is not 3-connected, and so there is a vertex y and there are non-empty sets X_1, X_2 such that $(X_1, X_2, \{u, v_1\}, \{y\})$ is a partition of $V(G)$ (where u is the other end of e), such that no vertex in X_1 is adjacent to any in X_2. Now $\{u, y\}$ is not a cut-set of G, and so v_1 is adjacent to a vertex in X_1 and a vertex in X_2. But this is impossible, since v_1 has neighbours v_2, v_3, u, and $u \notin X_1$, X_2, and v_2, v_3 are adjacent.

We deduce that $v_1 \in V(H)$, and similarly v_2, $v_3 \in V(H)$. Now v_1, v_2, v_3

are not all on the same line of H because H was chosen so that $|E(H)|$ was minimum. By (11.4) there exists $F_2' \subseteq E(H)$ so that $H/F_2' \cong K_{3,3}$ and no two of v_1, v_2, v_3 are identified under contraction of F_2'.

Choose $F_1 \subseteq E(G) - E(H)$ maximal so that the graph $G \backslash F_1$ is connected, and put $F_2 = (E(G) - F_1) - (E(H) - F_2')$. The result follows by applying (11.3) to $G \backslash F_1/F_2$. (This complicated choice of F_1, F_2 is to avoid isolated vertices in the minor.)

(11.5) *Let G be a graph, and let H be a subgraph with the same vertex-set. Suppose that H is a subdivision of $K_{3,3}$, and that every vertex v of valency 2 in H is adjacent in G to a vertex not in the line of H which passes through v. Then G is 3-connected.*

Proof. Suppose that x, y are distinct vertices of G, that X, $Y \subseteq V(G)$ are non-empty, and that $(X, Y, \{x, y\})$ is a partition of $V(G)$, and that no vertex in X is adjacent in G to any vertex in Y. Since $K_{3,3}$ is 3-connected, one of X, Y (X, say) contains no vertex with valency 3 in H, and indeed X is a subset of a line L of H, and contains neither of the ends of L. Thus x, y are also in L. Choose $v \in X$; and then v is not adjacent to any vertex not in L, contrary to hypothesis.

12. Extensions of 3-Connected Graphs

In this section we prove the following theorem.

(12.1) *Let M be a binary matroid, and let $M \backslash e \cong \mathcal{M}(G)$, where G is a 3-connected graph with a $K_{3,3}$ minor. Then either M is graphic or M has a minor isomorphic to F_7^*, R_{10}, or R_{12}.*

The proof will be by means of grafts. We have $M \backslash e \cong \mathcal{M}(G)$, and so there exists $T \subseteq V(G)$ so that $M \cong \mathcal{M}(G, T)$. If $|T| = 0$ or 2 or is odd then M is graphic. The result is therefore implied by the next theorem.

(12.2) *Let G be a 3-connected graph with a $K_{3,3}$ minor, and let $T \subseteq V(G)$, with $|T|$ even and $|T| \geqslant 4$. Then for some disjoint subsets F_1, F_2 of $E(G)$, $(G \backslash F_1/F_2, T/F_2)$ is one of the three grafts of Fig. 3 which correspond to F_7^*, R_{10}, and R_{12}.*

(We see that this is a purely graph-theoretic statement.)

Proof. If possible, let (G, T) be a counterexample, with $|E(G)|$ as small as possible. Clearly G is simple. Let H be a subgraph of G which is a subdivision of $K_{3,3}$, and choose H so that $|V(H)|$ is as small as possible. We show first that $V(H) = V(G)$. Let us suppose that this is not true.

(1) *Let e be an edge of G, with ends u, v and suppose that $v \notin V(H)$. If G/e is 3-connected then $u, v \in T$ and $|T| = 4$. If $G \circ e$ is 3-connected, then one of the following holds:*

(i) *u is cubic, $|T| = 4$, and T contains u and both its neighbours different from v,*

(ii) *v is cubic, $|T| = 4$, and T contains v and both its neighbours different from u,*

(iii) *u, v are cubic, $|T| = 6$, and T contains u, v and all their neighbours.*

For if G/e is 3-connected, then $u, v \in T$ and $|T| = 4$, because otherwise $(G/e, T/e)$ would be a smaller counterexample. Similarly if $G \circ e$ is 3-connected, one of (i), (ii), (iii) holds. (Recall that in the definition of $G \circ e$, it was not stipulated which edge incident with u should be contracted when u is cubic; thus we may choose this edge, if possible, so that it does not join two vertices in T.)

(2) *Each $v \in V(G) - V(H)$ is in T and is cubic.*

For if $v \notin T$ then by (1), $|T| = 4$, and T contains all neighbours of v and all their neighbours (except v), which is impossible from 3-connectedness. Thus $v \in T$. If v is not cubic, then by (1), $|T| = 4$ and T contains all neighbours of v, which is impossible by counting.

(3) *$|T| = 4$, provided that $V(H) \neq V(G)$.*

For suppose that $|T| \geqslant 6$. Then by (1), $|T| = 6$, and T contains $v \in V(G) - V(H)$, its neighbours, and all their neighbours, which is impossible by 3-connectedness.

Choose $v \in V(G) - V(H)$, and let u_1, u_2, u_3 be its neighbours.

(4) *T does not contain all of u_1, u_2, u_3.*

For if it does, then by (3), $T = \{v, u_1, u_2, u_3\}$. Choose $v' \notin T$. Choose three paths P_1, P_2, P_3 between v and v', vertex-disjoint except for v, v'. Then each P_i uses just one of u_1, u_2, u_3, u_i, say. Choose $F_1 \subseteq E(G)$, maximal such that F_1 contains no edge of any P_i, and such that the graph $(V(G), E(G) - F_1)$ is connected. Let $F_2 \subseteq E(G)$ contain those edges not in F_1 and not in any P_i, and also those edges in P_i not incident with u_i ($i = 1, 2, 3$). Then $(G \backslash F_1/F_2, T/F_2)$ is the graft corresponding to F_7^*, a contradiction.

We assume that $u_3 \notin T$. Let e be the edge joining v and u_3.

(5) $u_1, u_2 \in T$.

For by applying (1) to u_3 and v, we see that T contains all neighbours of v different from u_3, and so $u_1, u_2 \in T$.

Now by (1) and (4), G/e is not 3-connected. Choose a vertex y and non-empty subsets X_1, X_2 of $V(G)$ so that $(X_1, X_2, \{u_3, v, y\})$ is a partition of $V(G)$ and such that no vertex in X_1 is adjacent to any in X_2. Now $\{u_3, y\}$ is not a cut-set, and so v is adjacent to a vertex in X_1 and a vertex in X_2; $u_1 \in X_1$, $u_2 \in X_2$, say.

Now $v \notin V(H)$, and so all vertices in H within one of $X_i \cup \{u_3, y\}$ $(i = 1, 2)$ $(X_2 \cup \{u_3, y\}$, say) are on the same line of H.

G is 3-connected, and so there are two paths linking u_2 to u_3, y, respectively, contained within $X_2 \cup \{u_3, y\}$ and vertex-disjoint except for u_2. Thus there are subsets Y_1, Y_2 of $X_2 \cup \{u_3, y\}$, such that $Y_1 \cap Y_2 = \emptyset$, $u_3 \in Y_1$, $y \in Y_2$, $u_2 \notin Y_1 \cup Y_2$, and for $i = 1, 2$, the restriction of G to Y_i is connected, and u_2 is adjacent to a vertex in Y_i. Choose Y_1, Y_2 with these properties so that $Y_1 \cup Y_2$ is maximal. Certainly $Y_1 \cup Y_2 \subseteq (X_2 - \{u_2\}) \cup \{u_3, y\}$, and we claim that equality holds. For if not, then some vertex $z \in X_2 - \{u_2\}$ is not in $Y_1 \cup Y_2$ but is adjacent to a vertex in $Y_1 \cup Y_2$ (in Y_1, say); then we may replace Y_1, Y_2 by $Y_1 \cup \{z\}, Y_2$, contrary to the maximality of $Y_1 \cup Y_2$. Hence (Y_1, Y_2) is a partition of $(X_2 - \{u_2\}) \cup \{u_3, y\}$.

Let F be the set of all edges with both ends in the same Y_i. Then G/F is the graph of Fig. 5. (Here y_1 and y_2 are the vertices obtained from Y_1, Y_2, respectively, after contraction of F. In the figure y_1 and y_2 may or may not be adjacent.)

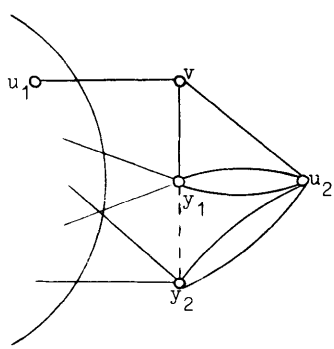

FIGURE 5

(6) *G/F is 3-connected.*

Let w be any vertex in X_1. Then there are three paths of G linking w with u_2, vertex-disjoint except for w and u_2; and so there are three paths Q_1, Q_2,

Q_3 of G/F, linking w with v, y_1, y_2, respectively, vertex-disjoint except for w. Thus w is 3-connected to u_2 in G/F. Also v, y_1, y_2 are 3-connected to u_2 because they are adjacent to it. Suppose therefore that (Z_1, Z_2, Z) is a partition of $V(G/F)$ with $|Z| \leqslant 2$ and Z_1, Z_2 non-empty, such that no vertex in Z_1 is adjacent to any in Z_2. Then we must have $u_2 \in Z$, because every other vertex is 3-connected to u_2. In particular, $Z - \{u_2\}$ is not a cut-set, and so u_2 has a neighbour in Z_1 and one in Z_2. Therefore, to show that G/F is 3-connected, it remains to show that the pairs v, y_1; v, y_2; y_1, y_2 are all 3-connected. v, y_1 are obviously 3-connected because they are adjacent. Now if $F = \varnothing$ we have $G/F = G$, and so G/F is 3-connected as required. We thus may assume that $F \neq \varnothing$, and hence one of $Y_1 - \{u_3\}$, $Y_2 - \{y\}$ is non-empty. If $Y_1 - \{u_3\} \neq \varnothing$ then because G is 3-connected, some vertex of $Y_1 - \{u_3\}$ is adjacent to a vertex not in $Y_1 - \{u_3\}$, different from u_3 and u_2 and hence in Y_2; thus y_1 and y_2 are adjacent in G/F. Similarly, y_1, y_2 are adjacent in G/F if $Y_2 - \{y\} \neq \varnothing$. Thus y_1, y_2 are 3-connected in G/F. It remains to check the pair v, y_2. Choose $w \in X_1$ and choose paths Q_1, Q_2, Q_3 as before; and then $Q_3 \cup Q_1$, and y_2, u_2, v, and y_2, y_1, v are three paths of G/F, linking y_2 to v, pairwise vertex-disjoint except for y_2 and v. Hence y_2 is 3-connected to v, and so G/F is 3-connected, as required.

(7) $F = \varnothing$ *and* $X_2 = \{u_2\}$.

For if not we have a contradiction to the minimality of G, since G/F has a $K_{3,3}$ minor and $|T/F| \geqslant 4$ (because v, u_1, $u_2 \in T/F$ and $|T/F|$ is even).

(8) $y \in T$.

For let f be the edge of G joining u_2 and y. Then $G \circ f$ is not 3-connected, because in it v is only adjacent to two vertices; and so by (11.1), G/f is 3-connected. Now G/f has a $K_{3,3}$ minor, and so by the minimality of G, $|T/f| \leqslant 2$ and so $y \in T$.

(9) $V(H) = V(G)$.

To complete the proof of (9), let G' be the graph obtained from G by deleting u_2 and v and adding three new edges joining the pairs (y, u_1), (y, u_3), (u_1, u_3), respectively. Clearly G' is 3-connected, and has a $K_{3,3}$ minor, and so by (11.2) there exist $F_1, F_2 \subseteq E(G')$, disjoint, so that $G' \backslash F_1/F_2$ is the graph of Fig. 4, where v_1, v_2, v_3 are relabelled as u_1, u_3, y. Moreover we can choose F_1, F_2 so that they contain none of the new edges; and then $(G \backslash F_1/F_2, T/F_2)$ is the graft of Fig. 3 corresponding to R_{12}, a contradiction.

We recall that H is a subgraph of G which is a subdivision of $K_{3,3}$, with $|V(H)|$ minimum. It follows that

(10) *Two vertices in the same line of H are adjacent in H if they are adjacent in G.*

For otherwise we could find a subdivision of $K_{3,3}$ with fewer vertices. We say that distinct vertices are *collinear* if they are in the same line of H.

(11) *For any edge e of H, in a line L of H, say, with ends u_1, u_2, say, one of the following is true*:

(i) *u_1, u_2 are both ends of L,*
(ii) *$u_1, u_2 \in T$ and $|T| = 4$,*
(iii) *One of u_1, u_2, say, u_1, is an end of L and there is a vertex v_e not on L which is cubic in G, adjacent to u_2 in G, and collinear with u_1.*
(We see that v_e necessarily has valency 2 in H.)

For suppose that u_2 is not an end of L. Then H/e is a subdivision of $K_{3,3}$, and so by the minimality of G, either $|T/e| = 2$ or G/e is not 3-connected. $|T/e| = 2$ gives alternative (ii); and G/e not 3-connected gives alternative (iii), by (11.5) and (10).

(12) $H \not\cong K_{3,3}$.

For if $T \subseteq V(K_{3,3})$ and $|T| = 4$ then we can make $(K_{3,3} \backslash F_1/F_2, T/F_2)$ the graft corresponding to F_7^* if we choose F_1, F_2 suitably. If $|T| = 6$ we have the graft corresponding to R_{10}; and by hypothesis $|T| \geqslant 4$ and is even.

Thus H has a vertex of valency 2. Let v be such a vertex, and let e_1, e_2 be edges of H incident with it, and let u_1, u_2 be their respective other ends. Let $e_3,\ldots, e_k$ be the other edges of G incident with v, and let $u_3,\ldots, u_k$ be their other ends. Let L be the line of H through v; then by (10), none of $u_3,\ldots, u_k$ is on L.

(13) *v has valency 3 in G (that is, $k = 3$).*

Now for each $i \geqslant 3$, $G\backslash e_i$ has a $K_{3,3}$ minor, and so by choice of G, $G\backslash e_i$ is not 3-connected. Suppose that $k \geqslant 4$. By (11.5) each u_i has valency 3 in G and 2 in H. Let x_i, y_i be the vertices adjacent to u_i in H. By applying (11) to x_i, u_i we see that $|T| = 4$ and $x_i, u_i \in T$ (case (iii) of (11) cannot occur, because v is the only vertex of G adjacent to u_i in G but not on the same line of H as u_i, and v is not cubic in G). Similarly, $y_i \in T$. Thus $\bigcup_{3 \leqslant i \leqslant k} \{x_i, u_i, y_i\} \subseteq T$, and yet $|T| = 4$. Let L' be the line of H containing x_3, u_3, y_3. Then for each $i \geqslant 3$, $|\{x_i, u_i, y_i\} \cup \{x_3, u_3, y_3\}| \leqslant 4$, and so at least two of x_i, u_i, y_i are on L'; thus all three are. Hence $u_3,\ldots, u_k$ are all on L'. Now $L' \neq L$ and so one of u_1, u_2 (u_1, say) is not on L'. We know that every vertex in T is on L', because $|T| \leqslant 4$ and $|\{x_3, u_3, y_3\} \cup$

$\{x_4, u_4, y_4\}| \geqslant 4$, and so $u_1, v \notin T$. Apply (11) to u_1, v; we deduce that u_1 is an end of L and there is a vertex w of G, cubic in G, adjacent to v and collinear with u_1. But then w must be on L', which is impossible.

(14) $|T| = 4$, *and* T *contains* v *and at least one of* u_1, u_2.

For u_3 is the only vertex of G adjacent to v and not on L, by (13). u_3 is not collinear with both u_1 and u_2; we may therefore assume that u_3 is not collinear with u_1. Thus there is no vertex except u_2 adjacent to v and collinear with u_1. We apply (11) to v, u_1, and deduce that $|T| = 4$ and v, $u_1 \in T$.

(15) $u_1, u_2 \in T$.

For suppose that $u_2 \notin T$. By (11) applied to v, u_2 we deduce that u_2 is an end of L, and u_3 is cubic in G, adjacent to v, and collinear with u_2. Let x, y be the vertices different from v, adjacent to u_3, where either $y = u_2$ or y is between $u_2$and u_3 in H. Now u_3 has valency 2 in H, and so T contains u_3 and one of x, y by (14). Suppose that $y \notin T$. Then u_3, $x \in T$ and $T = \{v, u_1, u_3, x\}$. Delete all edges except those in H and the edge e_3; contract all edges in the line through u_3 except the two edges incident with u_3; contract all edges in L except e_1 and e_2; contract all edges in the third line of H passing through u_2; and contract all except one edge in every other line of H. This produces the graft corresponding to R_{12}, a contradiction.

Thus $y \in T$, and $T = \{v, u_1, u_3, y\}$. (So $y \neq u_2$, since $u_2 \notin T$.) Delete e_1 and all edges of G not in H, except e_3; and contract all remaining edges except e_2, e_3, the three edges of H incident with y or u_3, and an edge incident with u_2 on the third line of H passing through u_2. This produces the graft corresponding to F_7^*, a contradiction.

(16) *There are at most two vertices with valency* 2 *in* H *and all such vertices lie on* L.

For by (15), T contains any such vertex and both its neighbours in H and yet $|T| = 4$.

(17) *All vertices in* T *lie on* L.

For we know that v, u_1, $u_2 \in T$ and lie on L. Suppose that $v' \in T$ and does not lie on L. By (16), v is the only vertex of H of valency 2, and so G is the graph of Fig. 6. (G has no other edges, because any other edge could be deleted without destroying 3-connectedness.) If $v' = u_3$ or x_1, delete the edges (u_3, x_2), (u_3, u_1), (u_2, x_1) and contract (u_3, x_1), (x_2, x_3); this produces the graft corresponding to F_7^*, a contradiction. Similarly $v' \neq x_2$, and so $v' = x_3$. But then we have the graft corresponding to R_{12}, a contradiction.

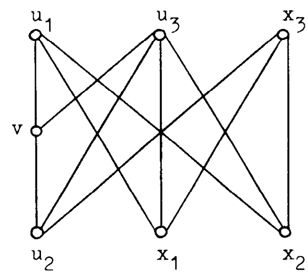

FIGURE 6

(18) *Conclusion.* Thus L contains four vertices, and H is the graph of Fig. 7, and $|T| = \{p, q, r, s\}$.

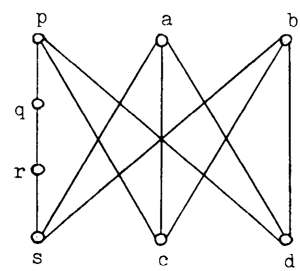

FIGURE 7

If q is adjacent to c in G, then adding the edge (q, c) to H and removing (p, c) gives an alternative choice of H which does not satisfy (17), which is impossible. Similarly q is not adjacent to d, and r is not adjacent to a or b. Thus, relabelling if necessary, we may assume that q is adjacent to a and r to c. (And G has no other edges, because they could be removed without destroying 3-connectedness.) But then deleting (r, s), (a, c), (a, d), (p, c) and contracting (b, d), (b, c), (a, s) gives the graft corresponding to F_7^*, a contradiction. This completes the proof of (12.2) and hence (12.1).

13. EXTENSIONS OF SUBDIVISIONS

We now prove the following.

(13.1) *Let G be a subdivision of a simple 3-connected graph H which has a $K_{3,3}$ minor, and let $T \subseteq V(G)$ be given. Then one of the following is true*:

(i) $\mathscr{M}(G, T)$ *is graphic,*

(ii) *there exists* $F \subseteq E(G)$ *such that* $|L - F| = 1$ *for each line* L *of* G *(so that* $G/F \cong H$*) and such that* $\mathscr{M}(G/F, T/F)$ *is non-graphic (that is,* $|T/F| \geqslant 4$ *and is even),*

(iii) *there exists* $F_1, F_2 \subseteq E(G)$*, disjoint, so that* $(G \backslash F_1/F_2, T/F_2)$ *is one of the grafts of Fig.* 8.

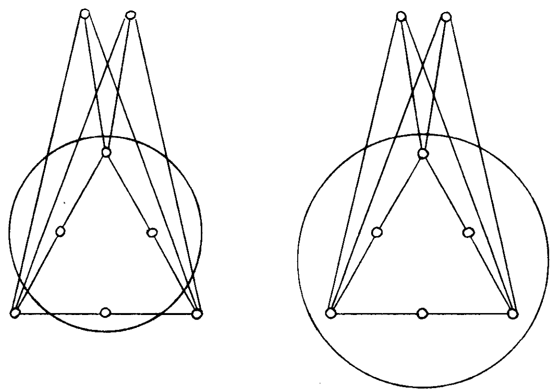

FIGURE 8

(13.1) may be deduced from the following lemma.

(13.2) *Let* G *be a subdivision of a simple* 3*-connected graph* H*, and let* $T \subseteq V(G)$ *be given. Then one of the following is true:*

(i) $\mathscr{M}(G, T)$ *is graphic,*

(ii) *there exists* $F \subseteq E(G)$ *such that* $|L - F| = 1$ *for each line* L *of* G *and such that* $|T/F| \geqslant 4$ *and is even,*

(iii) *there are three lines* L_1, L_2, L_3 *of* G *forming a circuit, with* $T \subseteq V(L_1) \cup V(L_2) \cup V(L_3)$*, and there exists* $F \subseteq E(G)$ *such that* $|L_i - F| = 2$ $(i = 1, 2, 3)$*,* $|L - F| = 1$ *for every other line* L *of* G*, and* T/F *contains all three of the vertices of* G/F *of valency* 2.

Proof of (13.1) *from* (13.2). Let G, T be as in (13.1). If (ii) of (13.2) holds then (ii) of (13.1) holds, by (10.3). Thus by (13.2), we may assume that alternative (iii) of (13.2) holds, and that $|T|$ is even. Let v_1, v_2, v_3 be the vertices in the corresponding triangle of H. By (11.2), there exists F_1', $F_2' \subseteq E(H)$, disjoint, so that $H \backslash F_1'/F_2'$ is the graph of Fig. 4. Let F_1'', F_2'' be the corresponding subsets (in the natural way) of $E(G/F)$. Put $F_1 = F_1''$,

$F_2 = F_2'' \cup F$. Then $(G\backslash F_1/F_2,\ T/F_2)$ is one of the grafts of Fig. 8, as required.

Proof of (13.2). If possible, choose a counterexample (G, T) with $|E(G)|$ minimum. Then clearly $|T| \geqslant 4$ and is even, for otherwise $\mathscr{M}(G, T)$ is graphic.

(1) *No two edges of G are in series in* $\mathscr{M}(G, T)$.

For if e_1, e_2 are in series in $\mathscr{M}(G, T)$, they are in series in G and hence are in the same line of G. We contract e_1. By the minimality of the counterexample we know that one of (i), (ii), (iii) holds for $(G/e_1, T/e_1)$. If (ii) or (iii) is true for $(G/e_1, T/e_1)$ then it is true for (G, T) which is impossible. Thus (i) holds and $\mathscr{M}(G, T)/e_1$ is graphic. But e_1 and e_2 are in series in $\mathscr{M}(G, T)$, and so $\mathscr{M}(G, T)$ is graphic, a contradiction.

(2) *Every vertex of valency* 2 *of G is in T. No two vertices of valency* 2 *are adjacent.*

For if v has valency 2 and $v \notin T$, it is easy to see that the edges incident with v are in series in $\mathscr{M}(G, T)$, contrary to (1). If v_1, v_2 both have valency 2 and are adjacent, then we know that $v_1, v_2 \in T$, and again it is easy to see that the two edges with just one end in $\{v_1, v_2\}$ are in series in $\mathscr{M}(G, T)$.

If v is a vertex of G of valency 2, let e, f be the edges incident with it, and let x, y be their other ends. Let G_v be the graph G with e, f exchanged. Thus $E(G_v) = E(G)$ and $G_v \cong G$. It is easy to check that $\mathscr{M}(G_v, T\Delta\{x, y\}) = \mathscr{M}(G, T)$.

(3) *If* $v_1,..., v_k$ *are vertices of G of valency* 2, *and* v_i *is adjacent to* x_i, y_i $(i = 1,..., k)$, *then*

$$|T\Delta\{x_1, y_1\}\Delta \cdots \Delta\{x_k, y_k\}| \geqslant 4.$$

For $\mathscr{M}(G', T') = \mathscr{M}(G, T)$, where $T' = T\Delta\{x_1, y_1\}\Delta \cdots \Delta\{x_k, y_k\}$ and G' is obtained from G by exchanging for each i the edges incident with v_i. But $\mathscr{M}(G, T)$ is not graphic, and so $|T'| \geqslant 4$.

(4) *There do not exist vertices* v_1, v_2, v_3 *of G of valency* 2 *and vertices* u_1, u_2, u_3 *of valency* $\geqslant 3$ *such that the following all hold*:

(a) u_1 *is adjacent to* v_1 *but not to* v_2, v_3,
(b) u_2 *is adjacent to* v_2 *but not to* v_3,
(c) u_3 *is adjacent to* v_3.

For suppose that they do exist. For $i = 1, 2, 3$, let e_i, f_i be the edges incident with v_i, with e_i incident with u_i. Choose $F' \subseteq E(G)$ so that F' contains none of $e_1, e_2, e_3, f_1, f_2, f_3$ and so that for every vertex $v \neq v_1, v_2, v_3$ of valency 2, F' contains just one edge incident with v. For $i = 1, 2, 3$, choose $g_i \in \{e_i, f_i\}$ as follows:

$$\begin{aligned} g_i &= f_i \qquad \text{if} \quad u_i \in T/(F' \cup \{g_j : 1 \leqslant j < i\}), \\ g_i &= e_i \qquad \text{otherwise.} \end{aligned}$$

Put $F = F' \cup \{g_1, g_2, g_3\}$; then $u_1, u_2, u_3 \in T/F$ (because of hypotheses (a), (b), (c)), and so $|T/F| \geqslant 4$ (since it is even) and (ii) is satisfied, which is impossible.

(5) *G has at least three vertices of valency* 2.

Let $v_1, \ldots, v_k$ be the vertices of G of valency 2. For each i, let e_i, f_i be the edges incident with v_i, and let x_i, y_i be their other ends.

Now $k > 0$, since (ii) is not satisfied with $F = \varnothing$.

If $k = 1$, then $x_1 \in T$, since (ii) is not satisfied with $F = \{e_1\}$; and similarly $y_1 \in T$. By (3), $|T - \{x_1, y_1\}| \geqslant 4$, and so $|T/e_1| \geqslant 4$; thus (ii) is satisfied with $F = \{e_1\}$, a contradiction.

Now suppose that $k = 2$. By relabelling if necessary we may assume that $x_1 \neq x_2, y_2$, and $x_2 \neq x_1, y_1$, although possibly $y_1 = y_2$. If there exists $w \neq x_1, x_2, y_1, y_2, v_1, v_2$ with $w \in T$, we define g_i for $i = 1, 2$ as follows:

$$\begin{aligned} &\text{if } x_i \notin T \qquad \text{put} \quad g_i = e_i, \\ &\text{if } x_i \in T \qquad \text{put} \quad g_i = f_i. \end{aligned}$$

Put $F = \{g_1, g_2\}$. Then $x_1, x_2, w \in T/F$ and so (ii) is satisfied, which is impossible. Thus $T \subseteq \{x_1, x_2, y_1, y_2, v_1, v_2\}$. Now if $x_1, x_2 \in T$ then by (3), $|T \Delta \{x_1, y_1\} \Delta \{x_2, y_2\}| \geqslant 4$, and so $y_1, y_2 \notin T$, and $y_1 \neq y_2$; but then (ii) is satisfied with $F = \{f_1, f_2\}$. We assume $x_1 \notin T$ without loss of generality. If $x_2 \notin T$, put $F = \{e_1, e_2\}$, and (ii) is satisfied. Thus $x_2 \in T$. If $y_2 \in T$, then $T \Delta \{x_2, y_2\} \subseteq \{v_1, v_2, y_1\}$ contrary to (3). Thus $y_2 \notin T$. But $|T| \geqslant 4$, and so $y_1 \neq y_2$, and $y_1 \in T$. Put $F = \{e_1, f_2\}$ and then (ii) is satisfied, a contradiction.

(6) *G has exactly three vertices of valency* 2, *and the corresponding edges of H form a triangle.*

For let L_1, L_2, L_3 be the lines of G passing through v_1, v_2, v_3, respectively. If $L_1 \cup L_2 \cup L_3$ does not include a circuit then u_1, u_2, u_3 can be chosen to contradict (4). Thus, since H is simple, $L_1 \cup L_2 \cup L_3$ is a circuit. This is true for every choice of v_1, v_2, v_3, and so $k = 3$.

(7) *Conclusion.* We may thus assume that $x_1 = y_2$, $x_2 = y_3$, $x_3 = y_1$. Now $v_1, v_2, v_3 \in T$, and so, by the falsity of (iii), there exists $w \in T$ with $w \neq x_1, x_2, x_3, v_1, v_2, v_3$. Choose $g_1 \in \{e_1, f_1\}$ so that $x_3 \in T/g_1$. Choose $g_2 \in \{e_2, f_2\}$ so that $x_1 \in T/\{g_1, g_2\}$. Put $g_3 = f_3$, and put $F = \{g_1, g_2, g_3\}$. Then $x_3, x_1, w \in T/F$, and so $|T/F| \geqslant 4$ and (ii) is satisfied, which is impossible. This completes the proof.

14. Proof of the Main Theorem

If (G, T) is one of the grafts of Fig. 8, then $\mathcal{M}(G, T)$ has an R_{12} minor, obtained by contracting the element Ω. Thus we may combine (12.1) and (13.1) to give the following.

(14.1) *Let M be a binary matroid, and let $M \backslash e \cong \mathcal{M}(G)$, where G is a subdivision of a graph H which is simple and 3-connected, and has a $K_{3,3}$ minor. Then either M is graphic or M has a minor isomorphic to one of F_7^*, R_{10}, and R_{12}.*

Proof. Choose $T \subseteq V(G)$ so that $M \cong \mathcal{M}(G, T)$. By (13.1), either M is graphic, or it has an R_{12} minor, or there exists $F \subseteq E(G)$ so that G/F is isomorphic to H and M/F is non-graphic. In the third case we apply (12.1) to M/F.

We use (14.1) for the third and final step in the proof of the following main theorem.

(14.2) *Let M be a 3-connected regular matroid. Then either M is graphic or cographic or M has a minor isomorphic to one of R_{10}, R_{12}.*

Proof. Let M be regular and 3-connected, and be neither graphic nor cographic. By Tutte's characterization of the graphic matroids [14], M has a minor isomorphic to one of $\mathcal{M}(K_{3,3})$, $\mathcal{M}(K_5)$. Suppose that M has no $\mathcal{M}(K_{3,3})$ minor. $M \not\cong \mathcal{M}(K_5)$ since M is not graphic; but $\mathcal{M}(K_5)$ is a splitter for the class of regular matroids without $\mathcal{M}(K_{3,3})$ minors, by (7.5), and so M is 1- or 2-separable, a contradiction. Thus M has a minor isomorphic to $\mathcal{M}(K_{3,3})$.

By (7.2), there exists $Z_0 \subseteq E(M)$ such that $M \times Z_0$ is a subdivision of $\mathcal{M}(K_{3,3})$. We may therefore choose $Z \subseteq E(M)$ with $|Z|$ maximum such that $M \times Z$ is graphic, is cyclically 3-connected, and has an $\mathcal{M}(K_{3,3})$ minor. By (5.4), there is an adjoinable Z-arc A. By (5.2), $M \times (Z \cup A)$ is cyclically 3-connected and has an $\mathcal{M}(K_{3,3})$ minor, and so by the maximality of Z,

$M \times (Z \cup A)$ is not graphic. Choose $e \in A$, and choose a connected graph G and $T \subseteq V(G)$ such that $M \times Z \cong \mathscr{M}(G)$ and

$$(M \times (Z \cup A))/(A - \{e\}) \cong \mathscr{M}(G, T).$$

Now $M \times (Z \cup A)$ is a subdivision of $\mathscr{M}(G, T)$ and so $\mathscr{M}(G, T)$ is not graphic. But $\mathscr{M}(G)$ is cyclically 3-connected, and so (by the remark after (11.1)) G is a subdivision of a simple 3-connected graph; and G has a $K_{3,3}$ minor. By (14.1), $\mathscr{M}(G, T)$, and hence M, has a minor isomorphic to one of F_7^*, R_{10}, or R_{12}, but F_7^* is impossible since M is regular. This completes the proof.

Now we can prove the main result.

(14.3) *Every regular matroid M may be constructed by means of* 1-, 2-, *and* 3-*sums, starting with matroids each isomorphic to a minor of M and each either graphic or cographic or isomorphic to R_{10}.*

Proof. We use induction on $|E(M)|$. Let M be a regular matroid. If M is graphic or cographic or isomorphic to R_{10} the result is true. Suppose not. Then by (14.2), if M is 3-connected then it has an R_{10} or R_{12} minor. By (7.4) if M has an R_{10} minor, it is 2-separable; and by (9.2) if M has an R_{12} minor then it has a 3-separation (X, Y) with $|X|, |Y| \geqslant 4$. By (2.10), M is expressible as a 1-, 2-, or 3-sum. By (2.1) and (2.6), if M is expressible as a 1- or 2-sum then the parts of the sum are isomorphic to minors of M. If M is not expressible as a 1- or 2-sum then it is 3-connected, and so by (4.1) the parts of the 3-sum are isomorphic to minors of M. Thus M is expressible as the 1-, 2-, or 3-sum of two matroids M_1, M_2, which are both isomorphic to minors of M. Thus M_1 and M_2 are both regular; and so, since they both have fewer elements than M, they may both be obtained in the required way, by induction. Hence so may M.

15. APPENDIX: CASE ANALYSIS

Now we give the detailed case-checking postponed from previous sections. The matroids concerned are all binary, and we represent them by matrices with entries over $GF(2)$, each element corresponding to a column, in the usual way. In order to show that a given matroid has another as a minor, we give matrix representations of both, list the columns of the first corresponding to elements which are to be deleted and contracted, and give a bijection from the remaining columns to the columns of the second matrix. The reader is presumed to be familiar with the matrix operations which correspond to matroid deletion and contraction, and to be able to test if a bijection between the column sets of two matrices over $GF(2)$ gives an isomorphism of the matroids they represent; thus the reader may verify that the bijection given is a matroid isomorphism in each case.

In Fig. 9 we list some matrix representations. The case analysis for (7.2), (7.4), (7.5), and (7.6) is given in the table of Fig. 10. To explain this table, let us take, for example, the fifth line. This deals with $\mathscr{M}^*(K_{3,3})$. If we examine the matrix for $\mathscr{M}^*(K_{3,3})$ in Fig. 9, we see that from symmetry there are only two non-isomorphic ways to add a column to this matrix which is non-zero and not the same as a column already in the matrix. These two possible columns are $(1, 0, 1, 0)^T$ and $(1, 1, 1, 0)^T$. (T denotes transpose.) We deal with these two cases simultaneously, using the symbol $*$, which indicates that the entry may be either 0 or 1. From the fifth line of Fig. 10 we learn the following. Let N_1 be a matroid isomorphic to $\mathscr{M}^*(K_{3,3})$, on a set $(1, 2,..., 9)$, where element i is represented by the ith column of the matrix in Fig. 9. Let M be a matroid on $\{1, 2,..., 9, e\}$, represented by the columns of the same matrix together with one additional column $(1, 0, 1, 0)^T$ or $(1, 1, 1, 0)^T$, so that $M\backslash e = N_1$. Let $N_2 \cong F_7$, and be on a set $\{1, 2,..., 7\}$, where again element i corresponds to the ith column of the matrix for F_7 in Fig. 9. Then

$$M\backslash Z_1/Z_2 \cong N_2,$$

where $Z_1 = \{5, 6\}$, $Z_2 = \{2\}$, and an isomorphism ϕ from $M\backslash Z_1/Z_2$ to N_2 is obtained by mapping elements 1, 3, 4, 7, 8, e, 9 of $M\backslash Z_1/Z_2$ to elements 1, 2, 3, 4, 5, 6, 7 of N_2, respectively. We deduce that if M is a binary matroid and $M\backslash e \cong \mathscr{M}^*(K_{3,3})$, then either M has an F_7 minor, or e is a loop, coloop, or parallel element of M. By taking duals, it follows that if $N \cong \mathscr{M}(K_{3,3})$ and $\mathscr{F}$ is the class of regular matroids, then the hypotheses of (7.1) are satisfied, and so (7.2) is true. The table similarly lists all the 1-element extensions necessary to verify (7.4), (7.5), and (7.6).

The table also contains information about R_{12}, which we use to verify that R_{12} satisfies the hypotheses of (9.1). However, the argument here needs more explanation. We follow the outline given after (9.2). M_1 and M_2 denote the matroids represented by the matrices obtained from the R_{12} matrix of Fig. 9 by adding the columns $(1, 1, 0, 0, 1, 1)^T$ and $(0, 0, 1, 1, 0, 0)^T$, respectively. The "R_{12}" entries in the table show the following.

(15.1) *If a non-zero new column is added to the R_{12} matrix of figure 9, and if the new matroid obtained is regular, then either the new column is the same as an old column, or it is one of $(1, 1, 0, 0, 0, 0)^T$, $(1, 1, 0, 0, 1, 1)^T$, $(0, 0, 0, 0, 1, 1)^T$, or $(0, 0, 1, 1, 0, 0)^T$. The first three of these give isomorphic matroids, the isomorphisms preserving the sets A, B and the new element.*

Now (9.1)(i) is easily verified—recall that B is the union of the two 3-element circuits of R_{12}, and so $A = \{3, 4, 7, 8, 11, 12\}$ and

$$F_7 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$F_7^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathcal{M}^*(K_{3,3}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathcal{M}(K_5) = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$\mathcal{M}^*(K_5) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$R_{10} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$R_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$M_2^* = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

FIGURE 9

N_1	e	Z_1	Z_2	N_2	ϕ
F_7	none				
F_7^*	$(1, 0, 0, 1)^T$	—	5	F_7	1, 2, 4, 3, e, 6, 7
	$(1, 1, 0, 0)^T$	—	3	F_7	1, 2, 4, 5, 6, e, 7
	$(1, 1, 1, *)^T$	—	4	F_7	1, 2, 3, 5, 6, 7, e
$\mathcal{M}^*(K_{3,3})$	$(1, *, 1, 0)^T$	5, 6	2	F_7	1, 3, 4, 7, 8, e, 9
$\mathcal{M}(K_5)$	$(1, 1, 1, *)^T$	7, 9, 10	4	F_7	1, 2, 3, 8, 6, 5, e
$\mathcal{M}^*(K_5)$	$(1, 1, 0, 0, 0, 0)^T$	—	4, 10	$\mathcal{M}^*(K_{3,3})$	1, 2, 6, 5, e, 9, 3, 8, 7
	$(1, 1, 0, 0, 1, 1)^T$	—	3, 4	$\mathcal{M}^*(K_{3,3})$	1,2,6,5,7,9,10,8,e
	$(*, *, *, 1, 1, 1)^T$	7	1, 2, 3	F_7	4, 5, 6, 10, 9, 8, e
	$(0, *, 1, 1, 0, 0)^T$	7	1, 2, 10	F_7	4, 5, 6, 3, 9, 8, e
R_{10}	$(1, 0, 1, *, *,)^T$	9, 10	4, 5	F_7	1, 2, 3, 8, e, 6, 7
	$(1, 1, 0, 0, 0)^T$	1, 2	4, 8	F_7	3, 5, 7, 10, e, 9, 6
	$(1, 1, 1, 1, 1)^T$	7, 10	1, 3	F_7	2, 4, 5, 9, 6, 8, e
R_{12}	$(1, 0, 1, *, *, 0)^T$	6, 9, 12	4, 5, 11	F_7	1, 2, 3, 10, e, 8, 7
	$(1, 0, 0, *, *, 1)^T$	8, 9, 12	4, 5, 7	F_7	2, 3, 6, 11, 10, 1, e
	$(1, *, 1, *, 1, *)^T$	8, 10, 12	2, 4, 6	F_7	1, 3, 5, 11, 9, 7, e
	$(1, *, 1, 1, 0, *)^T$	5, 9, 10	2, 6, 12	F_7	1, 3, 4, 11, 8, 7, e
	$(1, 1, 0, 0, 1, 1)^T$	—	—	M_1	1,..., 12, e
	$(0, 0, 1, 1, 0, 0)^T$	—	—	M_2	1,..., 12, e
	$(1, 1, 0, 0, 0, 0)^T$	—	—	M_1	9, 10, 11, 12, 5, 6, 7, 8, 1, 2, 3, 4, e
M_2^*	$(1, 1, *, 0, 0, 0, 1)^T$	1, 2, 6, 12	5, 9, 10	F_7^*	
	$(1, 1, 0, 0, 1, 1, 1)^T$	1, 5, 6, 7	2, 9, 10	F_7^*	8, 7, 11, x, 3, 4, y
	$(0, 0, 1, *, 0, 0, 1)^T$	2, 3, 6, 9	1, 5, 10	F_7^*	12, 11, 8, x, 3, 4, y
					7, 12, x, 8, 4, y, 11

FIGURE 10

$B = \{1, 2, 5, 6, 9, 10\}$. Both $(1, 1, 0, 0, 1, 1)^T$ and $(0, 0, 1, 1, 0, 0)^T$ are spanned by columns corresponding to elements in A and so, from the table, (9.1)(ii) is true. (9.1)(iii) is true since $R_{12} \cong R_{12}^*$, and it remains to verify (9.1)(iv) and (v).

First we verify (9.1)(v), since it is easier. By duality we must show the following:

(15.2) *If M is regular and $M \backslash y/x \cong R_{12}$, and if there is a cocircuit D of M with $\{x, y\} \subset D \subseteq A \cup \{x, y\}$, then either there is a circuit C of M with $y \in C \subseteq A \cup \{x, y\}$, or y is parallel in M/x to an element of B.*

(This is equivalent to (9.1)(v) because $R_{12} \cong R_{12}^*$, and under such an isomorphism A and B are interchanged.) Now $\{x, y\} \subset D$, and so y is not a coloop of M/x. If y is a loop in M/x then either $\{y\}$ or $\{x, y\}$ is a circuit of M and (15.2) is true. If y, z are parallel in M/x (for some $z \neq x, y$) then (15.2) is true if $z \in B$, and if $z \in A$ then either $\{y, z\}$ or $\{x, y, z\}$ is a circuit of M and again (15.2) is true. Thus we may assume that y is not a loop, coloop, or parallel element of M/x. However, $M/x\backslash y \cong R_{12}$, and so $M/x \cong M_1$ or M_2. In either case y is spanned in M/x by A, and so (15.2) is true. This verifies (9.1)(v).

Finally, we verify (9.1)(iv). We must show the following:

(15.3) *If M is regular and $M\backslash x/y \cong R_{12}$, and if there is a cocircuit D of M with $\{x, y\} \subset D \subseteq B \cup \{x, y\}$, then either there is a circuit C of M with $x \in C \subseteq B \cup \{x, y\}$, or x is parallel in M/y to an element of A.*

As before, we may assume that x is not a loop, coloop, or parallel element of M/y, and so $M/y \cong M_1$ or M_2. If $M/y \cong M_1$, then x is spanned in M/y by B, and so (15.3) is true. We therefore assume that $M/y \cong M_2$, so that $M^*\backslash y \cong M_2^*$, and is represented by the last matrix of Fig. 9, columns 1–12 representing elements 1–12, and column 13 representing x. Add a column $c = (c_1, ..., c_7)^T$ corresponding to y, to make a matrix representing M^*. Now by hypothesis there is a circuit C of M^* with $\{x, y\} \subset C \subseteq B \cup \{x, y\}$, and $B = \{1, 2, 5, 6, 9, 10\}$; thus, $c_7 = 1$, $c_1 = c_2$, and $c_5 = c_6$; and not all $c_1, ..., c_6$ are zero. Moreover, $(M^*/x)\backslash y \cong R_{12}$, and so, by (15.1), either y is a loop, coloop, or parallel element of M^*/x, or $M^*/x \cong M_1$ or M_2. Thus, either $(c_1, ..., c_7)^T$ agrees with one of the columns of our matrix representing M_2^* in Fig. 9 in the first six rows, or (by (15.1)) it is one of $(1, 1, 0, 0, 0, 0, 1)^T$, $(1, 1, 0, 0, 1, 1, 1)^T$, $(0, 0, 0, 0, 1, 1, 1)^T$, $(0, 0, 1, 1, 0, 0, 1)^T$. Using symmetry and the results $c_1 = c_2$, etc., we may assume that c is one of $(1, 1, 0, 0, 0, 0, 1)^T$, $(1, 1, 0, 0, 1, 1, 1)^T$, $(0, 0, 1, 1, 0, 0, 1)^T$, $(1, 1, 1, 0, 0, 0, 1)^T$, $(0, 0, 1, 0, 0, 0, 1)^T$. But for each of these choices of c, M^* has an F_7^* minor as shown by the last lines of the table. This completes the proof.

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