The James constant, the Jordan–von Neumann constant, weak orthogonality, and fixed points for multivalued mappings ♦

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Abstract

We give some sufficient conditions for the Domínguez–Lorenzo condition in terms of the James constant, the Jordan–von Neumann constant, and the coefficient of weak orthogonality. As a consequence, we obtain fixed point theorems for multivalued nonexpansive mappings.

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1. Introduction

In 1969, Nadler [14] extended the Banach Contraction Principle to multivalued contractive mappings in complete metric spaces. Since then some classical fixed point theorems for single valued nonexpansive mappings have been extended to multivalued nonexpansive mappings. Let $X$ be a Banach space and let $E$ be a nonempty bounded closed and convex subset of $X$. In 1974, Lim [13], using Edelstein’s method of asymptotic centers, proved the existence of a fixed point for a nonempty compact-valued nonexpansive self-mapping $T : E \rightarrow K(E)$ where $X$ is uniformly convex. Kirk and Massa [12] in 1990 extended Lim’s theorem by proving that every

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multivalued nonexpansive self-mapping $T : E \to K(E)$ has a fixed point for a space $X$ on which every asymptotic center in $E$ of each bounded sequence of $X$ is nonempty and compact. In 2001, Xu [19] extended Kirk–Massa’s theorem to a nonself-mapping $T : E \to KC(X)$ which satisfies the inwardness condition.

In 2004, Domínguez Benavides and Lorenzo [6] obtained a certain relationship between the Chebyshev radius of the asymptotic center of a bounded sequence and the modulus of noncompactness. With this result and a modification of the proof in [19], they were able to solve an open problem in [18] by proving that every nonempty compact and convex valued nonexpansive self-mapping $T : E \to KC(E)$ has a fixed point where $X$ is a nearly uniformly convex Banach space. Their method was generalized by Dhompongsa, Kaewcharoen, and Kaewkhao [4], and by Dhompongsa et al. [3]. In [4] the authors defined the Domínguez–Lorenzo condition ((DL)-condition, in short) and proved the existence of a fixed point for a multivalued nonexpansive and $(1 - \chi)$-contractive mapping $T : E \to KC(X)$ such that $T(E)$ is a bounded set and $T$ satisfies the inwardness condition, where $E$ is a nonempty bounded closed convex separable subset of a reflexive Banach space $X$ which satisfies the (DL)-condition. Very recently, the (DL)-condition has been studied by Wiśnicki and Wośko [17], Domínguez Benavides and Gavira [5], and Seajung [15]. It is worth to mention the main results of the first two of these papers. Wiśnicki and Wośko [17] introduced an ultrafilter version of the (DL)-condition. Their approach enables them to drop the separability condition in [4]. Domínguez Benavides and Gavira [5] proved that every uniformly smooth Banach space satisfies the (DL)-condition and hence has the weak multivalued fixed point property (see [5, Theorem 2]).

In this paper we give two sufficient conditions for the (DL)-condition in terms of the James constant, the Jordan–von Neumann constant, and the weak orthogonality coefficient. Consequently, we obtain two fixed point theorems for multivalued nonexpansive mappings.

2. Preliminaries

In this section we are going to recall some concepts and results which will be used in the following sections. For more details the reader may consult, for instance, [1,9].

Let $X$ be a Banach space and $E$ a nonempty subset of $X$. We shall denote by $FB(E)$ the family of nonempty bounded closed subsets of $E$, by $KC(E)$ the family of nonempty compact convex subsets of $E$. Let $H(\cdot, \cdot)$ be the Hausdorff distance on $FB(X)$, i.e.,

$$H(A, B) := \max \left\{ \sup_{x \in A} \inf_{y \in B} \|x - y\|, \sup_{y \in B} \inf_{x \in A} \|x - y\| \right\}, \quad A, B \in FB(X).$$

A multivalued mapping $T : E \to FB(X)$ is said to be a contraction if there exists a constant $k < 1$ such that

$$H(Tx, Ty) \leq k\|x - y\|, \quad x, y \in E,$$

and $T$ is said to be nonexpansive if

$$H(Tx, Ty) \leq \|x - y\|, \quad x, y \in E.$$

Let

$$\chi(A) = \inf\{d > 0: A \text{ can be covered by finitely many balls of radii } \leq d\}$$

denote the Hausdorff measure of noncompactness of a bounded set $A$. 

A multivalued mapping \( F : E \to 2^X \) is said to be \((1 - \chi)\)-contraction if, for each bounded subset \( A \) of \( E \) with \( \chi(A) > 0 \), \( F(A) \) is bounded and
\[
\chi(F(A)) \leq \chi(A).
\]
Here \( F(A) = \bigcup_{x \in A} Fx \).

The inward set of \( E \) at \( x \in E \) is defined by
\[
I_E(x) = \{ x + \lambda(y - x) : \lambda \leq 1, \ y \in E \}.
\]

Throughout the paper we let \( B_X \) and \( S_X \) denote, respectively, the closed unit ball and the unit sphere of \( X \). Let \( A \) be a nonempty bounded set in \( X \). The number \( r(A) = \inf \{ \sup_{y \in A} \| x - y \| : x \in A \} \) is called the Chebyshev radius of \( A \). The number \( \text{diam}(A) = \sup \{ \| x - y \| : x, y \in A \} \) is called the diameter of \( A \). A Banach space \( X \) has normal structure (respectively weak normal structure) if
\[
r(A) < \text{diam}(A)
\]
for every bounded closed (respectively weakly compact) convex subset \( A \) of \( X \) with \( \text{diam}(A) > 0 \).

The property WORTH was introduced by B. Sims in [16] as follows: \( X \) is said to satisfy property WORTH if for any \( x \in X \) and any weakly null sequence \( \{ x_n \} \) in \( X \),
\[
\limsup_{n \to \infty} \| x_n - x \| = \limsup_{n \to \infty} \| x_n + x \|.
\]

In [10], A. Jiménez-Melado and E. Llorens-Fuster defined the coefficient of weak orthogonality \( \mu(X) \), which is defined as the infimum of the set of the real numbers \( r > 0 \) such that
\[
\limsup_{n \to \infty} \| x + x_n \| \leq r \limsup_{n \to \infty} \| x - x_n \|
\]
for all \( x \in X \) and for all weakly null sequences \( \{ x_n \} \) in \( X \). It is known that \( X \) satisfies property WORTH if and only if \( \mu(X) = 1 \).

For a Banach space \( X \), the James constant, or the nonsquare constant was defined by Gao and Lau [7] as
\[
J(X) = \sup \{ \| x + y \| \wedge \| x - y \| : x, y \in B_X \}.
\]

The Jordan–von Neumann constant \( C_{NJ}(X) \) of \( X \), introduced by Clarkson [2], is defined by
\[
C_{NJ}(X) = \sup \left\{ \frac{\| x + y \|^2 + \| x - y \|^2}{2(\| x \|^2 + \| y \|^2)} : x, y \in X \text{ not both zero} \right\}.
\]

The following method and results deal with the concept of asymptotic centers. Let \( E \) be a nonempty bounded subset of \( X \) and \( \{ x_n \} \) be a bounded sequence in \( X \). We use \( r(E, \{ x_n \}) \) and \( A(E, \{ x_n \}) \) to denote the asymptotic radius and the asymptotic center of \( \{ x_n \} \) in \( E \), respectively, i.e.,
\[
r(E, \{ x_n \}) = \inf \left\{ \limsup_{n \to \infty} \| x_n - x \| : x \in E \right\}, \quad A(E, \{ x_n \}) = \left\{ x \in E : \limsup_{n \to \infty} \| x_n - x \| = r(E, \{ x_n \}) \right\}.
\]

It is known that \( A(E, \{ x_n \}) \) is a nonempty weakly compact convex set whenever \( E \) is [9].

Let \( \{ x_n \} \) and \( E \) be as above. Then \( \{ x_n \} \) is called regular relative to \( E \) if \( r(E, \{ x_n \}) = r(E, \{ x_n \}) \) for all subsequences \( \{ x_{n_i} \} \) of \( \{ x_n \} \) and \( \{ x_n \} \) is called asymptotically uniform relative to \( E \) if \( A(E, \{ x_n \}) = A(E, \{ x_{n_i} \}) \) for all subsequences \( \{ x_{n_i} \} \) of \( \{ x_n \} \). Furthermore, \( \{ x_n \} \) is
called regular asymptotically uniform relative to $E$ if $\{x_n\}$ is regular and asymptotically uniform relative to $E$. There always exists a subsequence of $\{x_n\}$ which is regular relative to $E$ (see [8,13]).

If $C$ is a bounded subset of $X$, the Chebyshev radius of $C$ relative to $E$ is defined by

$$r_E(C) = \inf \left\{ \sup_{y \in C} \| x - y \| : x \in E \right\}.$$ 

The Domínguez–Lorenzo condition introduced in [4] is defined as follows:

**Definition 2.1.** (See [4, Definition 3.1].) A Banach space $X$ is said to satisfy the Domínguez–Lorenzo condition ((DL)-condition, in short) if there exists $\lambda \in [0,1)$ such that for every weakly compact convex subset $E$ of $X$ and for every bounded sequence $\{x_n\}$ in $E$ which is regular relative to $E$,

$$r_E(A(E, \{x_n\})) \leq \lambda r(E, \{x_n\}).$$

**Theorem 2.2.** (See [4, Theorem 3.3].) Let $X$ be a reflexive Banach space satisfying the (DL)-condition and let $E$ be a nonempty bounded closed convex separable subset of $X$. If $T : E \to KC(X)$ is a nonexpansive and $(1 - \chi)$-contractive mapping such that $T(E)$ is a bounded set which satisfies the inwardness condition:

$$Tx \subset I_E(x) \quad \text{for all } x \in E,$$

then $T$ has a fixed point.

3. The James constant

We are going to give a sufficient condition for the (DL)-condition in terms of the James constant and the coefficient of weak orthogonality. It is an easy consequence of the following important inequality.

**Theorem 3.1.** Let $X$ be a Banach space and let $E$ be a weakly compact convex subset of $X$. Assume that $\{x_n\}$ is a bounded sequence in $E$ which is regular relative to $E$. Then

$$r_E(A(E, \{x_n\})) \leq \left( \frac{J(X)}{1 + \mu(X)} \right) r(E, \{x_n\}).$$

**Proof.** Denote $r = r(E, \{x_n\})$ and $A = A(E, \{x_n\})$. Since $\{x_n\} \subset E$ is bounded and $E$ is a weakly compact set, we can assume, by passing through a subsequence if necessary, that $x_n$ converges weakly to some element in $E$, say $x$. We note that since $\{x_n\}$ is regular, $r(E, \{x_n\}) = r(E, \{y_n\})$ for any subsequence $\{y_n\}$ of $\{x_n\}$. Let $z \in A$. Then we have

$$\limsup_n \| x_n - z \| = r. \quad (3.1)$$

Since $(x_n - x) \overset{w}{\to} 0$ and by the definition of $\mu(X)$ (for short $\mu = \mu(X)$), we have the following

$$\limsup_n \| x_n - 2x + z \| = \limsup_n \| (x_n - x) + (z - x) \|$$

$$\leq \mu \limsup_n \| (x_n - x) - (z - x) \|$$

$$= \mu r. \quad (3.2)$$
Conveycity of $E$ implies that $\frac{2}{\mu+1}x + \frac{\mu-1}{\mu+1}z \in E$ and thus we obtain

$$\lim_{n} \sup \left\| x_n - \left( \frac{2}{\mu+1}x + \frac{\mu-1}{\mu+1}z \right) \right\| \geq r. \quad (3.3)$$

On the other hand, by the weak lower semicontinuity of the $\| \cdot \|$, we have

$$\lim_{n} \inf \left\| \left( 1 - \frac{1}{\mu} \right)(x_n - x) - \left( 1 + \frac{1}{\mu} \right)(z - x) \right\| \geq \left( 1 + \frac{1}{\mu} \right)\|z - x\|. \quad (3.4)$$

Fix $\varepsilon > 0$ sufficiently small. Then, using (3.1)–(3.4), we obtain an integer $N$ such that

1. $\|x_N - z\| \leq r + \varepsilon$.
2. $\|x_N - 2x + z\| \leq \mu(r + \varepsilon)$.
3. $\|x_N - \left( \frac{2}{\mu+1}x + \frac{\mu-1}{\mu+1}z \right)\| \geq r - \varepsilon$.
4. $\| (1 - \frac{1}{\mu})(x_N - x) - (1 + \frac{1}{\mu})(z - x) \| \geq (1 + \frac{1}{\mu})\|z - x\| \left( \frac{r - \varepsilon}{r} \right)$.

Now, put $u = \frac{1}{r+\varepsilon}(x_N - z)$ and $v = \frac{1}{\mu(r+\varepsilon)}(x_N - 2x + z)$ and use the above estimates to conclude that $u, v \in B_X$, and so that

$$\|u + v\| = \left\| \frac{x_N - x}{r+\varepsilon} - \frac{z - x}{r+\varepsilon} + \frac{x_N - x}{\mu(r+\varepsilon)} + \frac{z - x}{\mu(r+\varepsilon)} \right\|$$

$$\geq \left( 1 + \frac{1}{\mu} \right) \left( \frac{r - \varepsilon}{r+\varepsilon} \right).$$

$$\|u - v\| = \left\| \frac{x_N - x}{r+\varepsilon} - \frac{z - x}{r+\varepsilon} - \frac{x_N - x}{\mu(r+\varepsilon)} - \frac{z - x}{\mu(r+\varepsilon)} \right\|$$

$$\geq \left( 1 + \frac{1}{\mu} \right) \left( \frac{\|z - x\|}{r} \right) \left( \frac{r - \varepsilon}{r+\varepsilon} \right).$$

Thus

$$J(X) \geq \|u + v\| \wedge \|u - v\| \geq \left( 1 + \frac{1}{\mu} \right) \left( \frac{r - \varepsilon}{r+\varepsilon} \right) \wedge \left( 1 + \frac{1}{\mu} \right) \left( \frac{\|z - x\|}{r} \right) \left( \frac{r - \varepsilon}{r+\varepsilon} \right).$$

By the weak lower semicontinuity of the $\| \cdot \|$ again we conclude that $\|z - x\| \leq r$ and hence

$$\left( 1 + \frac{1}{\mu} \right) \left( \frac{r - \varepsilon}{r+\varepsilon} \right) \wedge \left( 1 + \frac{1}{\mu} \right) \left( \frac{\|z - x\|}{r} \right) \left( \frac{r - \varepsilon}{r+\varepsilon} \right) = \left( 1 + \frac{1}{\mu} \right) \left( \frac{\|z - x\|}{r} \right) \left( \frac{r - \varepsilon}{r+\varepsilon} \right).$$
Therefore \( J(X) \geq (1 + \frac{1}{\mu})\left(\frac{\|z-x\|}{r}\right)(\frac{r-\varepsilon}{r+\varepsilon}) \). Since \( \varepsilon \) is arbitrary small, we obtain

\[
J(X) \geq \left(1 + \frac{1}{\mu}\right)\frac{\|z-x\|}{r}.
\]

This holds for arbitrary \( z \in A \). Hence we have

\[
\sup_{z \in A} \|x-z\| \leq \left(\frac{J(X)}{1 + \frac{1}{\mu}}\right)r,
\]

and therefore,

\[
r_E(A) \leq \left(\frac{J(X)}{1 + \frac{1}{\mu}}\right)r.
\]

From the above theorem we immediately have the following

**Corollary 3.2.** If \( X \) is a Banach space with \( J(X) < 1 + \frac{1}{\mu(X)} \), then \( X \) satisfies the \((DL)\)-condition.

By applying Theorem 2.2, we obtain

**Corollary 3.3.** Let \( X \) be a Banach space with \( J(X) < 1 + \frac{1}{\mu(X)} \) and let \( E \) be a nonempty bounded closed convex separable subset of \( X \). If \( T : E \rightarrow KC(X) \) is a nonexpansive and \((1 - \chi)\)-contractive mapping such that \( T(E) \) is a bounded set which satisfies the inwardness condition:

\[
Tx \subset I_E(x) \quad \text{for all } x \in E,
\]

then \( T \) has a fixed point.

**Proof.** Observe that \( J(X) < 2 \) since \( \mu \geq 1 \). Thus, \( X \) is reflexive, and then every bounded closed convex set is weakly compact. Now Theorem 2.2 and Corollary 3.2 can be applied to obtain a fixed point. \( \square \)

**Remark 3.4.** Corollaries 3.2 and 3.3 cover Corollaries 3.5 and 3.6 of Dhompongsa, Kaewcharoen, and Kaewkhao [4], respectively. To see this, we point that the condition of being uniformly nonsquare and having property \( WORTH \) of \( X \) implies the condition \( J(X) < 1 + \frac{1}{\mu(X)} \).

**Remark 3.5.** In [11, Theorem 2], Jiménez-Melado, Llorens-Fuster, and Saejung proved that if \( X \) is a Banach space with \( J(X) < 1 + \frac{1}{\mu(X)} \), then \( X \) has normal structure, and it is proved in [4, Theorem 3.2] that the \((DL)\)-condition implies the weak normal structure. Thus our Corollary 3.2 is stronger than Theorem 2 of [11].

4. The Jordan–von Neumann constant

In this section, we are going to give a sufficient condition for the \((DL)\)-condition in terms of the Jordan–von Neumann constant and the coefficient of weak orthogonality. Again, as in Section 3, we need a corresponding inequality.
Theorem 4.1. Let $X$ be a Banach space and let $E$ be a weakly compact convex subset of $X$. Assume that $\{x_n\}$ is a bounded sequence in $E$ which is regular relative to $E$. Then

$$r_E\left(A\left(E, \{x_n\}\right)\right) \leq \left(\sqrt{\frac{2\mu(X)^2C_{NJ}(X)}{\mu(X)^2 + 1}} - 1\right)r\left(E, \{x_n\}\right).$$

Proof. Let $r$, $A$, $\{x_n\}$, $x$, $z$ and $\mu$ be as in the proof of the previous theorem. Thus,

$$\limsup_n \|x_n - z\| = r \quad (4.1)$$

and

$$\limsup_n \|x_n - 2x + z\| \leq \mu r. \quad (4.2)$$

Since $\frac{2}{\mu^2 + 1}x + \frac{\mu^2 - 1}{\mu^2 + 1}z \in E$ and by the definition of $r$, we obtain

$$\limsup_n \left\| x_n - \left(\frac{2}{\mu^2 + 1}x + \frac{\mu^2 - 1}{\mu^2 + 1}z\right) \right\| \geq r. \quad (4.3)$$

The semicontinuity of the $\| \cdot \|$ yields the following:

$$\liminf_n \left\| \left(\mu^2 - 1\right)(x_n - x) - \left(\mu^2 + 1\right)(z - x) \right\| \geq \left(\mu^2 + 1\right)\|z - x\|. \quad (4.4)$$

Now, fix $\varepsilon > 0$ sufficiently small. Then, using (4.1)–(4.4), we obtain an integer $N$ such that

1. $\|x_N - z\| \leq r + \varepsilon.$
2. $\|x_N - 2x + z\| \leq \mu(r + \varepsilon).$
3. $\|x_N - \left(\frac{2}{\mu^2 + 1}x + \frac{\mu^2 - 1}{\mu^2 + 1}z\right)\| \geq r - \varepsilon.$
4. $\left(\mu^2 - 1\right)(x_N - x) - \left(\mu^2 + 1\right)(z - x)\| \geq \left(\mu^2 + 1\right)\|z - x\|\left(\frac{r - \varepsilon}{r}\right).$

Next, put $u = \mu^2(x_N - z)$ and $v = (x_N - 2x + z)$ and use the previous estimates to obtain $\|u\| \leq \mu^2(r + \varepsilon)$, $\|v\| \leq \mu(r + \varepsilon)$, and so that

$$\|u + v\| = \|\mu^2((x_N - x) - (z - x)) + (x_N - x) + (z - x)\|$$

$$= \left(\mu^2 + 1\right)\left\| (x_N - x) - \frac{\mu^2 - 1}{\mu^2 + 1}(z - x) \right\|$$

$$= \left(\mu^2 + 1\right)\left\| x_N - \left(\frac{2}{\mu^2 + 1}x + \frac{\mu^2 - 1}{\mu^2 + 1}z\right) \right\|$$

$$\geq \left(\mu^2 + 1\right)(r - \varepsilon),$$

$$\|u - v\| = \|\mu^2((x_N - x) - (z - x)) - ((x_N - x) + (z - x))\|$$

$$= \left(\mu^2 - 1\right)(x_N - x) - \left(\mu^2 + 1\right)(z - x)\|$$

$$\geq \left(\mu^2 + 1\right)\|z - x\|\left(\frac{r - \varepsilon}{r}\right).$$

By the definition of $C_{NJ}(X)$ we see that
\[ C_{NJ}(X) \geq \frac{\|u + v\|^2 + \|u - v\|^2}{2(\|u\|^2 + \|v\|^2)} \]
\[ \geq \left( \frac{\mu^2 + 1}{2\mu^2} \right) \left( 1 + \left( \frac{\|z - x\|}{r} \right)^2 \right) \left( \frac{r - \varepsilon}{r + \varepsilon} \right)^2. \]

Letting \( \varepsilon \to 0^+ \) we obtain that \( C_{NJ}(X) \geq \left( \frac{\mu^2 + 1}{2\mu^2} \right) (1 + \left( \frac{\|z - x\|}{r} \right)^2) \). Then we have
\[ \|z - x\| \leq \left( \sqrt{\frac{2\mu^2 C_{NJ}(X)}{\mu^2 + 1} - 1} \right) r. \]
This holds for arbitrary \( z \in A \), hence we have
\[ r_x(A) \leq \left( \sqrt{\frac{2\mu^2 C_{NJ}(X)}{\mu^2 + 1} - 1} \right) r \]
and therefore,
\[ r_E(A) \leq \left( \sqrt{\frac{2\mu^2 C_{NJ}(X)}{\mu^2 + 1} - 1} \right) r. \]

As a consequence of Theorem 4.1 we obtain the following corollary.

**Corollary 4.2.** Let \( X \) be a Banach space. If \( C_{NJ}(X) < 1 + \frac{1}{\mu(X)^2} \), then \( X \) satisfies the (DL)-condition.

Apply Theorem 2.2 and Corollary 4.2 to obtain the following corollary.

**Corollary 4.3.** Let \( X \) be a Banach space with \( C_{NJ}(X) < 1 + \frac{1}{\mu(X)^2} \) and let \( E \) be a nonempty bounded closed convex separable subset of \( X \). If \( T : E \to KC(X) \) is a nonexpansive and \((1 - \chi)\)-contractive mapping such that \( T(E) \) is a bounded set which satisfies the inwardness condition:
\[ Tx \subset I_E(x) \quad \text{for all } x \in E, \]
then \( T \) has a fixed point.

**Remark 4.4.** It is shown in [15, Theorem 5] that if \( C_{NJ}(X) < \frac{4}{1 + \mu(X)^2} \), then \( X \) satisfies the (DL)-condition. Clearly, \( \frac{4}{1 + \mu(X)^2} \leq 1 + \frac{1}{\mu(X)^2} \). Thus our Corollary 4.2 is better than Theorem 5 of [15].

**Remark 4.5.** Dhompongsa et al. proved in [3] that a Banach space \( X \) satisfies property (D), which is implied by the (DL)-condition, whenever \( C_{NJ}(X) \leq c_0 = 1.273 \ldots \). If we compare this result with Corollary 4.2, we observe that for those spaces \( X \) with \( \mu(X) \) close to 1, the result in [3] does not apply but our Corollary 4.2 still gives information on the (DL)-condition of \( X \).

**Remark 4.6.** As in Remark 3.5, Corollary 4.2 covers Theorem 1 of [11].

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