Oscillation of Even Order Linear Functional Differential Equations with Deviating Arguments of Mixed Type

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1. Introduction

This paper is concerned with the oscillatory behavior of linear functional differential equations of the form

\[ x^{(n)}(t) = p(t) x(g(t)), \] (1)

where \( n \) is even, \( p: [0, \infty) \rightarrow \mathbb{R} \) and \( g: [0, \infty) \rightarrow \mathbb{R} \) are continuous, \( p(t) > 0 \), \( g(t) \) is nondecreasing and \( \lim_{t \to \infty} g(t) = \infty \).

By a proper solution of Eq. (1) we mean a function \( x: [T_x, \infty) \rightarrow \mathbb{R} \) which satisfies (1) for all sufficiently large \( t \) and \( \sup \{ |x(t)| : t > T_x \} > 0 \) for any \( T > T_x \). We make the standing hypothesis that (1) does possess proper solutions. A proper solution is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory.

If \( x(t) \) is a nonoscillatory solution of (1), then there exist an even integer \( l \in \{0, 2, \ldots, n\} \) and a \( T_x \geq T_x \) such that

\[ (-1)^{l-1} x(t) x^{(l)}(t) > 0 \quad \text{on} \quad [T_x, \infty) \quad \text{for} \quad l \leq i \leq n. \] (2)

Such an \( x(t) \) is said to be a (nonoscillatory) solution of degree \( l \), and the totality of solutions of degree \( l \) of (1) is denoted by \( \mathcal{M}_l \). If we denote the set of all nonoscillatory solutions of (1) by \( \mathcal{M} \), then we have

\[ \mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_2 \cup \cdots \cup \mathcal{M}_n. \] (3)

It is known that in the case where \( g(t) = t \) Eq. (1) always has solutions of degrees 0 and \( n \), that is, \( \mathcal{M}_0 \neq \emptyset \) and \( \mathcal{M}_n \neq \emptyset \) for an ordinary differential equation \( x^{(n)} = p(t) x \) with \( p(t) > 0 \). The situation for the case where \( g(t) \neq t \) is different. In fact, it may happen that \( \mathcal{M}_0 = \emptyset \) or \( \mathcal{M}_n = \emptyset \) for (1) when the
deviating argument \( g(t) \) is retarded \( (g(t) < t) \) or advanced \( (g(t) > t) \) and the deviation \( |t - g(t)| \) is large enough; see, for example, Ladas [5] and Koplatadze and Šanturija [3].

We are interested in the situation in which \( \mathcal{I} = \emptyset \) for (1), that is, all proper solutions of (1) are oscillatory. Very recently the present author [4] has shown that such a situation occurs for functional differential equations involving both advanced and retarded arguments of the form

\[
x^{(n)}(t) = p(t) x(g(t)) + q(t) x(h(t)).
\] (4)

The deviating argument \( g(t) \) is said to be of mixed type if its advanced part

\[
\mathcal{A}_g = \{ t \in [0, \infty): g(t) > t \}
\] (5)

and its retarded part

\[
\mathcal{R}_g = \{ t \in [0, \infty): g(t) < t \}
\] (6)

are both unbounded subsets of \([0, \infty)\). It is natural to expect that the presence of a single deviating argument of mixed type will be sufficient to force all proper solutions of (1) to oscillate. The purpose of this paper is to show that this conjecture is correct. As a result we are able to prove that all proper solutions of the equation \( x^{(n)}(t) = px(t + \sin t) \) are oscillatory provided the constant \( p > 0 \) is sufficiently large.

We note that first-order equations with deviating arguments of mixed type have been extensively studied by Ivanov and Ševelo [1], Kitamura and Kusano [2] and Ševelo and Ivanov [6].

2. MAIN RESULT

The main result of this paper is the following.

**THEOREM 1.** Let \( g(t) \) be of mixed type and suppose that there is a constant \( \varepsilon > 0 \) such that

\[
\int_{-\infty}^{\infty} |g^{\varepsilon}(t)|^{n-1-\varepsilon} p(t) \, dt = \infty,
\] (7)

where \( g^{\varepsilon}(t) = \min\{g(t), t\} \). Suppose moreover that there exist two sequences \( \{t_k\}, \{\tau_k\} \) such that \( t_k \in \mathcal{A}_g, t_k \to \infty \) as \( k \to \infty \), \( \tau_k \in \mathcal{R}_g, \tau_k \to \infty \) as \( k \to \infty \),

\[
\int_{t_k}^{\tau_k} |g(s) g(t_k)|^{n-1} p(s) \, ds \geq (n-1)!
\] (8)
and
\[
\int_{x(t_k)}^{x(t_k)} [g(t_k) - g(s)]^{n-1} p(s) \, ds \geq (n-1)!
\] (9)
for all \(k = 1, 2, \ldots\). Then all proper solutions of (1) are oscillatory.

Proof. We first show that condition (7) guarantees that \(I_i = \emptyset\) for \(i \in \{2, 4, \ldots, n - 2\}\). Suppose to the contrary that there exists a positive solution \(x \in I_i\) for some \(i \in \{2, 4, \ldots, n - 2\}\). Note that (2) holds for \(t \geq t_0\).
Choose a \(t_1 > t_0\) so that \(g_*(t) \geq t_0\) for \(t \geq t_1\). We claim that
\[
x'(t) \geq \frac{(t - t_1)^{l-1} - 1}{(l-1)!} \int_t^{\infty} \frac{(s - t)^{n-1}}{(n-l-1)!} p(s) x(g(s)) \, ds, \quad t \geq t_1.
\] (10)
Observe that
\[
x^{(l)}(t) = \sum_{i=1}^{n-l} \frac{(t - T)^{l-1}}{(l-1)!} x^{(i)}(T) + (-1)^{n-l-1} \int_t^{T} \frac{(s - t)^{n-1}}{(n-l-1)!} x^{(n)}(s) \, ds
\]
for any \(t, T \geq t_1\). Using (2) and the fact that \(n-l\) is even, we have from the above
\[
x^{(l)}(t) \geq \int_t^{T} \frac{(s - t)^{n-1}}{(n-l-1)!} x^{(n)}(s) \, ds \quad \text{for } T \geq t \geq t_1,
\]
which in view of (1) gives
\[
x^{(l)}(t) \geq \int_t^{\infty} \frac{(s - t)^{n-1}}{(n-l-1)!} p(s) x(g(s)) \, ds, \quad t \geq t_1.
\] (11)
On the other hand, from the equation
\[
x'(t) = \sum_{i=1}^{l-1} \frac{(t - t_1)^{l-1}}{(l-1)!} x^{(i)}(t_1) + \int_{t_1}^{t} \frac{(t - s)^{l-2}}{(l-2)!} x^{(l)}(s) \, ds
\]
it follows that
\[
x'(t) \geq \int_{t_1}^{t} \frac{(t - s)^{l-2}}{(l-2)!} x^{(l)}(s) \, ds \geq \frac{(t - t_1)^{l-1}}{(l-1)!} x^{(l)}(t), \quad t \geq t_1.
\] (12)
where we have used (2) and the decreasing nature of \(x^{(l)}(t)\). Combining (11) with (12) yields (10) as claimed.
Let us now take a \(t_2 > t_1\) so that \(g_*(t) \geq t_1\) for \(t \geq t_2\). Since \(x^{(n-1)}(t) < 0\)
for \( t \geq t_1 \), there is a constant \( a \geq 1 \) such that \( x(t) \leq a t^{n-2} \) for \( t \geq t_1 \), and hence if \( s \geq t_2 \), then
\[
x(t) \leq a \left| g_*(s) \right|^{n-2} \quad \text{for} \quad t_1 \leq t \leq g_*(s). \tag{13}
\]
Dividing both sides of (10) by \( \left| x(t) \right|^{1+\delta} \), \( \delta = \varepsilon/(n-2) \), and integrating it over \([t_1, t_3]\), \( t_3 > t_2 \), we obtain
\[
\int_{t_1}^{t_3} \frac{x'(t)}{\left| x(t) \right|^{1+\delta}} dt \geq \int_{t_1}^{t_3} \frac{(t-t_1)^{-1}}{(l-1)! \left| x(t) \right|^{1+\delta}} \frac{(s-t)^{n-l-1}}{(n-l-1)!} p(s) x(g(s)) ds dt
\]
\[
= \int_{t_1}^{t_3} \frac{(s-t)^{n-l-1}(t-t_1)^{-1}}{(n-l-1)! (l-1)!} \frac{p(s) x(g(s))}{\left| x(t) \right|^{1+\delta}} dt ds \tag{14}
\]
\[
\geq \int_{t_2}^{t_3} \frac{\left| g_*(s) - t \right|^{n-l-1}(t-t_1)^{-1}}{(n-l-1)! (l-1)!} \frac{p(s) x(g(s))}{\left| x(t) \right|^{1+\delta}} dt ds.
\]
Noting that \( x(t) \) is increasing and taking (13) into account, we see that if \( t_2 \leq s \leq t_3 \), then
\[
\frac{p(s) x(g(s))}{\left| x(t) \right|^{1+\delta}} \geq \frac{x(g_*(s))}{\left| x(t) \right|^{1+\delta}} \frac{p(s)}{a^\delta \left| g_*(s) \right|^{(n-2)\delta}} \geq \frac{p(s)}{a^\delta (n-1)! \left| \delta x(t) \right|^{\delta}}.
\]
for \( t_1 \leq t \leq g_*(s) \). From (14) and (15) we see that
\[
\int_{t_2}^{t_3} \frac{\left| g_*(s) - t_1 \right|^{n-l-1}}{\left| g_*(s) \right|^{(n-2)\delta}} p(s) ds \leq a^\delta (n-1)! \int_{t_1}^{t_3} \frac{x'(t)}{\left| x(t) \right|^{1+\delta}} dt < \frac{a^\delta (n-1)!}{\delta \left| x(t) \right|^{\delta}}.
\]
Letting \( t_3 \to \infty \) in the above, we conclude that
\[
\int_{t_2}^{\infty} \left| g_*(s) \right|^{n-l-\delta} p(s) ds < \infty,
\]
which contradicts (7). A parallel argument holds if we assume that (1) has a negative solution of degree \( l \in \{2, 4, \ldots, n-2\} \).

Next we show that condition (8) does not allow Eq. (1) to have nonoscillatory solutions of degree \( n \). Suppose there is a solution \( x \in \mathcal{I}_n \). Without loss of generality we may assume that \( x(t) \) is positive. From (2) we then have
\[
x^{(i)}(t) > 0 \quad \text{on} \quad [t_0, \infty) \quad \text{for} \quad 0 \leq i \leq n. \tag{16}
\]
Let \( t_1 > t_0 \) be as before. From the equation
\[
x(u) = \sum_{i=0}^{n-1} \frac{(u-t)^i}{i!} x^{(i)}(t) + \frac{1}{(n-1)!} \int_t^u (u-t)^{n-1} x^{(n)}(t) dt \tag{17}
\]
and (16) it follows that
\[ x(u) \geq \frac{(u-v)^{n-1}}{(n-1)!} x^{(n-1)}(v) \quad \text{for } u \geq v \geq t_i. \]

Let \( t_k \in \mathcal{R}_x \) and put \( u = g(s) \) and \( v = g(t_k) \) in the above, where \( t_k \leq s \leq g(t_k) \).

Then
\[ x(g(s)) \geq \frac{[g(s) - g(t_k)]^{n-1}}{(n-1)!} x^{(n-1)}(g(t_k)), \quad t_k \leq s \leq g(t_k). \] (18)

Using (18) in (1) and integrating over \([t_k, g(t_k)]\), we have
\[ x^{(n-1)}(g(t_k)) - x^{(n-1)}(t_k) \geq \int_{t_k}^{g(t_k)} \frac{[g(s) - g(t_k)]^{n-1}}{(n-1)!} p(s) \, ds \cdot x^{(n-1)}(g(t_k)) \]
or
\[ x^{(n-1)}(g(t_k)) \left[ 1 - \int_{t_k}^{g(t_k)} \frac{[g(s) - g(t_k)]^{n-1}}{(n-1)!} p(s) \, ds \right] \geq x^{(n-1)}(t_k). \]

But this combined with (8) leads to the conclusion that \( x^{(n-1)}(t) \) must take on nonpositive values for arbitrarily large \( t \), a contradiction to (16).

Finally, we show that condition (9) precludes nonoscillatory solutions of degree 0. Suppose (1) has a positive solution \( x \in I_0 \). Then from (2)
\[ (-1)^i x^{(i)}(t) > 0 \quad \text{on } [t_0, \infty) \quad \text{for } 0 \leq i \leq n. \] (19)

Let \( t_i \) be as before. From (17) and (19) we easily obtain the inequality
\[ x(u) \geq \frac{(u-v)^{n-1}}{(n-1)!} x^{(n-1)}(v) \quad \text{for } v \geq u \geq t_i. \]

Let \( \tau_k \in \mathcal{R}_x \) and put \( u = g(s) \) and \( v = g(\tau_k) \) in the above, where \( g(\tau_k) \leq s \leq \tau_k \). We then have
\[ x(g(s)) \geq \frac{[g(\tau_k) - g(s)]^{n-1}}{(n-1)!} (-1)^{n-1} x^{(n-1)}(g(\tau_k)), \quad g(\tau_k) \leq s \leq \tau_k. \] (20)

From (20) and (1) we obtain
\[ x^{(n-1)}(\tau_k) - x^{(n-1)}(g(\tau_k)) \geq \int_{g(t_k)}^{\tau_k} \frac{[g(\tau_k) - g(s)]^{n-1}}{(n-1)!} p(s) \, ds \]
\[ \cdot (-1)^{n-1} x^{(n-1)}(g(\tau_k)) \]
or

\[ x^{(n-1)}(t_k) \geq (-1)^n - x^{(n-1)}(g(t_k)) \left[ \int_{t_k}^{x_k} \frac{|g(s) - g(t)|^{n-1}}{(n-1)!} p(s) \, ds - 1 \right]. \]

In view of (9) this contradicts the fact that \( x^{(n-1)}(t) \) is negative (see (19)). Similarly, (1) admits no negative solution of degree 0. This completes the proof of Theorem 1.

We have so far been concerned with the even order case of Eq. (1). We conclude with some comments on odd order equations of the form (1). Let \( n \geq 3 \) be odd. If \( x(t) \) is a nonoscillatory solution, then there is an odd integer \( l \in \{1, 3, ..., n\} \) such that the inequalities in (2) hold. Therefore the classification relation corresponding to (3) now becomes

\[ \mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_n, \]

so that the class \( \mathcal{C}_0 \) is absent. This makes the treatment even easier, and arguing exactly as in the first and the second part of the proof of Theorem 1, we are able to prove the following oscillation theorem for odd order linear equations with deviating arguments of mixed type.

**Theorem 2.** Let \( n \) be odd. Suppose that (7) holds for some \( \varepsilon > 0 \) and that there is a sequence \( \{t_k\} \) such that \( t_k \in \mathcal{C} \), \( t_k \to \infty \) as \( k \to \infty \) and (8) hold. Then all proper solutions of (1) are oscillatory.

**Example.** Consider the equation

\[ x^{(n)}(t) = px(t + \sin t), \quad (21) \]

where \( p > 0 \) is a constant. Clearly, the deviating argument \( g(t) = t + \sin t \) is of mixed type. Put

\[ \phi_{n-1}(t) = \int_{t}^{t + \sin t} \left[ (s + \sin s) - (t + \sin t) \right]^{n-1} ds. \]

It is easy to see that if \( t_k = (\pi/2) + 2k\pi, \ k = 1, 2, ..., \) then \( t_k \in \mathcal{C} \) and \( \phi_1(t_k) = \sin 1 - (1/2) > 0 \), and that by Jensen's inequality

\[ \phi_{n-1}(t_k) \geq |\phi_1(t_k)|^{n-1} = (\sin 1 - 1/2)^{n-1}. \]

Note that if \( n \) is even, then

\[ \phi_{n-1}(t) = \int_{t}^{t + \sin t} \left[ (t + \sin t) - (s + \sin s) \right]^{n-1} ds, \]
and that if \( \tau_k = (-\pi/2) + 2k\pi, k = 1, 2, \ldots \), then \( \tau_k \in \mathcal{R}_k \) and
\[
\phi_{n-1}(\tau_k) \geq |\phi_1(\tau_k)|^{n-1} = (\sin 1 - \frac{1}{2})^{n-1}.
\]

From Theorems 1 and 2 it follows that all proper solutions of (20) are oscillatory regardless of the parity of \( n \) provided \( p \) is so large that \( (\sin 1 - (1/2))^{n-1} p \geq (n - 1)! \).

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