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On the Diameter of a Graph Related to p-Regular Conjugacy Classes of Finite Groups

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1. INTRODUCTION

Let G be a finite group and p a fixed prime. Denote by $G_{p'}$ the set of p-regular elements of G and $\mathrm{Con}(G_{p'})$ the set of all conjugacy classes of G in $G_{p'}$. Define: $\rho'_p(G) = \{q : q \text{ prime}, \ q \mid |g^G|, \ g \in G_{p'}\}$. We obtain the following graph $\Gamma'_p = \Gamma'_p(G)$: the vertices of Γ'_p are primes in $\rho'_p(G)$, two vertices r, q are connected, if there exists a conjugacy class $C \in \mathrm{Con}(G_{p'})$ such that $rq \mid |C|$. If $p \nmid |G|$, then the graph $\Gamma'_p(G)$ is just the graph $\Gamma'(G)$ defined in [6].

Our aim is to investigate the arithmetical properties of p-regular conjugacy classes of G and the relationship between irreducible p-modular characters and p-regular conjugacy classes of G. In the last 20 years, the analogy between conjugacy classes and irreducible characters has been widely studied. We refer readers to [1–8].

Let $n(\Gamma'_p) = n(\Gamma'_p(G))$ denote the number of the connected components of the graph $\Gamma'_p(G)$ and $d(\Gamma'_p(G))$ denote the diameter of the graph $\Gamma'_p(G)$. Our main results are the following 5 theorems.

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THEOREM 1. *If G is a finite p-solvable group, then $n(\Gamma'_p(G)) \leq 2$.*

One natural question is whether we can decide the diameter of the graph $\Gamma'_p(G)$. If G is p-solvable, then we can prove the following theorem, but our methods cannot apply to general finite groups.

THEOREM 2. *Suppose that G is a finite p-solvable group.*

If $n(\Gamma'_p(G)) = 1$, then $d(\Gamma'_p(G)) \leq 6$.
If $n(\Gamma'_p(G)) = 2$, then each connected component has diameter at most 3.

The modular character degree graph has similar properties [10, Chap. VI, Theorem 20.3, p. 260]. A modular character degree graph Γ_p is defined as follows. Its vertices are primes in $\rho_p(G)$ (the set of all primes which divide some irreducible modular character of G), and two vertexes r, q are connected, if rq divides some irreducible modular character degree. One has the following results:

If $n(\Gamma_p(G)) = 2$, then $d(\Gamma_p(G)) \leq 5$.
If $n(\Gamma_p(G)) = 1$, then $d(\Gamma_p(G)) \leq 3$.

Next we discuss the properties of G with $n(\Gamma'_p(G)) = 2$.

THEOREM 3. *Suppose that G is a solvable group with $n(\Gamma'_p(G)) = 2$. Let X_1, X_2 be the connected components and write $\pi_i = \{q : q$ prime, $q \in X_i \setminus \{p\}\}$, $i = 1, 2$. Then G has a metabelian Hall π_1- or π_2-subgroup.*

The following theorem shows that the π_i-length of G can be bounded in terms of the p-length of G.

THEOREM 4. *Suppose that G is a finite solvable group with $n(\Gamma'_p(G)) = 2$. Let X_1, X_2 be the connected components and write $\pi_i = \{q : q$ prime, $q \in X_i \setminus \{p\}\}$, $i = 1, 2$. Then $l_{\pi_i}(G) \leq l_p(G) + 1$, $i = 1, 2$.*

The following result shows that in some sense the graph $\Gamma'_p(G)$ is rich in edges.

THEOREM 5. *Let G be a solvable group and choose $r, s \in \rho'_p(G) \setminus \{p\}$. If there exists no conjugacy class $C \in \mathrm{Con}(G_{p'})$ such that $rs \mid |C|$, then $l_r(G) \leq 1$, or $l_s(G) \leq 1$.*

2. PROOFS OF THEOREMS

The following lemma plays an important role in the proof of the theorems.

LEMMA 1. *Suppose that G is a p-separable group and choose $B = b^G$, $C = c^G \in \mathrm{Con}(G_{p'})$. If $(|B|, |C|) = 1$, then*:

(1) $C_G(b)C_G(c) = G$.

(2) *BC is a conjugacy class in* $\mathrm{Con}(G_{p'})$.

(3) $|BC| \mid |B|\,|C|$.

(4) *Suppose that there is no conjugacy class $D \in \mathrm{Con}(G_{p'})$ such that $(|D|, |B|) \geq 1$ and $(|D|, |C|) \geq 1$. If $|B| < |C|$, then $|BC| = |C|$ and $B^{-1}BC = C$.*

Proof. (1) Since $|G : C_G(b) \cap C_G(c)| = |G : C_G(b)|\,|C_G(b) : C_G(b) \cap C_G(c)| = |G : C_G(b)|\,|C_G(b)C_G(c) : C_G(c)| \leq |G : C_G(b)|\,|G : C_G(c)|$, by assumption $(|G : C_G(b)|, |G : C_G(c)|) = 1$, then $|G : C_G(b)|\,|G : C_G(c)| \mid |G : C_G(b) \cap C_G(c)|$. Consequently $|G : C_G(b)|\,|G : C_G(c)| = |G : C_G(b) \cap C_G(c)|$, then $|C_G(b)C_G(c) : C_G(c)| = |G : C_G(c)|$. Hence $C_G(b)C_G(c) = G$.

(2) We first prove that BC is a G-conjugacy class. It is obvious that we need only to prove that for any $g, h \in G$, $b^g c^h$ is conjugate to bc. By (1), $gh^{-1} \in G = C_G(b)C_G(c)$; then there exist $x \in C_G(b)$, $y \in C_G(c)$ such that $gh^{-1} = x^{-1}y$. Then $xg = yh$, and moreover $b^g c^h = b^{xg} c^{yh} = (bc)^{xg}$. In order to prove that BC is a conjugacy class in $G_{p'}$, we need only to find a element in BC belonging to $G_{p'}$. Let H be a Hall p'-subgroup of G; then there exist elements $g, h \in G$ such that $b^g, c^h \in H$. Then $b^g c^h \in BC$ and $b^g c^h$ is a p'-element.

(3) By (2), $BC = (bc)^G$. Since $C_G(b) \cap C_G(c) \subseteq C_G(bc)$, then $|BC| = |G : C_G(bc)| \mid |G : C_G(b) \cap C_G(c)| = |G : C_G(b)|\,|G : C_G(c)| = |B|\,|C|$, as desired.

(4) By (2), BC is a conjugacy class, if $|BC| > |C|$, then by (3), $(|BC|, |B|) > 1$ and $(|BC|, |C|) > 1$, a contradiction. Thus $|BC| = |C|$. Again by (3), $B^{-1}BC$ is a conjugacy class containing C; thus $B^{-1}BC = C$.

Now we prove Theorem 1.

Proof of Theorem 1. Suppose that $n(\Gamma_p'(G)) \geq 3$. Then there exists conjugacy classes A, B, C in $\mathrm{Con}(G_{p'})$ such that the prime divisors of $|A|$, $|B|$, and $|C|$ belong respectively to different connected components of $\Gamma_p'(G)$. Then $|A|$, $|B|$, $|C|$ are coprime to each other and any two satisfy the condition of Lemma 1(4). Without loss of generality, we can assume that $|A| > |B| > |C|$. Then $ACC^{-1} = A$, $BCC^{-1} = B$. Thus $A = A\langle CC^{-1}\rangle$, $B = B\langle CC^{-1}\rangle$. Thus A (resp. B) is a union of some cosets of normal subgroup $\langle CC^{-1}\rangle$. Then $|\langle CC^{-1}\rangle| \mid |A|$ and $|\langle CC^{-1}\rangle| \mid |B|$, a contradiction, as required.

Next we prove our main Theorem 2.

Proof of Theorem 2. Suppose that $n(\Gamma_p'(G)) = 1$, but $d(\Gamma_p'(G)) \geq 7$. Let $a, b \in \Gamma_p'(G)$ such that $d(a, b) = 7$. Thus we can choose the shortest path from a to b as $a - r - s - l - m - u - v - b$. Thus there exists conjugacy classes $C_1, C_2, C_3, C_4, C_5, C_6, C_7 \in \Gamma_p'(G)$ such that $ar \mid |C_1|, rs \mid |C_2|, sl \mid |C_3|, lm \mid |C_4|, mu \mid |C_5|, uv \mid |C_6, vb \mid |C_7|$. Then $|C_1|, |C_4|, |C_7|$ are coprime to each other and any two satisfy the condition of Lemma 1(4). Without loss of generality, we assume that $|C_1| < |C_4| < |C_7|$. Then $C_4 C_1 C_1^{-1} = C_4$, $C_7 C_1 C_1^{-1} = C_7$. Then $|\langle C_1 C_1^{-1}\rangle| \mid |C_4|, |\langle C_1 C_1^{-1}\rangle| \mid |C_7|$, a contradiction.

Suppose that $n(\Gamma_p'(G)) = 2$, but $d(\Gamma_p') \geq 4$. Choose a, b to belong to the same connected components $d(a, b) = 4$, let $a - r - s - t - b$ be the shortest path from a to b, and then choose conjugacy classes $C_1, C_2, C_3, C_4 \in \Gamma_p'(G)$ such that $ar \mid |C_1|, rs \mid |C_2|, st \mid |C_3|, tb \mid |C_4|$. But on the other hand, one can choose $C \in \Gamma_p'(G)$ such that the prime divisors of $|C|$ and $|C_1|$ belong respectively to different components. Then $|C|, |C_1|, |C_4|$ are coprime to each other and any two satisfy the condition of Lemma 1(4). As discussed above, we can also get a contradiction. We are done.

LEMMA 2. *Suppose that N is a normal subgroup of G.*

(1) *For any $x \in N$, $|x^N| \mid |x^G|$.*

(2) *For any $(xN)^{G/N} \in \mathrm{Con}((G/N)_{p'})$, there exists a conjugacy class $y^G \in \mathrm{Con}(G_{p'})$ such that $|(xN)^{G/N}| \mid |y^G|$.*

Proof. (1) It is obvious.

(2) Let $x = x_{p'} x_p$, x_p and $x_{p'}$ denote respectively the p-part and p'-part of x. Since the order of xN is not divisible by p, then $xN = x_{p'}N$ and $(xN)^{G/N} = (x_{p'}N)^{G/N}$. It is clear that $|(x_{p'}N)^{G/N}| \mid |(x_{p'})^G|$; then $|(xN)^{G/N}| \mid |(x_{p'})^G|$. Set $y^G = (x_{p'})^G$. We are done.

LEMMA 3 [6, Lemma 2]. *Suppose G is a finite solvable group. If the length of each conjugacy class is not divisible by r, then G has a central Sylow r-subgroup.*

Recall that $\rho_p'(G) = \{q \text{ prime: } q \mid |g^G|, \text{ for some } g \in G_{p'}\}$.

PROPOSITION 1. *Suppose that G is a finite group. Let π be a set of some primes.*

(1) *$p \notin \rho_p'(G)$; then $G = P \times K$, with P a Sylow p-subgroup of G.*

(2) *If G is p-solvable and $\pi \cap (\rho_p'(G) \cup \{p\}) = \emptyset$, then G has an abelian Hall π-subgroup.*

Proof. (1) Let $P \in \mathrm{Syl}_p(G)$. Then by the assumption we have $G = \bigcup_{x \in G} (PC_G(P))^x$. So $G = PC_G(P)$, and thus (1) follows.

(2) By induction on $|G|$ we may assume that $O_p(G) = 1$. Thus $O_{p'}(G) \neq 1$, as G is p-solvable. If there exists no prime $q \in \pi$ dividing the order of $O_{p'}(G)$, then by induction we are done. Thus there exists at least a prime $q \in \pi$ such that $O_{p'}(G)$ has a nontrivial Sylow q-subgroup Q. By Lemma 3, $O_{p'}(G) = K \times H$, where $1 \neq K$ is an abelian Hall π-subgroup of $O_{p'}(G)$. By induction we can assume that $O^p(G) = G$ and $H = 1$. Thus $K \leq Z(G)$, as K centralizes all p'-elements of G. Thus $G = K$, and we are done.

Proof of Theorem 3. It is obvious that we can assume that $|\pi_i| \geq 1$, $i = 1, 2$. Given a conjugacy class $C \in \mathrm{Con}(G_{p'})$, we say C belongs to X_i if the prime divisors of $|C|$ belong to X_i. Then for any conjugacy class $C \in \mathrm{Con}(G_{p'})$, C belongs to either X_1 or X_2. Let D_1 be a conjugacy class in $\mathrm{Con}(G_{p'})$ with the biggest class length. Without loss of generality, we assume that $D_1 \in X_2$. Set $M = \langle B \mid B \in X_1 \rangle$ and $N = \langle BB^{-1} \mid B \in X_1 \rangle$.

Step 1. If A and B belong to different connected components, then $AB = BA$ is also a conjugacy class of the p'-element and $|AB|$ equals the biggest in $|A|$ and $|B|$. If $|A|$ is the biggest, then $ABB^{-1} = A$.

Proof. It is obvious by Lemma 1(4).

Step 2. M is a proper subgroup of G.

Proof. Let D be the set of all conjugacy classes in $\mathrm{Con}(G_{p'})$ with the biggest class length. Then D is contained in X_2. Choose $C \in D$ and any $B \in X_1$. By Step 1, BC is a p'-conjugacy class and $|BC| = |C|$; then $BC \in D$. Thus $BD = D$, and moreover $MD = D$; then $|M| \mid |D|$. Thus M is a proper subgroup of G.

Step 3. There exists a p'-conjugacy class C in $G \setminus M$, and $|N| \mid |C|$.

Proof. If there is no p'-element in $G \setminus M$, then G/M is a p-group. By Step 2, Lemma 2, and induction on M the theorem is true. Thus we can assume that there exists a p'-conjugacy class C in $G \setminus M$. Let $B \in X_1$. By the definition of M, we know that $C \in X_2$. Then by Step 1, BC is a p'-conjugacy class in X_2 and $|BC| = |C|$. Moreover $CBB^{-1} = C$; thus $|CN| = |C|$, whence $|N| \mid |C|$.

Step 4. $N \leq Z(M)$.

Proof. Let $B \in X_1$, $b \in B$. By Step 3, $(|N|, |G : C_G(b)|) = (|N|, |B|) = 1$, but $|N : C_N(b)| \mid |(|N|, |G : C_G(b)|)$; thus $N = C_N(b)$. Since $M = \langle B \mid B \in X_1 \rangle$, then N is in the center of M.

Step 5. $M = P \times M_1$ and M_1 is abelian, where P is a Sylow p-subgroup of M.

Proof. By the definition of M and N, it is easy to see that M/N is in the center of G/N, whence, by Step 4, M is nilpotent. Thus $M = P \times M_1$. Write $M_1 = R \times Z$, with Z the largest Hall subgroup of M_1 which is contained in $Z(G)$. Let q be a prime divisor of $|R|$ and choose Q to be a Sylow q-subgroup of R. Thus $Q \trianglelefteq G$ and $N = [M, G] \geq [R, G] \geq [Q, G]$. As $[Q, G] \neq 1$, it follows that $q \mid |N|$. Thus if $q \mid |R|$, then $q \mid |N|$. Let $B = b^G \in X_1$; by Step 3 we have $(|Q|, |B|) = 1$. Since $|Q : C_Q(b)| \mid (|Q|, |B|)$ and noting that $M = \langle B \mid B \in X_1 \rangle$, we get $Q \leq Z(M)$. Thus $R \leq Z(M)$ and M_1 is abelian.

Step 6. Let $r \mid |B|$, where B is any conjugacy class in X_1. Then for any p'-element $a \in G \setminus M$, $C_G(a)$ contains a Sylow r-subgroup of G.

Proof. Since a^G belongs to X_2, then $|a^G|$ and r are coprime; that is, $C_G(a)$ contains a Sylow r-subgroup of G.

Step 7. G has a metabelian Hall π_1-subgroup.

Proof. If $r \in \pi_1$, then by Step 6, for each p'-element $a \in G \setminus M$, $C_G(a)$ contains a Sylow r-subgroup of G. Thus $\forall r \in \pi_1$, $r \notin \rho_p'(G/M)$, and by Proposition 1, G/M has an abelian Hall π_1-subgroup H/M. Thus by Step 5, G has a metabelian Hall π_1-subgroup, as desired.

Proof of Theorem 4. Suppose $l_p(G) = n$ and let $1 \leq N_1 \leq N_2 \leq \cdots \leq N_{n+1} = G$ be the p-chain of G (with $N_1 = O_{p'}(G)$, $N_2 = O_{p',p}(G)$, $N_3 = O_{p',p,p'}(G) \dots$). If N_i/N_{i-1} is a p'-group, then by Lemma 2, $\rho_p'(N_i/N_{i-1}) \subseteq \rho_p'(G)$.

(1) If $n(\Gamma(N_i/N_{i-1})) = 1$, then by Lemma 3, $N_i/Ni - 1$ has a central Hall π_1-subgroup, or a central Hall π_2-subgroup. Then $l_{\pi_i}(N_i/N_{i-1}) \leq 1$, $i = 1, 2$.

(2) If $n(\Gamma(N_i/N_{i-1})) = 2$, write Δ_1, Δ_2, respectively, for the vertex sets of $\Gamma(N_i/N_{i-1})$ in different connected component. We have $\Delta_i \subseteq \pi_i$, $i = 1, 2$. Since $p \dagger |N_i/N_{i-1}|$, by [6, Theorem 4], $l_{\Delta_i} \leq 1$. But on the other hand, by Lemma 3, N_i/N_{i-1} has a central Hall $\pi_i \setminus \Delta_i (i = 1, 2)$-subgroup; then $l_{\pi_j}(N_i/N_{i-1}) \leq 1$, $j = 1, 2$.

From (1) and (2) above, if N_i/N_{i-1} is a p'-group, then $l_{\pi_i} \leq 1$, $i = 1, 2$. Thus $l_{\pi_i}(G) \leq l_p(G) + 1$, $i = 1, 2$.

Proof of Theorem 5. We use induction on $|G|$. Let $\pi = \{r, s\}$.

We will prove the theorem in five steps.

Step 1. First we can assume that $\Phi(G) = 1$, $O_{\pi'}(G) = 1$, and $O^{\pi'}(G) = G$.

Proof. Clearly we can assume that $O_{\pi'}(G) = 1$, $O^{\pi'}(G) = G$. Suppose that $\Phi(G) > 1$. If r or s does not belong to $\rho_{p'}(G/\Phi(G))$, then by

Proposition 1, $G/\Phi(G)$ has an abelian Sylow r- or abelian Sylow s-subgroup. Then $l_r(G/\Phi(G)) \leq 1$ or $l_s(G/\Phi(G)) \leq 1$, and by [9, Chap. VI, Theorem 6.4], $l_r(G) \leq 1$ or $l_s(G) \leq 1$. The theorem is correct. Thus we can assume that $r, s \in \rho_{p'}(G/\Phi(G))$, so by induction $l_r(G/\Phi(g)) \leq 1$ or $l_s(G/\Phi(G)) \leq 1$. Thus we have $l_r(G) \leq 1$ or $l_s(G) \leq 1$, as desired. Hence we can assume that $\Phi(G) = 1$.

Step 2. $F(G) = O_r(G)O_s(G)$, and moreover $O_r(G) > 1$, $O_s(G) > 1$.

Proof. By Step 1, $F(G) = O_r(G)O_s(G)$. Assume that $O_r(G) = 1$. Thus $C_G(O_s(G)) \leq O_s(G)$, so for any p'-element $a \in G \setminus O_s(G)$, $s \mid |a^G|$, and $r \dagger |a^G|$. Thus $r \notin \rho_p'(G/O_s(G))$, and by Proposition 1, $G/O_s(G)$ has an abelian Sylow r-subgroup. Thus $l_r(G) \leq 1$, as desired. So we can assume that $O_r(G) > 1$, $O_s(G) > 1$.

Step 3. G has only two minimal normal subgroups $O_r(G)$ and $O_s(G)$.

Proof. First we prove that $O_r(G)$ and $O_s(G)$ are minimal normal subgroups of G. Since $\Phi(G) = 1$, $O_r(G)$ and $O_s(G)$ are direct sums of minimal normal subgroups of G. Suppose $M, N \leq O_r(G)$ are two different minimal normal subgroups of G. If r or s does not belong to $\rho_{p'}(G/M)$, then by Proposition 1, G/M has an abelian Sylow r- or abelian Sylow s-subgroup. If the Sylow s-subgroup of G/M is abelian, then $l_s(G/M) \leq 1$ and thus $l_s(G) \leq 1$, and the theorem is correct. So we can assume that G/M has an abelian Sylow r-subgroup. It follows that $l_r(G/M) \leq 1$. If $r, s \in \rho_{p'}(G/M)$, then by induction $l_r(G/M) \leq 1$ or $l_s(G/M) \leq 1$, and we can also assume that $l_r(G/M) \leq 1$. Similarly we can assume that $l_r(G/N) \leq 1$. By [9, Chap. VI, Theorem 6.4 (d)], $l_r(G) \leq 1$, and the theorem is correct. Thus we may assume that $O_r(G)$ and $O_s(G)$ are minimal normal subgroups of G.

Step 4. $O_r(G) \leq Z(G)$ or $O_s(G) \leq Z(G)$.

Proof. Suppose $C_G(O_r(G)) < G$ and $C_G(O_s(G)) < G$. Set $H = C_G(O_r(G))C_G(O_s(G))$. If H is a proper subgroup of G, then G/H is not a p-group, since $O^{\pi'}(G) = G$. Let $x \in G \setminus H$ be a p'-element. Then $rs \mid |x^G|$, a contradiction. Hence $H = G$. We may assume that there exists a p'-element $x \in C_G(O_r(G))$ with $x \notin C_G(O_s(G))$. Otherwise $C_G(O_r(G))$ or $C_G(O_s(G))$ contains a Hall π-subgroup of G, but since $O^{\pi'}(G) = G$, then $C_G(O_r(G)) = G$ or $C_G(O_s(G)) = G$, a contradiction. Similarly we may assume that there exists a p'-element $y \in C_G(O_s(G))$ with $y \notin C_G(O_r(G))$. We may assume that x, y belong to a Hall p'-subgroup of G (otherwise replace y by a suitable conjugate of y); thus xy is a p'-element of G, and $xy \in G \setminus (C_G(O_r(G)) \cup C_G(O_s(G)))$. Thus $rs \mid |G : C_G(xy)|$, a contradiction.

Step 5. Conclusion.

Proof. Without loss of generality, we assume that $O_r(G) \leq Z(G)$. Thus either $G = F(G)$ or $s \mid |b^G|$ for any p'-element b in $G \setminus F(G)$. If $G = F(G)$, the theorem is obviously correct. In the second case, $r \notin \rho'_p(G/F(G))$, so by Proposition 1(2), $G/F(G)$ has an abelian Sylow r-subgroup $RF(G)/F(G)$. Thus $R' = [R, R] \leq O_r(G) \leq Z(G)$, and by [9, Chap. VI, Theorem 6.10], $l_r(G) \leq 1$. This proves the theorem.

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