Journal of Functional Analysis **164**, 163–180 (1999)
Article ID jfan.1999.3388, available online at http://www.idealibrary.com on IDEAL®

Lie Isomorphisms of Nest Algebras

Laurent W. Marcoux*

Department of Mathematical Sciences, University of Alberta, Edmonton, Alberta, Canada T6G 2G1
E-mail: L.Marcoux@ualberta.ca

and

Ahmed R. Sourour*

Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia, Canada V8W 3P4
E-mail: sourour@math.uvic.ca

Received July 1, 1998; revised September 22, 1998; accepted December 29, 1998

In this paper we characterize linear maps φ between two nest algebras $\mathscr{T}(\mathscr{N})$ and $\mathscr{T}(\mathscr{M})$ which satisfy the property that $\varphi(AB-BA)=\varphi(A)\,\varphi(B)-\varphi(B)\,\varphi(A)$ for all $A, B \in \mathscr{T}(\mathscr{N})$. In particular, it is shown that such isomorphisms only exist if $\mathscr{N}$ is similar to $\mathscr{M}$ or $\mathscr{M}^{\perp}$.

1. INTRODUCTION

1.1. Let $\mathscr{A}$ be an associative algebra over $\mathbb{C}$. By considering a new product $[a, b]=ab-ba$, called the *Lie bracket*, $\mathscr{A}$ takes on the structure of a Lie algebra. In fact, it is a standard result (see, e.g. [Hu, Chapter V]) that every Lie algebra $\mathfrak{g}$ can be embedded as a subalgebra of an associate algebra—the universal enveloping algebra of $\mathfrak{g}$—in such a way that the product in $\mathfrak{g}$ is sent to the bracket in the enveloping algebra. A *Lie homomorphism* of $\mathscr{A}$ into a second associative algebra $\mathscr{B}$ is a linear map φ: $\mathscr{A} \to \mathscr{B}$ which preserves this new multiplication. That is, $\varphi[a, b]=[\varphi(a), \varphi(b)]$ for all $a, b \in \mathscr{A}$.

Lie homomorphisms between rings and between self-adjoint operator algebras have received a fair amount of attention ([Ma1, Ma2, Mi, CJR]).

We note that if α is an (associative) homomorphism or the negative of an anti-homomorphism from $\mathscr{A}$ to $\mathscr{B}$ and β is a linear map from $\mathscr{A}$ into

* Research supported in part by NSERC (Canada).

0022-1236/99 $30.00

the centre of $\mathscr{B}$, such that $\beta(ab-ba)=0$ for every a and b in $\mathscr{A}$, then $\alpha+\beta$ is a Lie homomorphism. It turns out that every Lie isomorphism between nest algebras is of such a form (Theorem 2.4).

1.2. The Lie automorphisms of $\mathscr{T}_n$, the algebra of $n\times n$ upper triangular complex matrices are characterized in [MS, Đ]. If we let J_n denote the $n\times n$ matrix $J_n=[\delta_{i,n+1-i}]$, where δ_{ij} denotes the Kronecker delta function, then the main result for Lie automorphisms is

THEOREM 1.3 [MS, Đ]. *Let $\varphi\colon \mathscr{T}_n\to\mathscr{T}_n$ be a linear map. Then φ is a Lie automorphism of $\mathscr{T}_n$ if and only if either*

(a) $\varphi(T)=S^{-1}TS+\tau(T)\,I$; *or*

(b) $\varphi(T)=-S^{-1}J_nT^tJ_nS+\tau(T)\,I$,

where $S\in\mathscr{T}_n$ is invertible, τ is a linear functional satisfying $\tau(AB-BA)=0$ for all $A, B\in\mathscr{T}_n$, and $\tau(I)\neq-1$.

1.4. In the present paper, we extend our analysis of Lie automorphisms of non-self- adjoint operator algebras to the infinite dimensional setting by characterizing Lie isomorphisms of nest algebras. The descriptions we obtain are similar in form to those which exist in the self-adjoint setting; nevertheless the two methods of proof share little in common.

2. PRELIMINARIES

2.1. A *nest* is a totally ordered set of closed subspaces of a Hilbert space $\mathscr{H}$ such that $\{0\}$, $\mathscr{H}\in\mathscr{N}$ and $\mathscr{N}$ is closed under the taking of arbitrary intersections and closed linear spans of its elements. By $\mathscr{N}^0$ we mean $\mathscr{N}\backslash\{\{0\}\ \mathscr{H}\}$. We set $\mathscr{P}(\mathscr{N})=\{P(N)\colon N\in\mathscr{N}\}$, where $P(N)$ denotes the orthogonal projection of $\mathscr{H}$ onto N. By $\mathscr{B}(\mathscr{H})$, we denote the set of bounded linear operators acting on $\mathscr{H}$, and if $\mathscr{H}_1$ and $\mathscr{H}_2$ are Hilbert spaces, then $\mathscr{B}(\mathscr{H}_1,\mathscr{H}_2)$ denotes the bounded linear maps from $\mathscr{H}_1$ into $\mathscr{H}_2$. For nonzero vectors x and y, we denote the rank one operator $u\mapsto\langle u, y\rangle\, x$ by $x\otimes y^*$.

The *nest algebra* associated to a nest $\mathscr{N}$ is the set $\mathscr{T}(\mathscr{N})=\{T\in\mathscr{B}(\mathscr{H})$: $TN\subseteq N$ for all $N\in\mathscr{N}\}$. In fact, if $\mathscr{L}$ is a lattice of closed subspaces of $\mathscr{H}$, we set $\operatorname{Alg}\mathscr{L}=\{T\in\mathscr{B}(\mathscr{H})\colon TL\subseteq L$ for all $L\in\mathscr{L}\}$, and if $\mathscr{A}$ is an algebra, we set $\operatorname{Lat}\mathscr{A}=\{M\subseteq\mathscr{H}\colon M$ is a closed subspace and $AM\subseteq M$ for all $A\in\mathscr{A}\}$. As such, $\mathscr{T}(\mathscr{N})=\operatorname{Alg}(\mathscr{N})$, and it is known ([R1, D1; Theorem 2.9]) that nests are *reflexive* in the sense that $\mathscr{N}=\operatorname{Lat}(\operatorname{Alg}(\mathscr{N}))$. From this it follows that $\mathscr{T}(\mathscr{N})$ is also *reflexive* in that $\operatorname{Alg}(\operatorname{Lat}(\mathscr{T}(\mathscr{N})))=\mathscr{T}(\mathscr{N})$. We refer the reader to [D1] for these facts as well as an excellent exposition of the general theory of nest algebras.

Given a nest $\mathcal{N}$ and $N \in \mathcal{N}$, we define

$$N_+ = \inf\{M \in \mathcal{N}: M > N\}$$

and

$$N_- = \sup\{M \in \mathcal{N}: M < N\}.$$

Then N_+ (resp. N_-) is referred to as the *immediate successor* (resp. *immediate predecessor*) of N. A space $N_+ \ominus N$ with $N_+ \neq N$ is referred to as an *atom* of $\mathcal{N}$. If $\mathcal{H}$ is densely spanned by the atoms of $\mathcal{N}$, then $\mathcal{N}$ is said to be *atomic*. If $\mathcal{N}$ has no atoms, then $\mathcal{N}$ is called *continuous*. In general, nests are neither atomic nor continuous. We write $\mathscr{E}(\mathcal{N})$ to denote the set of idempotents in $\mathcal{T}(\mathcal{N})$. Unlike $\mathscr{P}(\mathcal{N})$, we do not require the range of an idempotent in $\mathscr{E}(\mathcal{N})$ to lie in $\mathcal{N}$. The commutant of a set $\mathscr{S} \subseteq \mathcal{T}(\mathcal{N})$ is $\mathscr{S}' = \{T \in \mathcal{T}(\mathcal{N})$: $TS = ST$ for all $S \in \mathscr{S}\}$. We mention in passing that $\mathcal{T}(\mathcal{N})' = \mathbb{C}I$ [D1, Corollary 19.5].

If $\mathcal{T}(\mathcal{N})$ is a nest algebra, one can readily verify that $\mathcal{T}(\mathcal{N})^* := \{T^*$: $T \in \mathcal{T}(\mathcal{N})\} = \mathcal{T}(\mathcal{N}^\perp)$, where $\mathcal{N}^\perp$ is the nest $\{N^\perp: N \in \mathcal{N}\}$. The intersection $\mathscr{D}(\mathcal{N}) := \mathcal{T}(\mathcal{N}) \cap \mathcal{T}(\mathcal{N})^*$ is called the *diagonal* of $\mathcal{T}(\mathcal{N})$, and $\mathscr{D}(\mathcal{N})$ is a von Neumann algebra.

For any algebra $\mathscr{A}$ and any invertible element $a \in \mathscr{A}$ we denote by Ad_a the map $b \mapsto a^{-1}ba$ and observe that this is an automorphism of $\mathscr{A}$.

2.2. One of the major theorems in the theory of nest algebras is the following result of Davidson [D2; D1, Theorem 13.20]. An important special case is due to Larson [L].

THE SIMILARITY THEOREM [D2]. *Let $\mathcal{N}$ and $\mathcal{M}$ be nests on a separable Hilbert space $\mathcal{H}$. Then there exists an invertible operator S in $\mathscr{B}(\mathcal{H})$ such that $S\mathcal{N} := \{SN: N \in \mathcal{N}\} = \mathcal{M}$ if and only if there is a dimension preserving order isomorphism of $\mathcal{N}$ onto $\mathcal{M}$.*

In this case, $\mathcal{N}$ and $\mathcal{M}$ are said to be *similar*.

2.3. In trying to generalise Theorem 1.3 to the case of nest algebras, one is confronted with the task of finding a suitable replacement for the map that sends a matrix $A = [a_{ij}]$ to its transpose $[a_{ji}]$. The following lemma describes one way of doing this.

LEMMA. *Let $\mathcal{N}$ be a nest on a Hilbert space $\mathcal{H}$. Then*

(a) *There exists a conjugate-linear involution J on $\mathcal{H}$ such that $J(\mathcal{N}^\perp) = \mathcal{N}^\perp$.*

(b) *The mapping* $T \to JT^*J$ *is a linear anti-isomorphism from* $\mathcal{T}(\mathcal{N})$ *onto* $\mathcal{T}(\mathcal{N}^\perp)$.

(c) *For every invertible* $X \in B(\mathcal{H})$, *the mapping*

$$T \to X^{-1}JT^*JX$$

is anti-isomorphism from $\mathcal{T}(\mathcal{N})$ *onto* $\mathcal{T}(\mathcal{M})$, *where* $\mathcal{M} = X^{-1}(\mathcal{N}^\perp)$.

Proof. Let us first identify $\mathcal{H}$ with $L^2(\mu)$ for an appropriate measure space. For any closed subspace M of $\mathcal{H}$, we define $M_c = \{\bar{f}: f \in M\}$, the space of all conjugates of functions in M. One can then verify that M_c is a closed subspace of $\mathcal{H}$ as well, and that if K is any closed subspace of $\mathcal{H}$, then the dimension of $K_c \cap M_c$ coincides with that of $K \cap M$. In particular, if $\mathcal{N}$ is a nest, then $\mathcal{N}_c = \{N_c: N \in \mathcal{N}\}$ is also a nest. The map $N \mapsto N_c$ is a dimension preserving order isomorphism, as is the map $N^\perp \mapsto N_c^\perp$. It follows from the Similarity Theorem above that there exists an invertible operator S in $B(\mathcal{H})$ such that $S\mathcal{N}^\perp = \mathcal{N}_c^\perp$. Let $J_1(f) = \bar{f}$ for each $f \in L^2(\mu)$, and $J = J_1 S$. A simple calculation now confirms that J is the desired conjugate-linear involution taking $\mathcal{N}^\perp$ onto $\mathcal{N}^\perp$. This proves (a). Equipped with this map J, it is straightforward to verify parts (b) and (c). ∎

Using this definition, we can now state our main result:

2.4. MAIN THEOREM. *Let* $\mathcal{H}$ *be a complex, separable Hilbert space. Let* $\mathcal{N}$ *and* $\mathcal{M}$ *be nests on* $\mathcal{H}$. *A linear map* $\varphi: \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{M})$ *is a Lie isomorphism if and only if either*

(a) $\mathcal{N}$ *is similar to* $\mathcal{M}$ *and there exists an invertible* $Y \in \mathcal{B}(\mathcal{H})$ *satisfying* $Y(\mathcal{M}) = \mathcal{N}$ *and a linear functional* β *on* $\mathcal{T}(\mathcal{N})$ *with* $\beta(I) \neq -1$ *and* $\beta(C) = 0$ *for every commutator* $C = AB - BA$ *such that*

$$\varphi(T) = Y^{-1}TY + \beta(T)\, I;$$

or

(b) $\mathcal{N}$ *is similar to* $\mathcal{M}^\perp$ *and there exists an invertible* $Y \in \mathcal{B}(\mathcal{H})$ *satisfying* $Y(\mathcal{M}) = \mathcal{N}^\perp$ *and* β *as above such that*

$$\varphi(T) = -Y^{-1}JT^*JY - \beta(T)\, I,$$

where J *is the conjugate linear involution in Lemma* 2.3.

Consequently, every Lie isomorphism is a sum $\alpha + \beta$, *where* α *is an isomorphism or the negative of an anti-isomorphism and* β *maps* $\mathcal{T}(\mathcal{N})$ *into the centre of* $\mathcal{T}(\mathcal{M})$.

2.5. In many notable cases, the sum of the commutators in $\mathcal{T}(\mathcal{N})$ turns out to be all of $\mathcal{T}(\mathcal{N})$, and consequently the linear functional β must be zero. For instance, recall that a weakly closed subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ is said to have *infinite multiplicity* if $\mathcal{A}$ is isomorphic to $\mathcal{A} \otimes \mathcal{B}(\mathcal{H})$. It is a consequence of the Similarity Theorem that a nest algebra $\mathcal{T}(\mathcal{N})$ has infinite multiplicity if and only if $\mathcal{N}$ has no finite dimensional atoms. In this case we say that $\mathcal{N}$ is a *nest of infinite multiplicity.*

2.6. PROPOSITION. *Suppose $\mathcal{A}$ is a weakly closed subalgebra of $\mathcal{B}(\mathcal{H})$ and that $\mathcal{A}$ has infinite multiplicity. Then every element of $\mathcal{A}$ can be written as a sum of two commutators in $\mathcal{A}$.*

Proof. The proof of this result is identical to that of [Ha, Problem 234], where Halmos demonstrates this for the special case where $\mathcal{A} = \mathcal{B}(\mathcal{H})$. ∎

2.7. COROLLARY. *Let $\mathcal{N}$ be a nest of infinite multiplicity. Then every $T \in \mathcal{T}(\mathcal{N})$ can be written as a sum of two commutators.*

We remark that it has been observed by Larson [L] and Larson and Pitts [LP] that in any nest algebra of infinite multiplicity, the set of commutators an invertible element, and thus $\mathcal{T}(\mathcal{N})$ coincides with its commutator ideal.

2.8. COROLLARY. *Let $\mathcal{N}$ and $\mathcal{M}$ be nests such that $\mathcal{N}$ has infinite multiplicity. Then every Lie isomorphism between $\mathcal{T}(\mathcal{N})$ and $\mathcal{T}(\mathcal{M})$ is either an associative isomorphism or the negative of an anti-isomorphism.*

2.9. With obvious modifications, the proof that follows establishes the results in the case where $\mathcal{N}$ and $\mathcal{M}$ are nests on two different Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. In this case, the operator Y is an operator from $\mathcal{H}_2$ onto $\mathcal{H}_1$. In particular, we have the following finite dimensional corollary.

2.10. COROLLARY. *Let $\mathcal{N}$ and $\mathcal{M}$ be nests on $\mathbb{C}^n$ and $\mathbb{C}^m$ and let φ be a Lie isomorphism from $\mathcal{T}(\mathcal{N})$ onto $\mathcal{T}(\mathcal{M})$. Then $m = n$ and either $\varphi(T) = Y^{-1}TY + \beta(T)\, I$, or $\varphi(T) = -Y^{-1}T^tY + \beta(T)\, I$, where Y and β are as in Theorem 2.4. Here T^t denotes the usual transpose map.*

3. THE PROOFS

First we observe that it is straightforward to verify that if φ is of the form (a) or (b) in the statement of the Theorem, then φ is indeed a Lie isomorphism. The remainder of this paper is devoted to proving the converse.

We shall assume that $\mathcal{H}$ is always a complex, separable Hilbert space, that $\mathcal{N}$ and $\mathcal{M}$ are nests on $\mathcal{H}$ and that $\varphi\colon \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{M})$ is a Lie isomorphism.

We start with a characterization of idempotents modulo scalars in $\mathcal{T}(\mathcal{N})$ in terms of commutators.

3.1. LEMMA. *Let $\mathcal{N}$ be a nest and $A \in \mathcal{T}(\mathcal{N})$. Then*

(a) *A is the sum of a scalar and an idempotent if and only if*

$$[A, [A, [A, T]]] = [A, T] \qquad \text{for every} \quad T \in \mathcal{T}(\mathcal{N}). \tag{1}$$

(b) *A is the sum of a scalar and an idempotent whose range belongs to $\mathcal{N}$ if and only if*

$$[A, [A, T]] = [A, T] \qquad \text{for every} \quad T \in \mathcal{T}(\mathcal{N}). \tag{2}$$

Proof. (a) One direction is trivial. Assume now that A satisfies Eq. (1), i.e.

$$(A^3 - A)\, T - 3A^2TA + 3ATA^2 - T(A^3 - A) = 0 \qquad \text{for all} \quad T \in \mathcal{T}(\mathcal{N}). \tag{3}$$

Choose $N \in \mathcal{N}$ such that $N_- \neq \mathcal{H}$. Let $0 \neq y \in (N_-)^\perp$. For every $x \in N$, the rank one operator $x \otimes y^* \in \mathcal{T}(\mathcal{N})$, [D1, Lemma 2.8]. Upon taking $T = x \otimes y^*$ in Eq. (3) and then applying this equation to a vector z, we get

$$\begin{aligned} &\langle z, y\rangle (A^3 - A)\, x - 3\langle Az, y\rangle\, A^2 x \\ &\qquad + 3\langle A^2 z, y\rangle\, Ax - 3\langle (A^3 - A)\, z, y\rangle\, x = 0. \end{aligned} \tag{4}$$

We now consider two cases:

Case 1. There is an $M \in \mathcal{N}$ such that $M_- \neq \mathcal{H}$, and $(I - P_{M_-})\, A = \lambda(I - P_{M_-})$ for a scalar λ. By translating by a scalar, we may assume that $\lambda = 0$. Thus for any $N > M$, and any $y \in (N_-)^\perp$, we have that $A^* y = 0$. Hence Eq. (4) yields $A^3 = A$. Therefore $A = E - F$, where E and F are the spectral idempotents corresponding to the points 1 and -1 in the spectrum of A. Now Eq. (3) yields $ETF = 0 = FTE$, for every $T \in \mathcal{T}(\mathcal{N})$. A short computation yields that either E or F is of the form $\left[\begin{smallmatrix} 0 & X \\ 0 & 0 \end{smallmatrix}\right]$, and being also an idempotent, it must be 0. It follows that either A or $A + I$ is an idempotent.

Case 2. $(I - P_{N_-})\, A$ is not a scalar multiple of $(I - P_{N_-})$ whenever $N_- \neq \mathcal{H}$. Thus the vector y may be chosen so that y and $A^* y$ are linearly independent and so z may be chosen so that $\langle z, y\rangle = 0$ and $\langle Az, y\rangle = 1$. Now Eq. (4) implies that there is a monic quadratic polynomial q_N such that $q_N(A_N)=0$, where $A_N=A|_N$. We will show that there exists a quadratic polynomial q such that $q(A)=0$. If $\mathscr{H}_- \neq \mathscr{H}$, we take $N=\mathscr{H}$ and we are done. Otherwise, it is obvious that unless A is a scalar, the polynomials $\{q_N: N<\mathscr{H}\}$ must eventually agree and so $q(A)=0$ for a quadratic q. Now translate A so that $A^2=cI$. Eq. (3) yields that either A is a scalar or that $c=\frac{1}{4}$ and hence $A+\frac{1}{2}I$ is an idempotent.

(b) If A satisfies Eq. (2), then it satisfies Eq. (1) as well, and so $A=\lambda I+E$, where E is an idempotent and $[E,T]=[E,[E,T]]$ for all $T\in\mathscr{T}(\mathscr{N})$. But then $TE=ETE$ for all $T\in\mathscr{T}(\mathscr{N})$, i.e., ran $E\in \mathrm{Lat}(\mathscr{T}(\mathscr{N}))=\mathscr{N}$, as nests are reflexive [R1; D1, Theorem 2.9].

3.2. Corollary. *Let $\mathscr{N}$ and $\mathscr{M}$ be nests and suppose that $\varphi\colon \mathscr{T}(\mathscr{N})\to\mathscr{T}(\mathscr{M})$ is a Lie isomorphism. Then*

(a) *$\varphi(I)=\kappa I$ where κ is a nonzero scalar;*

(b) *If $E=E^2\in\mathscr{T}(\mathscr{N})$, it follows that $\varphi(E)=\alpha_E I+F$ where $\alpha_E\in\mathbb{C}$ and F is an idempotent in $\mathscr{T}(\mathscr{M})$;*

(c) *If E is an idempotent in $\mathscr{T}(\mathscr{N})$ and* ran $E\in\mathscr{N}$, *then $\varphi(E)=\alpha_E I+F$, where $\alpha_E\in\mathbb{C}$, F is an idempotent in $\mathscr{T}(\mathscr{M})$ and* ran $F\in\mathscr{M}$.

Furthermore, if $0\neq E\neq I$, then both the scalar α_E and the idempotent F occurring above are uniquely determined.

Proof. Part (a) follows from the fact that the centre of $\mathscr{T}(\mathscr{N})$ is the scalars. Parts (b) and (c) are immediate consequences of Lemma 3.1. The last assertion is obvious.

We shall henceforth adopt the convention of denoting the unique scalar associated to $\varphi(E)$ by α_E.

3.3. We are now in a position to prove the exceptional case where the nest $\mathscr{N}$ is the trivial nest $\{\{0\},\mathscr{H}\}$. Corollary 3.2 applied to φ^{-1} yields that $\mathscr{M}$ is also trivial and so φ is a Lie automorphism of $B(\mathscr{H})$. Since $B(\mathscr{H})$ is a prime ring, it follows from [Ma1, Ma2] that $\varphi=\alpha+\beta$ where β is a central map and α is either an algebra automorphism or an anti-automorphism. It is well known that automorphisms of $B(\mathscr{H})$ are inner. This establishes Theorem 2.4 in this case. We note that if dim $\mathscr{H}$ is infinite, then $\beta=0$.

In the foregoing, we shall assume that the nest $\mathscr{N}\neq\{\{0\},\mathscr{H}\}$. This is crucial in the discussion that follows 3.20.

3.4. We now define two auxiliary functions which we shall associate to a given Lie isomorphism between two nest algebras. Recall that $\mathscr{N}^0=\mathscr{N}\backslash\{\{0\},\mathscr{H}\}$.

Definition. Let $\mathcal{N}$ and $\mathcal{M}$ be nests and suppose $\varphi: \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{M})$ is a Lie isomorphism. We define a function

$$\hat{\varphi}: \mathcal{E}(\mathcal{N})\backslash\{0, I\} \to \mathcal{E}(\mathcal{N})\backslash\{0, I\}$$

by

$$\hat{\varphi}(E) = \varphi(E) - \alpha_E I.$$

We also define

$$\tilde{\varphi}: \mathcal{N}^0 \to \mathcal{M}^0$$

by

$$N \mapsto \operatorname{ran}(\varphi(P(N)) - \alpha_{P(N)} I),$$

i.e., $\tilde{\varphi}(N) = \operatorname{ran} \hat{\varphi}(P(N))$.

Next, we characterize idempotents having the same range in terms of commutators.

3.5. Lemma. *Let E, F be nonzero idempotent linear maps on a vector space over a field of characteristic different from 2. The following are equivalent*:

(a) $EF - FE = F - E + \lambda I$ *for a scalar* λ;

(b) $EF - FE = F - E$;

(c) $EF = F$ *and* $FE = E$.

Proof. That (c) implies (b) and (b) implies (a) are trivial.

To prove that (a) implies (c) assume that (a) is satisfied and consider

$$0 = (I - E)(EF - FE + F)\, E = (I - E)(F - E + \lambda I + F)\, E$$
$$= 2(I - E)\, FE$$

and

$$0 = E(EF - FE)\, E = E(F - E + \lambda I)\, E = EFE - (1 - \lambda)\, E.$$

Combining these, we get

$$FE = EFE = (1 - \lambda)\, E.$$

Thus, if $x \in \operatorname{ran} E$, then $Fx = (1-\lambda)\, x$, proving that $(1-\lambda)$ is an eigenvalue of F and so $\lambda = 0$ or 1. Similarly

$$0 = (I-F)(EF-FE-E)\, F = (I-F)(F-E+\lambda I-E)\, F$$

$$= -2(I-F)\, EF;$$

$$0 = F(EF-FE)\, F = F(F-E+\lambda I)\, F = (1+\lambda)\, F - FEF$$

yielding $EF = (1+\lambda)\, F$, and so $(1+\lambda)$ is an eigenvalue for E. This together with the above gives us that $\lambda = 0$ and $EF = F$, $FE = E$. ∎

3.6. PROPOSITION. *Let $\mathcal{N}$ and $\mathcal{M}$ be nests and $\varphi\colon \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{M})$ a Lie isomorphism. If E, F are idempotents in $\mathcal{T}(\mathcal{N})$ such that $0 \neq E \neq I$, $0 \neq F \neq I$ and* $\operatorname{ran} E = \operatorname{ran} F$, *then $\alpha_E = \alpha_F$ and* $\operatorname{ran} \hat{\varphi}(E) = \operatorname{ran} \hat{\varphi}(F)$.

Proof. Since $\operatorname{ran} E = \operatorname{ran} F$, we have $EF = F$ and $FE = E$, and hence $[E, F] = F - E$. It follows that $[\varphi(E), \varphi(F)] = \varphi(F) - \varphi(E)$ or equivalently

$$[\hat{\varphi}(E), \hat{\varphi}(F)] = \hat{\varphi}(F) - \hat{\varphi}(E) + (\alpha_F - \alpha_E)\, I.$$

By Lemma 3.5, we obtain that $\alpha_E = \alpha_F$, that $\hat{\varphi}(E)\, \hat{\varphi}(F) = \hat{\varphi}(F)$ and $\hat{\varphi}(F)\, \hat{\varphi}(E) = \hat{\varphi}(E)$. That is, $\operatorname{ran} \hat{\varphi}(E) = \operatorname{ran} \hat{\varphi}(F)$. ∎

Our next goal is to show that $\hat{\varphi}$ preserves order on $\mathcal{E}(\mathcal{N})$. We start with a definition.

3.7. DEFINITION. Let E_1 and E_2 be two idempotents. We say that $E_1 \leqslant E_2$ if $E_1 E_2 = E_1 = E_2 E_1$, or equivalently E_1 and E_2 commute and $\operatorname{ran} E_1 \subseteq \operatorname{ran} E_2$. We say that $E_1 < E_2$ if $E_1 \leqslant E_2$ and $E_1 \neq E_2$.

3.8. LEMMA. *Let $\mathcal{N}$ and $\mathcal{M}$ be nests, and suppose E_1, E_2 are commuting idempotents in $\mathcal{T}(\mathcal{N})$ such that $0 < E_1 < E_2 < I$. Set $F_i = \hat{\varphi}(E_i)$, $i = 1, 2$. Then either $0 < F_1 < F_2 < I$, or $0 < F_2 < F_1 < I$.*

Proof. First, we observe that F_1 and F_2 commute since $[E_1, E_2] = 0$. Now $E_1, E_2, E_2 - E_1 \notin \mathbb{C}I$ implies $F_1, F_2, F_2 - F_1 \notin \mathbb{C}I$, and in particular, $F_1 \neq F_2$.

Now $E := E_1 + (I - E_2)$ is an idempotent in $\mathcal{T}(\mathcal{N})$. By Corollary 3.2, $\varphi(E) \in \mathbb{C}I + \mathcal{E}(\mathcal{M})$, and so $T := (F_1 - F_2) \in \mathbb{C}I + \mathcal{E}(\mathcal{M})$, and $\sigma(F_1 - F_2)$ consists of two points λ and $\lambda + 1$. We may chose a Hamel basis that diagonalizes F_1 and F_2 simultaneously. Now if F_1 and F_2 are not comparable, then $\sigma(F_1 - F_2)$ contains 1 and -1, which is impossible. ∎

3.9. LEMMA. *Let $\mathcal{N}$ and $\mathcal{M}$ be nests, and E_1, E_2 and E_3 be commuting idempotents in $\mathcal{T}(\mathcal{N})$ satisfying $0 < E_1 < E_2 < E_3 < I$. Suppose φ: $\mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{M})$ is a Lie isomorphism. Set $F_i = \hat{\varphi}(E_i)$, $i = 1, 2, 3$.*

(a) *If $F_1 < F_2$, then $F_1 < F_2 < F_3$.*

(b) *If $F_1 > F_2$, then $F_1 > F_2 > F_3$.*

Consequently, the map $\hat{\varphi}$ (and also $\tilde{\varphi}$) is either order preserving or order reversing.

Proof. (a) The idempotents F_1, F_2, and F_3 are distinct and mutually comparable by Lemma 3.8. Now $E_1 + E_3 - E_2 \in \mathcal{E}(\mathcal{N})$ and so $F_1 + F_3 - F_2 \in \mathbb{C}I + \mathcal{E}(\mathcal{N})$. However, if $F_1 < F_3 < F_2$ or if $F_3 < F_1$, then $\sigma(F_1 + F_3 - F_2) = \{-1, 0, 1\}$. This is a contradiction, proving (a). Part (b) is similar. ∎

We shall refer to φ itself as being *order preserving* or *order reversing* according as $\hat{\varphi}$ is order preserving or order reversing respectively.

We may now extend the definition of $\hat{\varphi}$ and $\tilde{\varphi}$ to all of $\mathcal{E}(\mathcal{N})$ and $\mathcal{N}$ respectively.

3.10. DEFINITION. If φ: $\mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{M})$ is an order preserving Lie isomorphism, then we define

$$\hat{\varphi}(0) = 0, \qquad \hat{\varphi}(I) = I, \qquad \tilde{\varphi}(\{0\}) = \{0\}, \qquad \tilde{\varphi}(\mathcal{H}) = \mathcal{H}.$$

If φ is order reversing, we define

$$\hat{\varphi}(0) = I, \qquad \hat{\varphi}(I) = 0, \qquad \tilde{\varphi}(\{0\}) = \mathcal{H}, \qquad \tilde{\varphi}(\mathcal{H}) = \{0\}.$$

3.11. THEOREM. *Let $\mathcal{N}$ and $\mathcal{M}$ be nests, and suppose that φ: $\mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{M})$ is a Lie isomorphism. Then either*

(a) *$\tilde{\varphi}$ is a dimension preserving order isomorphism from $\mathcal{N}$ onto $\mathcal{M}$,*

or

(b) *the map $\tilde{\varphi}^{\perp}$ defined via $\tilde{\varphi}^{\perp}(N) = (\tilde{\varphi}(N))^{\perp}$ is a dimension preserving order isomorphism from $\mathcal{N}$ onto $(\mathcal{M}^{\perp})$.*

Proof. We have already established the order properties of $\tilde{\varphi}$. Furthermore, it is obvious that $\tilde{\varphi}^{-1} = \widetilde{\varphi^{-1}}$, thus $\tilde{\varphi}$ is a bijection. All that remains to be done is the dimension argument.

Towards that end, let $N_+ \ominus N$ denote an atom on $\mathcal{N}$ and assume that $\tilde{\varphi}$ is order preserving. If $k < \dim(N_+ \ominus N)$, we can find commuting projections $P_1, P_2, \ldots, P_k$ in $\mathcal{E}(\mathcal{N})$ such that

$$P_0 := P(N) < P_1 < P_2 < \cdots < P_k < P(N_+) =: P_{k+1}.$$

Let $F_i = \hat{\varphi}(P_i)$, $i = 0, 1, 2, ..., k+1$. By Lemma 3.9, we have that

$$F_0 < F_1 < F_2 < \cdots < F_k < F_{k+1}.$$

In particular, $\dim \operatorname{ran}(F_{k+1} - F_0) > k$, and so $\dim(\tilde{\varphi}(N_+) \ominus \tilde{\varphi}(N)) \geqslant \dim(N_+ \ominus N)$.

Applying the identical argument to φ^{-1} proves the reverse inequality and hence $\tilde{\varphi}$ is dimension preserving, as claimed.

The argument for case (ii) is completely analogous and is omitted. ▮

3.12. We shall reduce our discussions to the case where the Lie isomorphism is order preserving. Indeed if φ is order reversing, we may consider instead the map $\psi: \mathcal{T}(\mathcal{N}^{\perp}) \to \mathcal{T}(\mathcal{M})$ define by $\psi(T) = -\varphi(JT^*J)$, where J is the conjugate-linear involution in Lemma 2.3. It is easy to verify that ψ is a Lie isomorphism and is order preserving.

3.13. Proposition. *Let $\mathcal{N}$ and $\mathcal{M}$ be nests and suppose $\varphi: \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{M})$ is an order preserving Lie isomorphism. If $\mathcal{B}$ is a Boolean algebra of idempotents in $\mathcal{T}(\mathcal{N})$, then $\hat{\varphi}$ restricted to $\mathcal{B}$ is a Boolean algebra isomorphism of $\mathcal{B}$ onto a boolean algebra of idempotents in $\mathcal{T}(\mathcal{M})$.*

Proof. We have already seen that $\hat{\varphi}$ is bijective. We observe that $\hat{\varphi}(\mathcal{B})$ is a commuting set of idempotents. Therefore, in order to verify that $\hat{\varphi}$ is a Boolean algebra isomorphism, it suffices to show that for each $E_1, E_2 \in \mathcal{B}$,

$$\text{(i)} \quad \hat{\varphi}(E_1 \wedge E_2) = \hat{\varphi}(E_1) \wedge \hat{\varphi}(E_2),$$

and

$$\text{(ii)} \quad \hat{\varphi}(I - E_1) = I - \hat{\varphi}(E - 1).$$

(i) Now $E_1 \wedge E_2 = E_1 E_2 \leqslant E_j$, for $j = 1, 2$. Since $\hat{\varphi}$ is increasing, it follows that $\hat{\varphi}(E_1 E_2) \leqslant \hat{\varphi}(E_j)$, and since $\hat{\varphi}(E_1)$ and $\hat{\varphi}(E_2)$ commute, we get

$$\hat{\varphi}(E_1 \wedge E_2) \leqslant \hat{\varphi}(E_1)\, \hat{\varphi}(E_2) = \hat{\varphi}(E_1) \wedge \hat{\varphi}(E_2).$$

The same argument applied to $\hat{\varphi}^{-1} = \widehat{\varphi^{-1}}$ proves the reverse inclusion, and assertion (i) is proved.

(iii) There is nothing to prove when $E_1 = 0$ or I. Otherwise, $\varphi(I - E_1) = \kappa I - \varphi(E_1)$. It follows that $\hat{\varphi}(I - E_1)$ and $I - \hat{\varphi}(E_1)$ are two nontrivial idempotents whose difference is a scalar and hence must be equal. ▮

Recall that an abstract Boolean algebra B is said to be *σ-complete* if each countable subset of B has a greatest lower bound and a least upper bound in B.

3.14. PROPOSITION. *Let $\mathcal{N}$ be a nest. There exists a σ-complete Boolean algebra of projections in $\mathcal{T}(\mathcal{N})$ containing $\mathcal{P}(\mathcal{N})$, the set of all projections on members of the nest.*

Proof. Let $\mathcal{A}$ be the von Neumann algebra generated by $\mathcal{P}(\mathcal{N})$, and let $\mathcal{B}$ be the Boolean algebra of all projections in $\mathcal{A}$. That $\mathcal{B}$ is σ-complete follows easily from standard facts about abelian von Neumann algebras, e.g. [T; Lemma 1.1]. ▮

Given a collection $\mathcal{Q}$ of commuting idempotents, we shall denote by $BA(\mathcal{Q})$ the Boolean algebra of idempotents generated by $\mathcal{Q}$.

3.15. PROPOSITION. *Let $\mathcal{N}$ and $\mathcal{M}$ be nests, and $\varphi\colon \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{M})$ be an order preserving Lie isomorphism. Then $\mathcal{B}_\varphi := BA(\hat{\varphi}(\mathcal{P}(\mathcal{N})))$ is a bounded Boolean algebra of idempotents.*

Proof. By Proposition 3.14, $\mathcal{P}(\mathcal{N})$ in contained in a σ-complete Boolean algebra $\mathcal{B}$, and by Proposition 3.13, $\hat{\varphi}$ is a Boolean isomorphism between $\mathcal{B}$ and $\hat{\varphi}(\mathcal{B})$. Consequently $\hat{\varphi}(\mathcal{B})$ is a σ-complete and by [DS, Lemma XVII.3.3], it must be bounded. Since $\mathcal{B}_\varphi$ in included in $\hat{\varphi}(\mathcal{B})$, it must also be bounded. ▮

3.17. We are now in a position to employ the following

LEMMA [DS, Lemma XV.6.2]. *Let $\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_k$ be a finite collection of commuting, bounded Boolean algebras of idempotents in a Hilbert space $\mathcal{H}$. Then there exists a bounded self-adjoint invertible operator $R \in \mathcal{B}(\mathcal{H})$ such that RER^{-1} is an orthogonal projection for every E in the algebra determined by the Boolean algebras $\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_k$.*

In fact, we shall only require the case where $k=1$ and $\mathcal{B}_1 = \mathcal{B}_\varphi = BA(\hat{\varphi}(\mathcal{P}(\mathcal{N})))$.

3.18. COROLLARY. *Let $\varphi\colon \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{M})$ be an order preserving Lie isomorphism of nest algebras. Then there exists an invertible operator R in $\mathcal{B}(\mathcal{H})$ which may be taken to be self-adjoint such that $R\mathcal{B}_\varphi R^{-1}$ is a Boolean algebra of orthogonal projections on $\mathcal{H}$.*

It follows immediately from this result that

$$R\tilde{\varphi}(\mathcal{N})\, R^{-1}$$

is a nest, which we shall denote by $\mathcal{M}_1$.

3.19. This is a good time to stand back and to take stock of the situation. By composing φ with the transpose map if necessary, we have obtained an order preserving Lie isomorphism from $\mathscr{T}(\mathscr{N})$ to $\mathscr{T}(\mathscr{M})$. Keeping in mind that $\mathrm{Ad}_R(T)=R^{-1}TR$, we therefore have the following maps,

$$\begin{array}{ccccc} \mathscr{P}(\mathscr{N}) & \xrightarrow{\hat{\varphi}} & \hat{\varphi}(\mathscr{P}(\mathscr{N})) & \xrightarrow{\mathrm{Ad}_R} & \mathscr{P}(\mathscr{M}_1) \\ BA(\mathscr{P}(\mathscr{N})) & \xrightarrow{\hat{\varphi}} & \mathscr{B}_\varphi & \xrightarrow{\mathrm{Ad}_R} & BA(\mathscr{P}(\mathscr{M}_1)) \\ \mathscr{T}(\mathscr{N}) & \xrightarrow{\varphi} & \mathscr{T}(\mathscr{M}) & \xrightarrow{\mathrm{Ad}_R} & \mathscr{T}(\mathscr{M}_2), \end{array}$$

where $\mathscr{M}_2=\{RM\colon M\in\mathscr{M}\}$. But if $E\in\hat{\varphi}(\mathscr{P}(\mathscr{N}))$, then $\operatorname{ran} E=M\in\mathscr{M}$, and it is readily verified that RER^{-1} is the orthogonal projection onto RM. It follows from the surjectivity of $\tilde{\varphi}\colon\mathscr{N}\to\mathscr{M}$ that $\mathscr{M}_2=\mathscr{M}_1$, and that

$$\nu := \mathrm{Ad}_R\circ\varphi$$

defines a Lie isomorphism of $\mathscr{T}(\mathscr{N})$ onto $\mathscr{T}(\mathscr{M}_1)$ satisfying $\nu(\mathscr{P}(\mathscr{N}))\subseteq \mathbb{C}I+\mathscr{P}(\mathscr{M}_1)$. Furthermore, the associated map $\tilde{\nu}$ is a dimension preserving order isomorphism of $\mathscr{N}$ onto $\mathscr{M}_1$.

By the Similarity Theorem [D1, Theorem 13.20], we have that $\mathscr{N}$ is similar to $\mathscr{M}_1$ via an invertible operator S. This yields a new map

$$\rho := \mathrm{Ad}_S\circ\nu=\mathrm{Ad}_S\circ\mathrm{Ad}_R\circ\varphi$$

such that $\rho\colon\mathscr{T}(\mathscr{N})\to\mathscr{T}(\mathscr{N})$ is an order preserving Lie automorphism of $\mathscr{T}(\mathscr{N})$ satisfying

$$\rho(P(N))=\lambda_{P(N)}I+P(N)$$

for each $N\in\mathscr{N}$. We have reduced the problem of classifying Lie isomorphisms between two nest algebras to that of characterizing Lie automorphisms of a fixed nest algebra which preserve the nest projections modulo the scalars. This is our next goal.

3.20. PROPOSITION. *Let $\mathscr{N}$ be a nest and $P\in\mathscr{P}(\mathscr{N})$. Suppose that $\rho\colon\mathscr{T}(\mathscr{N})\to\mathscr{T}(\mathscr{N})$ is a Lie automorphism and that for each $Q\in\mathscr{P}(\mathscr{N})$,*

$$\rho(Q)=\lambda_Q I+Q$$

for some $\lambda_Q\in\mathbb{C}$. Then

(a) $\rho(P\mathscr{T}(\mathscr{N})(I-P))=P\mathscr{T}(\mathscr{N})(I-P)$;

(b) $\rho(P\mathscr{T}(\mathscr{N})\,P)+\mathbb{C}I=P\mathscr{T}(\mathscr{N})\,P+\mathbb{C}I.$

Proof. (a) If $Z \in P\mathcal{T}(\mathcal{N})(I-P)$, then $Z=[P, Z]$, and hence $\rho(Z) = [\rho(P), \rho(Z)] = [\lambda_P I + P, \rho(Z)] = [P, \rho(Z)]$, implying that $\rho(Z) \in P\mathcal{T}(\mathcal{N})(I-P)$. This proves the inclusion $\rho(P\mathcal{T}(\mathcal{N})(I-P)) \subseteq P\mathcal{T}(\mathcal{N})(I-P)$. Applying the same argument to ρ^{-1} proves the reverse inclusion.

(b) Let $Q \in \mathcal{P}(\mathcal{N})$ be any projection satisfying $Q \geqslant P$. Then for all $T \in P\mathcal{T}(\mathcal{N})\, P$, we have $[T, Q]=0$ and so $[\rho(T), \rho(Q)]=0$. But $\rho(Q) = \lambda_Q I + Q$, and so we have $[\rho(T), Q]=0$. Since this is true for every $Q \in \mathcal{P}(\mathcal{N})$ satisfying $Q \geqslant P$, we see that

$$(I-P)\,\rho(T)(I-P) \in \mathcal{D}(\mathcal{N}), \tag{1}$$

the diagonal of $\mathcal{N}$. Observe also that $[T, P]=0$ implies that P commutes with $\rho(T)$ and so we also have

$$P\rho(T)(I-P)=0.$$

Since $\mathcal{L}_P := P\mathcal{T}(\mathcal{N})$ is a Lie ideal in $\mathcal{T}(\mathcal{N})$, its image $\rho(\mathcal{L}_P)$ under ρ is a Lie ideal in $\mathcal{T}(\mathcal{N})$, and we see from part (a) and (1) above that

$$(I-P)\,\rho(\mathcal{L}_P)(I-P) \subseteq \mathcal{D}(\mathcal{N}).$$

Therefore the compression $\mathcal{L}_1$ of $\rho(\mathcal{L}_P)$ to $(I-P)\,\mathcal{H}$ is a Lie ideal in the restricted nest algebra $\mathcal{T}((I-P)\,\mathcal{N})$ on $(I-P)\,\mathcal{H}$, and $\mathcal{L}_1$ is contained in the diagonal of $\mathcal{T}((I-P)\,\mathcal{N})$. By [HMS; Theorem 12] applied to the closure of $\mathcal{L}_1$ in the weak operator topology we see that every operator in $\mathcal{L}_1$ is a scalar multiple of the identity on $(I-P)\,\mathcal{H}$. Thus for $T \in P\mathcal{T}(\mathcal{N})\, P$, we get $\rho(T) = \lambda_T I + S$, where $S \in P\mathcal{T}(\mathcal{N})\, P$. In other words,

$$\rho(P\mathcal{T}(\mathcal{N})\, P) + \mathbb{C}I \subseteq P\mathcal{T}(\mathcal{N})\, P + \mathbb{C}I.$$

Applying the same argument to ρ^{-1} we obtain the reverse inequality, and equality follows, as claimed. ∎

3.21. Proposition. *Let $\mathcal{N}$ be a nest and $P \in \mathcal{P}(\mathcal{N})$, $P \neq 0, I$. Suppose that $\rho\colon \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{N})$ is an order preserving Lie automorphism of $\mathcal{T}(\mathcal{N})$, and that for all $Q \in \mathcal{P}(\mathcal{N})$, $\rho(Q) = \lambda_Q I + Q$. Then for every $T \in P\mathcal{T}(\mathcal{N})\, P$, we can write*

$$\rho(T) = \psi_1(T) + \gamma_1(T)\, I,$$

where ψ_1 is an algebra automorphism of $P\mathcal{T}(\mathcal{N})\, P$ and γ_1 is a linear functional on $P\mathcal{T}(\mathcal{N})\, P$ that annihilates all commutators $[A, B]$, for $A, B \in P\mathcal{T}(\mathcal{N})\, P$.

Proof. By Proposition 3.20, we see that if $T \in P\mathcal{T}(\mathcal{N})\,P$, then $\rho(T) \in P\mathcal{T}(\mathcal{N})\,P + \mathbb{C}I$. We define $\gamma_1(T)$ to be the scalar that appears in $(I-P)\,\rho(T)(I-P)$, and set $\psi_1(T) = \rho(T) - \gamma_1(T)\,I$. Clearly ψ_1 and γ_1 are linear, and by Proposition 3.20, we get that ψ_1 maps $P\mathcal{T}(\mathcal{N})\,P$ onto itself.

It remains only to show that ψ_1 is an algebra homomorphism. First let $T \in P\mathcal{T}(\mathcal{N})\,P$ and $X \in P\mathcal{T}(\mathcal{N})(I-P)$. From Proposition 3.20, we have that $\rho(X) \in P\mathcal{T}(\mathcal{N})(I-P)$. Thus

$$\rho(TX) = \rho([T, X]) = [\rho(T), \rho(X)] = \psi_1(T)\,\rho(X).$$

Thus if $T_1, T_2 \in P\mathcal{T}(\mathcal{N})\,P$, we obtain

$$\psi_1(T_1T_2)\,\rho(X) = \rho(T_1T_2X) = \psi_1(T_1)\,\rho(T_2X) = \psi_1(T_1)\,\psi_1(T_2)\,\rho(X).$$

Since $X \in P\mathcal{T}(\mathcal{N})(I-P)$ is arbitrary and since by Proposition 3.19, ρ maps $P\mathcal{T}(\mathcal{N})(I-P)$ onto itself, we get

$$\psi_1(T_1T_2) = \psi_1(T_1)\,\psi_1(T_2). \quad \blacksquare$$

3.22. *Remark.* Similarly, we can show that

$$(I-P)\,\rho(I-P) = \psi_2 + \gamma_2 I,$$

where ψ_2 is an algebra automorphism of $(I-P)\,\mathcal{T}(\mathcal{N})(I-P)$ and γ_2 is a linear functional annihilating all commutators in $(I-P)\,\mathcal{T}(\mathcal{N})(I-P)$.

3.23. By combining the results in 3.20, 3.21, 3.22, we see that with respect to the decomposition $\mathcal{H} = P\mathcal{H} \oplus (I-P)\,\mathcal{H}$, we have

$$\rho\begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} = \begin{bmatrix} \psi_1(T_{11}) & \xi(T_{12}) \\ 0 & \psi_2(T_{22}) \end{bmatrix} + \beta\left(\begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}\right) I,$$

where $\beta\colon \mathcal{T}(\mathcal{N}) \to \mathbb{C}$ is a linear functional that annihilates $P\mathcal{T}(\mathcal{N})(I-P)$ and ξ is linear. It is now easy to verify that β annihilates all commutators in $\mathcal{T}(\mathcal{N})$. Indeed if $T = \left[\begin{smallmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{smallmatrix}\right]$ is a commutator, then it is obvious that T_{11} and T_{22} are commutators in the corresponding restricted algebras and so $\beta(T) = \gamma_1(T_{11}) + \gamma_2(T_{22}) = 0$.

Since ψ_1 and ψ_2 are nest algebra isomorphisms, they are spatially implemented [R2; Corollary 17.13], and hence there exist invertible operators $V \in \mathcal{B}(P\mathcal{H})$ and $W \in \mathcal{B}((I-P)\,\mathcal{H})$ such that

$$\psi_1(T_1) = V^{-1}T_1V, \qquad \text{for} \quad T_1 \in P\mathcal{T}(\mathcal{N})\,P$$

$$\psi_2(T_2) = W^{-1}T_2W, \qquad \text{for} \quad T_2 \in (I-P)\,\mathcal{T}(\mathcal{N})(I-P).$$

Now let $U = V \oplus W$ and $\rho_0 = (\mathrm{Ad}_U)^{-1}\rho$. It follows that $\rho_0(T) = \pi(T) + \beta(T)I$, where

$$\pi \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} = \begin{bmatrix} T_{11} & \mu(T_{12}) \\ 0 & T_{22} \end{bmatrix}$$

for a linear map μ. Since β annihilates all commutators, we get that π also is a Lie homomorphism. It is not hard to see that it is also bijective.

3.24. LEMMA. *Let $\mathcal{N}$ be a nest, $P \in \mathcal{P}(\mathcal{N})$, $P \neq 0, I$, and with respect to the decomposition $\mathcal{H} = P\mathcal{H} \oplus (I-P)\mathcal{H}$, let*

$$\pi \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \begin{bmatrix} A & \mu(B) \\ 0 & C \end{bmatrix}$$

be a Lie isomorphism of $\mathcal{T}(\mathcal{N})$. Then

(a) *μ is a left $\mathcal{T}(P\mathcal{N})$-module map, and a right $\mathcal{T}((I-P)\mathcal{N})$-module map.*

(b) *π is an algebra automorphism of $\mathcal{T}(\mathcal{N})$.*

Proof. (a) Let $A \in \mathcal{T}(P\mathcal{N})$, $B \in \mathcal{B}((I-P)\mathcal{H}, P\mathcal{H})$. Then

$$\begin{bmatrix} 0 & \mu(AB) \\ 0 & 0 \end{bmatrix} = \pi \begin{bmatrix} 0 & AB \\ 0 & 0 \end{bmatrix} = \pi \left[\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \right]$$

$$= \left[\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \mu(B) \\ 0 & 0 \end{bmatrix} \right] = \begin{bmatrix} 0 & A\mu(B) \\ 0 & 0 \end{bmatrix}.$$

Thus $\mu(AB) = A\mu(B)$. Similarly, $\mu(BD) = \mu(B)D$ for $D \in \mathcal{T}((I-P)\mathcal{N})$. A straightforward computation now shows that π is an automorphism. ∎

3.25. *Observation.* The fact that μ is a module map as above now implies that π is an algebra isomorphism. Again, by [R2], all automorphisms of nest algebras are spatial, and hence

$$\pi(T) = \mathrm{Ad}_Z(T) = Z^{-1}TZ$$

for some invertible operator $Z \in \mathcal{B}(\mathcal{H})$.

3.26. We are now in a position to complete the proof of the Main Theorem.

Proof. Working backwards, we shall piece together Y from the succession of similarities we have employed in the preceding lemmas and propositions.

From Observation 3.25, we have $\pi(T) = Z^{-1}TZ$, while from paragraph 3.23, we get

$$\rho(T) = U^{-1}\pi(T)\,U + \beta(T)\,I,$$

Hence $\rho(T) = U^{-1}Z^{-1}TZU + \beta(T)\,I$. In paragraph 3.19, we saw that $\rho = \mathrm{Ad}_S \circ \mathrm{Ad}_R \circ \varphi$, and hence

$$\varphi(T) = R^{-1}S^{-1}U^{-1}Z^{-1}TZUSR + \beta(T)\,I.$$

We simply let $Y = ZUSR$ to get

$$\varphi(T) = Y^{-1}TY + \beta(T)\,I,$$

as claimed. As for the structure of β, a simple computation shows that it must annihilate commutators, and that for φ to be injective, $\beta(I) \neq -1$. Also, since the scalars belong to the range of φ, we obtain

$$\mathcal{T}(\mathcal{M}) = \operatorname{ran}\varphi = Y^{-1}\mathcal{T}(\mathcal{N})\,Y = \mathcal{T}(Y^{-1}\mathcal{N}).$$

By the reflexivity of nests (see [R1] or [D1, Theorem 2.9]), we conclude that Y takes $\mathcal{M}$ onto $\mathcal{N}$. ∎

ACKNOWLEDGMENTS

The authors thank Ken Davidson for several useful remarks which helped improve this paper.

This research was conducted while the first author spent part of his sabbatical at the University of Victoria. He takes this opportunity to thank the Department of Mathematics and Statistics of the University of Victoria for their hospitality during his stay.

REFERENCES

[CJR] M. D. Choi, A. A. Jafarian, and H. Radjavi, Linear maps preserving commutativity, *Linear Algebra Appl.* **87** (1987), 227–241.

[D1] K. R. Davidson, “Nest Algebras,” Pitman Research Notes in Mathematics, Vol. 191, Longman, London/New York, 1988.

[D2] K. R. Davidson, Similarity and compact perturbations of nest algebras, *J. Reine Angew. Math.* **348** (1984), 286–294.

[Đ] D. Đoković, Automorphisms of the Lie algebra of upper triangular matrices over a connected commutative ring, *J. Algebra* **170** (1994), 101–110.

[DS] N. Dunford and J. T. Schwartz, “Linear Operators III,” Interscience, New York, 1971.

[HMS] T. D. Hudson, L. W. Marcoux, and A. R. Sourour, Lie ideals in triangular operator algebras, *Trans. Amer. Math. Soc.* **350**, 3321–3339.

[Ha] P. R. Halmos, "A Hilbert Space Problem Book," 2nd ed., Graduate Texts in Mathematics, Vol. 19, Springer-Verlag, New York/Heidelberg/Berlin, 1982.

[Hu] J. E. Humphries, "Introduction to Lie Algebras and Representation Theory," Graduate Texts in Mathematics, Vol. 9, Springer-Verlag, New York/Heidelberg/Berlin, 1972.

[L] D. R. Larson, Nest algebras and similarity transformations, *Ann. Math.* **121** (1985), 409–427.

[LP] D. R. Larson and D. Pitts, Idempotents in Nest algebras, *J. Funct. Anal.* **97** (1991), 162–193.

[MS] L. W. Marcoux and A. R. Sourour, Commutativity preserving linear maps and Lie automorphisms of triangular matrix algebras, *Linear Algebra Appl.* **288** (1999), 89–104.

[Ma1] W. S. Martindale, Lie isomorphisms of primitive rings, *Proc. Amer. Math. Soc.* **14** (1963), 909–916.

[Ma2] W. S. Martindale, Lie isomorphisms of simple rings, *J. London Math. Soc.* **44** (1969), 213–221.

[Mi] C. R. Miers, Lie isomorphisms of operator algebras, *Pacific J. Math.* **38** (1971), 717–735.

[R1] J. R. Ringrose, On some algebras of operators, *Proc. London Math. Soc.* **15** (1965), 61–83.

[R2] J. R. Ringrose, On some algebras of operators II, *Proc. London Math. Soc.* **16** (1966), 395–402.

[T] D. M. Topping, "Lectures on von Neumann Algebras," Van Nostrand, London/New York, 1971.

Printed in Belgium