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Minimal Quadrangulations of Nonorientable Surfaces

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1. INTRODUCTION

Let N be a compact 2-manifold. A polyhedron on N is called a quadrangulation if each face of the polyhedron is a quadrangle (square) with four distinct vertices; no two vertices are joined by more than one edge, and the intersection of any two distinct squares is either empty or at most one edge and at most three vertices. A quadrangulation of N is called *minimal* if the number of squares is minimal. For instance, Fig. 1 shows a minimal quadrangulation of the projective plane. We shall denote the number of squares in a minimal quadrangulation of N by $\psi(N)$. We shall construct quadrangular embeddings of K_n for $n \equiv 1 \pmod 4$ and the general octahedral graph. Both of these embeddings determine polyhedra which are minimal quadrangulations of the surfaces.

2. LOWER BOUND

Let N be a surface. Let Q be a quadrangulation of N, and let V, E, F be the number of vertices, edges, and faces of Q, respectively. Then

$$V - E + F = E(N) \tag{1}$$

is the Euler characteristic of N. Since Q is a quadrangulation, each pair of vertices is joined by at most one edge. So Q has no more edges than pairs of vertices. That is,

$$E \leqslant \binom{V}{2}. \tag{2}$$

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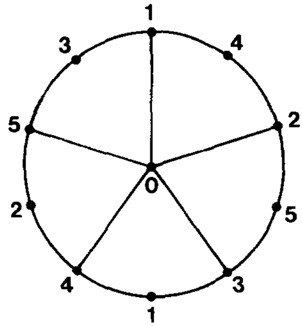

FIG. 1. A minimal quadrangulation of the projective plane.

Since each face is a square, and each edge is incident with two squares,

$$4F = 2E. \tag{3}$$

Together with (1) we obtain

$$2V - E = 2E(N). \tag{4}$$

From (2) it follows that

$$4V - V(V-1) \leqslant 4E(N). \tag{5}$$

This quadratic inequality has the solution

$$V \geqslant \left\lceil \frac{5 + \sqrt{25 - 16E(N)}}{2} \right\rceil. \tag{6}$$

From (1) and (3) we obtain

$$F = V - E(N). \tag{7}$$

Now we assume Q is a minimal quadrangulation of N. Then $F = \psi(N)$ and from (6) and (7) we obtain

$$\psi(N) \geqslant \left\lceil \frac{5 + \sqrt{25 - 16E(N)}}{2} \right\rceil - E(N).$$

Since $E(N_q) = 2 - q$, it follows that

$$\psi(N_q) \geqslant \left\lceil \frac{1 + \sqrt{16q - 7}}{2} \right\rceil + q. \tag{8}$$

3. Small Cases

THEOREM 1. *The number of squares in a minimal quadrangulation of the projective plane is five.*

Proof. According to the lower bound formula, $\psi(N_1) \geqslant 3$. Assume $\psi(N_1) = 3$. Then from (7)

$$F = 3, \qquad V = 4.$$

Then any two different squares have all four vertices in common; thus this is not a quadrangulation.

Now assume $\psi(N_1) = 4$. Then from (3) and (7) we obtain

$$F = 4, \qquad V = 5, \qquad E = 6.$$

So the 1-skeleton of the quadrangulation Q is a graph with five vertices and six edges only. That means there is at least one vertex of degree at most two. But this implies there are at least two squares with two edges in common, which contradicts the definition of quadrangulation. Thus $\psi(N_1) \geqslant 5$, and Fig. 1 shows a quadrangulation of N_1 with five squares; therefore $\psi(N_1) = 5$. ∎

THEOREM 2. *The number of squares in a minimal quadrangulation of Klein's bottle is six.*

Proof. According to the lower bound formula, $\psi(N_2) \geqslant 5$. Assume $\psi(N_2) = 5$, then from (3) and (2) we obtain

$$F = 5, \qquad V = 5, \qquad E = 10.$$

So the 1-skeleton of Q must be the complete graph K_5.

Consider the neighborhood of vertex 1 in Fig. 2. The unlabelled vertices are 5, 4, 3, 2 in clockwise cyclic order beginning at left above (or at right above). That means there are the counterclockwise oriented squares (1 3 5 4) (1 2 4 3) (1 5 3 2) (1 4 2 5) and one more, namely (2 5 4 3). But

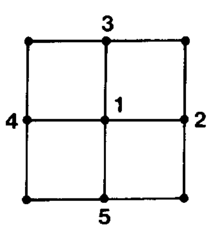

FIGURE 2

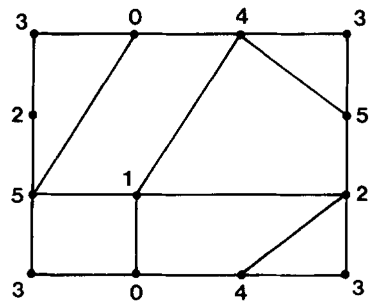

FIG. 3. A minimal quadrangulation of Klein's bottle.

these squares form an orientable polyhedron, which must be a polyhedron on the torus.

Thus $\psi(N_2) \geqslant 6$, and Fig. 3 shows a quadrangulation of N_2 with six squares; therefore $\psi(N_2) = 6$. ▮

Figures 4 and 5 show minimal quadrangulations of N_3 and N_4, respectively, which achieve the lower bound. If we read off the neighbors of vertex 0 in Fig. 4 cyclically we obtain the row

0. 1, 4, 5, 2, 6.

A *combinatorial scheme* for the embedding consists of such listings for each vertex. The scheme for the embedding in Fig. 4 is

0. 1, 4, 5, 2, 6
1. 2, 0, 3, 6
2. 4, 1, 0, 6, 5
3. 1, 6, 5, 4
4. 0, 2, 6, 3
5. 3, 2, 6, 0
6. 2, 5, 4, 1, 0, 3.

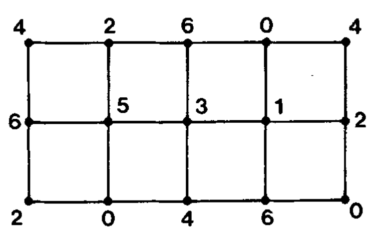

FIGURE 4

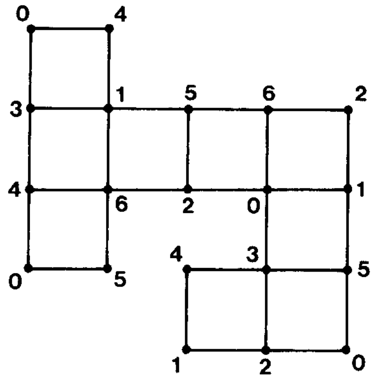

FIGURE 5

This scheme satisfies the following rule:

RULE Q.

If in row i.	$\ldots, j, k, \ldots$
or i.	$\ldots, k, j, \ldots$
and in row k.	$\ldots, i, l, \ldots$
or k.	$\ldots, l, i, \ldots,$
then in row j.	$\ldots, l, i, \ldots$
or j.	$\ldots, i, l, \ldots$
and in row l.	$\ldots, k, j, \ldots$
or l.	$\ldots, j, k, \ldots$.

If Σ is a combinatorial scheme of a graph G which satisfies rule Q, then there exists a quadrangular embedding of G into a surface, and the scheme of the embedding is precisely Σ (*Edmonds' technique*).

4. QUADRANGULAR EMBEDDINGS OF K_n FOR $n \equiv 1 \pmod 4$

The following is the scheme for a quadrangular embedding of K_{13} into a nonorientable surface of genus 28:

$$
\begin{array}{rllllllllllll}
0. & 1, & 9, & 11, & 4, & 5, & 10, & 8, & 6, & 12, & 7, & 2, & 3 \\
1. & 2, & 10, & 12, & 5, & 6, & 11, & 9, & 7, & 0, & 8, & 3, & 4 \\
2. & 3, & 11, & 0, & 6, & 7, & 12, & 10, & 8, & 1, & 9, & 4, & 5 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
12. & 0, & 8, & 10, & 3, & 4, & 9, & 7, & 5, & 11, & 6, & 1, & 2.
\end{array} \tag{9}
$$

Note that in scheme (9) row i is obtained from row 0 by adding i to each entry of row 0, the addition being that in $\mathbb{Z}_{13}$. Thus once we know row 0, we have all the information necessary. How did we obtain row 0? We use the powerful technique of current graphs. The theory of current graphs is treated in detail in Ringel's book [4]. To obtain row 0 of scheme (9) we use the current graph pictured in Fig. 6. This current graph is called a *cascade*, since there are broken arcs. Each arc (directed edge) carries an element of $\mathbb{Z}_{13}$ called the *current* of the arc. The vertices may be black (clockwise rotation) or white (counterclockwise rotation).

Now let a traveller walk along an arc, say the arc with current 1, in the direction of the arrow. She is supposed to continue her walk by taking the arc to the right when she reaches a white vertex and the left arc when she reaches a black vertex. This is indicated by the solid line in Fig. 6. However, if the traveller passes through a broken arc, all directions are reversed. So after passing through the arc labelled 1, she will take the arc to the left when she reaches a white vertex and to the right when she reaches a black vertex, until she passes through another broken arc. This "opposite" behavior is indicated by the broken line in Fig. 6, right. Thus as in Fig. 6 the traveller completes a walk (circuit) after traversing arcs with currents 1, -4, -2, 4, 5, -3, -5, 6, -1, -6, 2, 3. Normally we record a positive number when the arc is traversed in the direction of the arrow and a negative number when the arc is traversed in the direction opposite the direction of the arrow. But these instructions are reversed after passing through a broken arc. Thus after passing through the arc labelled 1 in Fig. 6, we record the next number as -4, even though we are traversing the arc in the same direction as the arrow. Passing through another broken arc cancels the opposite instructions. The cycle $(1, -4, -2, 4, 5, -3, -5, 6, -1, -6, 2, 3)$ is called the *log* of the circuit. When we replace each negative number by its positive counterpart in $\mathbb{Z}_{13}$, we obtain row 0 of scheme (9). A choice of clockwise and counterclockwise vertices in a current graph which induces just one circuit is called a *circular rotation*.

The cascades we use all satisfy the following:

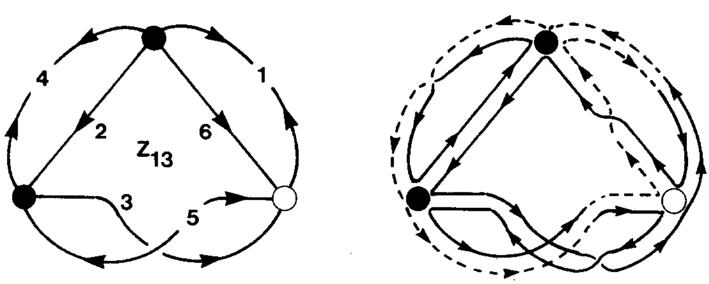

FIGURE 6

(C1). Each vertex has valence 4.

(C2). Each of the elements 1, 2, ..., $\lfloor n/2 \rfloor$ is a current of exactly one arc if $\mathbb{Z}_n$ is the group in use.

(C3). At each vertex the sum of the inward flowing currents equals the sum of the outward flowing currents (Kirchoff's current law).

(C4). The given rotation is circular.

(C5). The half arcs of a broken arc are both oriented toward the midpoint of the arc or both away from the midpoint.

(C6). The sum of all the currents of the broken arcs is 0 in $\mathbb{Z}_n$.

Properties C1 and C3 ensure that each face of the embedding is a square (i.e., that the resulting scheme satisfies rule Q), property C2 ensures that each vertex in the embedding is adjacent to every other vertex exactly once, and property C4 ensures that every line in the scheme is just one cycle.

THEOREM 3. *If* $n \equiv 1 (\mathrm{mod}\ 4)$ *then there exists a quadrangular embedding of* K_n *into a nonorientable surface, and the polyhedron determined by the embedding is a minimal quadrangulation of the surface.*

Proof. The cascade in Fig. 7 generates row 0 of the scheme for a

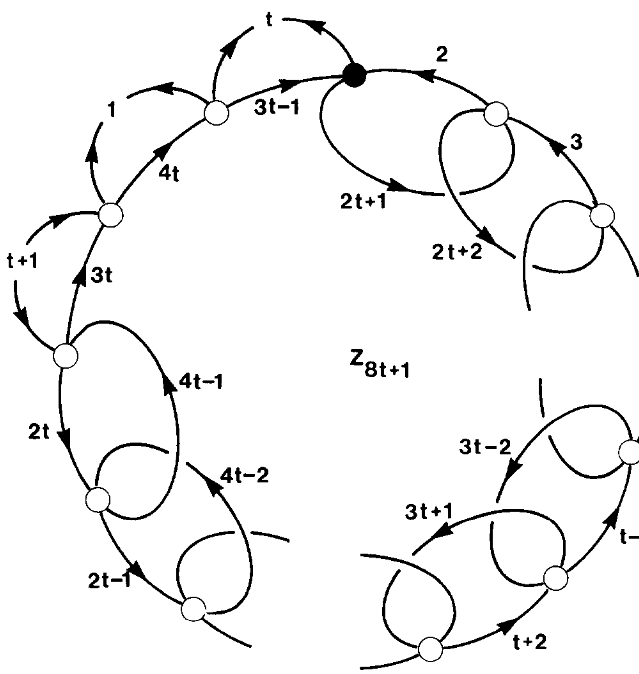

FIGURE 7

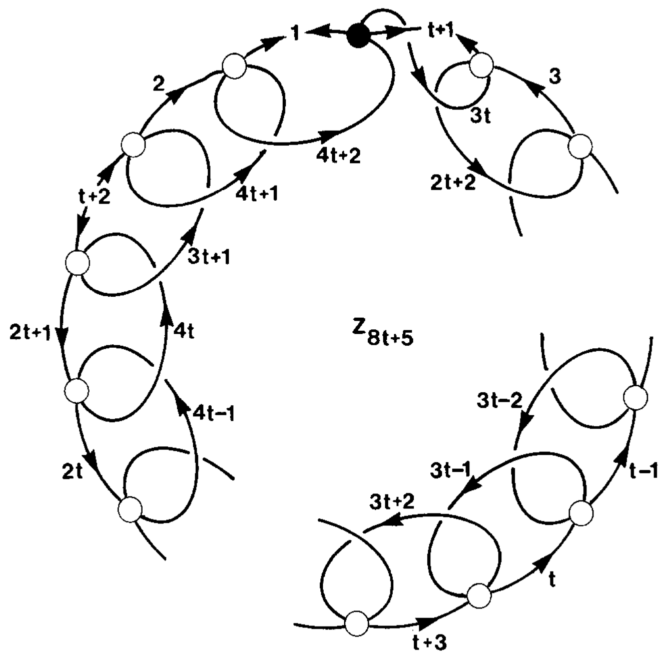

FIGURE 8

quadrangular embedding of K_n for $n=8t+1$ into a nonorientable surface of genus $16t^2-6t+1$ when $t \geqslant 2$. For $t \geqslant 2$ and $n=8t+5$, the cascade in Fig. 8 generates row 0 of the scheme for a quadrangular embedding of K_n into a nonorientable surface of genus $16t^2+10t-2$. By a theorem of Jungerman [2], the generated embedding must be nonorientable if the group used for the currents of a cascade has odd order. For the polyhedra determined by these embeddings, the inequalities (2), (5), (6), and (8) are in fact equalities, and thus the polyhedra are minimal quadrangulations of the nonorientable surfaces of genus $16t^2-6t+1$ and genus $16t^2+10t-2$, respectively. ∎

5. QUADRANGULAR EMBEDDINGS OF THE GENERAL OCTAHEDRAL GRAPH

The *general octahedral graph*, O_{2n}, is the graph with $2n$ vertices denoted by $0, 1, 2, ..., 2n-1$, and two vertices i and j are adjacent if and only if $|i-j| \neq n$. An alternative description is that O_{2n} is the complete n-partite graph with two vertices in each partite set.

Figure 9 shows a quadrangular embedding of O_8 into N_6. The scheme of the embedding is

$$\begin{array}{ll}
0. & 1, 2, 5, 6, 3, 7 \\
4. & 1, 5, 2, 6, 3, 7 \\
1. & 0, 4, 2, 6, 3, 7 \\
5. & 0, 4, 2, 6, 3, 7 \\
2. & 0, 4, 1, 5, 3, 7 \\
6. & 0, 4, 1, 5, 3, 7 \\
3. & 0, 4, 1, 5, 2, 6 \\
7. & 0, 4, 1, 2, 5, 6.
\end{array}$$

In rows 0 and 7 the transposition of the elements 2 and 5 provide a crosscap, ensuring that the embedding is into a nonorientable surface.

THEOREM 4. *There exists a quadrangular embedding of O_{2n} into a nonorientable surface of genus n^2-3n+2 and the polyhedron determined by the embedding is a minimal quadrangulation of the surface.*

Proof. The following scheme is the scheme of an embedding of O_{2n} into a nonorientable surface of genus n^2-3n+2:

$$\begin{array}{rllllllll}
0. & 1, & 2, & n+1, n+2, & 3, & n+3, ..., & n-1, & 2n-1 \\
n. & 1, n+1, & 2, & n+2, & 3, & n+3, ..., & n-1, & 2n-1 \\
1. & 0, & n, & 2, & n+2, & 3, & n+3, ..., & n-1, 2n-1 \\
n+1. & 0, & n, & 2, & n+2, & 3, & n+3, ..., & n-1, 2n-1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \quad \vdots \\
n-1. & 0, & n, & 1, & n+1, & 2, & n+2, ..., & n-2, 2n-2 \\
2n-1. & 0, & n, & 1, & 2, & n+1, & n+2, ..., & n-2, 2n-2.
\end{array}$$

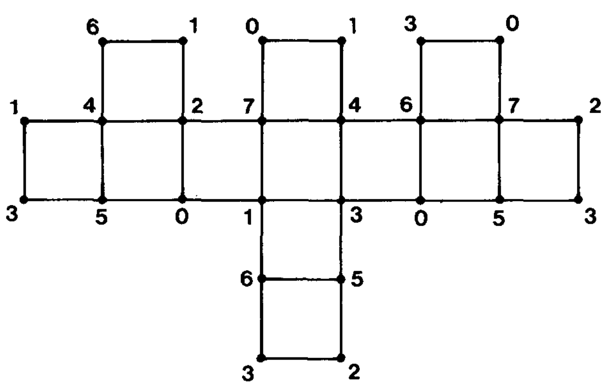

FIGURE 9

The transpositions in rows 0 and $2n-1$ of the elements 2 and $n+1$ provide a crosscap. The number of squares in the embedding is n^2-n. Now we must prove that the polyhedron determined by the embedding is a minimal quadrangulation. If $n \geqslant 4$, then the inequality

$$16n^2-56n+49<16n^2-48n+25<16n^2-40n+25$$

holds. It follows that

$$4n-7<\sqrt{16(n^2-3n+2)-7}<4n-5$$

or

$$-2<\sqrt{16(n^2-3n+2)-7}-4n+5<0,$$

which implies that

$$n^2-n-1<\tfrac{1}{2}\sqrt{16(n^2-3n+2)-7}+\tfrac{1}{2}+n^2-3n+2<n^2-n,$$

and thus

$$\lceil \tfrac{1}{2}(1+\sqrt{(16(n^2-3n+2)-7)}\rceil+n^2-3n+2=n^2-n.$$

Since $q=n^2-3n+2$, the left-hand side of the equation is identical to the right-hand side of (8), and therefore the quadrangulation is minimal. ∎

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