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An elementary proof that the length of $X_1^4 + X_2^4 + X_3^4 + X_4^4$ is 4

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Abstract

An elementary proof is given, to show that the quartic form $X_1^4 + X_2^4 + X_3^4 + X_4^4$ cannot be written as a sum of three squares of real quadratic forms.

MSC: 11E25

1. Introduction

Reznick [3] conjectured that in the polynomial ring $A := \mathbb{R}[X_1, X_2, \ldots, X_{2^k}]$ in 2^k indeterminates, the length of the form

$$S_k := X_1^{2^k} + X_2^{2^k} + \cdots + X_{2^k}^{2^k}$$

is 2^k, i.e., the form $S_k = \sum_{i=1}^{2^k} X_i^{2^k}$ cannot be written as a sum of fewer than 2^k squares of homogeneous polynomials over $\mathbb{R}$. This conjecture is evidently true for $k = 1$. Yiu [5] establish its validity for $k = 2$:

Theorem 0 (cf. Yiu [5]). *The quartic form $X_1^4 + X_2^4 + X_3^4 + X_4^4$ cannot be written as a sum of three squares of real quadratic forms.*

The proof of the above theorem in [5] is a geometric argument. The purpose of this note is to give an elementary proof.

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2. The length of forms in $\mathbb{R}[X_1, X_2, \ldots, X_m]$

Let $A_m := \mathbb{R}[X_1, X_2, \ldots, X_m]$ be the polynomial ring over $\mathbb{R}$ in m indeterminates. The *forms* in A_m mean homogeneous polynomials in A_m.

Given that a non-zero form f in A_m is a sum of squares of forms in A_m, the *length* $l(f)$ of f in A_m is defined as the smallest $n \in \mathbb{N}$ such that $f = \sum_{i=1}^{n} f_i^2$, where $f_1, f_2, \ldots, f_n$ are forms in A_m; if $f = 0$, we define $l(f) := 0$ and if f is not a sum of squares of forms in A_m then we define $l(f) := \infty$.

Remark (cf. Scharlau [4, Theorem 4.3.4]). Let f be a quadratic form in A_m and $\bar{f} = f + X_{m+1}^2$. Then the length of $\bar{f}$ in A_{m+1} is $l(f) + 1$.

It is a motivation for us to consider the length of $f + X_{m+1}^4$ in $A_{m+1} = A_m[X_{m+1}]$, where f is a quartic form in A_m.

Clearly, $l(f) \leqslant l(f + X_{m+1}^4) \leqslant l(f) + 1$.

Example. Let $f := (X_1^2 + X_2^2)(X_1^2 + X_2^2 + X_3^2) \in \mathbb{R}[X_1, X_2, X_3] = A_3$. Then $l(f + X_4^4) = 3 = l(f)$.

Proof. It follows from $f = (X_1^2 + X_2^2)^2 + (X_1X_3)^2 + (X_2X_3)^2$ that $l(f) \leqslant 3$. Suppose that $l(f) < 3$. Then $f = g^2 + h^2$ for some forms $g, h \in A_3$. Since in $\mathbb{R}(X_1, X_2, X_3)$

$$X_1^2 + X_2^2 + X_3^2 = \frac{f}{X_1^2 + X_2^2} = \left(\frac{1}{X_1^2 + X_2^2}\right)^2 \cdot [(gX_1 + hX_2)^2 + (gX_2 - hX_1)^2],$$

then $X_1^2 + X_2^2 + X_3^2$ is a sum of two squares in the quotient field $\mathbb{R}(X_1, X_2, X_3)$ of $\mathbb{R}[X_1, X_2, X_3]$. By [1], Corollary to Theorem A, $X_1^2 + X_2^2 + X_3^2$ is a sum of two squares in $\mathbb{R}[X_1, X_2, X_3]$, but this is impossible. Therefore, $l(f) = 3$.

From

$$f + X_4^4 = (\sqrt{2}X_1X_4 + X_2X_3)^2 + (\sqrt{2}X_2X_4 - X_1X_3)^2 + (X_1^2 + X_2^2 - X_4^2)^2$$

it follows that $l(f + X_4^4) = 3 = l(f)$. □

In the above example, $f = (X_1^2 + X_2^2)(X_1^2 + X_2^2 + X_3^2)$ is reducible in $\mathbb{R}[X_1, X_2, X_3]$. Our main result is:

Theorem 1. *Let f be a quartic form in $A_m = \mathbb{R}[X_1, \ldots, X_m]$. If $l(f) = 3$ and f is irreducible in A_m, then the length of $f + X_{m+1}^4$ in A_{m+1} is $4 = l(f) + 1$.*

To prove the above theorem, we need the following lemma.

Lemma 2. *Let f be a quartic form in A_m with length $l(f) = n > 1$. Then the length of $f + X_{m+1}^4$ in A_{m+1} is n if and only if there exist $U := (a_1, \ldots, a_{n-1})$* and *$V := (b_1, \ldots, b_{n-1})$ in A_m^{n-1}, where a_i are quadratic forms and b_i are linear forms,* *such that*

$$f = (U \cdot U) + \tfrac{1}{4}(V \cdot V)^2 \quad \text{and} \quad U \cdot V = 0.$$

where $U \cdot V := a_1b_1 + a_2b_2 + \cdots + a_{n-1}b_{n-1}$.

Proof. If $l(f + X_{m+1}^4) = n$, then there is an equation

$$f + X_{m+1}^4 = p_1^2 + \cdots + p_n^2, \quad \text{for some quadratic forms } p_i \in A_{m+1}.$$

We may write $p_i = a_i + b_iX_{m+1} + c_iX_{m+1}^2$, where a_i are quadratic forms in A_m, b_i are linear forms in A_m and $c_i \in \mathbb{R}$.

Set $U_0 := (a_1, \dots, a_n)$, $U_1 := (b_1, \dots, b_n)$ and $U_2 := (c_1, \dots, c_n)$. Comparing coefficients of the above equation we get the following system of equations in A_m:

$$f = U_0 \cdot U_0, \quad 2U_1 \cdot U_0 = 0, \quad 2U_2 \cdot U_0 + U_1 \cdot U_1 = 0,$$

$$2U_2 \cdot U_1 = 0, \quad U_2 \cdot U_2 = 1. \tag{$*$}$$

Since $U_2 \cdot U_2 = 1$, there is an orthogonal transformation which maps the vector U_2 to $(0, \dots, 0, -1)$ (cf. [2]). Therefore, we may assume that $U_2 = (0, \dots, 0, -1)$.

Set $U := (a_1, \dots, a_{n-1})$ and $V := (b_1, \dots, b_{n-1})$. Then $(*)$ reduces to

$$f = U \cdot U + a_n^2, \quad U \cdot V = 0, \quad 2a_n = V \cdot V, \quad b_n = 0.$$

Therefore,

$$f = (U \cdot U) + \tfrac{1}{4}(V \cdot V)^2.$$

Conversely, assume there exist $U := (a_1, \dots, a_{n-1})$, $V := (b_1, \dots, b_{n-1})$ in A_m^{n-1}, where a_i are quadratic forms and b_i are linear forms, such that

$$f = U \cdot U + \tfrac{1}{4}(V \cdot V)^2 \quad \text{and} \quad U \cdot V = 0.$$

Then we may write

$$f + X_{m+1}^4 = \sum_{i=1}^{n-1} (a_i + b_iX_{m+1})^2 + \left[\frac{1}{2} \left(\sum_{i=1}^{n-1} b_i^2 \right) - X_{m+1}^2 \right]^2.$$

Hence $n = l(f) \leqslant l(f + X_{m+1}^4) = n$ and $l(f + X_{m+1}^4) = n$. □

Theorem 3. *Let f be a quartic form in $\mathbb{R}[X_1, \dots, X_m]$. If the length $l(f) = 2$, then $l(f + X_{m+1}^4) = 3 = l(f) + 1$.*

Proof. Suppose that $l(f + X_{m+1}^4) < 3$. By Lemma 2 there exist a quadratic form a_1 and a linear form b_1 in A_m, such that $f = a_1^2 + \frac{1}{4}b_1^4$ and $a_1b_1 = 0$, which implies $a_1 = 0$ or $b_1 = 0$, a contradiction. Therefore, $l(f + X_{m+1}^4) = l(f) + 1$. □

Proof of Theorem 1. Suppose that $l(f + X_{m+1}^4) < 4$. Then $l(f + X_{m+1}^4) = l(f) = 3$ because of $l(f) = 3 \leqslant l(f + X_{m+1}^4) \leqslant l(f) + 1 = 4$. By Lemma 2, there exist quadratic forms a_1, a_2 and linear forms b_1, b_2 in A_m, such that

$$f = a_1^2 + a_2^2 + \tfrac{1}{4}(b_1^2 + b_2^2)^2 \tag{1}$$

and

$$a_1 b_1 + a_2 b_2 = 0. \tag{2}$$

Claim. *There is no* $(0,0) \neq (\mu_1, \mu_2) \in \mathbb{R}^2$ *such that* $\mu_1 b_1 + \mu_2 b_2 \neq 0$.

Proof. Suppose that there exists some $(0,0) \neq (\mu_1, \mu_2) \in \mathbb{R}^2$ with $\mu_1 b_1 + \mu_2 b_2 = 0$. We may assume that $\mu_1 \neq 0$. From (2) it follows that $a_1(-\mu_2/\mu_1 b_2) + a_2 b_2 = 0$, which implies that $b_2 = 0$ or $a_2 = \mu_2/\mu_1 a_1$.

If $b_2 = 0$, then $a_1 b_1 = 0$ by (2), which yields that $l(f) \leqslant 2$, a contradiction.

If $a_2 = \mu_2/\mu_1 a_1$, then

$$f = \left(1 + \frac{\mu_2^2}{\mu_1^2}\right) a_1^2 + \frac{1}{4}(b_1^2 + b_2^2)^2.$$

Thus $l(f) \leqslant 2$, this is a contradiction. Hence, our claim has been proved. □

Proof of Theorem 1 (Conclusion). Multiplying both sides of (1) by b_1^2, we obtain

$$f b_1^2 = a_1^2 b_1^2 + a_2^2 b_1^2 + \tfrac{1}{4}(b_1^2 + b_2^2)^2 b_1^2.$$

From (2) it follows that:

$$4 f b_1^2 = 4(-a_2 b_2)^2 + 4a_2^2 b_1^2 + (b_1^2 + b_2^2)^2 b_1^2 = (b_1^2 + b_2^2)(4a_2^2 + b_1^4 + b_1^2 b_2^2). \tag{3}$$

Since f is irreducible then $f \mid (b_1^2 + b_2^2)$ or $f \mid (4a_2^2 + b_1^4 + b_1^2 b_2^2)$. But $\deg(b_1^2 + b_2^2) = 2 < \deg(f) = 4$ then $f \mid (4a_2^2 + b_1^4 + b_1^2 b_2^2)$, i.e., there exists some $0 \neq \lambda \in \mathbb{R}$ such that

$$f = \lambda(4a_2^2 + b_1^4 + b_1^2 b_2^2). \tag{4}$$

From (3) and (4) it follows that $4b_1^2\lambda = (b_1^2 + b_2^2)$, i.e., $(4\lambda - 1)b_1^2 = b_2^2$. Thus $4\lambda - 1 \geqslant 0$ and $b_2 \pm \sqrt{4\lambda - 1}\, b_1 = 0$. This is a contradiction to the above claim. Therefore, $l(f + X_{m+1}^4) = 4$. □

3. Proof of Theorem 0

Lemma 4. *If* $m \geqslant 3$ *then the form* $p_m := X_1^4 + \cdots + X_m^4$ *is irreducible in* $\mathbb{C}[X_1, \ldots, X_m]$.

Proof. We proceed by induction on m.

$m = 3$: Since $X_1^4 + X_2^4 = \prod_{\xi \in \mathbb{C}, \xi^4 = -1}(X_1 - \xi X_2)$, the form p_3 is irreducible by Eisenstein's criterion.

$m \to m + 1$: As $p_{m+1} = p_m + X_{m+1}^4$ and p_m is irreducible, i.e., prime, then by Eisenstein's criterion again, p_{m+1} is irreducible. □

Proof of Theorem 0. It is easily seen that the length of $X_1^4 + X_2^4$ is 2. From Theorem 3 it follows that the length of $X_1^4 + X_2^4 + X_3^4$ is 3. By Lemma 4, we get that $X_1^4 + X_2^4 + X_3^4 + X_4^4$ is irreducible. Hence, the length of $X_1^4 + X_2^4 + X_3^4 + X_4^4$ is 4 by Theorem 1. □

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