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Existence results for boundary value problems of nonlinear fractional differential equations☆

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ABSTRACT

In this paper, we consider the existence of solutions for the nonlinear fractional differential equation

$$^{C}D_{0+}^{\alpha}u(t) + r\,^{C}D_{0+}^{\alpha-1}u(t) = f(t, u(t)), \quad t \in (0, 1)$$

with the boundary value conditions

$$u(0) = u(1), u(\xi) = \eta, \quad \xi \in (0, 1),$$

where $^{C}D_{0+}^{\alpha}$ and $^{C}D_{0+}^{\alpha-1}$ are the standard Caputo derivative with $1 < \alpha \leq 2, r \neq 0$. By using the contraction mapping principle and the Schauder fixed point theorem, some existence results are obtained. In addition, Lemma 2.6 in this paper is a valuable tool in seeking solvability of the fractional differential equations.

1. Introduction

In this paper, we are concerned with the existence and uniqueness of solutions for the fractional differential equation

$$^{C}D_{0+}^{\alpha}u(t) + r\,^{C}D_{0+}^{\alpha-1}u(t) = f(t, u(t)), \quad t \in (0, 1) \tag{1.1}$$

with the boundary value conditions

$$u(0) = u(1), \qquad u(\xi) = \eta, \quad \xi \in (0, 1), \tag{1.2}$$

where $^{C}D_{0+}^{\alpha}$ and $^{C}D_{0+}^{\alpha-1}$ are the standard Caputo derivative with $1 < \alpha \leq 2$, and $r \neq 0$.

Recently, differential equations of fractional order have proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetism, etc. (see [1–5]). There has been a significant development in the study of fractional differential equations and inclusions in recent years; see the monographs of Kilbas et al. [6], Lakshmikantham et al. [7], Podlubny [4], Samko, et al. [8], and the survey by Agarwal et al. [9]. For some recent contributions on fractional differential equations, see [9–22] and the references therein.

However, no contributions exist, as far as we know, concerning the existence of solutions for problem (1.1)–(1.2). Since the boundary value condition $u(0) = u(1)$ in (1.2) involves the periodicity, we cannot expect to transform problem (1.1)–(1.2) into integral equations directly as in the literature mentioned above. So we shall introduce a suitable substitution of the variable to overcome the difficulty. By using the contraction mapping principle and the Schauder fixed point theorem, the existence and uniqueness of solution (1.1)–(1.2) are obtained.

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Another important contribution of the present paper is Lemma 2.6 in this paper, which can be employed conveniently in seeking the solvability of fractional differential equations in order to transform fractional differential equations into integral equations. Therefore, Lemma 2.6 in this paper is a valuable tool. Previously, Shuqin Zhang obtained a result similar to Lemma 2.6 (see Lemma 2.3 in [17]). In recent years, Zhang's Lemma 2.3 has been used in a great deal of literature. However, the conditions in Zhang's Lemma 2.3 are not clear, in other words, Zhang has not definitely given any condition in his Lemma 2.3. In addition, Zhang has not given a clear proof for Lemma 2.3 in [17]. In fact, in order to guarantee that the conclusions in Zhang's Lemma 2.3 are true, certain extra conditions are necessary. In the present paper, we definitely put forward the conditions to ensure that the conclusion in Lemma 2.6 (or in Zhang's Lemma 2.3) is true, and give a clear proof.

The organization of the paper is as follows. In Section 2, we present some necessary definitions and preliminary results that will be used to prove our main results. The proofs of our main results are given in Section 3. Finally, we will give two examples to demonstrate our main results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Let $\mathbb{N}$ be the set of positive integers, and $\mathbb{R}$ be the set of real numbers.

Definition 2.1 (*[6]*). The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $y : (a, b] \to \mathbb{R}$ is given by

$$I_{a+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s) ds, \quad t \in (a, b].$$

Definition 2.2 (*[6]*). The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a function $y : (a, b] \to \mathbb{R}$ is given by

$$D_{a+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{y(s)}{(t-s)^{\alpha-n+1}} ds, \quad t \in (a, b],$$

where $n = [\alpha] + 1$, $[\alpha]$ denote the integer part of α.

Definition 2.3 (*[6]*). The Caputo fractional of order $\alpha > 0$ of function y on $(a, b]$ is defined via the above Riemann–Liouville derivatives by

$$({}^C D_{a+}^{\alpha} y)(x) = \left(D_{a+}^{\alpha}\left[y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!}(t-a)^k\right]\right)(x), \quad x \in (a, b],$$

where $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$; $n = \alpha$ for $\alpha \in \mathbb{N}$.

Lemma 2.1 (*[6]*). *Let n be a positive integer, $\alpha \in (n-1, n]$. If $y \in C^n[a, b]$, then*

$$(I_{a+}^{\alpha}\, {}^C D_{a+}^{\alpha} y)(x) = y(x) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!}(x-a)^k$$

holds on $[a, b]$.

Lemma 2.2 (*[6]*). *Let $m \in \mathbb{N}$, $\alpha \in (m-1, m)$. If $y \in C^m[a, b]$, then*

$${}^C D_{a+}^{\alpha} y(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^x \frac{y^{(m)}(t)}{(x-t)^{\alpha-m+1}} dt$$

holds on $[a, b]$.

Lemma 2.3 (*[6]*). *Let $k \in \mathbb{N}$, $\alpha > 0$. If $(D_a^{\alpha} y)(x)$ and $(D_{a+}^{\alpha+k} y)(x)$ exist, then*

$$(D^k D_{a+}^{\alpha} y)(x) = (D_{a+}^{\alpha+k} y)(x).$$

Lemma 2.4 (*[6]*). *If $\alpha > 0$, $\beta > 0$, $\alpha + \beta > 1$, then*

$$(I_{a+}^{\alpha} I_{a+}^{\beta} y)(x) = (I_{a+}^{\alpha+\beta} y)(x)$$

satisfies at any point on $[a, b]$ for $y \in L_p(a, b)$, $1 \le p \le \infty$.

Lemma 2.5 (*[6]*). *Let $\alpha > 0$ and $y \in C[a, b]$. Then*

$$({}^C D_{a+}^{\alpha} I_{a+}^{\alpha} y)(x) = y(x)$$

holds on $[a, b]$.

The following lemma not only is fundamental in this paper, but also can be applied in other fractional differential equations.

Lemma 2.6. *Let $n \in \mathbb{N}$ with $n \geq 2$, $\alpha \in (n-1, n]$, If $y \in C^{n-1}[a,b]$ and ${}^C D^{\alpha}_{a+} y \in C(a,b)$, then*

$$(I^{\alpha}_{a+} {}^C D^{\alpha}_{a+} y)(x) = y(x) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!}(x-a)^k$$

holds on (a,b).

Proof. If $\alpha = n$, then noting that ${}^C D^{\alpha}_{a+} y = y^{(n)}$ in this case, it is easy to verify that Lemma 2.6 is true. So, we need only give the proof in the case $n-1 < \alpha < n$.

By Definition 2.3, we have

$$({}^C D^{\alpha-1}_{a+} y)(x) = \left(D^{\alpha-1}_{a+} \left[y(t) - \sum_{k=0}^{n-2} \frac{y^{(k)}(a)}{k!}(t-a)^k \right] \right)(x), \quad x \in (a,b]$$

and

$$\begin{aligned} ({}^C D^{\alpha}_{a+} y)(x) &= \left(D^{\alpha}_{a+} \left[y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!}(t-a)^k \right] \right)(x) \\ &= D^{\alpha}_{a+} \left[y(t) - \sum_{k=0}^{n-2} \frac{y^{(k)}(a)}{k!}(t-a)^k \right](x) - \frac{y^{(n-1)}(a)}{(n-1)!}[D^{\alpha}_{a+}(t-a)^{n-1}](x) \\ &= \left(D^{\alpha}_{a+} \left[y(t) - \sum_{k=0}^{n-2} \frac{y^{(k)}(a)}{k!}(t-a)^k \right] \right)(x) - \frac{y^{(n-1)}(a)}{\Gamma(n-\alpha)}(x-a)^{n-\alpha-1} \end{aligned} \tag{2.1}$$

for any $x \in (a,b)$.

On the other hand, in view of Lemma 2.3, we have

$$\begin{aligned} (D^1 {}^C D^{\alpha-1}_{a+} y)(x) &= \left(D^1 D^{\alpha-1}_{a+} \left[y(t) - \sum_{k=0}^{n-2} \frac{y^{(k)}(a)}{k!}(t-a)^k \right] \right)(x) \\ &= \left(D^{\alpha}_{a+} \left[y(t) - \sum_{k=0}^{n-2} \frac{y^{(k)}(a)}{k!}(t-a)^k \right] \right)(x), \quad x \in (a,b). \end{aligned} \tag{2.2}$$

From Lemma 2.2 and the fact that $y \in C^{(n-1)}[0,1]$, we know that ${}^C D^{\alpha-1}_{a+} y \in C[0,1]$ and

$$\lim_{x \to a+0} ({}^C D^{\alpha-1}_{a+} y)(x) = \lim_{x \to a+0} \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{y^{(n-1)}(t)}{(x-t)^{\alpha-n+1}} dt = 0.$$

Since ${}^C D^{\alpha}_{a+} y \in C(a,b)$, it is easy to see that $D^{\alpha}_{a+} \left[y(t) - \sum_{k=0}^{n-2} \frac{y^{(k)}(a)}{k!}(t-a)^k \right] \in C(a,b)$ from (2.1), and so, $D^1 {}^C D^{\alpha-1}_{a+} \in C(a,b)$ from (2.2). Consequently, it follows that

$$\begin{aligned} (I^1_{a+} D^1 {}^C D^{\alpha-1}_{a+} y)(x) &= \int_a^x ({}^C D^{\alpha-1}_{a+} y)'(t) dt \\ &= ({}^C D^{\alpha-1}_{a+} y)(x) - \lim_{t \to a+0} ({}^C D^{\alpha-1}_{a+} y)(t) \\ &= ({}^C D^{\alpha-1}_{a+} y)(x), \quad x \in (a,b). \end{aligned}$$

From the above equality, keeping in mind that ${}^C D^{\alpha-1}_{a+} y \in C[0,1]$, we can easily deduce that $D^1 {}^C D^{\alpha-1}_{a+} y \in L(a,c)$ for any $c \in (a,b)$. Thus, by Lemma 2.4, it follows that

$$(I^{\alpha}_{a+} D^1 {}^C D^{\alpha-1}_{a+} y)(x) = (I^{\alpha-1}_{a+} I^1_{a+} D^1 {}^C D^{\alpha-1}_{a+} y)(x) = (I^{\alpha-1}_{a+} {}^C D^{\alpha-1}_{a+} y)(x), \quad x \in (a,b). \tag{2.3}$$

Thus, by Lemma 2.1 and (2.3), we have

$$(I^{\alpha}_{a+} D^1 {}^C D^{\alpha-1}_{a+} y)(x) = y(x) - \sum_{k=0}^{n-2} \frac{y^{(k)}(a)}{k!}(x-a)^k, \quad x \in (a,b). \tag{2.4}$$

Note that the condition $({}^C D^{\alpha}_{a+}y)(x) \in C(a,b)$ guarantees that $(D^{\alpha}_{a+}y)(x)$ and $(D^{(\alpha-1)}_{a+}y)(x)$ exist on (a,b). So, by formula (2.1), Lemma 2.3 and Definition 2.3, taking into account that $y \in C^{n-1}[a,b]$, we have

$$({}^C D^{\alpha}_{a+}y)(x) = D^1 D^{\alpha-1}_{a+}\left[y(t) - \sum_{k=0}^{n-2}\frac{y^{(k)}(a)}{k!}(t-a)^k\right](x) - \frac{y^{(n-1)}(a)}{\Gamma(n-\alpha)}(x-a)^{n-\alpha-1}$$
$$= [D^{1C} D^{\alpha-1}_{a+}y](x) - \frac{y^{(n-1)}(a)}{\Gamma(n-\alpha)}(x-a)^{n-\alpha-1}, \quad x \in (a,b). \tag{2.5}$$

Thus, (2.5) together with (2.4) implies

$$(I^{\alpha}_{a+}{}^C D^{\alpha}_{a+}y)(x) = (I^{\alpha}_{a+}D^{1C}D^{\alpha-1}y)(x) - \frac{y^{(n-1)}(a)}{\Gamma(n-\alpha)}[I^{\alpha}_{a+}(t-a)^{n-\alpha-1}](x)$$
$$= y(x) - \sum_{k=0}^{n-2}\frac{y^{(k)}(a)}{k!}(x-a)^k - \frac{y^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1}$$
$$= y(x) - \sum_{k=0}^{n-1}\frac{y^{(k)}(a)}{k!}(x-a)^k, \quad x \in (a,b).$$

Then the proof is completed. □

Remark 2.1. Lemma 2.6 is not only an improvement of Lemma 2.1, but also a more convenient tool than Lemma 2.1 in application (this is very important).

Consider the following boundary value problem of a fractional differential equation

$${}^C D^{\alpha}_{0+}u(t) + r\,{}^C D^{\alpha-1}_{0+}u(t) = h(t), \quad t \in (0,1), \tag{2.6}$$
$$u(0) = u(1), \qquad u(\xi) = \eta, \quad \xi \in (0,1). \tag{2.7}$$

We have the following lemma which plays an important role in the proof of our main results.

Lemma 2.7. *Let $1 < \alpha \le 2$, $h \in C[0,1]$, $r \ne 0$. If $u \in C^1[0,1]$ is a solution of BVP (2.6) and (2.7), then $v = ue^{rt}$ satisfies*

$$v(t) = \eta e^{rt} + \frac{1-e^{r(t-\xi)}}{(e^r-1)\Gamma(\alpha-1)}\int_0^1 e^{rs}ds\int_0^s(s-\tau)^{\alpha-2}h(\tau)d\tau - \frac{e^{r(t-\xi)}}{\Gamma(\alpha-1)}\int_0^{\xi} e^{rs}ds\int_0^s(s-\tau)^{\alpha-2}h(\tau)d\tau$$
$$+\frac{1}{\Gamma(\alpha-1)}\int_0^t e^{rs}ds\int_0^s(s-\tau)^{\alpha-2}h(\tau)d\tau, \quad t \in [0,1]. \tag{2.8}$$

Conversely, If v is given by (2.8), then $u = ve^{-rt} \in C^1[0,1]$ and u is a solution of BVP (2.6) and (2.7).

Proof. Let $u \in C^1[0,1]$ be a solution of BVP (2.6) and (2.7). Then by Lemma 2.2, it is easy to see that ${}^C D^{\alpha-1}_{0+}u \in C[0,1]$ because $u' \in C[0,1]$, and so ${}^C D^{\alpha}_{0+}u \in C(0,1)$ from the relations ${}^C D^{\alpha}_{0+}u = h(t) - r\,{}^C D^{\alpha-1}_{0+}u$ and $h \in C[0,1]$. Thus, by Lemma 2.6, we have

$$I^{\alpha}_{0+}{}^C D^{\alpha}_{0+}u(t) = u(t) - b_0 - b_1 t, \quad t \in (0,1) \tag{2.9}$$

and

$$I^{\alpha-1}_{0+}{}^C D^{\alpha-1}_{0+}u(t) = u(t) - d_1 \in (0,1),$$

respectively.

Hence, in view of Lemma 2.4, it follows that

$$I^{\alpha}_{0+}{}^C D^{\alpha-1}_{0+}u(t) = I^1_{0+}I^{\alpha-1}_{0+}{}^C D^{\alpha-1}_{0+}u(t) = \int_0^t u(s)ds - d_0 - d_1 t, \quad t \in (0,1). \tag{2.10}$$

So, formulas (2.6), (2.9) and (2.10) imply

$$u(t) + r\int_0^t u(s)ds = c_0 + c_1 t + I^{\alpha}_{0+}h(t), \quad t \in (0,1), \tag{2.11}$$

where $c_1, c_2 \in \mathbb{R}$, and $I^{\alpha}_{0+}h(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds$, $t \in (0,1)$.

Let $\phi(t) = \int_0^t (t-s)^{\alpha-2}h(s)ds$. Now we show that $\phi \in C[0,1]$.

Indeed, for any given $t \in (0, 1)$, taking $\delta > 0$ with $t + \delta \in (0, 1)$, it follows from $h \in C[0, 1]$ and $1 < \alpha \leq 2$ that

$$|\phi(t+\Delta t) - \phi(t)| \leq \int_0^t ((t-s)^{\alpha-2} - (t+\Delta t - s)^{\alpha-2})|h(s)|ds + \int_t^{t+\Delta t} (t+\Delta t - s)^{\alpha-2}|h(s)|ds$$
$$\leq \frac{m}{\alpha-1}[t^{\alpha-1} - (t+\Delta t)^{\alpha-1} + 2(\Delta t)^{\alpha-1}] \quad (2.12)$$

holds for any $\Delta t \in (0, \delta)$, where $m = \max_{s\in[0,1]} |h(s)|$. From (2.12), we conclude that $\lim_{\Delta t\to 0+} \phi(t+\Delta t) = \phi(t)$. Similarly, we can obtain that $\lim_{\Delta t\to 0-} \phi(t + \Delta t) = \phi(t)$. Thus $\phi \in C(0, 1)$. In addition, it is easy to know that ϕ is continuous at $t = 0, 1$. So $\phi \in C[0, 1]$. Therefore, keeping in mind that $u' \in C[0, 1]$, it follows from (2.11) that

$$u'(t) + ru(t) = c_1 + \frac{d}{dt} I_{0+}^{\alpha} h(t)$$
$$= c_1 + \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} h(s)ds, \quad t \in [0, 1]. \quad (2.13)$$

Thus, setting $v = ue^{rt}$, from (2.13), we immediately have

$$v'(t) = c_1 e^{rt} + \frac{e^{rt}}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} h(s)ds, \quad t \in [0, 1]. \quad (2.14)$$

By integrating both sides of (2.14) on $[0, t]$, we have

$$v(t) = v(0) + \frac{c_1}{r}(e^{rt} - 1) + \frac{1}{\Gamma(\alpha-1)} \int_0^t e^{rs} ds \int_0^s (s-\tau)^{\alpha-2} h(\tau)d\tau, \quad t \in [0, 1]. \quad (2.15)$$

Since $v(1) = e^r v(0)$ by condition $u(0) = u(1)$ (cf. (2.7)), formula (2.15) gives

$$v(0) = \frac{c_1}{r} + \frac{1}{(e^r - 1)\Gamma(\alpha-1)} \int_0^1 e^{rs} ds \int_0^s (s-\tau)^{\alpha-2} h(\tau)d\tau. \quad (2.16)$$

Substituting (2.16) into (2.15), it follows that

$$v(t) = \frac{c_1}{r} e^{rt} + \frac{1}{(e^r - 1)\Gamma(\alpha-1)} \int_0^1 e^{rs} ds \int_0^s (s-\tau)^{\alpha-2} h(\tau)d\tau$$
$$+ \frac{1}{\Gamma(\alpha-1)} \int_0^t e^{rs} ds \int_0^s (s-\tau)^{\alpha-2} h(\tau)d\tau, \quad t \in [0, 1]. \quad (2.17)$$

Furthermore, the condition $u(\xi) = \eta$ in (2.7) implies that $v(\xi) = \eta e^{r\xi}$. Thus, it follows from (2.17) that

$$c_1 = \eta r - \frac{re^{-r\xi}}{(e^r - 1)\Gamma(\alpha-1)} \int_0^1 e^{rs} ds \int_0^s (s-\tau)^{\alpha-2} h(\tau)d\tau$$
$$- \frac{re^{-r\xi}}{\Gamma(\alpha-1)} \int_0^{\xi} e^{rs} ds \int_0^s (s-\tau)^{\alpha-2} h(\tau)d\tau. \quad (2.18)$$

Now, substituting (2.18) into (2.17), we have the relation (2.8).

Conversely, assume that v is given by (2.8). By the previous proof, we know that $\phi = \int_0^t (t-s)^{\alpha-2} h(s)ds$ is continuous on $[0, 1]$ from the fact that $h \in C[0, 1]$. Thus, by differentiating both sides of (2.8), we get

$$v'(t) = c_1 e^{rt} + \frac{1}{\Gamma(\alpha-1)} e^{rt} \int_0^t (t-s)^{\alpha-2} h(s)ds, \quad t \in [0, 1], \quad (2.19)$$

where c_1 is described as in (2.18), and so $v \in C^1[0, 1]$. Furthermore, formula (2.19) together with (2.8) and (2.18) ensure that (2.17) holds on $[0, 1]$, and

$$v(\xi) = \eta e^{r\xi}, \qquad v(1) = v(0)e^r. \quad (2.20)$$

Let $u = ve^{-rt}$. Then $u' \in C[0, 1]$ from the fact that $v \in C^1[0, 1]$. Moreover it follows from (2.19) that

$$u' + ru = c_1 + I_{0+}^{\alpha-1} h(t), \quad t \in [0, 1]. \quad (2.21)$$

Thus, in view of Lemma 2.5, we have

$${}^C D_{0+}^{\alpha-1} u'(t) + r\,{}^C D_{0+}^{\alpha-1} u(t) = {}^C D_{0+}^{\alpha-1} I_{0+}^{\alpha-1} h(t) = h(t), \quad t \in (0, 1). \quad (2.22)$$

Now, we show that ${}^C D_{0+}^{\alpha-1} u'(t) = {}^C D_{0+}^{\alpha} u(t)$, $t \in (0, 1)$.

In fact, owing to the fact that $u' \in C[0,1]$, $1 < \alpha \le 2$, we have

$$\begin{aligned} {}^{C}D^{\alpha}_{0+}u(t) &= [D^{\alpha}_{0+}(u(x) - u(0) - u'(0)x)](t) \\ &= [D^{\alpha}_{0+}u](t) - \frac{u(0)}{\Gamma(1-\alpha)}t^{-\alpha} - \frac{u'(0)}{\Gamma(2-\alpha)}t^{1-\alpha}, \quad t \in (0,1) \end{aligned} \tag{2.23}$$

from Definition 2.3, and

$$\begin{aligned} [D^{\alpha}_{0+}u](t) &= \frac{1}{\Gamma(2-\alpha)}\left(\frac{d}{dt}\right)^2 \int_0^t (t-x)^{1-\alpha}u(x)dx \\ &= \frac{1}{(\alpha-2)\Gamma(2-\alpha)}\left(\frac{d}{dt}\right)^2 \int_0^t u(x)d(t-x)^{2-\alpha} \\ &= \frac{1}{(2-\alpha)\Gamma(2-\alpha)}\left(\frac{d}{dt}\right)^2 \left[u(0)t^{2-\alpha} + \int_0^t (t-x)^{2-\alpha}u'(x)dx\right] \\ &= \frac{1}{\Gamma(2-\alpha)}\frac{d}{dt}\left[u(0)t^{1-\alpha} + \int_0^t (t-x)^{1-\alpha}u'(x)dx\right] \\ &= \frac{1-\alpha}{\Gamma(2-\alpha)}u(0)t^{-\alpha} + \frac{1}{\Gamma(2-\alpha)}\frac{d}{dt}\int_0^t (t-x)^{1-\alpha}u'(x)dx \\ &= \frac{1-\alpha}{\Gamma(2-\alpha)}u(0)t^{-\alpha} + [D^{\alpha-1}_{0+}u'(x)](t), \quad t \in (0,1) \end{aligned} \tag{2.24}$$

from Definition 2.2.

Now, formulas (2.23) and (2.24) imply

$$\begin{aligned} [{}^{C}D^{\alpha}_{0+}u(x)](t) &= [D^{\alpha-1}_{0+}u'(x)](t) - \frac{u'(0)}{\Gamma(2-\alpha)}t^{1-\alpha} \\ &= [D^{\alpha-1}_{0+}(u'(x) - u'(0))](t) \\ &= [{}^{C}D^{\alpha-1}_{0+}u'(x)](t), \quad t \in (0,1). \end{aligned}$$

Thus, it follows from (2.22) that

$${}^{C}D^{\alpha}_{0+}u(t) + r\,{}^{C}D^{\alpha-1}_{0+}u(t) = h(t), \quad t \in (0,1).$$

On the other hand, formula (2.20) with $u = ve^{-rt}$ implies that $u(0) = u(1)$ and $u(\xi) = \eta$.

To summarize, $u \in C^1[0,1]$ is a solution of BVP (2.6) and (2.7). □

Finally, for the remainder of this section, we give some hypotheses which will be used in the paper.

(H_1) $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$, and there exists a constant $L > 0$ such that

$$|f(t,x) - f(t,y)| \le L|x-y|$$

for each $t \in [0,1]$ and all $x, y \in \mathbb{R}$, where L satisfies the condition: if $r > 0$, then $L < \frac{\Gamma(\alpha)r}{2(1-e^{-r})}$; if $r < 0$, then $L < \frac{\Gamma(\alpha)|r|}{2(e^{|r|}-1)}$.

(H_1) $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$, and there exists two nonnegative constants A, B such that

$$|f(t,x)| \le A + B|x|$$

for each $t \in [0,1]$ and all $x \in \mathbb{R}$, where B satisfies the condition: if $r > 0$, then $0 < B < \frac{\Gamma(\alpha)r}{2(1-e^{-r})}$; if $r < 0$, then $0 < B < \frac{\Gamma(\alpha)|r|}{2(e^{|r|}-1)}$.

3. Main results

We are now in a position to state our main theorems.

Theorem 3.1. *Assume* (H_1) *holds. Then BVP (1.1)–(1.2) has a unique solution u with $u \in C^1[0,1]$.*

Proof. We first introduce some notations. Let $X = C[0,1]$ be a Banach space with the usual norm $\|u\|_C = \max_{t\in[0,1]}\{|u(t)|\}$. We also equip the space X with the norm $\|u\|_r = \max_{t\in[0,1]}\left\{|u(t)|e^{-rt}\right\}$, where r is described as in (1.1)–(1.2). It is well known that the norm $\|u\|_r$ is equivalent to the norm $\|u\|_C$.

Denote the operator on X by

$$\begin{aligned} Av = {} & \eta e^{rt} + \frac{1-e^{r(t-\xi)}}{(e^r-1)\Gamma(\alpha-1)}\int_0^1 e^{rs}ds\int_0^s (s-\tau)^{\alpha-2} f(\tau, ve^{-r\tau})d\tau \\ & - \frac{e^{r(t-\xi)}}{\Gamma(\alpha-1)}\int_0^\xi e^{rs}ds\int_0^s (s-\tau)^{\alpha-2} f(\tau, ve^{-r\tau})d\tau \\ & + \frac{1}{\Gamma(\alpha-1)}\int_0^t e^{rs}ds\int_0^s (s-\tau)^{\alpha-2} f(\tau, ve^{-r\tau})d\tau, \quad t\in[0,1]. \end{aligned} \tag{3.1}$$

From (H_1), the function $f(t, v(t)e^{-rt})$ is continuous on $[0, 1]$ for any $v \in X$. Clearly, the operator A maps X into X. In view of Lemma 2.7, the operator A has a fixed point $v \in X$ if and only if $u = ve^{-rt}$ is a solution of BVP (1.1)–(1.2) with $u \in C^1[0, 1]$. Therefore, we only need to seek a fixed point of operator A in X.

For any $v_1, v_2 \in X$, by (H_1), we have

$$\begin{aligned} |f(s, v_2(s)e^{-rs}) - f(s, v_1(s)e^{-rs})| & \le L|v_2(s)e^{-rs} - v_1(s)e^{-rs}| \\ & \le L\|v_2 - v_1\|_r, \quad s \in [0, 1]. \end{aligned}$$

Thus, it follows from (3.1) that

$$\begin{aligned} |(Av_2 - Av_1)(t)| & \le \frac{L}{\Gamma(\alpha-1)}\|v_2 - v_1\|_r \left[\frac{|1-e^{r(t-\xi)}|}{|e^r-1|}\int_0^1 e^{rs}ds\int_0^s (s-\tau)^{\alpha-2}d\tau \right. \\ & \quad \left. + e^{r(t-\xi)}\int_0^\xi e^{rs}ds\int_0^s (s-\tau)^{\alpha-2}d\tau + \int_0^t e^{rs}ds\int_0^s (s-\tau)^{\alpha-2}d\tau\right] \\ & \le \frac{L\|v_2 - v_1\|_r}{\Gamma(\alpha)|r|}[|1-e^{r(t-\xi)}| + e^{r(t-\xi)}|e^{r\xi}-1| + |e^{rt}-1|] \end{aligned} \tag{3.2}$$

for all $t \in [0, 1]$.

Hence, if $r > 0$, then from (3.2) and $0 < \xi < 1$, the following inequality

$$|(Av_2 - Av_1)(t)| \le \frac{2L}{\Gamma(\alpha)r}\|v_2 - v_1\|_r e^{rt}(1-e^{-r}), \quad t \in [0, 1]$$

holds. So, $\|Av_2 - Av_1\|_r \le \frac{2L(1-e^{-r})}{\Gamma(\alpha)r}\|v_2 - v_1\|_r$. Moreover, $\frac{2L(1-e^{-r})}{\Gamma(\alpha)r} < 1$ from hypothesis (H_1).

Similarly, if $r < 0$, then $\|Av_2 - Av_1\|_r \le \frac{2L(e^{|r|}-1)}{\Gamma(\alpha)|r|}\|v_2 - v_1\|_r$ and $\frac{2L(e^{|r|}-1)}{\Gamma(\alpha)|r|} < 1$.

Hence, whether $r > 0$ or $r < 0$, we always conclude that the operator A has a unique fixed point v in X by the Banach contraction principle. Then $u = e^{-rt}v$ is a unique solution of BVP (1.1)–(1.2) with $u \in C^1[0, 1]$ from Lemma 2.7. □

Theorem 3.2. *Let* (H_2) *holds. Then BVP* (1.1)–(1.2) *has at least one solution* u *with* $u \in C^1[0, 1]$.

Proof. Let $v_0 = \eta e^{rt}$. Take $R > \frac{2A(1-e^{-r})}{\Gamma(\alpha)r-2B(1-e^{-r})}$ if $r > 0$ or $R > \frac{2A(e^{|r|}-1)}{\Gamma(\alpha)|r|-2B(e^{|r|}-1)}$ if $r < 0$. Put $\Omega_R = \{x \in X : \|v - v_0\|_r \le R\}$. Define the operator A as (3.1). We first show that $A\Omega_R \subset \Omega_R$.

In fact, by (H_2), for any $v \in \Omega_R$, we have

$$|f(s, v(s)e^{-rs})| \le A + B|v(s)e^{-rs}| \le A + B\|v\|_r \le A + BR, \quad s \in [0, 1].$$

Thus, doing argument analogous to that of formula (3.2), we have

$$|(Av - v_0)(t)| \le \frac{A+BR}{\Gamma(\alpha)|r|}[|1-e^{r(t-\xi)}| + e^{r(t-\xi)}|e^{r\xi}-1| + |e^{rt}-1|], \quad t \in [0, 1].$$

Therefore, if $r > 0$, then the following inequality

$$\begin{aligned} \|Av - v_0\|_r & \le \frac{2(A+BR)(1-e^{-r})}{\Gamma(\alpha)r} \\ & = \frac{2A(1-e^{-r})}{\Gamma(\alpha)r} + \frac{2B(1-e^{-r})}{\Gamma(\alpha)r}R < R \end{aligned}$$

holds from (H_2) and the choice of R.

Similarly, if $r < 0$, then

$$\|Av - v_0\|_r \le \frac{2(A+BR)(e^{|r|}-1)}{\Gamma(\alpha)|r|} = \frac{2A(e^{|r|}-1)}{\Gamma(\alpha)|r|} + \frac{2B(e^{|r|}-1)}{\Gamma(\alpha)|r|}R < R.$$

Thus A maps Ω_R into Ω_R.

Now, we show that A is completely continuous on Ω_R. In what follows, we will assume that $r > 0$; for the case $r < 0$, the proof is similar.

Owing to the equivalence on the norms $\|u\|_r$ and $\|u\|_C$, we will give the proof in the case that X is equipped with the norm $\|u\|_C$.

First, from $A\Omega_R \subset \Omega_R$, it follows that $\|Av\|_C \le \|Av\|_r e^r \le Re^r$ for any $v \in \Omega_R$, and so $\{w | w \in A\Omega_R\}$ is uniformly bounded.

Second, we show that $A\Omega_R$ is equicontinuous.

Indeed, for any $v \in \Omega_R$, formula (3.1) implies

$$\begin{aligned}(Av)'(t) = \eta r e^{rt} &- \frac{re^{r(t-\xi)}}{(e^r-1)\Gamma(\alpha-1)}\int_0^1 e^{rs}ds\int_0^s (s-\tau)^{\alpha-2} f(\tau, ve^{-r\tau})d\tau \\ &- \frac{re^{r(t-\xi)}}{\Gamma(\alpha-1)}\int_0^\xi e^{rs}ds\int_0^s (s-\tau)^{\alpha-2} f(\tau, ve^{-r\tau})d\tau \\ &+ \frac{1}{\Gamma(\alpha-1)}e^{rt}\int_0^t (t-\tau)^{\alpha-2} f(\tau, ve^{-r\tau})d\tau, \quad t \in [0,1].\end{aligned}$$

Thus, by (H_2), it follows that

$$|(Av)'(t)| \le \eta r e^r + \frac{3e^r(A+BR)}{\Gamma(\alpha)}, \quad t \in [0,1].$$

Thus, $A\Omega_R$ is equicontinuous. By the Arzela–Ascoli theorem, $A\Omega_R$ is relatively compact.

Finally, we show that A is continuous on Ω_R.

Let $\{v_n\}$ be an arbitrary sequence in Ω_R with $\|v_n - v\|_C \to 0 (n \to \infty)$, $v \in \Omega_R$. Then $\|v_n - v\|_r \to 0 (n \to \infty)$, and so, there exist two constants D_1, D_2 such that $v(t)e^{-rt} \in [D_1, D_2]$ and $v_n(t)e^{-rt} \in [D_1, D_2]$, $n = 1, 2 \ldots$ for each $t \in [0,1]$. Furthermore, the uniform continuity of f on $[0,1] \times [D_1, D_2]$ implies that for any $\varepsilon > 0$, there exists $\delta > 0$, when ever $|x_1 - x_2| < \delta$, $x_1, x_2 \in [D_1, D_2]$, the following inequality

$$|f(t, x_2) - f(t, x_1)| < G\varepsilon \tag{3.3}$$

holds for any $t \in [0,1]$, where $G = \frac{\Gamma(\alpha)r}{2(e^r-1)}$.

Since $v_n \to v$, we have that there exists $N \ge 1$ such that the following inequality

$$|v_n(t)e^{-rt} - v(t)e^{-rt}| < \delta$$

holds for all $t \in [0,1]$ and $n \ge N$. Hence, by (3.1) together with (3.3), we have

$$\begin{aligned}|(Av_n - Av)(t)| &\le \frac{G\varepsilon}{\Gamma(\alpha-1)}\left[\frac{|1-e^{r(t-\xi)}|}{e^r-1}\int_0^1 e^{rs}ds\int_0^s (s-\tau)^{\alpha-2}d\tau \right.\\ &\quad \left. + e^{r(t-\xi)}\int_0^\xi e^{rs}ds\int_0^s (s-\tau)^{\alpha-2}d\tau + \int_0^t e^{rs}ds\int_0^s (s-\tau)^{\alpha-2}d\tau\right] \\ &\le \frac{G\varepsilon}{\Gamma(\alpha)r}[|1-e^{r(t-\xi)}| + e^{r(t-\xi)}(e^{r\xi}-1) + (e^{rt}-1)] \\ &\le \frac{2G\varepsilon}{\Gamma(\alpha)r}(e^r - 1), \quad t \in [0,1]\end{aligned}$$

for any $n \ge N$.

Thus

$$\|Av_n - Av\|_C \le \frac{2G\varepsilon}{\Gamma(\alpha)r}(e^r-1) = \varepsilon$$

for any $n \ge N$.

Summing up the above analysis, we obtain that $A : \Omega_R \to \Omega_R$ is completely continuous. Thus, by the Schauder fixed point theorem, there exists a point $v \in \Omega_R$ with $v = Av$. In view of Lemma 2.7, we know that $u = e^{-rt}v$ is a solution of BVP (1.1)–(1.2) with $u \in C^1[0,1]$. This completes the proof. □

Example 3.1. Consider the following boundary value problem

$$\begin{cases} {}^C D_{0+}^{\frac{3}{2}} u(t) + r\, {}^C D_{0+}^{\frac{1}{2}} u(t) = h(t)\sin u + g(t), & t \in (0,1], \\ u(0) = u(1), \quad u(\xi) = \eta, & \xi \in (0,1), \end{cases}$$

where $r > 0$ and $h, g \in C[0,1]$ with $\max_{t\in[0,1]} |h(t)| < \frac{\sqrt{\pi} r}{4(1-e^{-r})}$.

Let $f(t,u) = h(t)\sin u + g(t),\ t \in [0,1]$. Then it is easy to see that the condition (H_1) holds. Therefore, the boundary value problem (1.1)–(1.2) has a unique solution by Theorem 3.1.

Example 3.2. Consider the following boundary value problem

$$\begin{cases} {}^C D_{0+}^{\frac{3}{2}} u(t) + r\, {}^C D_{0+}^{\frac{1}{2}} u(t) = h(t)\dfrac{u}{1+|u|} + g(t), & t \in (0,1], \\ u(0) = u(1), \quad u(\xi) = \eta, & \xi \in (0,1), \end{cases}$$

where $r > 0$ and $h, g \in C[0,1]$ with $\max_{t\in[0,1]} |h(t)| < \frac{\sqrt{\pi} r}{4(1-e^{-r})}$.

Let $f(t,u) = h(t)\frac{u}{1+|u|} + g(t),\ t \in [0,1]$. Then it is easy to see that the hypothesis (H_2) is satisfied, and so the boundary value problem (1.1)–(1.2) has a solution by Theorem 3.2.

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