Dynamic faults have small effect on broadcasting in hypercubes

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Received 1 October 1999; received in revised form 23 January 2002; accepted 19 October 2002

Abstract

We consider a broadcasting problem in the \(n\)-dimensional hypercube in the shouting communication mode, i.e. any node of a network can inform all its neighbours in one time step. In addition, during any time step a number of links of the network can be faulty. Moreover, the faults are dynamic. Given a number \(m \leq n - 1\), the problem is to determine the minimum broadcasting time if at most \(m\) faults are allowed in any step. The case \(m = n - 1\) was studied in Chlebus et al. (Networks 27 (1996) 309), De Marco and Vaccaro (Inform. Process. Lett. 66 (1998) 321), Fraigniaud and Lazard (Inform. Process. Lett. 39 (1991) 115) and completely solved in Dobrev and Vr\v{r}o (Inform. Process. Lett. 71 (1999) 81). A related problem, what is the maximal \(m\) s.t. the minimum broadcasting time remains \(n\) was proposed in De Marco and Vaccaro (Inform. Process. Lett. 66 (1998) 321). We prove that for \(m \leq n - 3\) the minimum broadcasting time is \(n\). If \(m = n - 2\) the broadcasting time is always at most \(n + 1\), for \(n > 3\), and the upper bound is the best possible. Our method is related to the isoperimetric problem in graphs and can be applied to other networks.

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Keywords: Broadcasting; Fault tolerance; Hypercube; Isoperimetric problem
1. Introduction

Broadcasting is the standard communication problem in interconnection networks when a node has to send a message to all other nodes. There are many applications of the broadcasting problem in parallel and distributed computing [5,7,8]. Recently, a lot of attention has been paid to fault-tolerant dissemination of information in networks [9]. In this paper, we consider the shouting communication mode in which any node can inform all its neighbours in one time step. The links are bidirectional. The computation is synchronous. In addition, we assume that during any time step a number of links of the network can be faulty. The faults are dynamic in the sense that the set of faulty links can change during the execution of the broadcast. This model was introduced by Santoro and Widmayer [10]. We study this problem for the $n$-dimensional hypercube network. Given a number $m \leq n - 1$, the problem is to determine the minimum broadcasting time if at most $m$ faults are allowed in any time step. Note that $n$ is the edge connectivity of the $n$-dimensional hypercube. The aim of this note is to summarize the partial results and describe the complete solution to the above problem.

The case $m = n - 1$ was studied in [2,3,6] and completely solved in [4]. The broadcasting time in that case is always at most $n + 2$, for $n \geq 2$, and cannot be improved in general. A related problem, what is the maximal $m$ s.t. the broadcasting time remains $n$, was proposed in [3]. Note that the diameter of the hypercube is $n$ so any broadcasting requires time at least $n$. We prove that for $m \leq n - 3$ the minimum broadcasting time is $n$. If $m = n - 2$ the broadcasting time is always at most $n + 1$, for $n \geq 3$, and cannot be improved in general.

Our method previously used in [4] is related to the isoperimetric problem in graphs [1] and can be applied to other networks.

2. A complete solution

**Theorem 1.** Let $n \geq 4$. The minimum broadcasting time $T(n)$ in the $n$-dimensional hypercube with $m \leq n - 1$ dynamic link faults satisfies

$$T(n) \leq \begin{cases} 
  n + 2 & \text{if } m = n - 1, \\
  n + 1 & \text{if } m = n - 2, \\
  n & \text{if } m \leq n - 3.
\end{cases}$$

The upper bounds are the best possible.

**Proof.** The first case was proved in our paper [4] and the corresponding worst-case lower bound was described in [3]. The upper bounds for the second and the third case can be proved using the same method as in [4]. Clearly, the upper bound for the third case is the best possible. In what follows we show that the upper bound in the second case is the best possible by exhibiting a choice of faulty links that yields a broadcasting time $n + 1$. 
Recall the definition of the hypercube. The vertices are labelled by all subsets of the set \( \{1,2,3,\ldots,n\} \). Two vertices are adjacent if one of the labels can be obtained from the other label by adding exactly one element. The nodes labelled by the \( i \)-element subsets are called the nodes of the \( i \)th level, for \( i = 0,1,2,\ldots,n \). The description of the faulty links in any time step follows. Assume that the node labelled by \( \emptyset \) contains the original message. In the first step we put faults on links between nodes labelled by \( \emptyset \) and \( \{i\} \), for \( i = 3,4,\ldots,n \). After the first step, only the nodes labelled by \( \emptyset,\{1\} \) and \( \{2\} \) know the message. In the second step we put faults on links between nodes labelled by \( \{2\} \) and \( \{2,i\} \) for \( i = 3,4,\ldots,n \). After the second step, only those nodes of the second level whose labels contain 1 know the message. From now on till the \( (n-2) \)-nd step we will let the message spread without putting faults. We claim that after the \( i \)th step, \( 2 \leq i \leq n-2 \), the nodes of the \( i \)th level which know the message are precisely the nodes whose labels contain 1. The claim holds for \( i = 2 \). Assume that the claim is true for some \( i \geq 2 \). After the \( (i+1) \)-st step all nodes of the \( (i+1) \)-st level with labels containing 1 will know the message. This follows from the definition of the hypercube. On the other hand assume that a node of the \( (i+1) \)-st level whose label does not contain 1 knows the message. This means that this node is adjacent to a node from the \( i \)th level whose label contains 1. This contradicts the definition of the hypercube.

Now consider the situation after the \( (n-2) \)-nd step. Consider the node \( u \) on the \( (n-1) \)-st level labelled by \( \{1,2,3,\ldots,n-1\} \). Observe that \( u \) has exactly \( n-2 \) neighbours which know the message after the \( (n-2) \)-nd step and 2 other neighbours that do not. In the \( (n-1) \)-st step we put faults on the links between \( u \) and those \( n-2 \) informed neighbours. So after the \( (n-1) \)-st step the vertex \( u \) will not know the message. There is another uninformed node on the \( (n-1) \)-st level after the \( (n-1) \)-st step, namely the node whose label does not contain 1, i.e. the node \( v \) labelled by \( \{2,3,4,\ldots,n\} \). In the \( n \)th step we put faults on \( n-2 \) links between the node labelled by \( \{1,2,3,\ldots,n\} \) and nodes on the \( (n-1) \)-st level except for the nodes \( u \) and \( v \). So after the \( n \)th step the node labelled by \( \{1,2,3,\ldots,n\} \) is uniformed and one additional step is needed to complete the broadcasting. □

References


