JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 29, 467–497 (1970)

On the Effective Determination of the Wave Operator from Given Spectral Data in the Case of a Difference Equation Corresponding to a Sturm—Liouville Differential Equation

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Submitted by R. Bellman

FOREWORD

The impulse to this article originates from some ideas in R. Bellman's article, "A note on an inverse problem in mathematical physics," in which he uses continued fractions to solve a set of second order difference equations, and points out that one can determine the coefficients of the equations from spectral data and the representation of continued fractions.

The approach in the present paper was suggested to me by Professor G. Borg, to whom I am in debt for support and advice.

To the differential equation $y''(t) + (\lambda r(t) - q(t))\,y(t) = 0$, there is a corresponding difference equation of the form

$$(1)\qquad \begin{cases} \dfrac{z_{i+1} - z_i}{h} = (q_{i+1} - \lambda r_{i+1})\, y_{i+1}; & h > 0,\ q_{i+1} \text{ and } r_{i+1} \text{ are real, } i = 1, 2, \ldots, n-1, \\[2ex] \dfrac{y_{i+1} - y_i}{h} = z_i\,, & \end{cases}$$

or in matrix notation

$$\begin{pmatrix} z_{i+1} \\ y_{i+1} \end{pmatrix} = \begin{pmatrix} 1 + h^2(q_{i+1} - \lambda r_{i+1}), & h(q_{i+1} - \lambda r_{i+1}) \\ h, & 1 \end{pmatrix} \begin{pmatrix} z_i \\ y_i \end{pmatrix},$$

$$\begin{pmatrix} z_{i+1} \\ y_{i+1} \end{pmatrix} = C_{i+1}(\lambda) \begin{pmatrix} z_i \\ y_i \end{pmatrix},$$ we note that $\det C_i \equiv 1$ and that $$C_i^{-1} = \begin{pmatrix} 1, & -h(q_i - \lambda r_i) \\ -h, & 1 + h^2(q_i - \lambda r_i) \end{pmatrix},$$

$$\begin{pmatrix} z_n \\ y_n \end{pmatrix} = C_n C_{n-1} \cdots C_2 C_1 \begin{pmatrix} z_0 \\ y_0 \end{pmatrix} = S_n(\lambda) \begin{pmatrix} z_0 \\ y_0 \end{pmatrix}.$$

409/29/3-1

By induction it is easy to show that $S_n(\lambda)$ has the form

$$S_n = \begin{pmatrix} S_{11}(\lambda), & S_{12}(\lambda) \\ S_{21}(\lambda), & S_{22}(\lambda) \end{pmatrix}$$

where S_{11} and S_{12} are polynomials of degree n in λ and S_{21} and S_{22} are polynomials of degree $n - 1$. Further we have det $S_n(\lambda) \equiv 1$.

Now the following question arises: Starting from a given matrix

$$S_n = \begin{pmatrix} S_{11}, & S_{12} \\ S_{21}, & S_{22} \end{pmatrix}$$

where $S_{ij}(\lambda)$ are polynomials in λ with degrees as above, and where det $S_n(\lambda) \equiv 1$, is it possible to consider $S_n(\lambda)$ as a solution matrix of a difference equation of the form (1), that is can we write $S_n(\lambda) = C_n C_{n-1} \cdots C_2 C_1$ where

$$C_i(\lambda) = \begin{pmatrix} 1 + h^2 Q_i, & hQ_i \\ h, & 1 \end{pmatrix}, \qquad Q_i = q_i - \lambda r_i?$$

What is required to ensure that all r_i have the same sign?

The following result will be proved. If $S_{11}(\lambda)$ and $S_{21}(\lambda)$ have all zeroes simple and real and if the zeroes of $S_{11}(\lambda)$ separate the zeroes S_{21} and vice versa, that is

$$\lambda_k^{11} < \lambda_k^{21} < \lambda_{k+1}^{11}, \qquad k = 1, 2, \ldots, n$$

(λ_k^{ij} denotes the k-th zero of $S_{ij}(\lambda)$) then there are uniquely defined matrices $C_i(\lambda)$ and

$$A = \begin{pmatrix} a, & b \\ 0, & 1/a \end{pmatrix}$$

so that $S_n(\lambda)$ can be written $S_n(\lambda) = C_n C_{n-1} \cdots C_2 C_1 A$.

All r_i $i = 1, 2, \ldots, n$ will have the same sign.

Nothing has to be assumed about the zeroes of S_{12} and S_{22}. However, the determinant condition will be enough to ensure that all their zeroes are real and simple and interlacing in the described sense.

We start with the following simple lemma:

LEMMA 1. *If in the equation* (1) q_i, r_i *are real and all* r_i *have the same sign, then the polynomials of* $S_n(\lambda)$ *have real zeroes.*

Proof.

$$Q_i = q_i - \lambda r_i\,,$$

$$z_{i+1} - z_i = hQ_{i+1}y_{i+1} \Rightarrow \sum_{i=0}^{n-1} (z_{i+1} - z_i)\bar{y}_{i+1} = \sum_{i=0}^{n-1} hQ_{i+1} \,|\, y_{i+1} \,|^2;$$

$$\sum_{i=0}^{n-1} z_{i+1}\bar{y}_{i+1} - \sum_{i=0}^{n-1} z_i(\bar{y}_{i+1} - \bar{y}_i) - \sum_{i=0}^{n-1} \bar{y}_i z_i = \sum_{i=0}^{n-1} hQ_{i+1} \,|\, y_{i+1} \,|^2.$$

$$z_n\bar{y}_n - z_0\bar{y}_0 - h \sum_{i=0}^{n-1} |\, z_i \,|^2 = h \sum_{i=0}^{n-1} q_{i+1} \,|\, y_{i+1} \,|^2 - \lambda \sum_{i=0}^{n-1} r_{i+1} \,|\, y_{i+1} \,|^2. \tag{2}$$

Further, from

$$\begin{pmatrix} z_n \\ y_n \end{pmatrix} = \begin{pmatrix} S_{11}\,, & S_{12} \\ S_{21}\,, & S_{22} \end{pmatrix} \begin{pmatrix} z_0 \\ y_0 \end{pmatrix}$$

follows that the difference equation has a nontrivial solution with boundary values $y_0 = 0$, $z_n = 0$ if and only if λ is a zero of $S_{11}(\lambda)$.

In the same way

$S_{12}(\lambda) = 0 \Leftrightarrow \lambda$ is eigenvalue corresponding to the boundary conditions $\begin{cases} z_0 = 0, \\ z_n = 0. \end{cases}$

$S_{21}(\lambda) = 0 \Leftrightarrow \lambda$ is eigenvalue corresponding to the boundary conditions $\begin{cases} y_0 = 0, \\ y_n = 0. \end{cases}$

$S_{22}(\lambda) = 0 \Leftrightarrow \lambda$ is eigenvalue corresponding to the boundary conditions $\begin{cases} z_0 = 0, \\ y_n = 0. \end{cases}$

Now if λ is a zero of some of the polynomials $S_{ij}(\lambda)$ and if $\binom{z_i}{y_i}$ is the corresponding eigensolution then we have $z_n\bar{y}_n - z_0\bar{y}_0 = 0$, and taking the imaginary part of equation (2) we obtain

$$\operatorname{Im} \lambda \cdot \sum_{i=0}^{n-1} r_{i+1} \,|\, y_{i+1} \,|^2 = 0 \Rightarrow \operatorname{Im} \lambda = 0.$$

Thus the zeroes of the polynomials are all real.

LEMMA 2. *Let*

$$S(\lambda) = \begin{pmatrix} S_{11}(\lambda), & S_{12}(\lambda) \\ S_{21}(\lambda), & S_{22}(\lambda) \end{pmatrix}$$

where

(i) Det $S(\lambda) = 1$,

(ii) S_{11} *and* S_{12} *are polynomials of degree* n,

(iii) S_{21} *and* S_{22} *are polynomials of degree* $n - 1$.

Now if

(a) *the zeroes of* S_{11} *and* S_{21} *are all real and interlacing in the sense that* $\lambda_i^{11} < \lambda_i^{21} < \lambda_{i+1}^{11}$ *for* $i = 1, 2,..., n - 1$, *then*

(b) *the zeroes of* S_{11} *and* S_{12} *are all real and interlacing,*

(c) *the zeroes of* S_{12} *and* S_{22} *are all real and interlacing,*

(d) *the zeroes of* S_{21} *and* S_{22} *are all real and interlacing.*

Proof. (b) Look at the interval $(\lambda_i^{11}, \lambda_{i+1}^{11})$.

$$S_{11}S_{22} - S_{12}S_{21} \equiv 1 \Rightarrow S_{12}(\lambda_i^{11})\, S_{21}(\lambda_i^{11}) = S_{12}(\lambda_{i+1}^{11})\, S_{21}(\lambda_{i+1}^{11}) = -1.$$

By our assumption S_{21} changes sign exactly once in the interval $(\lambda_i^{11}, \lambda_{i+1}^{11})$, which implies that $S_{12}(\lambda_i^{11})$ and $S_{12}(\lambda_{i+1}^{11})$ have different signs. It follows that S_{12} has an odd number of zeroes on each interval $(\lambda_i^{11}, \lambda_{i+1}^{11})$. But as the number of zeroes of S_{12} can't exceed n, S_{12} must have exactly one zero in each interval $(\lambda_i^{11}, \lambda_{i+1}^{11})$, $i = 1, 2,..., n - 1$. Q.E.D.

(c) and (d) are proved in a similar way.

In the same way, of course, one can prove that (b) $\Rightarrow$ (a), (c) and (d) and that (c) $\Rightarrow$ (*a*), (b) and (d). However, (d) does not necessarily $\Rightarrow$ (a), (b), and (c).

LEMMA 3. *The same as Lemma 2 except that all polynomials are supposed to have same degree. The proof is analogous.*

REMARK. We have the following general theorem of algebra. If P_n and P_m are polynomials of degree n and m, respectively, and with no common factor then there are uniquely defined polynomials Q_{n-1} and Q_{m-1}, respectively, such that $P_m Q_{n-1} - P_n Q_{m-1} \equiv 1$. One also has

$$P_m(Q_{n-1} + A(\lambda)\, P_n) - P_n(Q_{m-1} + A(\lambda)\, P_m) \equiv 1$$

for all polynomials $A(\lambda)$.

Lemma 2 and the above remark then tell us that if S_{11} and S_{12} are given with n real interlacing zeroes then S_{21} and S_{22} are uniquely determined by the conditions: (i) det $S \equiv 1$ and (ii) the degree of S_{21} and S_{22} is $n - 1$. Furthermore it follows that the zeroes of S_{12} and S_{22} are all real and interlacing (Lemma 2).

THEOREM I. *If there is given a matrix*

$$S_n(\lambda) = \begin{pmatrix} S_{11}, S_{12} \\ S_{21}, S_{22} \end{pmatrix}$$

with $\det S_n \equiv 1$ *and whose elements* S_{ij} *are polynomials with interlacing zeroes as described earlier, we can write in a unique way* $S_n = C_n C_{n-1}, \ldots, C_2 C_1 A$ *where*

$$C_i = \begin{pmatrix} 1 + h^2 Q_i, hQ_i \\ h, \quad 1 \end{pmatrix}, \qquad A = \begin{pmatrix} a, & b \\ 0, & 1/a \end{pmatrix},$$

$Q_i = q_i - \lambda r_i$, *and a and b are certain constants.*

Proof. Suppose that the theorem is true for $n \leqslant p$. For $n = p + 1$ we introduce the following notations:

$$S_{p+1}(\lambda) = \begin{pmatrix} a_{p+1}\lambda^{p+1} + a_p\lambda^p + \cdots, b_{p+1}\lambda^{p+1} + b_p\lambda^p + \cdots \\ c_p\lambda^p + c_{p-1}\lambda^{p-1} + \cdots, d_p\lambda^p + d_{p-1}\lambda^{p-1} + \cdots \end{pmatrix} = \begin{pmatrix} S_{11}, S_{12} \\ S_{21}, S_{22} \end{pmatrix}.$$

If the theorem is true, then $S_p = C_{p+1}^{-1} S_{p+1}$ with

$$C_{p+1}^{-1} = \begin{pmatrix} 1, & -h(q_{p+1} - \lambda r_{p+1}) \\ -h, & 1 + h^2(q_{p+1} - \lambda r_{p+1}) \end{pmatrix}.$$

We now multiply $S_{p+1}(\lambda)$ from the left with a matrix

$$C^{-1} = \begin{pmatrix} 1, & -h(q - \lambda r) \\ -h, & 1 + h^2(q - \lambda r) \end{pmatrix}$$

and try to decide q and r so that the new matrix $C^{-1}S_{p+1}$ has the same properties as S_{p+1} but with polynomials of lower degree than those of S_{p+1}.

$$\begin{pmatrix} 1, & -h(q - \lambda r) \\ -h, & 1 + h^2(q - \lambda r) \end{pmatrix} S_{p+1}$$

$$= \begin{pmatrix} (a_{p+1} + hrc_p)\lambda^{p+1} + (a_p - hqc_p + hrc_{p-1})\lambda^p + \cdots, \cdots \\ (-ha_{p+1} - h^2rc_p)\lambda^{p+1} + (-ha_p + (1 + h^2q)c_p - h^2rc_{p-1})\lambda^p + \cdots, \cdots \end{pmatrix}.$$

We see that we can choose r so that $a_{p+1} + hrc_p = 0$, $-ha_{p+1} - h^2rc_p = 0$. Then we can choose q so that $-ha_p + (1 + h^2q)c_p - h^2rc_{p-1} = 0$. Then $a_p' = a_p - hqc_p + hrc_{p-1} = c_p/h \neq 0$. We thus het

$$S_p = C^{-1}S_{p+1} = \begin{pmatrix} S_{11}', S_{12}' \\ S_{21}', S_{22}' \end{pmatrix}$$

where we know that S'_{11} has degree p and S'_{21} has degree less than or equal to $p-1$. Now $\det S_{p+1} \equiv 1 \Rightarrow a_{p+1}d_p - b_{p+1}c_p = 0$, i.e., $a_{p+1}/c_p = b_{p+1}/d_p$, which implies that $b_{p+1} + hrd_p = 0$. Thus S'_{12} is of degree less than or equal to p. Further, $\det S_p \equiv 1$, which implies that the degree of S'_{22} cannot exceed $p-1$.

This gives

$$- hb_p + (1 + h^2q)\, d_p - h^2rd_p = 0$$

or

$$d_p - h(b_p - hqd_p + hrd_{p-1}) = 0.$$

But $b_p' = b_p - hqd_p + hrd_{p-1}$ so that $b_p' = d_p/h \neq 0$, i.e., the degree of S'_{12} is exactly p.

So far the "separation properties" of the zeroes have not been used. Those properties imply that the degrees of S'_{21} and S'_{22} are exactly $p-1$ as will be seen.

We have

$$S_p = \begin{pmatrix} 1, & 0 \\ -h, & 1 \end{pmatrix} \begin{pmatrix} 1, & -hQ \\ 0, & 1 \end{pmatrix} \begin{pmatrix} S_{11}, & S_{12} \\ S_{21}, & S_{22} \end{pmatrix},$$

$$\begin{pmatrix} 1, & -hQ \\ 0, & 1 \end{pmatrix} \begin{pmatrix} S_{11}, & S_{12} \\ S_{21}, & S_{22} \end{pmatrix} = \begin{pmatrix} S_{11} - hQS_{21}, & S_{12} - hQS_{22} \\ S_{21}, & S_{22} \end{pmatrix} = \begin{pmatrix} S'_{11}, & S'_{12} \\ S_{21}, & S_{22} \end{pmatrix}.$$

The zeroes of S_{21} and S_{22} are real and interlacing. Lemma 3 then implies that the same is true for S'_{11} and S'_{12}. ($S'_{11}S_{22} - S'_{12}S_{21} \equiv 1$.) Now

$$S_p = \begin{pmatrix} 1, & 0 \\ -h, & 0 \end{pmatrix} \begin{pmatrix} S'_{11}, & S'_{12} \\ S_{21}, & S_{22} \end{pmatrix} = \begin{pmatrix} S'_{11} & S'_{12} \\ S_{21} - hS'_{11}, & S_{22} - hS'_{12} \end{pmatrix} = \begin{pmatrix} S'_{11}, & S'_{12} \\ S'_{21}, & S'_{22} \end{pmatrix}.$$

If λ_i and λ_{i+1} are two consecutive zeroes of S'_{11} then it follows from Lemma 2 that S'_{21} changes sign exactly once on the interval $(\lambda_i, \lambda_{i+1})$. But as there are $p-1$ such intervals, S'_{21} is of degree not less than $p-1$. The previous result then implies that degree of S'_{21} is exactly $p-1$. The same conclusion is valid for S'_{22}.

Lemma 2 $\Rightarrow$ that S_p has the same separation properties of the zeroes as S_{p+1} has.

Theorem I now follows by induction after we have checked the case $p = 1$. We have

$$S_1 = \begin{pmatrix} a_1\lambda + a_0, & b_1\lambda + b_0 \\ c_0, & d_0 \end{pmatrix},$$

$$C^{-1}S_1 = \begin{pmatrix} 1, & -h(q - \lambda r) \\ -h, & 1 + h^2(q - \lambda r) \end{pmatrix} \begin{pmatrix} a_1\lambda + a_0, & b_1\lambda + b_0 \\ c_0, & d_0 \end{pmatrix}$$

$$= \begin{pmatrix} (a_1 + hrc_0)\, \lambda + a_0 - hqc_0, & \cdots \\ (-ha_1 - h^2rc_0)\, \lambda + (1 + h^2q)\, c_0 - ha_0, & \cdots \end{pmatrix}.$$

We have supposed that a_1 and b_1 are not equal to zero. This and $\det S_1 \equiv 1$ implies that c_0 and d_0 are not equal to zero. In the same way as before we can choose first $r = -a_1/hc_0 = -b_1/hd_0$ and then q so that $(1 + h^2q)\,c_0 - ha_0 = 0$. Then

$$C^{-1}S_1 = \begin{pmatrix} a_0 - hqc_0\,, b_0 - hqd_0 \\ 0, -hb_0 + (1 + h^2q)\,d_0 \end{pmatrix} = \begin{pmatrix} a, & b \\ 0, & 1/a \end{pmatrix} = A$$

for $\det C^{-1}S_1 = \det C^{-1} \det A \equiv 1$. Thus $S_1 = CA$ and the proof of Theorem I is complete.

For the sake of clarity we rewrite the equations which determine the coefficients r_i and q_i. If

$$\begin{aligned} S_i &= C_iC_{i-1} \cdots C_2C_1A \\ &= \begin{pmatrix} a_{i,i}\lambda^i + a_{i,i-1}\lambda^{i-1} + \cdots + a_{i,0}\,, & b_{i,i}\lambda^i + b_{i,i-1}\lambda^{i-1} + \cdots + b_{i,0} \\ c_{i,i-1}\lambda^{i-1} + c_{i,i-2}\lambda^{i-2} + \cdots + c_{i,0}\,, & d_{i,i-1}\lambda^{i-1} + \cdots \quad d_{i,0} \end{pmatrix}, \end{aligned}$$

then

$$a_{i,i} + hr_ic_{i,i-1} = 0,$$

$$-ha_{i,i-1} + (1 + h^2q_i)\,c_{i,i-1} - h^2r_ic_{i,i-2} = 0.$$

$$a_{i,i} = \frac{c_{i+1,i}}{h}, \qquad b_{i,i} = \frac{d_{i+1,i}}{h}.$$

If starting with a matrix $S_n = C_nC_{n-1} \cdots C_1A$, the computation of r_n and q_n is quite simple, but as, of course, the coefficients $a_{i,j}$ and $c_{i,j}$ depend on $a_{i+1,j}$, $c_{i+1,j}$, $c_{i+1,j-1}$, r_{i+1} and q_{i+1} the numerical difficulties will be considerable already after a few steps if we try to consecutively determine the coefficients r_i and q_i for $i = n, n-1, \ldots, 1$.

Returning to our matrices we have

$$A^{-1} = \begin{pmatrix} a, & b \\ 0, & 1/a \end{pmatrix}^{-1} = \begin{pmatrix} 1/a, & -b \\ 0, & a \end{pmatrix},$$

$$\begin{pmatrix} S_{11}\,, & S_{12} \\ S_{21}\,, & S_{22} \end{pmatrix} A^{-1} = \begin{pmatrix} S_{11}/a, & -bS_{11} + aS_{12} \\ S_{21}/a, & -bS_{21} + aS_{22} \end{pmatrix} = C_nC_{n-1} \cdots C_1\,.$$

If we now inspect the proof of Theorem I we notice that for the computation of the coefficients r_i and q_i only knowledge of the polynomials S_{11} and S_{21} is needed. We thus conclude that if the zeroes of S_{11} and S_{21} are given (interlacing) and if the ratio a_{n+1}/c_n between the highest degree coefficients of these polynomials is fixed ($2n$ given parameters) then there is a unique defined constant a and two uniquely defined polynomials S_{12} and S_{22} so th

$$\begin{pmatrix} S_{11}/a, & S_{12} \\ S_{21}/a, & S_{22} \end{pmatrix} = C_n C_{n-1} \cdots C_2 C_1 .$$

There are $2n$ parameters q_i and r_i .

The next theorem shows how to compute r_i and q_i if the zeroes of S and S_{12} are given.

THEOREM II. *If*

$$S_n = \begin{pmatrix} S_{11}, & S_{12} \\ S_{21}, & S_{22} \end{pmatrix}$$

is given as in Theorem I *and if* $S_{11} - hS_{12}$ *is of degree not higher than* n — (*i.e., if* $a^n = hb^n$) *then* S_n *can be written*

$$S_n = \begin{pmatrix} a, & 0 \\ 0, & 1/a \end{pmatrix} C_n C_{n-1} \cdots C_2 C_1$$

and the factorization is unique.

Proof. Suppose that the proposition is true for $n \leqslant p$. For $n = p +$ we have

$$S_p = \begin{pmatrix} S_{11}, & S_{12} \\ S_{21}, & S_{22} \end{pmatrix} \begin{pmatrix} 1, & -h(q - \lambda r) \\ -h, & 1 + h^2(q - \lambda r) \end{pmatrix} = S_{p+1} C^{-1}$$

$$= \begin{pmatrix} S_{11} - hS_{12}, & S_{12} - hQ(S_{11} - hS_{12}) \\ S_{21} - hS_{22}, & S_{22} - hQ(S_{21} - hS_{22}) \end{pmatrix} = \begin{pmatrix} S'_{11}, & S'_{12} \\ S'_{21}, & S'_{22} \end{pmatrix}.$$

One easily realises that $S'_{11} = S_{11} - hS_{12}$ is exactly of degree p (see Figure and that the zeroes of S'_{11} separates the zeroes both S_{11} and S_{12} .

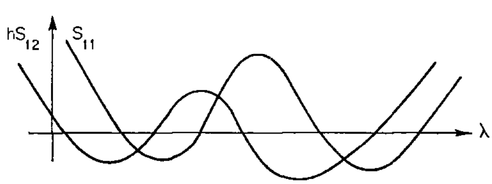

FIGURE 1

With the same notations as before we get

$$S'_{12} = S_{12} - hQ(S_{11} - hS_{12})$$
$$= b_{p+1}\lambda^{p+1} + b_p\lambda^p + \cdots - h(q - \lambda r)$$
$$\times \{\underbrace{(a_{p+1} - hb_{p+1})}_{=0} \lambda^{p+1} + \underbrace{(a_p - hb_p)}_{\neq 0} \lambda^p + \cdots\}$$
$$= \{\underbrace{b_{p+1} + hr(a_p - hb_p)}_{b'_{p+1}}\} \lambda^{p+1}$$
$$+ \{\underbrace{b_p - hq(a_p - hb_p) + hr(a_{p-1} - hb_{p-1})}_{b_p'}\} \lambda^p + \cdots$$
$$S'_{11} = \underbrace{(a_p - hb_p)}_{a_p' \neq 0} \lambda^p + \cdots .$$

First we can choose $r = - b_{p+1}/h(a_p - hb_p)$ so that $b'_{p+1} = 0$ and then q so that $a_p' - hb_p' = 0$:

$\det S_{p+1} \equiv 1 \Rightarrow c_p - hd_p = 0$ and that $S'_{21} = S_{21} - hS_{22}$ is exactly of degree $p - 1$ (for the zeroes of S_{21} and S_{22} are real and interlacing).

$$\det S_p \equiv 1 \Rightarrow S'_{22} \text{ is of degree } p - 1.$$

We also have to prove that, for example, S'_{11} and S'_{12} have interlacing zeroes. Suppose that λ_i and λ_{i+1} are two consecutive zeroes of $S'_{11} = S_{11} - hS_{12}$. The fact that the zeroes of S'_{11} separates those of $S_{12} \Rightarrow$ that $S_{12}(\lambda_i)$ and $S_{12}(\lambda_{i+1})$ have different signs. But that means that

$$S'_{12}(\lambda_i) = S_{12}(\lambda_i) - hQ(\lambda_i)\, \underbrace{(S_{11}(\lambda_i) - hS_{12}(\lambda_i))}_{=0} = S_{12}(\lambda_i)$$

and

$$S'_{12}(\lambda_{i+1}) = S_{12}(\lambda_{i+1})$$

have different signs, which implies that S'_{11} and S'_{12} have interlacing zeroes. (And so S'_{11}, S'_{21} and S'_{12}, S'_{22} and S'_{21}, S'_{22} by Lemma 2.)

The proof will be completed by checking the case $p = 1$.

$$\begin{pmatrix} a_1\lambda + a_0 , & b_1\lambda + b_0 \\ c_0 & d_0 \end{pmatrix} \begin{pmatrix} 1, & - h(q - r\lambda) \\ - h, & 1 + h^2(q - r\lambda) \end{pmatrix}$$

$= a_1 - hb_1$

$= 0 \Rightarrow c_0 - hd_0$

$= 0$

$$= \begin{pmatrix} \underbrace{(a_1 - hb_1)}_{=0}\lambda + a_0 - hb_0\,,\ \underbrace{(a_1hr - b_1h^2r)}_{=0}\lambda^2 \\ \quad + (a_0hr - hqa_1 + (1+h^2q)\,b_1 - h^2rb_0)\,\lambda - hqa_0 + (1+h^2q)\,b_0 \\ \underbrace{c_0 - hd_0}_{=0}\,,\ \underbrace{(hrc_0 - h^2rd_0)}_{=0}\lambda + (-hqc_0 + (1+h^2q)\,d_0) \end{pmatrix}$$

$$= \begin{pmatrix} a_0 - hb_0\,, & [-hq\overbrace{(a_1 - hb_1)}^{=0} + b_1 + hr(a_0 - hb_0)]\,\lambda + b_0 - hq(a_0 - hb_0) \\ 0, & d_0 - hq\underbrace{(c_0 - hd_0)}_{=0} \end{pmatrix}$$

$$= \begin{pmatrix} a_0 - hb_0\,, & [b_1 + hr(a_0 - hb_0)]\,\lambda + b_0 - hq(a_0 - hb_0) \\ 0, & d_0 \end{pmatrix}.$$

Now $\det S_1 \equiv 1 \Rightarrow a_0 - hb_0 \neq 0$ for $a_0 - hb_0 = 0$ would give $a_1\lambda + a_0 = h(b_1\lambda + b_0)$ or

$$\det S_1 = (a_1\lambda + a_0)\,d_0 - \frac{1}{h}(a_1\lambda + a_0)\,c_0 = \underbrace{\left(d_0 - \frac{c_0}{h}\right)}_{=0}(a_1\lambda + a_0) \equiv 1,$$

which is impossible.

Thus we can choose r so that $b_1 + hr(a_0 - hb_0) = 0$, and then q so that $b_0 - hq(a_0 - hb_0) = 0$:

$$S_1C^{-1} = \begin{pmatrix} a_0 - hb_0\,, & 0 \\ 0, & d_0 \end{pmatrix} = \begin{pmatrix} a, & 0 \\ 0, & 1/a \end{pmatrix},$$

and Theorem II is proved.

$$\begin{pmatrix} S_{11}\,, & S_{12} \\ S_{21}\,, & S_{22} \end{pmatrix} = \begin{pmatrix} a, & 0 \\ 0, & 1/a \end{pmatrix} C_nC_{n-1} \cdots C_2C_1\,.$$

$$\begin{pmatrix} 1/a, & 0 \\ 0, & a \end{pmatrix}\begin{pmatrix} S_{11}\,, & S_{12} \\ S_{21}\,, & S_{22} \end{pmatrix} = \begin{pmatrix} 1/a \cdot S_{11}\,, & 1/aS_{12} \\ aS_{21}\,, & aS_{22} \end{pmatrix} = C_nC_{n-1} \cdots C_2C_1\,.$$

In the proof of Theorem II we needed only S_{11} and S_{12} to compute r_i and q_i. Thus we see that the matrices C_i are uniquely defined by the $2n$ zeroes of S_{11} and S_{12}.

We have

$$r_i = \frac{-b_n}{h(a_{n-1} - hb_{n-1})} = \frac{1}{h^2 \sum_{i=1}^{n} (\lambda_i^{11} - \lambda_i^{12})}.$$

THEOREM III.

$$S_n(\lambda) = \begin{pmatrix} S_{11}, S_{12} \\ S_{21}, S_{22} \end{pmatrix}$$

is assumed to have the same properties as in the previous theorems. Then all r_i, $i = 1, 2, \ldots, n$ *will have the same sign.*

Proof. We have

$$S_n = \begin{pmatrix} S_{11}, S_{12} \\ S_{21}, S_{22} \end{pmatrix} = C_n \cdots C_1 \cdot A, \qquad A = \begin{pmatrix} a, & b \\ 0, & 1/a \end{pmatrix},$$

$$C_n C_{n-1} \cdots C_2 C_1 = \begin{pmatrix} S_{11}, S_{12} \\ S_{21}, S_{22} \end{pmatrix} \begin{pmatrix} 1/a, & -b \\ 0, & a \end{pmatrix}$$

$$= \begin{pmatrix} S_{11} \cdot 1/a, & -bS_{11} + aS_{12} \\ S_{21} \cdot 1/a, & -bS_{21} + aS_{22} \end{pmatrix} = \begin{pmatrix} S'_{11}, S'_{12} \\ S'_{21}, S'_{22} \end{pmatrix} = S_n'.$$

Lemma 2 implies that the polynomials of S_n' fulfill the requirement of having interlacing sets of zeroes. Also naturally det $S_n' \equiv 1$. Further we have seen in the proofs of Theorems I and II that those properties are conserved when we multiply from the left or from the right with C_n^{-1} and C_1^{-1}, respectively. Repeating this procedure we deduce that the matrix $C_{i+1} \cdot C_i$ has the same properties. Now

$$C_{i+1} C_i = \begin{pmatrix} 1 + h^2 Q_{i+1}, hQ_{i+1} \\ h, \quad 1 \end{pmatrix} \begin{pmatrix} 1 + h^2 Q_i, hQ_i \\ h, \quad 1 \end{pmatrix}$$

$$= \begin{pmatrix} (1 + h^2 Q_{i+1})(1 + h^2 Q_i) + h^2 Q_{i+1}, & h\{Q_{i+1} + Q_i + h^2 Q_{i+1} Q_i\} \\ h(2 + h^2 Q_i), & 1 + h^2 Q_i \end{pmatrix}.$$

We now know that the zero of the first degree polynomial $1 + h^2 Q_i$ lies between the zeroes of the second degree polynomial $h(Q_{i+1} + Q_i + h^2 Q_{i+1} Q_i)$.

$$1 + h^2 Q_i(\lambda_0) = 0 \Rightarrow h\{Q_{i+1} + Q_i + h^2 Q_{i+1} Q_i\} = h\{Q_{i+1}[1 + h^2 Q_i] + Q_i\}$$

$$= hQ_i(\lambda_0) = -\frac{1}{h} < 0 \qquad \text{for} \qquad \lambda = \lambda_0. \text{ (See Figure 2.)}$$

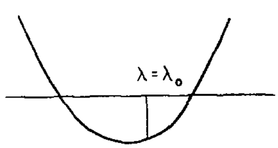

FIGURE 2

Now this obviously means that $h\{h^2Q_iQ_{i+1} + Q_i + Q_{i+1}\} \to +\infty$ when $\lambda \to \infty$, or, in other words, the coefficient of λ^2 must be positive. The coefficient of λ^2 in the polynomial $h[Q_{i+1} + Q_i + Q_{i+1}Q_ih^2]$ is

$$h^3(-r_{i+1})(-r_i) = h^3 r_{i+1} r_i > 0,$$

i.e., $r_{i+1}r_i > 0$ for $i = 1, 2, \ldots, n-1$. But this means that all r_i must have the same sign. Q.E.D.

REMARK.

$$r_i = \frac{1}{h^2 \sum_{i=1}^{n} (\lambda_i^{11} - \lambda_i^{12})}$$

gives that $r_i > 0$ if $\lambda_\nu^{11} > \lambda_\nu^{12}$ and $r_i < 0$ if $\lambda_\nu^{11} < \lambda_\nu^{12}$.

The next theorem is a converse of Theorem III.

THEOREM IV. *If* $S_n = C_nC_{n-1} \cdots C_2C_1$ *where all* r_i *have the same sign then* $\det S^n \equiv 1$, *and*

the zeroes of S_{11} *and* S_{12} *are all real and interlacing,*
the zeroes of S_{11} *and* S_{21} *are all real and interlacing,*
the zeroes of S_{12} *and* S_{22} *are all real and interlacing,*
the zeroes of S_{21} *and* S_{22} *are all real and interlacing.*

Proof. By induction it is easy to prove that

$$S_n = \begin{pmatrix} \lambda^n \cdot h^{2n} \cdot \prod_{i=1}^{n} (-r_i) + \cdots, & \lambda^n h^{2n-1} \prod_{i=1}^{n} (-r_i) + \cdots \\ \lambda^{n-1} \cdot h^{2n-1} \cdot \prod_{i=1}^{n-1} (-r_i) + \cdots, & \lambda^{n-1} \cdot h^{2n-2} \cdot \prod_{i=1}^{n-1} (-r_i) + \cdots \end{pmatrix}.$$

By the latter part of the proof of Theorem III we see that Theorem IV is true for $n = 2$.

Suppose it is true for $n = p$:

$$S_p = C_pC_{p-1} \cdots C_2C_1 = \begin{pmatrix} S_{11}, & S_{12} \\ S_{21}, & S_{22} \end{pmatrix},$$

$$S_{p+1} = \begin{pmatrix} 1 + h^2Q, & hQ \\ h, & 1 \end{pmatrix} \begin{pmatrix} S_{11}, & S_{12} \\ S_{21}, & S_{22} \end{pmatrix}$$

$$= \begin{pmatrix} S_{11} + hQ(S_{11}h + S_{21}), & S_{12} + hQ(hS_{12} + S_{22}) \\ S_{11}h + S_{21}, & hS_{12} + S_{22} \end{pmatrix} = \begin{pmatrix} S'_{11}, & S'_{12} \\ S'_{21}, & S'_{22} \end{pmatrix}.$$

(a) All $r_i < 0$, $i = 1, 2, \ldots, p, p+1$. The highest degree coefficients are positive, S_{ij} and $S'_{ij} \to +\infty$ when $\lambda \to +\infty$ (Figure 3).

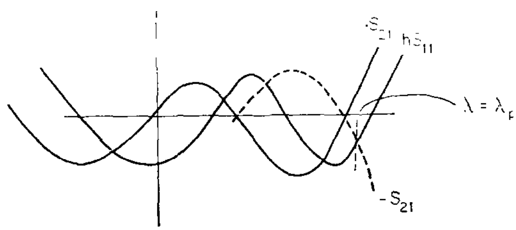

FIGURE 3

It is enough to show that the zeroes of S'_{11} and S'_{21} separate each other (Lemma 2).

Let λ_i and λ_{i+1} be two consecutive zeroes of $S'_{21} = hS_{11} + S_{21}$. Then

$$S'_{11}(\lambda_i) = S_{11}(\lambda_i) + \underbrace{hQ(S_{11}h + S_{21})}_{=0} \qquad \text{and} \qquad S'_{11}(\lambda_{i+1}) = S_{11}(\lambda_{i+1})$$

have different signs (for $hS_{11} + S_{21}$ and S_{11} have interlacing zeroes), which again means that the interval $(\lambda_i, \lambda_{i+1})$ contains an odd number of zeroes of S'_{11}.

Suppose now that some interval $(\lambda_i, \lambda_{i+1})$ contains three or more zeroes of S'_{11}. Then if λ_p is the greatest zero of S'_{21} we can have no zero of S'_{11} for $\lambda > \lambda_p$. But $\lambda = \lambda_p \Rightarrow S'_{11}(\lambda_p) = S_{11}(\lambda_p) < 0$ (see Figure 3).

$S'_{11} \to +\infty$ when $\lambda \to +\infty \Rightarrow$ there actually is a zero of S'_{11} for $\lambda > \lambda_p$. Thus each interval $(\lambda_i, \lambda_{i+1})$ contains exactly one zero of S'_{11}. Q.E.D.

(b) All $r_i > 0$ (see Figure 4, S'_{ij} and $S_{ij} \to +\infty$ when $\lambda \to -\infty$).

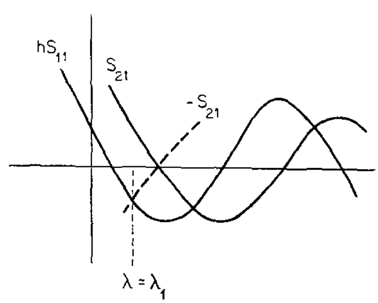

FIGURE 4

Let λ_1 be the smallest zero of S'_{21}. Then $S'_{11}(\lambda_1) = S_{11}(\lambda_1) < 0$. $S_{11} \to +\infty$ when $\lambda \to -\infty \Rightarrow$ that S'_{11} has a zero for $\lambda < \lambda_1 \Rightarrow$ in the same way as in (a) that every interval $(\lambda_i, \lambda_{i+1})$ contains exactly one zero of S'_{11}, which completes Theorem IV.

As is to be expected it is also easy to prove Theorem IV by a method analogous to that used in the classical proof of Sturm's oscillation theorem for a differential equation.

Alternative proof of Theorem IV. We start with the difference equations

$$\text{(I)} \quad \begin{cases} \dfrac{v_{i+1} - v_i}{h} = (q_{i+1} - r_{i+1}\lambda_2)\, u_{i+1}, & u_0, v_0 \text{ given}, \\[2ex] \dfrac{u_{i+1} - u_i}{h} = v_i, \end{cases}$$

$$\text{(II)} \quad \begin{cases} \dfrac{z_{i+1} - z_i}{h} = (q_{i+1} - \lambda_1 r_{i+1})\, y_{i+1}, & y_0, z_0 \text{ given}, \\[2ex] \dfrac{y_{i+1} - y_i}{h} = z_i. \end{cases}$$

If we multiply equations (I) and (II) by y_{i+1} and u_{i+1}, respectively, and if we subtract we obtain

$$\frac{z_{i+1} - z_i}{h} \cdot u_{i+1} - \frac{v_{i+1} - v_i}{h} \cdot y_{i+1} = (\lambda_2 - \lambda_1)\, r_{i+1} y_{i+1} u_{i+1},$$

$$\frac{1}{h} \sum_{i=0}^{n-1} [z_{i+1}u_{i+1} - z_i(u_{i+1} - u_i) - u_i z_i]$$

$$-\frac{1}{h} \sum_{i=0}^{n-1} [v_{i+1}y_{i+1} - v_i(y_{i+1} - y_i) - v_i y_i] = (\lambda_2 - \lambda_1) \sum_{i=0}^{n-1} r_{i+1} y_{i+1} u_{i+1},$$

$$z_n u_n - z_0 u_0 - (v_n y_n - v_0 y_0) = (\lambda_2 - \lambda_1) \sum_{i=1}^{n} r_i y_i u_i h,$$

$$\begin{pmatrix} z_n \\ y_n \end{pmatrix} = \begin{pmatrix} S_{11}, S_{12} \\ S_{21}, S_{22} \end{pmatrix}_{\lambda = \lambda_1} \begin{pmatrix} z_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} S_{11}(\lambda_1) \cdot z_0 + S_{12}(\lambda_1)\, y_0 \\ S_{21}(\lambda_1)\, z_0 + S_{22}(\lambda_1)\, y_0 \end{pmatrix},$$

$$\begin{pmatrix} v_n \\ u_n \end{pmatrix} = \begin{pmatrix} S_{11}(\lambda_2)\, v_0 + S_{12}(\lambda_2)\, u_0 \\ S_{21}(\lambda_2)\, v_0 + S_{22}(\lambda_2)\, u_0 \end{pmatrix}$$

$$\Rightarrow (S_{11} \cdot z_0 + S_{12} y_0)_{\lambda=\lambda_1} \cdot (S_{21} v_0 + S_{22} u_0)_{\lambda=\lambda_2} - z_0 u_0$$

$$- (S_{11} v_0 + S_{12} \cdot u_0)_{\lambda=\lambda_2} (S_{21} z_0 + S_{22} y_0)_{\lambda=\lambda_1} + v_0 y_0$$

$$= (\lambda_2 - \lambda_1) \sum_{i=1}^{n} r_i y_i u_i h,$$

$$(\lambda_2 - \lambda_1) \sum_{i=1}^{n} r_i y_i u_i h$$

$$= (S_{11} z_0 + S_{12} y_0)_{\lambda=\lambda_1} \cdot \{(S_{21}(\lambda_2) - S_{21}(\lambda_1))\, v_0 + (S_{22}(\lambda_2) - S_{22}(\lambda_1))\, u_0\}$$

$$+ (S_{11} z_0 + S_{12} y_0)_{\lambda=\lambda_1} \cdot (S_{21} v_0 + S_{22} u_0)_{\lambda=\lambda_1} - z_0 u_0$$

$$- \{(S_{11}(\lambda_2) - S_{11}(\lambda_1))\, v_0 + (S_{12}(\lambda_2) - S_{12}(\lambda_1))\, u_0\} \cdot (S_{21} z_0 + S_{22} y_0)_{\lambda=\lambda_1}$$

$$- (S_{11} v_0 + S_{12} u_0)_{\lambda=\lambda_1} \cdot (S_{21} z_0 + S_{22} y_0)_{\lambda=\lambda_1} + v_0 y_0\,,$$

$$\lambda_2 \to \lambda_1 = \lambda \Rightarrow \sum_{i=1}^{n} r_i y_i u_i h = (z_0 S_{11} + y_0 S_{12}) \cdot (v_0 S'_{21} + u_0 S'_{22})$$

$$- (v_0 S'_{11} + u_0 S'_{12})\,(z_0 S_{21} + y_0 S_{22}) + z_0 v_0 \{S_{11} S_{21} - S_{11} S_{21}\}$$
$$+ z_0 u_0 \{S_{11} S_{22} - S_{12} S_{21} - 1\} + y_0 v_0 \{S_{12} S_{21} - S_{11} S_{22} + 1\}$$
$$+ y_0 u_0 (S_{12} S_{22} - S_{12} S_{22}),$$

$$\sum_{i=1}^{n} r_i y_i u_i h = z_0 v_0 (S_{11} S'_{21} - S'_{11} S_{21}) + z_0 u_0 (S_{11} S'_{22} - S'_{12} S_{21})$$

$$+ y_0 v_0 (S_{12} S'_{21} - S'_{11} S_{22}) + y_0 u_0 (S_{12} S'_{22} - S'_{12} S_{22}),$$

or if we put $u_0 = y_0$, $v_0 = z_0$,

$$\sum_{i=1}^{n} r_i y_i^2 h = z_0^2 (S_{11} S'_{21} - S'_{11} S_{21})$$

$$+ z_0 y_0 \{S_{11} S'_{22} - S'_{11} S_{22} + S_{12} S'_{21} - S'_{12} S_{21}\}$$

$$+ y_0^2 (S_{12} S'_{22} - S'_{12} S_{22}).$$

Now let us take $y_0 = 0$ and $z_0 = 1$. Then

$$\sum_{i=1}^{n} r_i y_i^2 h = S_{11}S'_{21} - S_{11}S'_{21}$$

and we have proved that $S_{11}S'_{21} - S'_{11}S_{21}$ has the same sign for all λ if the coefficients r_i $i = 1,..., n$ have the same sign for all i.

If, for example, $r_i > 0$, then $S_{11}S'_{21} - S'_{11}S_{21} > 0$ for all λ:

$$S_{11}S'_{21} - S'_{11}S_{21} = S_{11}^2 \left(\frac{S_{21}}{S_{11}}\right)' \qquad \text{if} \qquad S_{11} \neq 0,$$

$$= - S_{21}^2 \left(\frac{S_{11}}{S_{21}}\right)' \qquad \text{if} \qquad S_{21} \neq 0$$

($S_{11}S_{22} - S_{12}S_{21} = 1 \Rightarrow$ that S_{11} and S_{21} are never zero for the same value of λ).

We now know that $S_{11}(\lambda)/S_{21}(\lambda) = z_n/y_n$ is a decreasing function of λ and hat $S_{21}/S_{11} = y_n/z_n$ is an increasing function of λ and that z^n and y^n are never zero at the same time. (The same conclusion is valid of course if n is replaced by any index i, $1 \leqslant i \leqslant n$.)

The fact that S_{21}/S_{11} is an increasing function of λ on every interval where the denominator S_{11} does not vanish implies that every interval $(\lambda_i^{11}, \lambda_{i+1}^{11})$ (λ_i^{11} and λ_{i+1}^{11} are two consecutive zeroes of S_{11}) contains at least one zero λ_i^{21} of $S_{21}(\lambda)$. For if $S_{21}(\lambda)$ has constant sign for $\lambda \in (\lambda_i^{11}, \lambda_i^{11})$ then

$$\lim_{\lambda \to \lambda_i^{11}} \frac{S_{21}}{S_{11}} = \lim_{\lambda \to \lambda_{i+1}^{11}} \frac{S_{21}}{S_{11}} = \pm \infty,$$

which is impossible.

We also conclude that if λ_1^{21} is the least and λ_ν^{21} is the largest zero of S_{21} then S_{11} has zeroes for both $\lambda < \lambda_1^{21}$ and $\lambda > \lambda_\nu^{21}$. If, namely, S_{21}/S_{11} has constant sign for $\lambda > \lambda_\nu^{21}$ or $\lambda < \lambda_\nu^{21}$ then the monotone function $S_{21}(\lambda)/S_{11}(\lambda)$ cannot vanish for $\lambda = \pm \infty$, i.e.,

$$\lim_{\lambda \to \pm\infty} \frac{S_{21}}{S_{11}} \neq 0,$$

which is a contradiction.

It now only remains to prove that S_{21} has exactly $n - 1$ real zeroes.

If we plot the point (y_i, i) in a Cartesian coordinate system and if we connect "consecutive" points by straight lines as in Figure 5 we obtain a continuous function $y(x)$. The zeroes of this function have the property that if we increase λ then the zeroes, which depend continuously upon λ, decrease ($r_i > 0$); see Figure 6.

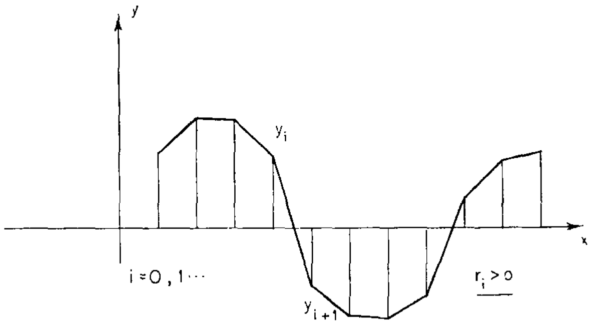

FIGURE 5

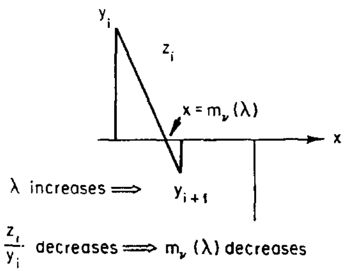

FIGURE 6

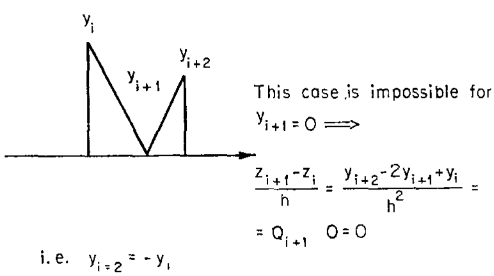

FIGURE 7

The case sketched in Figure 7 is impossible, which implies that the zeroes cannot appear in the interior of the interval $(0, n)$ when changing λ. The zeroes will appear one by one at the end point $x = n$ and then travel towards the left end point $x = 0$.

If we now take λ large enough then we know that y_i and y_{i+1} have different signs for all i, $1 \leqslant i \leqslant n-1$ ($r_i > 0$). We have, namely,

$$\frac{z_{i+1} - z_i}{h} = Q_{i+1} y_{i+1} \Rightarrow y_{i+2} = \underbrace{(2 + h^2 Q_{i+1})}_{<0} y_{i+1} + y_i$$

with $2 + h^2 Q_{i+1} < 0$ for all i and sufficiently large λ.

$$y_0 = 0 \Rightarrow y_2 y_1 < 0,$$

$$y_{i+1} y_i < 0 \Rightarrow y_{i+2} y_{i+1} = \underbrace{(2 + h^2 Q_{i+1})}_{<0} y_{i+1}^2 + \underbrace{y_{i+1} y_i}_{<0} < 0.$$

Induction gives the desired relation for all i.

On the other hand if we take λ sufficiently negative then all elements of the matrices

$$C_i = \begin{pmatrix} 1 + h^2 Q_i, & hQ_i \\ h & 1 \end{pmatrix}$$

are positive and so all the y_i, $i \geqslant 1$ will be positive.

We finally conclude that as λ tends from $-\infty$ to $+\infty$ then $y_n(\lambda) = S_{21}$ changes sign exactly $n - 1$ times.

By Lemma 2, Theorem IV follows.

Some Notes on the Question of Convergence

Consider now the inhomogenous system

$$(1) \qquad \begin{cases} \dfrac{z_{i+1} - z_i}{h} = (q_{i+1} - r_{i+1}\lambda) y_{i+1} + f_{i+1}, \\ \\ \dfrac{y_{i+1} - y_i}{h} = z_i, \end{cases} \qquad i = 0, 1, \ldots, n,$$

and suppose $r_i > 0$. After multiplying Eq. (1) by $\bar{y}_{i+1}$ and summing over i one obtains as in Lemma 1:

$$z_{n+1} \bar{y}_{n+1} - z_0 \bar{y}_0 - h \sum_{i=0}^{n} |z_i|^2$$

$$= h \sum_{i=0}^{n} q_{i+1} |y_{i+1}|^2 - \lambda \sum_{i=0}^{n} r_{i+1} |y_{i+1}|^2 h + \sum_{i=0}^{n} f_{i+1} \bar{y}_{i+1} h.$$

We choose boundary conditions so that $z_{n+1}\bar{y}_{n+1} - z_0\bar{y}_0 = 0$. If we take the imaginary part of the equation we obtain

$$\operatorname{Im}\lambda \cdot \sum r_{i+1} \mid y_{i+1} \mid^2 h = \operatorname{Im}\left(\sum_{i=0}^{n} f_{i+1}\bar{y}_{i+1}h\right)$$

or

$$\mid \operatorname{Im}\lambda \mid \cdot \sum r_{i+1} \mid y_{i+1} \mid^2 h \leqslant \left| \sum_{i=0}^{n} f_{i+1}\bar{y}_{i+1}h \right|$$

$$\leqslant \left\{\sum_{i=0}^{n} \mid f_{i+1} \mid^2 h\right\}^{1/2} \cdot \left\{\sum_{i=0}^{n} \mid y_{i+1} \mid^2 h\right\}^{1/2},$$

i.e.,

$$(2) \quad \sum r_{i+1} \mid y_{i+1} \mid^2 h \leqslant \frac{1}{\mid \operatorname{Im}\lambda \mid} \left\{\sum \mid f_{i+1} \mid^2 h\right\}^{1/2} \cdot \left\{\sum \mid y_{i+1} \mid^2 h\right\}^{1/2}.$$

Now choose $h = 1/(n + 1)$ and consider the sequences of functions on $(0, 1)$

$$\left.\begin{aligned} y_{n+1}(t) &= y_i^{(n+1)}, \\ r_{n+1}(t) &= r_i^{(n+1)}, \\ q_{n+1}(t) &= q_i^{(n+1)}, \end{aligned}\right\} \quad \text{for} \quad (i-1)\,h \leqslant t < i \cdot h, \qquad i = 1, 2,..., n + 1.$$

where $r_i^{(n+1)}$, $q_i^{(n+1)}$ are the coefficients corresponding to the sets of zeroes λ_i^{11}, λ_i^{12}, $i = 1, 2,..., n + 1$, and where $y_i^{(n+1)}$, $i = 0, 1, 2,..., n + 1$, is a solution of the system (1) with coefficients $r_i^{(n+1)}$, $q_i^{(n+1)}$ and a boundary condition that makes $z_{n+1}\bar{y}_{n+1} - z_0\bar{y}_0 = 0$.

If now the number $n + 1$ of zeroes tends to infinity one might put the questions: What happens with the functions $y_{n+1}(t)$, $r_{n+1}(t)$, and $q_{n+1}(t)$? Is it possible to find conditions on the sequences of zeroes that make the sequences of functions $y_{n+1}(t)$, $r_{n+1}(t)$, and $q_{n+1}(t)$ in any sense convergent? Will the limiting functions satisfy the differential equation corresponding to difference equation?

If we could show a relation of the form $r_i^{(n+1)} > \delta > 0$, $i = 1, 2,..., n + 1$, then the inequality (2) would give

$$\delta \sum_{i=0}^{n} \mid y_{i+1} \mid^2 h \leqslant \sum_{i=0}^{n} r_{i+1} \mid y_{i+1} \mid^2 h$$

$$\leqslant \frac{1}{\mid \operatorname{Im}\lambda \mid} \left\{\sum_{i=0}^{n} \mid f_{i+1} \mid^2 h\right\}^{1/2} \left\{\sum_{i=0}^{n} \mid y_{i+1} \mid^2 h\right\}^{1/2}$$

i.e.,

$$\left\{\sum_{i=0}^{n} |y_{i+1}|^2 h\right\}^{1/2} \leqslant \frac{1}{\delta \cdot |\operatorname{Im} \lambda|} \cdot \left\{\sum_{i=0}^{n} |f_{i+1}|^2 h\right\}^{1/2}$$

or

$$\left\{\int_0^1 |y_{n+1}(t)|^2 \, dt\right\}^{1/2} \leqslant \frac{1}{\delta \, |\operatorname{Im} \lambda|} \cdot \left\{\int_0^1 |f(t)|^2 \, dt\right\}^{1/2}$$

or in Hilbert notations

$$\| y_{n+1} \| \leqslant \frac{1}{\delta \, |\operatorname{Im} \lambda|} \| f \| .$$

The sequence y_n being bounded in norm will then contain a weakly convergent subsequence:

$$y_{n_i} \to y \text{ weakly.}$$

Or if we introduce the operator $G_n(\lambda)$ defined by $y_n = G_n(\lambda) f$:

$$\| G_n(\lambda) f \| \leqslant \frac{1}{|\operatorname{Im} \lambda| \, \delta} \cdot \| f \| \Rightarrow \| G_n(\lambda) \| \leqslant \frac{1}{|\operatorname{Im} \lambda| \, \delta},$$

which means that the sequence $G_n(\lambda)$ contains a weakly convergent subsequence $G_{n_i}(\lambda) \to G(\lambda)$. That is,

$$(G_{n_i} \varphi, \Psi) \to (G(\lambda) \varphi, \Psi) \qquad \text{for all} \qquad \varphi, \Psi \in L^2(0, 1).$$

(We can of course choose the subsequence so that $1/r_{n_i}(t) < 1/\delta$ becomes weakly convergent.)

After that of course remains to examine the operator $G(\lambda)$. What can be said about the spectrum of $G(\lambda)$? What about the range of $G(\lambda)$? As a preliminary attempt to tackle the problem of finding an estimation of the coefficients r_i, I will show various theorems that might be useful for a closer examination of the structure of our matrices.

THEOREM V. *If* $S_n(\lambda) = C_n C_{n-1} \cdots C_1$ *we have*

$$\sum_{i=1}^{n} \lambda_i^{11} = \sum_{i=1}^{n-1} \frac{2 + h^2 q_i}{h^2 r_i} + \frac{1 + h^2 q_n}{h^2 r_n},$$

$$\sum_{i=1}^{n} \lambda_i^{12} = \frac{1 + h^2 q_1}{h^2 r_1} + \sum_{i=2}^{n-1} \frac{2 + h^2 q_i}{h^2 r_i} + \frac{1 + h^2 q_n}{h^2 r_n}.$$

Proof. We can write

$$C_i(\lambda) = \begin{pmatrix} 1 + h^2q_i\,, & hq_i \\ h, & 1 \end{pmatrix} - \lambda r_i h \begin{pmatrix} h, & 1 \\ 0, & 0 \end{pmatrix} = A_i - \lambda r_i hB$$

$$S_n = (A_n - \lambda r_n hB) \cdots (A_i - \lambda r_i hB) \cdots (A_i - \lambda r_1 hB)$$

$$= \lambda^n h^n \prod_{i=1}^{n} (-r_i) \cdot B^n - \lambda^{n-1} h^{n-1} \prod_{i=1}^{n} (-r_i) \Big\{ \frac{1}{r_n} A_n B^{n-1}$$

$$+ \frac{1}{r_{n-1}} \cdot BA_{n-1}B^{n-2} + \cdots \frac{1}{r_i} B^{n-i}A_iB^{i-1} + \cdots \frac{1}{r_1} B^{n-1}A_1 \Big\} + \cdots$$

Now

$$B^2 = \begin{pmatrix} h, & 1 \\ 0, & 0 \end{pmatrix}^2 = \begin{pmatrix} h^2, & h \\ 0, & 0 \end{pmatrix} = hB,$$

$$B^\nu = h^{\nu-1}B; \qquad A_iB = \begin{pmatrix} h(1 + h^2q_i), & 1 + h^2q_i \\ h^2 & h \end{pmatrix},$$

$$BA_i = \begin{pmatrix} h, & 1 \\ 0, & 0 \end{pmatrix} \begin{pmatrix} 1 + h^2q_i\,, & hq_i \\ h, & 1 \end{pmatrix} = \begin{pmatrix} h(2 + h^2q_i), & 1 + h^2q_i \\ 0, & 0 \end{pmatrix},$$

$$BA_iB = \begin{pmatrix} h(2 + h^2q_i), & 1 + h^2q_i \\ 0, & 0 \end{pmatrix} \begin{pmatrix} h, & 1 \\ 0, & 0 \end{pmatrix} = \begin{pmatrix} h^2(2 + h^2q_i), & h(2 + h^2q_i) \\ 0, & 0 \end{pmatrix}$$

$$= h(2 + h^2q_i) \cdot B,$$

$$S_n = \lambda^n h^{2n-1} \prod_{i=1}^{n} (-r_i)\, B - \lambda^{n-1} h^{n-1} \prod_{i=1}^{n} (-r_i) \Big\{ h^{n-2} \frac{1}{r_n} A_nB$$

$$+ \cdots h^{n-2} \frac{1}{r_i} (2 + h^2q_i)\, B + \cdots + \frac{h^{n-2}}{r_i} \cdot BA_1 \Big\} + \cdots,$$

$$S_{11} = \lambda^n h^{2n} \prod_{i=1}^{n} (-r_i) - \lambda^{n-1} \prod_{i=1}^{n} (-r_i)\, h^{2n-2} \Big\{ \frac{1 + h^2q_n}{r_n}$$

$$+ \cdots \frac{2 + h^2q_i}{r_i} + \cdots + \frac{2 + h^2q_1}{r_1} \Big\} + \cdots,$$

$$S_{12} = \lambda^n h^{2n-1} \prod_{i=1}^{n} (-r_i) - \lambda^{n-1} \prod_{i=1}^{n} (-r_i)\, h^{2n-3} \Big\{ \frac{1 + h^2q_n}{r_n}$$

$$+ \cdots \frac{2 + h^2q_i}{r_i} + \cdots \frac{1 + h^2q_1}{r_1} \Big\} + \cdots.$$

This implies that:

$$\sum_{i=1}^{n} \lambda_i^{11} = \frac{1 + h^2 q_n}{h^2 r_n} + \sum_{i=1}^{n-1} \frac{2 + h^2 q_i}{h^2 r_i},$$

$$\sum_{i=1}^{n} \lambda_i^{12} = \frac{1 + h^2 q_n}{h^2 r_n} + \sum_{i=2}^{n-1} \frac{2 + h^2 q_i}{h^2 r_i} + \frac{1 + h^2 q_1}{h^2 r_1}.$$

The next theorem tells us what happens if we reverse the order of multiplication in the matrix $S_n = C_n C_{n-1} \cdots C_1$.

THEOREM VI. *If* $S_n = C_n C_{n-1} \cdots C_2 C_1$ *and* $S_n' = C_1 C_2 \cdots C_{n-1} C_n$ *then* $S_{12}' = S_{12}$ *and* $hS_{11} + S_{21} = hS_{11}' + S_{21}'$, $S_{11} = S_{22}' + hS_{12}'$, $S_{11}' = S_{22} + hS_{12}$.

Proof. The theorem is trivial for $n = 1$. Suppose it is true for $n = p$. For $n = p + 1$ we get:

$$S_{p+1} = C_{p+1} C_p \cdots C_1 = C_{p+1} S_p,$$

$$S_{p+1}' = C_1 C_2 \cdots C_p C_{p+1} = S_p' C_{p+1},$$

$$S_p = \begin{pmatrix} S_{11}, & S_{12} \\ S_{21}, & S_{22} \end{pmatrix}, \qquad S_p' = \begin{pmatrix} S_{11}', & S_{12}' \\ S_{21}', & S_{22}' \end{pmatrix}, \qquad C_{p+1} = \begin{pmatrix} 1 + h^2 Q, & hQ \\ h, & 1 \end{pmatrix},$$

$$S_{p+1} = \begin{pmatrix} 1 + h^2 Q, & hQ \\ h, & 1 \end{pmatrix} \begin{pmatrix} S_{11}, & S_{12} \\ S_{21}, & S_{22} \end{pmatrix}$$

$$= \begin{pmatrix} (1 + h^2 Q) S_{11} + hQS_{21}, & (1 + h^2 Q) S_{12} + hQS_{22} \\ hS_{11} + S_{21}, & hS_{12} + S_{22} \end{pmatrix}$$

$$= \begin{pmatrix} S_{11}(p + 1), & S_{12}(p + 1) \\ S_{21}(p + 1), & S_{22}(p + 1) \end{pmatrix},$$

$$S_{p+1}' = \begin{pmatrix} S_{11}', & S_{12}' \\ S_{21}', & S_{22}' \end{pmatrix} \begin{pmatrix} 1 + h^2 Q, & hQ \\ h, & 1 \end{pmatrix}$$

$$= \begin{pmatrix} S_{11}'(1 + h^2 Q) + hS_{12}', & hQS_{11}' + S_{12}' \\ S_{21}'(1 + h^2 Q) + hS_{22}', & hQS_{21}' + S_{22}' \end{pmatrix}$$

$$= \begin{pmatrix} S_{11}'(p + 1), & S_{12}'(p + 1) \\ S_{21}'(p + 1), & S_{22}'(p + 1) \end{pmatrix}.$$

By straightforward computation we now get:

$$\begin{aligned} S'_{12}(p+1) &= hQS'_{11} + S'_{12} = hQ(S_{22} + hS_{12}) + S_{12} \\ &= (1 + h^2Q)\,S_{12} + hQS_{22} = S_{12}(p+1), \end{aligned}$$

$$\begin{aligned} \underline{hS'_{11}(p+1)} + S'_{21}(p+1) &= h \cdot S'_{11}(1 + h^2Q) + h^2S'_{12} + S'_{21}(1 + h^2Q) + hS'_{22} \\ &= (1 + h^2Q)\,(hS'_{11} + S'_{21}) + h[hS'_{12} + S'_{22}] \\ &= (1 + h^2Q)\,(hS_{11} + S_{21}) + hS_{11} \\ &= h[(1 + h^2Q)\,S_{11} + hQS_{21}] + hS_{11} + S_{21} \\ &= \underline{hS_{11}(p+1) + S_{21}(p+1)}, \end{aligned}$$

$$\begin{aligned} \underline{S_{11}(p+1)} &= (1 + h^2Q)\,S_{11} + hQS_{21} = S_{11} + hQ(hS_{11} + S_{21}) \\ &= S'_{22} + hS'_{12} + hQ(hS'_{11} + S'_{21}) \\ &= hQS'_{21} + S'_{22} + h[hQS'_{11} + S'_{12}] \\ &= \underline{S'_{22}(p+1) + hS'_{12}(p+1)}; \end{aligned}$$

$$\begin{aligned} S'_{11}(p+1) &= S'_{11}(1 + h^2Q) + hS'_{12} \\ &= (S_{22} + hS_{12})\,(1 + h^2Q) + hS_{12} \\ &= hS_{12} + S_{22} + h[(1 + h^2Q)\,S_{12} + hQS_{22}] \\ &= \underline{S_{22}(p+1) + hS_{12}(p+1)}, \end{aligned}$$

and the proof is completed.

As in the theory of differential equations we might consider the problem of finding stationary values of the function

$$F(y_0, y_1, \ldots, y_n) = \sum_{i=0}^{n-1} z_i^2 h + \sum_{i=1}^{n} q_i y_i^2 h$$

under the conditions

(1) $\sum_{i=1}^{n} r_i y_i^2 h = 1$ and

and

(2) $z_0 = 0$

or

(3) $y_0 = 0$

(suppose all $r_i > 0$) ($z_i = (y_{i+1} - Y_i)/h$, $i = 0, 1, \ldots, n-1$).

By use of Lagrange's method we obtain

$$\frac{\partial}{\partial y_\nu}\left(F - \lambda \sum_{i=1}^{n} r_i y_i^2 h\right)$$

$$= 2z_{\nu-1}\frac{1}{h}\cdot h + 2z_\nu \cdot \left(-\frac{1}{h}\right)\cdot h + 2q_\nu y_\nu h - \lambda \cdot 2r_\nu y_\nu h = 0,$$

$\nu = 1, 2, \ldots, n-1,$

i.e.,

$$\frac{z_\nu - z_{\nu-1}}{h} = (q_\nu - r_\nu\lambda)\, y_\nu\,,$$

$$\nu = 0 \Rightarrow \frac{\partial}{\partial y_0}\left(F - \lambda \sum_{i=1}^{n} r_i y_i^2 h\right) = 2z_0 \cdot \left(-\frac{1}{h}\right)\cdot h = 0,$$

i.e.,

$$z_0 = 0,$$

$$\nu = n \Rightarrow \frac{\partial}{\partial y_n}\left(F - \lambda \sum_{i=1}^{n} r_i y_i^2 h\right) = 2z_{n-1}\frac{1}{h}\cdot h + 2q_n \cdot y_n h - \lambda \cdot 2r_n y_n h = 0,$$

$$\frac{0 - z_{n-1}}{h} = (q_n - r_n\lambda)\, y_n\,.$$

We thus see that the sequence $y_0\,, y_1\,, \ldots, y_n$ makes F stationary under the conditions (1) and (2) or (3) if and only if it satisfies our difference equation under the conditions $z_0 = 0$ or $y_0 = 0$ and $z_n = 0$ and $\sum r_i y_i^2 h = 1$.

The parameter λ must then be a zero of $S_{11}(\lambda)$ or S_{12}.

We also have

$$F = \sum_{i=0}^{n-1} z_i^2 h + \sum_{i=1}^{n} q_i y_i^2 h = \lambda \sum_{i=1}^{n} r_i y_i^2 h + z_n y_n - z_0 y_0 = \lambda.$$

The existence of a greatest and a least value of F under the given boundary conditions is obvious. Thus

$$\lambda_1^{12} \leqslant \sum_{i=0}^{n-1} z_i^2 h + \sum_{i=1}^{n} q_i y_i^2 h \leqslant \lambda_n^{12}$$

for all sequences $y_0\,, \ldots, y_n$ such that

$$\sum_{i=1}^{n} r_i y_i^2 h = 1 \qquad \text{and} \qquad z_0 = 0, \qquad \text{i.e.,} \qquad y_0 = y_1\,.$$

And

$$\lambda_1^{11} \leqslant \sum_{i=0}^{n-1} z_i^2 h + \sum_{i=1}^{n} q_i y_i^2 h \leqslant \lambda_n^{11}$$

for all sequences $y_0, y_1, \ldots, y_n$ such that

$$\sum_{i=1}^{n} r_i y_i^2 h = 1 \qquad \text{and} \qquad y_0 = 0.$$

With the aid of these inequalities we can now easily deduce a number of estimations of the coefficients r_i and q_i.

LEMMA 4. (a) *Taking*

$$y_\nu = y_{\nu+1} = k, \qquad \nu \geqslant 1,$$
$$y_i = 0, \quad i \neq \nu, \quad \nu + 1,$$

we obtain

$$r_\nu k^2 h + r_{\nu+1} k^2 h = 1, \qquad z_{\nu-1} = \frac{k}{h}, \qquad z_\nu = 0, \qquad z_{\nu+1} = -\frac{k}{h},$$

$$\lambda_1^{11} \leqslant \frac{2k^2}{h^2} \cdot h + q_\nu k^2 h + q_{\nu+1} k^2 h \leqslant \lambda_n^{11},$$

$$k^2 h = \frac{1}{r_\nu + r_{\nu+1}} \Rightarrow \underline{\lambda_1^{11} \leqslant \frac{2 + h^2(q_\nu + q_{\nu+1})}{h^2(r_\nu + r_{\nu+1})} \leqslant \lambda_n^{11}}, \qquad 1 \leqslant \nu < n-1,$$

$$\lambda_1^{11} \leqslant \frac{1 + h^2(q_{n-1} + q_n)}{h^2(r_{n-1} + r_n)} \leqslant \lambda_n^{11}.$$

(b)

$$y_\nu = k = -y_{\nu+1}, \qquad y_i = 0 \quad for \quad i \neq \nu, \nu = 1, \nu \geqslant 1,$$

$$z_{\nu-1} = \frac{k}{h}, \qquad z_\nu = -\frac{2k}{h}, \qquad z_{\nu+1} = \frac{k}{h}$$

$$\Rightarrow \lambda_1^{11} \leqslant \frac{k^2}{h} + \left(\frac{2k}{h}\right)^2 \cdot h + \frac{k^2}{h} + k^2 h(q_\nu + q_{\nu+1}) \leqslant \lambda_n^{11},$$

$$(r_\nu + r_{\nu+1}) k^2 h = 1,$$

$$\underline{\Rightarrow \lambda_1^{11} \leqslant \frac{6 + h^2(q_\nu + q_{\nu+1})}{h^2(r_\nu + r_{\nu+1})} \leqslant \lambda_n^{11}}, \qquad 1 \leqslant \nu < n-1,$$

$$\lambda_1^{11} \leqslant \frac{5 + h^2(q_{n-1} + q_n)}{h^2(r_{n-1} + r_n)} \leqslant \lambda_n^{11}.$$

(c) (a) *and* (b)

$$\Rightarrow \frac{6 + h^2(q_\nu + q_{\nu+1})}{h^2(r_\nu + r_{\nu+1})} - \frac{2 + h^2(q_\nu + q_{\nu+1})}{h^2(r_\nu + r_{\nu+1})} \leqslant \lambda_n^{11} - \lambda_1^{11}$$

or

$$\underline{r_\nu + r_{\nu+1} \geqslant \frac{4}{h^2(\lambda_n^{11} - \lambda_1^{11})}}, \qquad n - 1 \geqslant \nu \geqslant 1.$$

(d) $y_\nu = k,\ y_i = 0,$ *if* $i \neq \nu,\ 1 \leqslant \nu < n$

$$\Rightarrow \underline{\lambda_1^{11} \leqslant \frac{2 + h^2 q_\nu}{h^2 r_\nu} \leqslant \lambda_n^{11}}.$$

(e) $y_0 = y_1 = k,\ y_i = 0,$ *if* $i > 1$

$$\Rightarrow \underline{\lambda_1^{12} \leqslant \frac{1 + h^2 q_1}{h^2 r_1} \leqslant \lambda_n^{12}}.$$

(f) $y_n = k,\ y_i = 0$ *for* $i < n$

$$\Rightarrow \underline{y_1^{11} < \frac{1 + h^2 q_n}{h^2 r_n} < \lambda_n^{12}}.$$

(d) *and* (e) *give, for* $\nu = 1$,

$$\left.\begin{aligned} \frac{2 + h^2 q_1}{h^2 r_1} &< \lambda_n^{11} \\ \frac{1 + h^2 q_1}{h^2 r_1} &> \lambda_1^{12} \end{aligned}\right\} \Rightarrow \frac{1}{h^2 r_1} < \lambda_n^{11} - \lambda_1^{12}.$$

(g) $$y_\nu = \frac{k}{r_\nu^{\frac{1}{2}}}, \qquad y_{\nu+1} = \pm\frac{k}{r_{\nu+1}^{\frac{1}{2}}}, \qquad 1 \leqslant \nu \leqslant n - 1,$$

$$y_i = 0, \qquad \text{if} \qquad i \neq \nu, \nu + 1,$$

$$r_\nu \cdot \frac{k^2}{r_\nu} \cdot h + r_{\nu+1} \cdot \frac{k^2}{r_{\nu+1}} h = 1 \Rightarrow 2k^2 h = 1,$$

$$z_{\nu-1}^2 = \frac{k^2}{r_\nu h^2}, \qquad z_\nu{}^2 = \frac{k^2}{h^2}\left(\pm\frac{1}{\sqrt{r_{\nu+1}}} + \frac{1}{\sqrt{r_\nu}}\right)^2,$$

$$z_{\nu+1}^2 = \frac{k^2}{h^2 r_{\nu+1}}, \qquad \nu < n - 1,$$

$$\Rightarrow \lambda_1^{11} \leqslant \frac{k^2}{hr_\nu} + \frac{k^2}{h}\left(\frac{1}{\sqrt{r_\nu}} \pm \frac{1}{\sqrt{r_{\nu+1}}}\right)^2 + \frac{k^2}{hr_{\nu+1}} + k^2h \cdot \left\{\frac{q_\nu}{r_\nu} + \frac{q_{\nu+1}}{r_{\nu+1}}\right\} \leqslant \lambda_n^{11}$$

$$\Rightarrow \lambda_1^{11} \leqslant k^2h\left\{\frac{2 + h^2q_\nu}{h^2r_\nu} + \frac{2 + h^2q_{\nu+1}}{h^2r_{\nu+1}} \pm \frac{2}{h^2\sqrt{r_\nu r_{\nu+1}}}\right\} \leqslant \lambda_n^{11}, \qquad k^2h = \frac{1}{2},$$

$$\lambda_1^{11} \leqslant \left\{\frac{2 + h^2q_{n-1}}{h^2r_{n-1}} + \frac{1 + h^2q_n}{h^2r_n} \pm \frac{2}{h^2\sqrt{r_nr_{n-1}}}\right\} k^2h < \lambda_n^{11},$$

$$\Rightarrow \frac{1}{2} \cdot \frac{4}{h^2\sqrt{r_\nu r_{\nu+1}}} \leqslant \lambda_n^{11} - \lambda_1^{11},$$

or

$$\sqrt{r_\nu r_{\nu+1}} \geqslant \frac{2}{h^2(\lambda_n^{11} - \lambda_1^{11})}, \qquad 1 \leqslant \nu \leqslant n - 1.$$

The estimations (c) and (g) in Lemma 4 tell us that at least every second r_i is greater than, for example,

$$\delta_n = \frac{1}{h^2(\lambda_n^{11} - \lambda_1^{11})}.$$

By letting

$$h = \frac{1}{n} \qquad \text{and} \qquad \underline{\lambda_n^{11} - \lambda_1^{11} \leqslant K \cdot n^2}$$

we get $\delta_n \geqslant 1/K = \delta > 0$, i.e., in that case $r_ir_{i+1} > 4\delta^2$. $r_i + r_{i+1} \geqslant 4\delta$. This is of course far from sufficient to obtain an inequality of the form

$$\frac{1}{|\operatorname{Im}\lambda|}\left(\sum f_i^2h\right)^{1/2}\left(\sum y_i^2h\right)^{1/2} \geqslant \sum_{i=1}^{n} r_iy_i^2h \geqslant \delta \cdot \sum y_i^2h.$$

To do that we would probably need to show that every $r_i > \delta$. To obtain such a relation it is probably necessary to impose further conditions on the sequences λ_i^{11} and λ_i^{12}, especially to ensure that the difference between λ_i^{11} and λ_i^{12} does not get to small. Perhaps an inequality of the form

$$0 < c_1 < \frac{\lambda_i^{11} - \lambda_i^{12}}{\lambda_{i+1}^{11} - \lambda_i^{11}} < c_2 < 1$$

would be sufficient.

Another interesting equality for a solution of the difference equation is the following

$$\text{(h)} \quad \sum_{i=1}^{n} r_i y_i^2 h = z_0^2\{S_{11}(\lambda)\, S'_{21}(\lambda) - S'_{11}(\lambda) \cdot S_{21}(\lambda)\}$$
$$+ y_0 z_0\{S_{12}S'_{21} - S'_{12}S_{21} + S_{11}S'_{22} - S'_{11}S_{22}\}$$
$$+ y_0^2\{S'_{22}S_{12} - S'_{12}S_{22}\}.$$

If we for instance let λ_k be the k-th zero of $S_{11}(\lambda)$ and if we let $y_0 = 0$ then

$$\sum r_i y_i^2 h = z_0^2 \cdot \{- S'_{11}(\lambda_k) \cdot S_{21}(\lambda_k)\} = z_0^2 \cdot S'_{11}(\lambda_k) \cdot \frac{1}{S_{12}(\lambda_k)}$$
$$= z_0^2 \cdot \left(\frac{S_{11}}{S_{12}}\right)'_{\lambda=\lambda_k}$$

$$(\text{For } S_{11}S_{22} - S_{12}S_{21} = 1 \Rightarrow S_{12}(\lambda_k)\, S_{21}(\lambda_k) = -1).$$

This relation might give useful information about the boundedness of $\sum r_i y_i^2 h$ when we let the degree of the polynomials S_{11} and S_{12} ten to infinity.

The relation above was obtained in the proof of Theorem IV (second version).

Some Further Remarks on the Problem of Convergence

We earlier proved the inequality

$$\sum_{i=1}^{n} r_i |y_i|^2 h \leqslant \frac{1}{|\operatorname{Im}\lambda|} \cdot \sum_{i=1}^{n} |f_i|\,|y_i|\, h$$

for a solution of the difference equation. Using Cauchy's inequality we can write:

$$\sum r_i |y_i|^2 h \leqslant \frac{1}{|\operatorname{Im}\lambda|} \cdot \sum_{i=1}^{n} \left|\frac{f_i}{\sqrt{r_i}}\right| \cdot \sqrt{r_i}\, |y_i|\, h$$
$$\leqslant \frac{1}{|\operatorname{Im}\lambda|} \left\{\sum_{i=1}^{n} \frac{|f_i|^2}{r_i} h\right\}^{1/2} \cdot \left\{\sum_{i=1}^{n} r_i |y_i|^2 h\right\}^{1/2},$$

i.e.,

$$\left\{\sum_{i=1}^{n} r_i |y_i|^2 h\right\}^{1/2} \leqslant \frac{1}{|\operatorname{Im}\lambda|} \cdot \left\{\sum_{i=1}^{n} \frac{|f_i|^2}{r_i} h\right\}^{1/2}.$$

Or, if we use the previous notations,

$$\int_0^1 r_n(t) \mid y_n(t) \mid^2 dt \leqslant \frac{1}{\mid \operatorname{Im} \lambda \mid^2} \cdot \int_0^1 \frac{\mid f(t) \mid^2}{r_n(t)} dt$$

or

$$\| \sqrt{r_n} y_n \| \leqslant \frac{1}{\mid \operatorname{Im} \lambda \mid} \cdot \left\| \frac{f}{\sqrt{r_n}} \right\| .$$

We may now consider the class F of functions f such that

$$\left\| \frac{f}{\sqrt{r_n}} \right\|$$

is bounded for every fixed function f of F. (F may of course consist of only the function $f \equiv 0$.) (Challenging problem: Determine the class F.)

Let us denote

$$\sqrt{r_n} y_n = G_n(\lambda) f.$$

Then

$$\| G_n(\lambda) f \| \leqslant \left\| \frac{f}{\sqrt{r_n}} \right\| \leqslant K_f , \qquad \text{for} \qquad f \in F.$$

Every operator $G_n(\lambda)$ is the resolvent of an operator T_n with spectrum located on the real axis. If now $G_n(\lambda)$ (or some subsequence $G_{n_i}(\lambda)$) has a weak limit $G(\lambda)$ so that $(G_n(\lambda) f, \varphi) \to (G(\lambda) f, \varphi)$ for every $f \in F$ and every $\varphi \in L^2(0, 1)$ and if $G(\lambda)$ is the resolvent of an operator T, then this operator T may have both residual and continuous spectrum for $\operatorname{Im} \lambda \neq 0$ if $\bar{F} \neq L^2(0, 1)$.

If $\bar{F} = L^2(0, 1)$ we can have only continuous spectrum for $\operatorname{Im} \lambda \neq 0$.

If $F = L^2(0, 1)$ one can show that there really exists a weakly convergent subsequence $G_n(\lambda)$ in the following manner: If $F = L^2(0, 1)$, then

$$\left\| \frac{f}{\sqrt{r_n}} \right\| \leqslant K_f \qquad \text{for all} \qquad f \in L^2(0, 1)$$

and

$$\| G_n f \| \leqslant K_f \qquad \text{for all} \qquad f \in L^2(0, 1).$$

For fixed φ let us introduce the linear functional

$$L_n f = (G_n f, \varphi) \qquad \text{defined on} \qquad L^2(0, 1).$$

Now

$$\mid L_n f \mid \leqslant \| G_n f \| \cdot \| \varphi \| \leqslant K_f \cdot \| \varphi \|$$

so that $|L_n f|$ is bounded for a fixed function $f \in L^2(0, 1)$. The uniform boundedness theorem then states that $|L_n f| \leqslant K \|f\|$ for all $f \in L^2(0, 1)$. (K is independent of f.) Now we can select a convergent subsequence $L_{n_i} f$ for a fixed f. If we repeat this for every f_i in some denumerable everywhere dense set A in $L^2(0, 1)$ and if we in the usual way select the "diagonal sequence," we obtain a subsequence L_{n_ν} so that $L_{n_\nu} f_i$ has a limit for every $f_1 \in A$ when $\nu \to \infty$.

$L_{n_\nu} f_i \to L f_i$.

$$|Lf_i| \leqslant \overline{\lim_{\nu \to \infty}} |L_{n_\nu} f_i| \leqslant K \|f_i\| \Rightarrow |Lf_i| \leqslant K \|f_i\|$$

so that L can be extended to the closure of A, i.e., to all of $L^2(0, 1)$

$$L_{n_\nu} f \to Lf \qquad \text{for all} \qquad f \in L^2(0, 1).$$

We have thus shown that for a fixed φ we can select a subsequence G_{n_ν} so that $(G_{n_\nu} f, \varphi)$ converges for all $f \in L^2(0, 1)$.

If we now, starting with the sequence G_{n_ν} make a similar diagonalization process letting φ be an element in a denumerable everywhere dense subset of $L^2(0, 1)$, we can construct a subsequence G_{n_μ} so that $(G_{n_\mu} f, \varphi)$ converges for arbitrary f and g in $L^2(0, 1)$. The sequence is, in other words, weakly convergent:

$$(G_{n_\mu} f, \varphi) \to (Gf, \varphi) \qquad \text{for all} \qquad f, g \in L^2(0, 1).$$

We note that we cannot immediately deduce that G is bounded for

$$|(Gf, \varphi)| \leqslant \overline{\lim_{\mu \to \infty}} |(G_{n_\mu} f, \varphi)| \leqslant \overline{\lim_{\mu \to \infty}} \|G_{n_\mu} f\| \cdot \|\varphi\| \leqslant K_f \cdot \|\varphi\|$$

is not enough to ensure boundedness for G.

It may thus still happen that we gave a continous spectrum for $\operatorname{Im} \lambda \neq 0$.

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