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Fuzzy Groups

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1. INTRODUCTION

The concept of a fuzzy set, introduced in [1], was applied in [2] to generalize some of the basic concepts of general topology. The present note constitutes a similar application to the elementary theory of groupoids and groups.

2. FUZZY SUBGROUPOIDS AND IDEALS

Let S be a groupoid, i.e., a set closed under a binary composition (which will be denoted multiplicatively). We recall that a fuzzy set in S is a function μ from S into $[0, 1]$.

DEFINITION 2.1. μ will be called a *fuzzy subgroupoid* of S if, for all x, y in S,

$$\mu(xy) \geqslant \min(\mu(x), \mu(y))$$

It will be called a *fuzzy left ideal*, if $\mu(xy) \geqslant \mu(y)$; a *fuzzy right ideal*, if $\mu(xy) \geqslant \mu(x)$; and a *fuzzy ideal*, if it is a fuzzy left and right ideal (or equivalently: if $\mu(xy) \geqslant \max(\mu(x), \mu(y))$.

Clearly a fuzzy (left, right) ideal is a fuzzy subgroupoid. Note that for any fuzzy subgroupoid in S we have $\mu(x^n) \geqslant \mu(x)$ for all $x \in S$, where x^n is any composite of x's. We also have

PROPOSITION 2.1. *For any $\theta \in [0, 1]$, $\{z \mid z \in S, \mu(z) \geqslant \theta\}$ is a subgroupoid or (left, right) ideal if μ is a fuzzy subgroupoid or fuzzy (left, right) ideal.*

PROPOSITION 2.2. *Let μ be into $\{0, 1\}$, so that μ is the characteristic function of a subset $T \subseteq S$. Then μ is a fuzzy subgroupoid or (left, right) ideal if and only if T is a subgroupoid or (left, right) ideal, respectively.*

Proof. If μ is into $\{0, 1\}$, then "$\mu(xy) \geqslant \min(\mu(x), \mu(y))$" is equivalent to "$\mu(x) = \mu(y) = 1$ implies $\mu(xy) = 1$", i.e., to "x, y in T implies xy in T". Similarly, "$\mu(xy) \geqslant \mu(y)$" is equivalent to "y in T implies xy in T". //

From now on we shall denote the characteristic function of T by φ_T.

3. The Lattices of Fuzzy Subgroupoids and Ideals

We recall that inclusion of fuzzy sets in S is defined as follows: $\mu \subseteq \nu$ means $\mu(x) \leqslant \nu(x)$ for all $x \in S$. Clearly the set of all fuzzy sets in S is a complete lattice $\mathscr{L}$ under this ordering. We shall denote the sup and inf in $\mathscr{L}$ by $\cup$ and $\cap$, respectively. The least and greatest elements of $\mathscr{L}$ are the constant functions 0 and 1. Note that these functions are just $\varphi_{\varnothing}$ and φ_S, so that they are fuzzy ideals (and in particular, fuzzy subgroupoids).

Proposition 3.1. *The $\cap$ of any set of fuzzy subgroupoids is a fuzzy subgroupoid.*

Proof.

$$[\cap \mu_i]\,(xy) = \inf[\mu_i(xy)] \geqslant \inf[\min(\mu_i(x), \mu_i(y))]$$
$$= \min(\inf \mu_i(x), \inf \mu_i(y)) = \min([\cap \mu_i]\,(x), [\cap \mu_i]\,(y)).$$ //

It follows (e.g., [3, Prop. 16.1]) that the fuzzy subgroupoids of S are also a complete lattice. In this lattice, the inf of a set of fuzzy subgroupoids μ_i is just $\cap \mu_i$, while their sup is the least μ (i.e., the $\cap$ of all μ's) which $\supseteq \cup \mu_i$. More generally, we have

Definition 3.1. The fuzzy subgroupoid (σ) *generated* by the fuzzy set σ is defined as the least fuzzy subgroupoid which $\supseteq \sigma$.

Proposition 3.2. $(\varphi_T) = \varphi_{(T)}$, *where (T) is the subgroupoid generated by T.*

Proof. If $\mu \supseteq \varphi_T$ we have $\mu = 1$ for all $x \in T$; but since μ is a fuzzy subgroupoid, this implies $\mu = 1$ for any composite of elements of T, so that $\mu \supseteq \varphi_{(T)}$. Thus $\varphi_{(T)} \subseteq$ the $\cap$ of all such μ's; while conversely, $\varphi_{(T)}$ itself is such a μ by Proposition 2.1. //

Thus the subgroupoid lattice of S can be regarded as a sublattice of the fuzzy subgroupoid lattice of S.

Proposition 3.3. *The $\cap$ or $\cup$ of any set of fuzzy (left, right) ideals is a fuzzy (left, right) ideal.*

Proof.

$$[\cap \mu_i](xy) = \inf[\mu_i(xy)] = \inf[\mu_i(y)] = [\cap \mu_i](y),$$

and similarly for $\cup$ and on the right. //

Thus the fuzzy (left, right) ideals of S are a complete sublattice of $\mathscr{L}$. The analog of Proposition 3.2 for ideals requires additional assumptions about S. For example, if S is a semigroup (i.e., its composition is associative) and has a left identity, then the left ideal generated by $T \subseteq S$ is just ST; in this case it is easily seen that the fuzzy left ideal generated by φ_T is just φ_{ST}.

4. Homomorphisms

We recall that if μ is a fuzzy set in S, and f is a function defined on S, then the fuzzy set ν in $f(S)$ defined by

$$\nu(y) = \sup_{x \in f^{-1}(y)} \mu(x) \qquad \text{for all } y \in f(S)$$

is called the *image* of μ under f. Similarly, if ν is a fuzzy set in $f(S)$, then the fuzzy set $\mu = f \circ \nu$ in S (i.e., the fuzzy set defined by $\mu(x) = \nu(f(x))$ for all $x \in S$) is called the *preimage* of ν under f. Readily, if $\mu = \varphi_T$, then the image of μ under f is just $\varphi_{f(T)}$; and if $\nu = \varphi_W$ (where $W \subseteq f(S)$), then the preimage of ν under f is just $\varphi_{f^{-1}(W)}$.

Proposition 4.1. *A homomorphic preimage of a fuzzy subgroupoid or (left, right) ideal is a fuzzy subgroupoid or (left, right) ideal, respectively.*

Proof.

$$\begin{aligned}\mu(xy) = \nu(f(xy)) = \nu(f(x)f(y)) \geqslant \min(\nu(f(x)), \nu(f(y))) \\ = \min(\mu(x), \mu(y)),\end{aligned}$$

and similarly for ideals. //

We say that a fuzzy set μ in S has the *sup property* if, for any subset $T \subseteq S$, there exists $t_0 \in T$ such that $\mu(t_0) = \sup_{t \in T} \mu(t)$. For example, if μ can take on only finitely many values (in particular, if it is a characteristic function), it has the sup property.

Proposition 4.2. *A homomorphic image of a fuzzy subgroupoid which has the sup property is a fuzzy subgroupoid, and similarly for (left, right) ideals.*

Proof. Given $f(x), f(y)$ in $f(S)$, let $x_0 \in f^{-1}(f(x))$, $y_0 \in f^{-1}(f(y))$ be such that

$$\mu(x_0) = \sup_{t \in f^{-1}(f(x))} \mu(t), \qquad \mu(y_0) = \sup_{t \in f^{-1}(f(y))} \mu(t),$$

respectively. Then

$$\nu(f(x)f(y)) = \sup_{Z \in f^{-1}(f(x)f(y))} \mu(Z) \geqslant \min(\mu(x_0), \mu(y_0))$$
$$= \min(\nu(f(x)), \nu(f(y))),$$

and similarly for ideals. //

If f is any function defined on S, and ν is any fuzzy set in $f(S)$, then the image of the preimage of ν under f is just ν itself, since

$$\sup_{x \in f^{-1}(y)} \nu(f(x)) = \nu(y) \qquad \text{for all } y \in f(S).$$

Conversely, if μ is any fuzzy set in S, then the preimage of the image of μ under f always $\supseteq \mu$, since

$$\sup_{z \in f^{-1}(f(x))} \mu(z) \geqslant \mu(x) \qquad \text{for all } x \in S.$$

We call μ *f-invariant* if $f(x) = f(y)$ implies $\mu(x) = \mu(y)$. Clearly if μ is f-invariant, then the preimage of its image under f is μ itself. It follows (e.g., [3, Theorem 18.4]) that f is a one-to-one correspondence between the f-invariant fuzzy sets in S and the fuzzy sets in $f(S)$. Similarly [3, Theorem 26.5] for the f-invariant fuzzy subgroupoids in S and the fuzzy subgroupoids in $f(S)$, provided that the former have the sup property.

5. Fuzzy Subgroups

Definition 5.1. If S is a group, a fuzzy subgroupoid μ of S will be called a *fuzzy subgroup* of S if $\mu(x^{-1}) \geqslant \mu(x)$ for all $x \in S$.

It is readily verified that

Proposition 5.1. *φ_T is a fuzzy subgroup if and only if T is a subgroup.*

Proposition 5.2. *The $\cap$ of any set of fuzzy subgroups is a fuzzy subgroup.*

PROPOSITION 5.3. *The fuzzy subgroup generated by the characteristic function of a set is just the characteristic function of the subgroup generated by the set.*

PROPOSITION 5.4. *Let μ be a fuzzy subgroup of S; then $\mu(x^{-1}) = \mu(x)$ and $\mu(x) \leqslant \mu(e)$ for all $x \in S$, where e is the identity element of S.*

Proof. $\mu(x) = \mu((x^{-1})^{-1}) \geqslant \mu(x^{-1}) \geqslant \mu(x)$; hence

$$\mu(e) = \mu(xx^{-1}) \geqslant \min(\mu(x), \mu(x^{-1})) = \mu(x). \quad //$$

COROLLARY. $\{x \mid \mu(x) = \mu(e)\}$ *is a subgroup.*

Proof. Use Proposition 2.1. //

We shall denote this subgroup by G_μ.

PROPOSITION 5.5. $\mu(xy^{-1}) = \mu(e)$ *implies* $\mu(x) = \mu(y)$.

Proof.

$$\begin{aligned}\mu(x) &= \mu((xy^{-1})\,y) \geqslant \min(\mu(e), \mu(y)) = \mu(y)\\ &= \mu((yx^{-1})\,x) \geqslant \min(\mu(e), \mu(x)) = \mu(x). \quad //\end{aligned}$$

COROLLARY. *μ is constant on each coset of G_μ.*

COROLLARY. *If G_μ has finite index, μ has the sup property.*

PROPOSITION 5.6. *μ is a fuzzy subgroup of S if and only if*

$$\mu(xy^{-1}) \geqslant \min(\mu(x), \mu(y)) \qquad \textit{for all } x, y \textit{ in } S.$$

Proof. If μ is a fuzzy subgroup we have

$$\mu(xy^{-1}) \geqslant \min(\mu(x), \mu(y^{-1})) = \min(\mu(x), \mu(y)).$$

Conversely, if $\mu(xy^{-1}) \geqslant \min(\mu(x), \mu(y))$, let $y = x$ to obtain $\mu(e) \geqslant \mu(x)$ for all $x \in S$; hence

$$\mu(y^{-1}) = \mu(ey^{-1}) \geqslant \min(\mu(e), \mu(y)) = \mu(y),$$

and it follows that

$$\mu(xy) = \mu(x(y^{-1})^{-1}) \geqslant \min(\mu(x), \mu(y^{-1})) \geqslant \min(\mu(x), \mu(y)). \quad //$$

PROPOSITION 5.7. *A group cannot be the $\cup$ of two proper fuzzy subgroups.*

Proof. Let μ, ν be proper fuzzy subgroups of S such that $\mu(x) = 1$ or $\nu(x) = 1$ for all $x \in S$. Let u, v in S be such that $\mu(u) = 1$, $\mu(v) < 1$, $\nu(u) < 1$, $\nu(v) = 1$, and consider uv. If $\mu(uv) = 1$, then since $\mu(u^{-1}) = 1$ we would have $\mu(v) = \mu(u^{-1}(uv)) \geqslant \min(\mu(u^{-1}), \mu(uv)) = 1$, contradiction; and a similar contradiction is obtained if $\nu(uv) = 1$. //

PROPOSITION 5.8. *A homomorphic image or preimage of a fuzzy subgroup is a fuzzy subgroup (in the former case, provided the sup property holds).*

Proof. For preimages,

$$\mu(x^{-1}) = \nu(f(x^{-1})) = \nu(f(x)^{-1}) \geqslant \nu(f(x)) = \mu(x).$$

For images, given $f(x) \in f(S)$, let $x_0 \in f^{-1}(f(x))$ be such that

$$\mu(x_0) = \sup_{t \in f^{-1}(f(x))} \mu(t);$$

then

$$\nu(f(x)^{-1}) = \sup_{z \in f^{-1}(f(x)^{-1})} \mu(z) \geqslant \mu(x_0^{-1}) \geqslant \mu(x_0) = \nu(f(x)). \quad //$$

PROPOSITION 5.9. *The fuzzy (left, right) ideals in a group are just the constant functions.*

Proof. Clearly if μ is a constant function, it is a fuzzy ideal, since $\mu(xy) = \mu(x) = \mu(y)$ for all x, y in S. Conversely, let S be a group and μ a fuzzy left ideal, so that $\mu(xy) \geqslant \mu(y)$ for all x, y. Putting $y = e$ gives $\mu(x) \geqslant \mu(e)$ for all x, while putting $x = y^{-1}$ gives $\mu(e) \geqslant \mu(y)$ for all y; thus $\mu = \mu(e)$ is a constant function. //

PROPOSITION 5.10. *Let G_p be the cyclic group of prime order p, and let μ be any fuzzy subgroup of G_p; then $\mu(x) = \mu(1) \leqslant \mu(0)$ for all $x \neq 0$ in G_p, and conversely any such μ is a fuzzy subgroup.*

Proof. For any such μ, $\mu(xy) \geqslant \min(\mu(x), \mu(y))$ is immediate since $00 = 0$, and $\mu(x^{-1}) \geqslant \mu(x)$ is immediate since $-0 = 0$. Conversely, for any $x \neq 0$ and $y \neq 0$ in G_p, x is a sum of y's and y a sum of x's, so that $\mu(x) \geqslant \mu(y) \geqslant \mu(x)$. //

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