Fuzzy Groups

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1. INTRODUCTION

The concept of a fuzzy set, introduced in [1], was applied in [2] to generalize some of the basic concepts of general topology. The present note constitutes a similar application to the elementary theory of groupoids and groups.

2. FUZZY SUBGROUPOIDS AND IDEALS

Let $S$ be a groupoid, i.e., a set closed under a binary composition (which will be denoted multiplicatively). We recall that a fuzzy set in $S$ is a function $p$ from $S$ into $[0, 1]$.

**Definition 2.1.** $p$ will be called a fuzzy subgroupoid of $S$ if, for all $x, y$ in $S$,

$$p(xy) \geq \min(p(x), p(y))$$

It will be called a fuzzy left ideal, if $p(y) \geq p(x)$; a fuzzy right ideal, if $p(xy) \geq p(x)$; and a fuzzy ideal, if it is a fuzzy left and right ideal (or equivalently: if $p(xy) \geq \max(p(x), p(y))$).

Clearly a fuzzy (left, right) ideal is a fuzzy subgroupoid. Note that for any fuzzy subgroupoid in $S$ we have $p(x^n) \geq p(x)$ for all $x \in S$, where $x^n$ is any composite of $x$'s. We also have

**Proposition 2.1.** For any $\theta \in [0, 1]$, $\{z \mid z \in S, p(z) \geq \theta\}$ is a subgroupoid or (left, right) ideal if $p$ is a fuzzy subgroupoid or fuzzy (left, right) ideal.

**Proposition 2.2.** Let $\mu$ be into $[0, 1]$, so that $\mu$ is the characteristic function of a subset $T \subseteq S$. Then $\mu$ is a fuzzy subgroupoid or (left, right) ideal if and only if $T$ is a subgroupoid or (left, right) ideal, respectively.
Proof. If $\mu$ is into $\{0, 1\}$, then "$\mu(xy) \geq \min(\mu(x), \mu(y))$" is equivalent to "$\mu(x) = \mu(y) = 1$ implies $\mu(xy) = 1$", i.e., to "$x, y$ in $T$ implies $xy$ in $T$". Similarly, "$\mu(xy) \geq \mu(y)$" is equivalent to "$y$ in $T$ implies $xy$ in $T$".

From now on we shall denote the characteristic function of $T$ by $\varphi_T$.

3. THE LATTICES OF FUZZY SUBGROUPOIDS AND IDEALS

We recall that inclusion of fuzzy sets in $S$ is defined as follows: $\mu \subseteq \nu$ means $\mu(x) \leq \nu(x)$ for all $x \in S$. Clearly the set of all fuzzy sets in $S$ is a complete lattice $\mathcal{L}$ under this ordering. We shall denote the sup and inf in $\mathcal{L}$ by $\cup$ and $\cap$, respectively. The least and greatest elements of $\mathcal{L}$ are the constant functions $0$ and $1$. Note that these functions are just $\varphi_S$ and $\varphi_S$, so that they are fuzzy ideals (and in particular, fuzzy subgroupoids).

**Proposition 3.1.** The $\cap$ of any set of fuzzy subgroupoids is a fuzzy subgroupoid.

**Proof.**

$$[\cap \mu_i] (xy) = \inf[\mu_i(xy)] \geq \inf[\min(\mu_i(x), \mu_i(y))]$$

$$= \min(\inf \mu_i(x), \inf \mu_i(y)) = \min([\cap \mu_i] (x), [\cap \mu_i] (y)).$$

It follows (e.g., [3, Prop. 16.1]) that the fuzzy subgroupoids of $S$ are also a complete lattice. In this lattice, the inf of a set of fuzzy subgroupoids $\mu_i$ is just $\cap \mu_i$, while their sup is the least $\mu$ (i.e., the $\cap$ of all $\mu_i$'s) which $\supseteq \cup \mu_i$.

More generally, we have

**Definition 3.1.** The fuzzy subgroupoid $(\sigma)$ generated by the fuzzy set $\sigma$ is defined as the least fuzzy subgroupoid which $\supseteq \sigma$.

**Proposition 3.2.** $(\varphi_T) = \varphi_{(T)}$, where $(T)$ is the subgroupoid generated by $T$.

**Proof.** If $\mu \supseteq \varphi_T$ we have $\mu = 1$ for all $x \in T$; but since $\mu$ is a fuzzy subgroupoid, this implies $\mu = 1$ for any composite of elements of $T$, so that $\mu \supseteq \varphi_{(T)}$. Thus $\varphi_{(T)} \subseteq \cap$ of all such $\mu$'s; while conversely, $\varphi_{(T)}$ itself is such a $\mu$ by Proposition 2.1.

Thus the subgroupoid lattice of $S$ can be regarded as a sublattice of the fuzzy subgroupoid lattice of $S$.

**Proposition 3.3.** The $\cap$ or $\cup$ of any set of fuzzy (left, right) ideals is a fuzzy (left, right) ideal.
Proof.

\[
[\cap \mu_i](xy) = \inf[\mu_i(xy)] = \inf[\mu_i(y)] = [\cap \mu_i](y),
\]
and similarly for \( \cup \) and on the right. \\

Thus the fuzzy (left, right) ideals of \( S \) are a complete sublattice of \( L \). The analog of Proposition 3.2 for ideals requires additional assumptions about \( S \). For example, if \( S \) is a semigroup (i.e., its composition is associative) and has a left identity, then the left ideal generated by \( T \subseteq S \) is just \( ST \); in this case it is easily seen that the fuzzy left ideal generated by \( \varphi_T \) is just \( \varphi_{ST} \).

4. Homomorphisms

We recall that if \( \mu \) is a fuzzy set in \( S \), and \( f \) is a function defined on \( S \), then the fuzzy set \( \nu \) in \( f(S) \) defined by

\[
\nu(y) = \sup_{x \in f^{-1}(y)} \mu(x) \quad \text{for all } y \in f(S)
\]

is called the image of \( \mu \) under \( f \). Similarly, if \( \nu \) is a fuzzy set in \( f(S) \), then the fuzzy set \( \mu = f \circ \nu \) in \( S \) (i.e., the fuzzy set defined by \( \mu(x) := \nu(f(x)) \) for all \( x \in S \)) is called the preimage of \( \nu \) under \( f \). Readily, if \( \mu = \varphi_T \), then the image of \( \mu \) under \( f \) is just \( \varphi_{f(T)} \); and if \( \nu = \varphi_W \) (where \( W \subseteq f(S) \)), then the preimage of \( \nu \) under \( f \) is just \( \varphi_{f^{-1}(W)} \).

**Proposition 4.1.** A homomorphic preimage of a fuzzy subgroupoid or (left, right) ideal is a fuzzy subgroupoid or (left, right) ideal, respectively.

Proof.

\[
\mu(xy) = \nu(f(xy)) = \nu(f(x)f(y)) \geq \min(\nu(f(x), \nu(f(y)))
\]

\[= \min(\mu(x), \mu(y)),\]

and similarly for ideals.

We say that a fuzzy set \( \mu \) in \( S \) has the sup property if, for any subset \( T \subseteq S \), there exists \( t_0 \in T \) such that \( \mu(t_0) = \sup_{t \in T} \mu(t) \). For example, if \( \mu \) can take on only finitely many values (in particular, if it is a characteristic function), it has the sup property.

**Proposition 4.2.** A homomorphic image of a fuzzy subgroupoid which has the sup property is a fuzzy subgroupoid, and similarly for (left, right) ideals.
Given \( f(x), f(y) \) in \( f(S) \), let \( x_0 \in f^{-1}(f(x)), y_0 \in f^{-1}(f(y)) \) be such that

\[
\mu(x_0) = \sup_{t \in f^{-1}(f(x))} \mu(t), \quad \mu(y_0) = \sup_{t \in f^{-1}(f(y))} \mu(t),
\]

respectively. Then

\[
v(f(x)f(y)) = \sup_{Z \in f^{-1}(f(x)f(y))} \mu(Z) \geq \min(\mu(x_0), \mu(y_0))
\]

and similarly for ideals. \( \Box \)

If \( f \) is any function defined on \( S \), and \( \nu \) is any fuzzy set in \( f(S) \), then the image of the preimage of \( \nu \) under \( f \) is just \( \nu \) itself, since

\[
\sup_{x \in f^{-1}(f(y))} \nu(f(x)) = \nu(y) \quad \text{for all } y \in f(S).
\]

Conversely, if \( \mu \) is any fuzzy set in \( S \), then the preimage of the image of \( \mu \) under \( f \) always \( \supseteq \mu \), since

\[
\sup_{x \in f^{-1}(f(x))} \mu(z) \supseteq \mu(x) \quad \text{for all } x \in S.
\]

We call \( \mu \) \( f \)-invariant if \( f(x) = f(y) \) implies \( \mu(x) = \mu(y) \). Clearly if \( \mu \) is \( f \)-invariant, then the preimage of its image under \( f \) is \( \mu \) itself. It follows (e.g., [3, Theorem 18.4]) that \( f \) is a one-to-one correspondence between the \( f \)-invariant fuzzy sets in \( S \) and the fuzzy sets in \( f(S) \). Similarly [3, Theorem 26.5] for the \( f \)-invariant fuzzy subgroupoids in \( S \) and the fuzzy subgroupoids in \( f(S) \), provided that the former have the sup property.

5. Fuzzy Subgroups

**Definition 5.1.** If \( S \) is a group, a fuzzy subgroupoid \( \mu \) of \( S \) will be called a fuzzy subgroup of \( S \) if \( \mu(x^{-1}) \supseteq \mu(x) \) for all \( x \in S \).

It is readily verified that

**Proposition 5.1.** \( \varphi_T \) is a fuzzy subgroup if and only if \( T \) is a subgroup.

**Proposition 5.2.** The \( \cap \) of any set of fuzzy subgroups is a fuzzy subgroup.
PROPOSITION 5.3. The fuzzy subgroup generated by the characteristic function of a set is just the characteristic function of the subgroup generated by the set.

PROPOSITION 5.4. Let $\mu$ be a fuzzy subgroup of $S$; then $\mu(x^{-1}) = \mu(x)$ and $\mu(x) \leq \mu(e)$ for all $x \in S$, where $e$ is the identity element of $S$.

Proof. $\mu(x) = \mu((x^{-1})^{-1}) \geq \mu(x^{-1}) \geq \mu(x)$; hence

$$\mu(e) = \mu(xx^{-1}) \geq \min(\mu(x), \mu(x^{-1})) = \mu(x).$$

COROLLARY. $\{x \mid \mu(x) = \mu(e)\}$ is a subgroup.

Proof. Use Proposition 2.1.

We shall denote this subgroup by $G_\mu$.

PROPOSITION 5.5. $\mu(xy^{-1}) = \mu(e)$ implies $\mu(x) = \mu(y)$.

Proof.

$$\mu(x) = \mu((xy^{-1})y) \geq \min(\mu(e), \mu(y)) = \mu(y)$$

$$= \mu((yx^{-1})x) \geq \min(\mu(e), \mu(x)) = \mu(x).$$

COROLLARY. $\mu$ is constant on each coset of $G_\mu$.

COROLLARY. If $G_\mu$ has finite index, $\mu$ has the sup property.

PROPOSITION 5.6. $\mu$ is a fuzzy subgroup of $S$ if and only if $\mu(xy^{-1}) \geq \min(\mu(x), \mu(y))$ for all $x, y$ in $S$.

Proof. If $\mu$ is a fuzzy subgroup we have

$$\mu(xy^{-1}) \geq \min(\mu(x), \mu(y^{-1})) = \min(\mu(x), \mu(y)).$$

Conversely, if $\mu(xy^{-1}) \geq \min(\mu(x), \mu(y))$, let $y = x$ to obtain $\mu(e) \geq \mu(x)$ for all $x \in S$; hence

$$\mu(y^{-1}) = \mu(ey^{-1}) \geq \min(\mu(e), \mu(y)) = \mu(y),$$

and it follows that

$$\mu(xy) = \mu(x(y^{-1})^{-1}) \geq \min(\mu(x), \mu(y^{-1})) \geq \min(\mu(x), \mu(y)).$$

PROPOSITION 5.7. A group cannot be the $\cup$ of two proper fuzzy subgroups.
Proof. Let \( \mu, \nu \) be proper fuzzy subgroups of \( S \) such that \( \mu(x) = 1 \) or \( \nu(x) = 1 \) for all \( x \in S \). Let \( u, v \) in \( S \) be such that \( \mu(u) = 1, \nu(v) < 1 \), \( \nu(v) = 1 \), and consider \( uv \). If \( \mu(uv) = 1 \), then since \( \mu(u^{-1}) = 1 \) we would have \( \mu(v) = \mu(u^{-1}(uv)) \geq \min(\mu(u^{-1}), \mu(uv)) = 1 \), contradiction; and a similar contradiction is obtained if \( \nu(uv) = 1 \).

PROPOSITION 5.8. A homomorphic image or preimage of a fuzzy subgroup is a fuzzy subgroup (in the former case, provided the sup property holds).

Proof. For preimages,

\[
\mu(x^{-1}) = \nu(f(x^{-1})) = \nu(f(x)) = \nu(f(x^{-1})) = \mu(x).
\]

For images, given \( f(x) \in f(S) \), let \( x_0 \in f^{-1}(f(x)) \) be such that

\[
\mu(x_0) = \sup_{t \in f^{-1}(f(x))} \mu(t);
\]

then

\[
\nu(f(x)^{-1}) = \sup_{z \in f^{-1}(f(x))} \mu(z) \geq \mu(x_0^{-1}) \geq \mu(x_0) = \nu(f(x)).
\]

PROPOSITION 5.9. The fuzzy (left, right) ideals in a group are just the constant functions.

Proof. Clearly if \( \mu \) is a constant function, it is a fuzzy ideal, since \( \mu(xy) = \mu(x) = \mu(y) \) for all \( x, y \) in \( S \). Conversely, let \( S \) be a group and \( \mu \) a fuzzy left ideal, so that \( \mu(xy) \geq \mu(y) \) for all \( x, y \). Putting \( y = e \) gives \( \mu(x) \geq \mu(e) \) for all \( x \), while putting \( x = y^{-1} \) gives \( \mu(e) \geq \mu(y) \) for all \( y \); thus \( \mu = \mu(e) \) is a constant function.

PROPOSITION 5.10. Let \( G_p \) be the cyclic group of prime order \( p \), and let \( \mu \) be any fuzzy subgroup of \( G_p \); then \( \mu(x) = \mu(1) \leq \mu(0) \) for all \( x \neq 0 \) in \( G_p \), and conversely any such \( \mu \) is a fuzzy subgroup.

Proof. For any such \( \mu, \mu(xy) \geq \min(\mu(x), \mu(y)) \) is immediate since \( 00 = 0 \), and \( \mu(x^{-1}) \geq \mu(x) \) is immediate since \( -0 = 0 \). Conversely, for any \( x \neq 0 \) and \( y \neq 0 \) in \( G_p \), \( x \) is a sum of \( y \)'s and \( y \) a sum of \( x \)'s, so that \( \mu(x) \geq \mu(y) \geq \mu(x) \).

REFERENCES