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Algebras of almost periodic functions with Bohr–Fourier spectrum in a semigroup: Hermite property and its applications ☆

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Abstract

It is proved that the unital Banach algebra of almost periodic functions of several variables with Bohr–Fourier spectrum in a given additive semigroup is an Hermite ring. The same property holds for the Wiener algebra of functions that in addition have absolutely convergent Bohr–Fourier series. As applications of the Hermite property of these algebras, we study factorizations of Wiener–Hopf type of rectangular matrix functions and the Toeplitz corona problem in the context of almost periodic functions of several variables.

Keywords: Almost periodic functions; Hermite rings; Wiener algebra; Matrix functions; Factorization; Toeplitz corona

1. Introduction

We let $\mathbb{R}$ be the real field, $\mathbb{R}^k$ the vector space of real k-dimensional vector columns, and AP^k the algebra of complex-valued almost periodic functions of k real variables $x = (x_1, x_2, \dots, x_k)$, i.e., the closed subalgebra of $L^\infty(\mathbb{R}^k)$ (with respect to the standard Lebesgue measure) generated by all the functions $e_\lambda(x) = e^{i\langle\lambda,x\rangle}$. Here the variable $x = (x_1, \dots, x_k)^{\mathrm{T}}$ and the parameter $\lambda =$

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$(\lambda_1, \dots, \lambda_k)^{\mathrm{T}}$ are in $\mathbb{R}^k$, the superscript T denotes the transposition operation on a vector (or a matrix), and

$$\langle \lambda, x\rangle = \sum_{j=1}^{k} \lambda_j x_j$$

is the standard inner product of λ and x. The norm in AP^k will be denoted by $\|\cdot\|_\infty$. The next proposition is standard (see, e.g., [33, Section 1.1]).

Proposition 1.1. *AP^k is a commutative unital C^*-algebra, and therefore can be identified with the algebra $C(\mathcal{B}_k)$ of complex-valued continuous functions on a certain compact Hausdorff topological space $\mathcal{B}_k$. Moreover, $\mathbb{R}^k$ is dense in $\mathcal{B}_k$.*

The space $\mathcal{B}_k$ is called the *Bohr compactification* of $\mathbb{R}^k$.

For any $f \in AP^k$ its *Bohr–Fourier series* is defined by the formal sum

$$\sum_{\lambda} f_\lambda e^{i\langle\lambda,t\rangle}, \quad t \in \mathbb{R}^k, \tag{1.1}$$

where

$$f_\lambda = \lim_{T\to\infty} \frac{1}{(2T)^k} \int_{[-T,T]^k} e^{-i\langle\lambda,x\rangle} f(x)\,dx, \quad \lambda \in \mathbb{R}^k,$$

and the sum in (1.1) is taken over the set $\sigma(f) = \{\lambda \in \mathbb{R}^k\colon\ f_\lambda \neq 0\}$, called the *Bohr–Fourier spectrum* of f. The Bohr–Fourier spectrum of every $f \in AP^k$ is at most a countable set. The *Bohr mean* $M\{f\}$ of $f \in AP^k$ is given by

$$M\{f\} := f_0 = \lim_{T\to\infty} \frac{1}{(2T)^k} \int_{[-T,T]^k} f(x)\,dx.$$

The *Wiener algebra* APW^k is defined as the set of all $f \in AP^k$ such that the Bohr–Fourier series (1.1) of f converges absolutely. The Wiener algebra is a Banach $*$-algebra with respect to the *Wiener norm* $\|f\|_W = \sum_{\lambda\in\mathbb{R}^k} |f_\lambda|$ (the multiplication in APW^k is pointwise). Note that APW^k is dense in AP^k. For the general theory of almost periodic functions of one and several variables we refer the reader to the books [13,28,29] and to [33, Chapter 1].

Let Δ be a non-empty subset of $\mathbb{R}^k$. Denote

$$AP^k_\Delta = \{f \in AP^k\colon\ \sigma(f) \subseteq \Delta\}, \qquad APW^k_\Delta = \{f \in APW^k\colon\ \sigma(f) \subseteq \Delta\}.$$

If Δ is an additive subset of $\mathbb{R}^k$, then AP^k_Δ (respectively APW^k_Δ) is a subalgebra of AP^k (respectively APW^k), which is unital if in addition $0 \in \Delta$.

A subset S of $\mathbb{R}^k$ is said to be a *halfspace* if it has the following properties:

(i) $\mathbb{R}^k = S \cup (-S)$;
(ii) $S \cap (-S) = \{0\}$;
(iii) if $\lambda, \mu \in S$ then $\lambda + \mu \in S$;
(iv) if $\lambda \in S$ and α is a nonnegative real number, then $\alpha\lambda \in S$.

A standard example (used extensively in [38,40]) of a halfspace is given by the set

$$E_k = \{(\lambda_1, \ldots, \lambda_k)^{\mathrm{T}} \in \mathbb{R}^k\colon\ \lambda_1 = \lambda_2 = \cdots = \lambda_{j-1} = 0,\ \lambda_j \neq 0\ \Rightarrow \lambda_j > 0\}.$$

This example is representative in the following sense.

Proposition 1.2. *A subset S of $\mathbb{R}^k$ is a halfspace if and only if there exists an invertible real $k \times k$ matrix Z such that*

$$S = ZE_k := \{Z\lambda\colon\ \lambda \in E_k\}.$$

Proof. The "if" part is obvious. For the "only if" part, note that the halfspace S induces a linear order on $\mathbb{R}^k$: $\lambda \preccurlyeq \mu$, $\lambda, \mu \in \mathbb{R}^k$, if and only if $\mu - \lambda \in S$. Now the proposition follows as a special case of basic results on linearly ordered vector spaces, see [16] or [17, Section IV.6]. □

Throughout the paper every ring is assumed to be commutative and with unity, which will be denoted e. Let $\mathcal{R}$ be a ring. An ordered n-tuple $(a_1, \ldots, a_n)$ of elements of $\mathcal{R}$ is said to be *unimodular*, if there exist $b_1, \ldots, b_n \in \mathcal{R}$ such that $a_1b_1 + \cdots + a_nb_n = e$. A ring $\mathcal{R}$ is called an *Hermite ring* if every unimodular row can be complemented; in other words, given elements $a_1, \ldots, a_m \in \mathcal{R}$ that generate $\mathcal{R}$ (as an ideal), there exist $m \times m$ matrices F and G with entries in $\mathcal{R}$ such that $a_1, \ldots, a_m$ form the first row of F and $FG = I$ (equivalently: $GF = I$). The Hermite property of various algebras is useful in control systems theory, see, e.g., [47].

Our central result is as follows.

Theorem 1.3. *Let $S \subset \mathbb{R}^k$ be a halfspace, and let $\Sigma \subseteq S$ be an additive semigroup: if $\lambda, \mu \in \Sigma$, then also $\lambda + \mu \in \Sigma$. Assume $0 \in \Sigma$. Then the unital algebras AP^k_Σ and APW^k_Σ are Hermite rings.*

The main ingredient of its proof—contractability of the maximal ideal spaces under the hypotheses of Theorem 1.3—is established in Section 3. Once the contractability is proved, Theorem 1.3 follows by application of V.Y. Lin's theorem [30]. Applications of the result on Hermite property to rectangular factorizations and Toeplitz corona problems are given in Sections 4 and 5, respectively. In the preliminary Section 2, some known results and constructions are recalled.

Throughout the paper, if X is a set (typically a Banach space or an algebra), then we denote by $(X)^{m\times n}$ the set of $m \times n$ matrices with entries in X.

2. Auxiliary results

2.1. Hermite rings and coprime factorizations

The defining property of Hermite rings is equivalent to its matrix analog:

Lemma 2.1. *If $\mathcal{R}$ is a Hermite ring, then every $k \times m$ right (respectively $m \times k$ left) invertible matrix F over $\mathcal{R}$ can be complemented, i.e., there is an $m \times m$ invertible matrix F_0 over $\mathcal{R}$ such that F forms the first k rows (respectively k columns) of F_0.*

For a proof, see [47, p. 345], for example.

An important application of Hermite rings $\mathcal{R}$ without divisors of zero has to do with coprime factorizations. Let $\mathcal{R}$ be a unital commutative ring without divisors of zero, and let $\mathcal{F}$ be its field of fractions. Two matrices G_1 and G_2 with entries in $\mathcal{R}$ are called *right coprime* if they have same number of columns and there exist matrices X_1 and X_2 with entries in $\mathcal{R}$ of suitable size such that $X_1G_1 + X_2G_2 = I$. Dually, matrices G_1 and G_2 with entries in $\mathcal{R}$ are called *left coprime* if they have same number of rows and there exist Y_1 and Y_2 with entries in $\mathcal{R}$ such that $G_1Y_1 + G_2Y_2 = I$. Now let G be a rectangular matrix with entries in $\mathcal{F}$. A *right coprime factorization* of G is, by definition, a representation of the form $G = NM^{-1}$, where N and M are matrices with entries in $\mathcal{R}$, M is of square size with nonzero determinant, and N and M are right coprime. A *left coprime factorization* of G is a representation of the form $G = Q^{-1}P$, where P and Q are matrices with entries in $\mathcal{R}$, Q is of square size with nonzero determinant, and P and Q are left coprime.

In general, the existence of a coprime factorization from one side does not imply existence of coprime factorization from the other side. However:

Proposition 2.2. *Assume $\mathcal{R}$ is a Hermite ring without divisors of zero. If G is a matrix with entries in the field of fractions of $\mathcal{R}$, and G admits a left (respectively, right) coprime factorization, then G admits also a right (respectively, left) coprime factorization.*

For a proof of Proposition 2.2 see, e.g., [47].

The converse of Proposition 2.2 holds as well; however, we will not use this fact in the present paper.

2.2. Maximal ideal spaces of commutative Banach algebras: polynomial convexity

We present here some well-known background information on commutative unital Banach algebra (all Banach algebras are assumed to be over the complex field $\mathbb{C}$). The book [46] is used as the main reference.

Let $\mathcal{A}$ be a commutative unital Banach algebra. The maximal ideal space of $\mathcal{A}$ will be denoted $\mathbb{M}(\mathcal{A})$. The elements of $\mathbb{M}(\mathcal{A})$ are nonzero multiplicative linear functionals $\phi : \mathcal{A} \to \mathbb{C}$. Every such functional is automatically bounded and has norm equal to 1. Clearly, every nonzero multiplicative linear functional belongs to $\mathcal{A}^*$, the dual space of $\mathcal{A}$ of continuous linear functionals. Thus, $\mathbb{M}(\mathcal{A}) \subseteq \mathcal{A}^*$. The topology in $\mathbb{M}(\mathcal{A})$ is induced by the weak* topology of $\mathcal{A}^*$. Note that $\mathbb{M}(\mathcal{A})$ is compact.

A set $\Lambda \subset \mathcal{A}$ is called a *set of generators* of $\mathcal{A}$ if $\mathcal{A}$ is the smallest closed unital subalgebra of $\mathcal{A}$ that contains Λ. We fix a set of generators Λ of a commutative unital Banach algebra $\mathcal{A}$. Let $X(\Lambda)$ be the set of functions $f : \Lambda \to \mathbb{C}$. We consider $X(\Lambda)$ in the product topology, i.e., in the weak* topology when Λ is treated as a discrete topological space.

There is a natural map

$$\chi : \mathbb{M}(\mathcal{A}) \longrightarrow X(\Lambda)$$

defined by

$$\big(\chi(\phi)\big)(\lambda) = \phi(\lambda), \quad \forall \lambda \in \Lambda.$$

Proposition 2.3. *The map χ is continuous, one-to-one, and $\mathbb{M}(\mathcal{A})$ is homeomorphic to the image $\chi(\mathbb{M}(\mathcal{A}))$ (in the induced topology from $X(\Lambda)$).*

Proof. The pre-base of open sets in $X(\Lambda)$ consists of the sets

$$\Omega_{\lambda_0,\varepsilon,f_0} := \big\{ f \in X(\Lambda)\colon \big| f(\lambda_0) - f_0(\lambda_0)\big| < \varepsilon \big\}, \quad \lambda_0 \in \Lambda,\ f_0 \in X(\Lambda),\ \varepsilon > 0.$$

The pre-image of the set $\Omega_{\lambda_0,\varepsilon,f_0}$ under the map χ is either empty, or it consists of all elements $\phi_0 \in \mathbb{M}(\mathcal{A})$ such that

$$\chi(\phi_0) \in \Omega_{\lambda_0,\varepsilon,f_0}.$$

In the latter case, together with ϕ_0, the pre-image $\chi^{-1}(\Omega_{\lambda_0,\varepsilon,f_0})$ contains the open set

$$\big\{\phi \in \mathbb{M}(\mathcal{A})\colon \big|\phi(\lambda_0) - \phi_0(\lambda_0)\big| < \varepsilon - \big|\phi_0(\lambda_0) - f_0(\lambda_0)\big|\big\}.$$

This shows that $\chi^{-1}(\Omega_{\lambda_0,\varepsilon,f_0})$ is open. Hence χ is continuous. If $\phi_1(\lambda) = \phi_2(\lambda)$ for every $\lambda \in \Lambda$, where $\phi_1, \phi_2 \in \mathbb{M}(\mathcal{A})$, then $\phi_1(x) = \phi_2(x)$ for every $x \in \mathcal{A}$ which can be expressed as a finite linear combination of products of finitely many elements of Λ. Since the set of all such linear combinations is dense in $\mathcal{A}$, by continuity of ϕ_1, ϕ_2 we obtain that $\phi_1(x) = \phi_2(x)$ for every $x \in \mathcal{A}$. Thus, χ is one-to-one.

Since $\mathbb{M}(\mathcal{A})$ is compact and χ is continuous, the image $\chi(\mathbb{M}(\mathcal{A}))$ is compact as well. Now clearly χ is a homeomorphism. □

For a fixed $\lambda \in \Lambda$, denote by $\pi_\lambda : X(\Lambda) \to \mathbb{C}$ the coordinate projection on the λ-component:

$$\pi_\lambda(f) = f(\lambda), \quad \forall f \in X(\Lambda).$$

Clearly, π_λ is continuous.

Let P_Λ be the smallest algebra (under pointwise multiplication, addition, and scalar multiplication) of functions $X(\Lambda) \to \mathbb{C}$ that contains constant functions and all projections π_λ. Thus, P_Λ consists of finite linear combinations of finite products of powers of the projections π_λ (including the powers with zero exponent that represent the constant function 1). The functions in P_Λ are obviously continuous. If

$$p = \sum_{j_1,\dots,j_k \geqslant 0} a_{j_1,\dots,j_k} \pi_{\lambda_1}^{j_1} \pi_{\lambda_2}^{j_2} \cdots \pi_{\lambda_k}^{j_k},$$

where the sum is finite and $a_{j_1,\dots,j_k} \in \mathbb{C}$, then for every $f \in X(\Lambda)$ we have

$$p(f) = \sum_{j_1,\dots,j_k \geqslant 0} a_{j_1,\dots,j_k} f(\lambda_1)^{j_1} f(\lambda_2)^{j_2} \cdots f(\lambda_k)^{j_k}. \tag{2.1}$$

The next statement is essentially Theorem 5.8 of [46].

Proposition 2.4. *The set $\chi(\mathbb{M}(\mathcal{A}))$ is polynomially convex, i.e.: if $f \in X(\Lambda)$ satisfies*

$$\bigl|p(f)\bigr| \leqslant \max\bigl\{\bigl|p(\alpha)\bigr|\colon \alpha \in \chi\bigl(\mathbb{M}(\mathcal{A})\bigr)\bigr\}, \quad \forall p \in P_\Lambda, \tag{2.2}$$

then $f \in \chi(\mathbb{M}(\mathcal{A}))$.

Note that because of compactness of $\chi(\mathbb{M}(\mathcal{A}))$, the maximum is attained in (2.2). Obviously, if $f \in \chi(\mathbb{M}(\mathcal{A}))$, then (2.2) is satisfied.

Proof. Let $f \in X(\Lambda)$ be such that (2.2) is satisfied.

Define a map $h : P_\Lambda \to \mathcal{A}_0$ by setting $h(\pi_\lambda) = \lambda$, $h(1) = 1$ (on the left here 1 denotes the constant function 1), and demanding that h be linear and multiplicative. The range of h is the smallest (not necessarily closed) unital subalgebra $\mathcal{A}_0$ that contains Λ. Thus $\mathcal{A}_0$ is dense in $\mathcal{A}$.

Define $\psi : \mathcal{A}_0 \to \mathbb{C}$ by setting

$$\psi\bigl(h(p)\bigr) = p(f), \quad \forall p \in P_\Lambda.$$

We verify that ψ is well defined. Indeed, assume $h(p) = h(q)$ for some $p, q \in P_\Lambda$. Write

$$p = \sum_{j_1,\dots,j_k \geqslant 0} a_{j_1,\dots,j_k} \pi_{\lambda_1}^{j_1} \pi_{\lambda_2}^{j_2} \cdots \pi_{\lambda_k}^{j_k}, \qquad q = \sum_{j_1,\dots,j_k \geqslant 0} a'_{j_1,\dots,j_k} \pi_{\lambda_1}^{j_1} \pi_{\lambda_2}^{j_2} \cdots \pi_{\lambda_k}^{j_k},$$

where the sums are finite, and

$$a_{j_1,\dots,j_k}, a'_{j_1,\dots,j_k} \in \mathbb{C}.$$

The condition $h(p) = h(q)$ means

$$\sum_{j_1,\dots,j_k \geqslant 0} a_{j_1,\dots,j_k} \lambda_1^{j_1} \lambda_2^{j_2} \cdots \lambda_k^{j_k} = \sum_{j_1,\dots,j_k \geqslant 0} a'_{j_1,\dots,j_k} {\lambda_1}^{j_1} {\lambda_2}^{j_2} \cdots {\lambda_k}^{j_k}.$$

Therefore, for every $\phi \in \mathbb{M}(\mathcal{A})$ we have $\chi(\phi) \in X(\Lambda)$ and using formula (2.1) we obtain (since ϕ is a multiplicative linear functional)

$$(p - q)\bigl(\chi(\phi)\bigr) = 0.$$

Now (2.2) implies that $(p - q)(f) = 0$, i.e., $p(f) = q(f)$, and ψ is well defined. Clearly, ψ is a nonzero (because $\psi(1) = 1$) linear multiplicative map. If $b = h(p) \in \mathcal{A}_0$, where $p \in P_\Lambda$, then

$$\begin{aligned}\bigl|\psi(b)\bigr| = \bigl|p(f)\bigr| &\leqslant \max\bigl\{\bigl|p\bigl(\chi(\phi)\bigr)\bigr|\colon \phi \in \mathbb{M}(\mathcal{A})\bigr\} \\ &= \max\bigl\{\bigl|\phi\bigl(h(p)\bigr)\bigr|\colon \phi \in \mathbb{M}(\mathcal{A})\bigr\} \leqslant \bigl\|h(p)\bigr\| = \|b\|.\end{aligned}$$

Thus, ψ can be extended by continuity to an element (again denoted by ψ) of $\mathbb{M}(\mathcal{A})$. But now for every $\lambda \in \Lambda$ we have

$$f(\lambda) = \pi_\lambda(f) = \psi\bigl(h(\pi_\lambda)\bigr) = \psi(\lambda),$$

and $f = \chi(\psi)$. $\square$

3. Proof of the main result

We now apply the construction of Section 2.2 to algebras of almost periodic functions. This will lead to the following result.

Theorem 3.1. *Let $S \subset \mathbb{R}^k$ be a halfspace, and let $\Sigma \subseteq S$ be an additive semigroup containing zero. Then the topological spaces $\mathbb{M}(AP^k_\Sigma)$ and $\mathbb{M}(APW^k_\Sigma)$ are contractible.*

A particular case of Theorem 3.1, namely for $\mathbb{M}(AP^1_\Sigma)$, was proved by A. Brudnyi [11]. In our proof of the theorem, we use some ideas and a lemma from [11].

Theorem 1.3 is now obtained from Theorem 3.1 by virtue of V.Y. Lin's theorem [30, Theorem 3], see also [47, Theorem 8.68], which we state below for the reader's convenience (in fact, Theorem 3.2 is a particular case of V.Y. Lin's theorem, but it will suffice for our purposes).

Theorem 3.2. *Let $\mathcal{A}$ be a commutative unital Banach algebra. If the space of maximal ideals of $\mathcal{A}$ is contractible, then $\mathcal{A}$ is a Hermite ring.*

The rest of this section is devoted to the proof of Theorem 3.1. We focus on the algebra AP^k_Σ first.

The following property of halfspaces will be needed.

Proposition 3.3. *Let $S \subset \mathbb{R}^k$ be a halfspace. Then there exists a unique vector $Y(S) \in \mathbb{R}^k$ of unit length such that $\langle x, Y(S)\rangle \geqslant 0$ for every $x \in S$.*

Proof. For the case $S = E_k$, the standard example of a halfspace, the proposition is obvious, with $Y(E_k) = (1, 0, \dots, 0)^{\mathrm{T}}$. We reduce the general case to the standard example. Indeed, if the $k \times k$ matrix Z is as in Proposition 1.2, we let

$$Y(S) = \left(Z^{-1}\right)^{\mathrm{T}} Y(E_k) / \left\| \left(Z^{-1}\right)^{\mathrm{T}} Y(E_k) \right\|. \qquad \square$$

Since for every invertible $k \times k$ matrix Z the algebras AP^k_Σ and $AP^k_{Z\Sigma}$ are isometrically isomorphic (the isometric isomorphism is induced by the map $e_\lambda \mapsto e_{Z\lambda}$, $\lambda \in \Sigma$, on elementary exponentials), in view of the proof of Proposition 3.3 we may (and do) assume that $S = E_k$ and $Y(E_k) = (1, 0, \dots, 0)^{\mathrm{T}}$. We denote by $V(E_k) = \{0\} \times \mathbb{R}^{k-1}$ the subspace orthogonal to $Y(E_k)$.

The set AP^k_Σ of almost periodic functions $f\colon \mathbb{R}^k \to \mathbb{R}$ with Bohr–Fourier spectrum in Σ is a unital commutative Banach algebra in the $L_\infty(\mathbb{R}^k)$ norm. Furthermore,

$$\Lambda := \{e_\lambda\colon \lambda \in \Sigma\}$$

is obviously a generating set of AP^k_Σ. In the notation of Section 2.2, we will work with the sets $X(\Lambda)$ of functions $\Lambda \to \mathbb{C}$ (in the weak* topology), P_Λ, and

$$\chi\left(\mathbb{M}\left(AP^k_\Sigma\right)\right) \subseteq X(\Lambda)$$

which is homeomorphic to $\mathbb{M}(AP^k_\Sigma)$.

Define the map

$$R : X(\Lambda) \times [0, 1] \longrightarrow X(\Lambda)$$

by

$$R(f, t)(e_\lambda) = t^{\langle \lambda, Y(E_k)\rangle} f(e_\lambda), \quad \lambda \in \Sigma,\ f \in X(\Lambda),\ t \in [0, 1].$$

It is understood that here $0^0 = 1$.

Lemma 3.4. *R is continuous.*

For the proof see the proof of Lemma 2.1 in [11] (here the equality $0^0 = 1$ is essential).

We will show that R maps $\chi(\mathbb{M}(AP^k_\Sigma)) \times [0, 1]$ into $\chi(\mathbb{M}(AP^k_\Sigma))$. Due to the continuity of R (Lemma 3.4) and compactness of $\chi(\mathbb{M}(AP^k_\Sigma))$ (Proposition 2.3), it suffices to show that R maps $\chi(\mathbb{M}(AP^k_\Sigma)) \times (0, 1]$ into $\chi(\mathbb{M}(AP^k_\Sigma))$. Arguing by contradiction assume that there is a $v \in \chi(\mathbb{M}(AP^k_\Sigma))$ and $t > 0$ such that $R(v, t) \notin \chi(\mathbb{M}(AP^k_\Sigma))$. By Proposition 2.4, there is a polynomial

$$p = \sum_{j_1,\dots,j_k \geqslant 0} a_{j_1,\dots,j_k} \pi_{\lambda_1}^{j_1} \pi_{\lambda_2}^{j_2} \cdots \pi_{\lambda_k}^{j_k} \in P_\Lambda, \quad \lambda_1, \dots, \lambda_k \in \Sigma,$$

such that

$$\bigl|p\bigl(R(v, t)\bigr)\bigr| > \max\bigl\{\bigl|p(\alpha)\bigr|: \alpha \in \chi\bigl(\mathbb{M}\bigl(AP^k_\Sigma\bigr)\bigr)\bigr\}.$$

We have

$$p\bigl(R(v, t)\bigr) = \sum_{j_1,\dots,j_k \geqslant 0} a_{j_1,\dots,j_k} t^{\langle j_1\lambda_1 + \dots + j_k\lambda_k, Y(E_k)\rangle} v(e_{\lambda_1})^{j_1} v(e_{\lambda_2})^{j_2} \cdots v(e_{\lambda_k})^{j_k}.$$

Define the polynomial

$$q := \sum_{j_1,\dots,j_k \geqslant 0} a_{j_1,\dots,j_k} t^{\langle j_1\lambda_1 + \dots + j_k\lambda_k, Y(E_k)\rangle} \pi_{\lambda_1}^{j_1} \pi_{\lambda_2}^{j_2} \cdots \pi_{\lambda_k}^{j_k} \in P_\Lambda.$$

Then $p(R(v, t)) = q(v)$, and we have

$$\bigl|q(v)\bigr| > \max\bigl\{\bigl|p(\alpha)\bigr|: \alpha \in \chi\bigl(\mathbb{M}\bigl(AP^k_\Sigma\bigr)\bigr)\bigr\}. \tag{3.1}$$

Consider for a fixed t, $0 < t \leqslant 1$, the following transformation: if $\alpha = \chi(\phi)$ for some $\phi \in \mathbb{M}(AP^k_\Sigma)$, we let $\alpha_t = \chi(\phi_t)$, where

$$\phi_t : APP^k_\Sigma \longrightarrow \mathbb{C}$$

is defined by the property that

$$\phi_t(e_\lambda) = t^{\langle \lambda, Y(E_k)\rangle} \phi(e_\lambda), \quad \forall \lambda \in \Lambda,$$

and extended by linearity to the algebra APP^k_{Σ} of all almost periodic polynomials (i.e., almost periodic functions whose Bohr–Fourier spectrum is a finite set) of several variables with Bohr–Fourier spectrum in Σ. The map ϕ_t is also multiplicative; one verifies this fact by taking advantage of the equality

$$\phi_t(e_\lambda e_\mu) = \phi_t(e_{\lambda+\mu}) = t^{\langle \lambda+\mu, Y(E_k)\rangle} \phi(e_{\lambda+\mu}) = \phi_t(e_\lambda)\phi_t(e_\mu)$$

for all $\lambda, \mu \in \Sigma$.

Next, we show that ϕ_t is bounded. To this end, we will use a proposition

Proposition 3.5. *Let $f \in AP^k_{\Omega}$, where*

$$\Omega \subseteq \textit{closure of } E_k := \{(\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k\colon \lambda_1 \geqslant 0\}.$$

Then, for every fixed $(k-1)$-tuple $(x_2', x_3', \dots, x_k') \in \mathbb{R}^{k-1}$, the function $\tilde{f}$ defined by $\tilde{f}(x') := f(x', x_2', \dots, x_k')$ has the following properties:

- (a) *$\tilde{f}$ is almost periodic with Bohr–Fourier spectrum in $\mathbb{R}_+$, the set of nonnegative real numbers*;
- (b) *$\tilde{f}$ admits a unique continuation, also denoted by $\tilde{f}$, into the open complex upper half-plane $\mathbb{C}_+$ such that the continuation of $\tilde{f}$ is continuous on $\mathbb{C}_+ \cup \mathbb{R}$ and analytic on $\mathbb{C}_+$*;
- (c) *for every fixed $y' > 0$, the slice $\tilde{f}_{y'}(x') := \tilde{f}(x' + iy')$ is an almost periodic function of x', and*

$$\|\tilde{f}_{y'}\|_\infty \leqslant \|\tilde{f}\|_\infty;$$

- (d) *for every fixed $y' > 0$, the function $\hat{f}_{y'}(x', x_2', \dots, x_k') := \tilde{f}(x' + iy')$ is an almost periodic function of the variables $(x', x_2', \dots, x_k')$, and $\sigma(\hat{f}_{y'}) \subseteq \Omega$*;
- (e) *if in addition $f \in APW^k_{\Omega}$, then for every fixed $y' > 0$, we have*:
  - *$\tilde{f}_{y'} \in APW^1_{\Omega}$*;
  - *the inequality $\|\tilde{f}_{y'}\|_W \leqslant \|\tilde{f}\|_W$ holds*;
  - *the function $\hat{f}_{y'}$ belongs to APW^k_{Ω} as a function of $(x', x_2', \dots, x_k') \in \mathbb{R}^k$.*

Proof. Part (a) follows easily from the fact that, given any set $\Delta \subseteq \mathbb{R}^k$, every function in AP^k_{Δ} can be uniformly approximated by almost periodic polynomials with Bohr–Fourier spectrum in Δ; this goes back to [9]. Part (b) is standard given that $\sigma(\tilde{f}) \subset \mathbb{R}_+$ by (a). For (d), we use the Poisson formula (see [22, Chapter 8] or [43, Chapter 5], for example):

$$\tilde{f}_{y'}(x') = \frac{y'}{\pi} \int_{-\infty}^{\infty} \frac{f(t, x_2', \dots, x_k')}{(t-x')^2 + (y')^2}\, dt, \quad y' > 0. \tag{3.2}$$

Let $h^{(1)}, h^{(2)}, \dots$ be a sequence of almost periodic polynomials with $\sigma(h^{(j)}) \subseteq \Omega$ and such that

$$\sup_{x \in \mathbb{R}^k} \left| h^{(j)}(x) - f(x) \right| < \frac{1}{j}, \quad j = 1, 2, \dots.$$

For every fixed $y' > 0$, the functions $\widehat{h^{(j)}}_{y'}(x', x'_2, \dots, x'_k)$, $j = 1, 2, \dots$, are obviously almost periodic polynomials with Bohr–Fourier spectrum in Ω. Now (3.2) gives for a fixed $x' \in \mathbb{R}$:

$$\sup_{(x'_2,\dots,x'_k)\in\mathbb{R}^{k-1}} \left|\hat{f}_{y'}(x', x'_2, \dots, x'_k) - \widehat{h^{(j)}}_{y'}(x', x'_2, \dots, x'_k)\right|$$
$$\leqslant \frac{y'}{\pi}\int_{-\infty}^{\infty} \frac{\sup_{(x'_2,\dots,x'_k)\in\mathbb{R}^{k-1}} |f(t, x'_2, \dots, x'_k) - h^{(j)}(t, x'_2, \dots, x'_k)|}{(t-x')^2 + (y')^2}\,dt \leqslant \frac{1}{j}, \tag{3.3}$$

hence

$$\sup_{(x', x'_2,\dots,x'_k)\in\mathbb{R}^{k}} \left|\hat{f}_{y'}(x', x'_2, \dots, x'_k) - \widehat{h^{(j)}}_{y'}(x', x'_2, \dots, x'_k)\right| \leqslant \frac{1}{j},$$

and (d) follows. Part (c) can be proved by using a string of inequalities analogous to (3.3). Finally, the part (e) is easy to verify taking into account that every almost periodic function in the Wiener algebra is the sum of its absolutely convergent Bohr–Fourier series. □

We have

$$\phi_t(e_\lambda) = \phi\left(t^{\langle\lambda, Y(E_k)\rangle} e^{i\langle\lambda, x\rangle}\right) = \phi\left(e^{i\langle\lambda, x - i\log t Y(E_k)\rangle}\right).$$

Note that $-\log t \geqslant 0$ because $t \leqslant 1$. So for any almost periodic polynomial f in AP^k_Σ we have

$$\phi_t\big(f(x)\big) = \phi\big(f\big(x - i\log t Y(E_k)\big)\big).$$

(Note that the right-hand side is well defined because of Proposition 3.5(d).) But

$$\sup_{x\in\mathbb{R}^k}\left|f\big(x - i\log t Y(E_k)\big)\right| = \sup_{(x_2,\dots,x_k)\in\mathbb{R}^{k-1}}\left\{\sup_{x'\in\mathbb{R}}\left|f(x' - i\log t, x_2, \dots, x_k)\right|\right\}$$
$$\leqslant \sup_{(x_2,\dots,x_k)\in\mathbb{R}^{k-1}}\left\{\sup_{x'\in\mathbb{R}}\left|f(x', x_2, \dots, x_k)\right|\right\} = \sup_{x\in\mathbb{R}^k}\left|f(x)\right|,$$

where the inequality follows from Proposition 3.5(c). This proves that ϕ_t is bounded on the set of almost periodic polynomials. Now we can extend ϕ_t by continuity to a bona fide $\phi_t \in \mathbb{M}(AP^k_\Sigma)$. From (3.1) we have

$$\left|q(v)\right| > \max\left\{\left|p(\alpha_t)\right|\colon \alpha \in \chi\left(\mathbb{M}\left(AP^k_\Sigma\right)\right)\right\} = \max\left\{\left|q(\alpha)\right|\colon \alpha \in \chi\left(\mathbb{M}\left(AP^k_\Sigma\right)\right)\right\},$$

a contradiction with $v \in \chi(\mathbb{M}(AP^k_\Sigma))$.

We have proved that R maps $\chi(\mathbb{M}(AP^k_\Sigma)) \times [0, 1]$ into $\chi(\mathbb{M}(AP^k_\Sigma))$. Note that the restriction of R to $\chi(\mathbb{M}(AP^k_\Sigma)) \times \{1\}$ is the identity map. Moreover, the range of the restriction of R to $\chi(\mathbb{M}(AP^k_\Sigma)) \times \{0\}$ is the maximal ideal space of the algebra $AP^k_{\Sigma\cap V(E_k)}$, or more precisely can be identified with $\chi(AP^k_{\Sigma\cap V(E_k)})$.

Let us verify this statement. Note that by construction

$$R(g,0)(e_\lambda) = \begin{cases} 0 & \text{if } \lambda \in \Sigma \text{ and } \langle \lambda, Y(E_k)\rangle \neq 0, \\ g(e_\lambda) & \text{if } \lambda \in V(E_k) \cap \Sigma. \end{cases} \tag{3.4}$$

Here $g \in \chi(\mathbb{M}(AP^k_\Sigma))$; thus, $g = \chi(\hat{g})$, $\hat{g} \in \mathbb{M}(AP^k_\Sigma)$, and we have $g(e_\lambda) = \hat{g}(e_\lambda)$, $\lambda \in \Sigma$. Since $AP^k_{\Sigma \cap V(E_k)}$ is a closed unital subalgebra of AP^k_Σ, the restriction $\hat{g}|_{AP^k_{\Sigma \cap V(E_k)}}$ of $\hat{g}$ to $AP^k_{\Sigma \cap V(E_k)}$ belongs to the maximal ideal space of $AP^k_{\Sigma \cap V(E_k)}$. Now (3.4) reads

$$R(g,0)(e_\lambda) = \begin{cases} 0 & \text{if } \lambda \in \Sigma \text{ and } \langle \lambda, Y(E_k)\rangle \neq 0, \\ \chi(\hat{g}|_{AP^k_{\Sigma \cap V(E_k)}})(e_\lambda) & \text{if } \lambda \in V(E_k) \cap \Sigma. \end{cases} \tag{3.5}$$

Identifying the right-hand side of (3.5) with $\hat{g}|_{AP^k_{\Sigma \cap V(E_k)}}$, we see that the range of R restricted to $\chi(\mathbb{M}(AP^k_\Sigma)) \times \{0\}$ is contained in the maximal ideal space of $AP^k_{\Sigma \cap V(E_k)}$.

Conversely, let $h_0 = \chi(\hat{h}_0)$, $\hat{h}_0 \in \mathbb{M}(AP^k_{\Sigma \cap V(E_k)})$. Define the map

$$h \colon APP^k_\Sigma \longrightarrow \mathbb{C}$$

by

$$h(e_\lambda) = \begin{cases} \hat{h}_0(e_\lambda) & \text{if } \lambda \in \Sigma \text{ and } \lambda \in V(E_k) \cap \Sigma, \\ 0 & \text{if } \lambda \in \Sigma \setminus V(E_k). \end{cases}$$

Since the functions $\{e_\lambda\}_{\lambda \in \Sigma}$ are linearly independent, we can extend h by linearity to a linear map, also denoted h, on APP^k_Σ. The map h is clearly unital. Note that the algebra APP^k_Σ is spanned as a linear space by the functions $\{e_\lambda\}_{\lambda \in \Sigma}$. Therefore, h is also multiplicative, as one easily verifies taking into account the multiplicativity of $\hat{h}_0$ and the property that

$$\lambda_1 + \lambda_2 \in \Sigma \setminus V(E_k)$$

provided $\lambda_1, \lambda_2 \in \Sigma$ and at least one of λ_1, λ_2 does not belong to $V(E_k)$. Moreover, h is bounded (in the $\|\cdot\|_\infty$ norm). Indeed, h is a composition of the bounded linear functional $\hat{h}_0$ and the linear projection P defined on APP^k_Σ by the equality

$$P(e_\lambda) = \begin{cases} e_\lambda & \text{if } \lambda \in V(E_k) \cap \Sigma, \\ 0 & \text{if } \lambda \in \Sigma \setminus V(E_k). \end{cases}$$

It is easy to see that

$$P(f) = \lim_{T \longrightarrow \infty} \frac{1}{2T} \int_{-T}^{T} f(x, x_2, \ldots, x_k)\,dx, \qquad f \in APP^k_\Sigma.$$

Thus, P is bounded, and hence so is h. Therefore, h extends by continuity to an element (also denoted by h) of $\mathbb{M}(AP^k_\Sigma)$. Then

$$R\bigl(\chi(h), 0\bigr)(e_\lambda) = \begin{cases} 0 & \text{if } \lambda \in \Sigma \text{ and } \langle \lambda, Y(E_k)\rangle \neq 0, \\ \chi(\hat{h}_0)(e_\lambda) & \text{if } \lambda \in V(E_k) \cap \Sigma, \end{cases} \tag{3.6}$$

which shows that $\mathbb{M}(AP^k_{\Sigma\cap V(E_k)})$ is contained in the range of the restriction of R to $\chi(\mathbb{M}(AP^k_\Sigma)) \times \{0\}$.

The algebra $AP^k_{\Sigma\cap V(E_k)}$ may be identified with $AP^k_{\Sigma'}$, where Σ' is a suitable semigroup of $\mathbb{R}^{k-1}$; in other words, we identify $V(E_k)$ with $\mathbb{R}^{k-1}$, and note that $E_k \cap V(E_k)$ is a half-space in $V(E_k)$. Now we apply an analogous procedure in the case of $k-1$ variables. After k such steps, we end up with a continuous map

$$\widetilde{R}: \chi\big(\mathbb{M}\big(AP^k_\Sigma\big)\big) \times [0,1] \longrightarrow \chi\big(\mathbb{M}\big(AP^k_\Sigma\big)\big),$$

where $\widetilde{R}$ is a composition of k contractions constructed as above, such that the map $\widetilde{R}$ is the identity when restricted to $\chi(\mathbb{M}(AP^k_\Sigma)) \times \{1\}$, and the range of its restriction to $\chi(\mathbb{M}(AP^k_\Sigma)) \times \{0\}$ is a singleton. This proves Theorem 3.1 for the algebra AP^k_Σ.

For the contractibility of $\mathbb{M}(APW^k_\Sigma)$ the proof is analogous; we have to use part (e) of Proposition 3.5 in that case.

4. AP factorizations of rectangular matrix functions

In this section we apply the Hermite properties established in Theorem 1.3 to factorizations of almost periodic matrix functions of several variables.

We start with the concept of factorization. Throughout this section, we fix a halfspace $S \subset \mathbb{R}^k$ and an additive subgroup $\Sigma \subseteq \mathbb{R}^k$. Let $G \in (AP^k_\Sigma)^{m\times n}$. A *left* $(AP_S)_\Sigma$ *factorization* of G is a representation of the form

$$G = G^+ \operatorname{diag}(e_{\lambda_1}, \dots, e_{\lambda_p}) G^-, \tag{4.1}$$

where $G^+ \in (AP^k_{\Sigma\cap S})^{m\times p}$, $G^- \in (AP^k_{\Sigma\cap(-S)})^{p\times n}$, G^+ has a right inverse in $(AP^k_{\Sigma\cap S})^{p\times m}$, G^- has a left inverse in $(AP^k_{\Sigma\cap(-S)})^{n\times p}$, and $\lambda_1, \dots, \lambda_p \in \Sigma$. The elements λ_j's are called the *factorization indices*; they are unique if we require $\lambda_j - \lambda_{j-1} \in S$ for $j = 2, \dots, n$ (we will assume that these inclusions hold). If APW is used in place of AP throughout in the above definition, then we say that (4.1) is a *left* $(APW_S)_\Sigma$ *factorization* of a matrix function $G \in (AP^k_\Sigma)^{m\times n}$. We say that a left $(AP_S)_\Sigma$ (or $(APW_S)_\Sigma$) factorization (4.1) is *canonical* if the factorization indices are zero, that is, if the middle factor in (4.1) is the identity matrix I_p. A right $(AP_S)_\Sigma$ (or $(APW_S)_\Sigma$) factorization is introduced analogously, with the roles of G_+ and G_+ interchanged, in other words, a right $(AP_S)_\Sigma$ (or $(APW_S)_\Sigma$) factorization coincides with a left $(AP_{-S})_\Sigma$ (or $(APW_{-S})_\Sigma$) factorization. Therefore, we focus on the left factorization only, and the adjective "left" will be often omitted. A matrix function $G \in (AP^k_\Sigma)^{m\times n}$ is said to be $(AP_S)_\Sigma$*-factorable* if it admits an $(AP_S)_\Sigma$-factorization; analogously $(APW_S)_\Sigma$-factorable matrix functions are introduced.

Left invertibility of G^- and right invertibility of G^+ in the formula (4.1) imply that $p \geqslant \max\{m,n\}$. On the other hand, if no upper bound is imposed on p then every matrix function $G \in (APW^k_\Sigma)^{m\times n}$ can be represented as $G = G^+G^-$ with

$$G^+ = [\Pi_S G \quad I_m], \qquad G^- = \begin{bmatrix} I_n \\ \Pi_{(-S)\setminus\{0\}} G \end{bmatrix}, \tag{4.2}$$

where for a nonempty set $\Lambda \subseteq \mathbb{R}^k$ we denote by Π_Λ the projection

$$\Pi_\Lambda f = \sum_{\lambda \in \Lambda \cap \sigma(f)} f_\lambda e_\lambda, \quad f \in \left(APW^k\right)^{m \times n}. \tag{4.3}$$

To exclude the triviality (4.2), we impose an additional condition

$$p = \max\{m, n\} \tag{4.4}$$

on the factorization (4.1).

Under condition (4.4), for square ($m = n$) matrix functions G the factors $G^{\pm}$ in (4.1) are also square and therefore two-sided invertible in the algebras $(AP^k_{\Sigma \cap (\pm S)})^{n \times n}$.

If $k = 1$, the only halfspaces are $\mathbb{R}_{\pm} := \{x \in \mathbb{R}\colon \pm x \geqslant 0\}$. Therefore, representation (4.1) with $m = n = p$ and $\Sigma = \mathbb{Z}$ up to a simple change of variable $\zeta = \exp(2\pi i x)$ is the classical Wiener–Hopf factorization on the unit circle $\mathbb{T}$. The pioneer work on this kind of factorization was done by I. Gohberg and M.G. Krein [19], see e.g. [12,18,20] for the further development. The Wiener–Hopf factorization of rectangular matrix functions with G_+ (G_-) being left (respectively, right) invertible was treated by M. Rakowski and one of the authors [35].

The case $k = 1$, $\Sigma = \mathbb{R}$ (again, with $m = n$) corresponds to the *AP* (or *APW*) factorization of matrix functions, introduced by Yu. Karlovich and one of the authors (see e.g. [25]) and considered further in subsequent publications [6–8,26,27,34]. A systematic exposition of the *AP* factorization and its properties can be found in [10]. *AP* factorizations with respect to arbitrary subgroups Σ of $\mathbb{R}$ were studied in [37], and the transition to $k > 1$ was accomplished in [38,41]. These papers by H.J. Woerdeman and the authors, along with [39,40], contain also applications of the $(AP_S)_\Sigma$ factorization to various extension and interpolation problems, as well as spectral estimation. We mention here for completeness that $(APW_S)_\Sigma$ factorization can be thought of as a particular case of the factorization in Wiener algebras in the setting of ordered Abelian groups; see [31,32,36] and [15] for recent developments in this direction.

To formulate a factorization criterion for square matrix functions, let us recall that the formula

$$\langle f, g \rangle = M\{f g^*\}, \quad f, g \in AP^k, \tag{4.5}$$

defines an inner product on AP^k. The completion of AP^k with respect to this inner product is called the *Besicovitch space* and is denoted by B^k. Thus B^k is a Hilbert space. The projection (4.3) extends by continuity to the orthogonal projection (also denoted Π_Λ) on B^k. We denote by B^k_Λ the range of Π_Λ, or, equivalently, the completion of AP^k_Λ with respect to the inner product (4.5). The vector valued Besicovitch space $(B^k)^{n \times 1}$ consists of $n \times 1$ columns with components in B^k, with the standard Hilbert space structure. Similarly, $(B^k_\Lambda)^{n \times 1}$ is the Hilbert space of $n \times 1$ columns with components in B^k_Λ.

For $F \in (AP^k_\Sigma)^{m \times n}$, where Σ is an additive subgroup of $\mathbb{R}^k$, the *Toeplitz operator* $T(F)_{S,\Sigma}$ acts from $(B^k_{S \cap \Sigma})^{n \times 1}$ to $(B^k_{S \cap \Sigma})^{m \times 1}$ according to the formula

$$T(F)_{S,\Sigma} \phi = \Pi_S(F\phi) = \Pi_{S \cap \Sigma}(F\phi).$$

Theorem 4.1. *Let Σ be an additive subgroup of $\mathbb{R}^k$ and let $G \in (APW^k_\Sigma)^{n \times n}$. Then G admits a canonical $(AP_S)_{\mathbb{R}^k}$ factorization if and only if the operator $T(G^{\mathrm{T}})_{S,\mathbb{R}^k}$ is invertible, where the superscript* T *indicates the transposed matrix. If this is the case, then the operator $T(G^{\mathrm{T}})_{S,\Sigma}$ is* *invertible as well, and any $(AP_S)_{\mathbb{R}^k}$ factorization of G automatically is its canonical $(APW_S)_\Sigma$ factorization.*

Theorem 4.1 is a subset of a lengthy set of equivalent statements established by J.A. Ball, Yu. Karlovich and the authors [3, Theorem 2.3] which in turn is based on previous results from [7,23,24,37,45]. We refer an interested reader to [3] for the complete proof and the detailed history of the matter. For non-square matrix functions, the following result holds.

Theorem 4.2. *Let $G \in (AP^k_\Sigma)^{m\times n}$ with $m < n$ (respectively, $m > n$). Then G is $(AP_S)_\Sigma$-factorable with indices $\lambda_1, \dots, \lambda_p$ if and only if G can be augmented by $n-m$ rows (respectively, $m-n$ columns) to a square $(AP_S)_\Sigma$-factorable matrix function with the same indices. In particular, G admits a canonical $(AP_S)_\Sigma$-factorization if and only if G can be augmented to a square size matrix function that admits a canonical $(AP_S)_\Sigma$-factorization.*

Proof. For the sake of definiteness let $m < n$; the case $m > n$ can be considered similarly.

Sufficiency. Suppose that F is a square $n \times n$ matrix the first m rows of which form G: $F = \left[\begin{smallmatrix} G \\ F_0 \end{smallmatrix}\right]$. Let $F = F^+ D F^-$ be an $(AP_S)_\Sigma$-factorization of F. Then $G = G^+ D F^-$, where G^+ is formed by the first m rows of F^+, is an $(AP_S)_\Sigma$-factorization of G.

Necessity. Let $G = G^+ D G^-$ be an $(AP_S)_\Sigma$-factorization of G. According to Theorem 1.3 and Lemma 2.1, a right invertible matrix function G^+ can be row-augmented to an invertible in $(AP^k_{S\cap\Sigma})^{n\times n}$ square matrix F^+. Let $F = F^+ D G^-$. Then F is a row augmentation of G which is obviously $(AP_S)_\Sigma$-factorable. $\square$

An $(APW^k_\Sigma)^{m\times n}$ analog of Theorem 4.2 is also valid, with essentially the same proof. Combining this analog (for the special case of canonical factorization) with Theorem 4.1, we obtain:

Corollary 4.3. *The following statements are equivalent for $G \in (APW^k_\Sigma)^{m\times n}$:*

(1) *G admits a canonical $(APW_S)_{\mathbb{R}^k}$ factorization;*
(2) *G can be augmented to a square size matrix function that admits a canonical $(APW_S)_{\mathbb{R}^k}$-factorization;*
(3) *G can be augmented to a matrix function $\widetilde{G} \in (APW^k_\Sigma)^{\max\{m,n\}\times\max\{m,n\}}$ such that the operator $T(\widetilde{G}^{\mathrm{T}})_{S,\mathbb{R}^k}$ is invertible.*

In each of the statements (1)–(3), $\mathbb{R}^k$ can be replaced with Σ.

For an $(APW_S)_\Sigma$-factorization to exist it is obviously necessary that

$$G \in \left(APW^k_\Sigma\right)^{m\times n} \quad \text{and} \quad G \text{ has one-sided inverse in } \left(APW^k_\Sigma\right)^{n\times m}. \tag{4.6}$$

These conditions are sufficient for existence of an $(APW_S)_\Sigma$-factorization in the case $m = n = 1$, but it is well known that for $m = n > 1$ these conditions are not sufficient, see [25] for the first example of this kind. We do not know what the situation is with regard to sufficiency of (4.6) when $m \neq n$, in particular, when $\min\{m, n\} = 1 < \max\{m, n\}$. However, if in the latter case one of the entries of G is invertible then G is $(APW_S)_\Sigma$-factorable. If, for example, $G = [g_1, \dots, g_n]$ with $g_j \in APW^k_\Sigma$, $j = 1, 2, \dots, n$, and invertible g_1, then

$$G = \left[g_1^+, h_2^+, \dots, h_n^+\right](e_\lambda I_n) \begin{bmatrix} g_1^- & h_2^- & \dots & h_n^- \\ & \ddots & & \\ 0 & \dots & 0 & g_1^- \end{bmatrix}$$

is the desired factorization. Here $g_1 = g_1^+ e_\lambda g_1^-$ is an $(AP_S)_\Sigma$-factorization of the scalar function g_1; note that an $(AP_S)_\Sigma$-factorization of a scalar almost periodic function with Bohr–Fourier spectrum in Σ exists provided the function is invertible, i.e., takes nonzero values bounded away from zero, a result proved in [15] (see [14] for special cases). Furthermore,

$$h_j^+ = g_1^+\left(\Pi_S g_1^{-1}\right) g_j, \quad j = 2, 3, \dots, n,$$

and

$$h_j^- = g_1^-\left(\Pi_{(-S)\setminus\{0\}} g_1^{-1}\right) g_j, \quad j = 2, \dots, n.$$

5. The Toeplitz corona problem

To state the main result of this section, we fix a halfspace $S \subset \mathbb{R}^k$ and denote by $\|X\|$ the operator norm of an $p \times q$ matrix X: $\|X\| = \max_{x \neq 0}\{\|Xx\|_2/\|x\|_2\}$, where $\|x\|_2$ is the standard Euclidean norm in $\mathbb{C}^q$. Also, we will use the notion of left coprimeness given in Section 2.1.

Theorem 5.1. *Suppose that we are given $A \in (APW^k_S)^{p\times m}$, $B \in (APW^k_S)^{p\times p}$ with B invertible in $(APW^k)^{p\times p}$, and a positive number γ. Assume in addition that A and B are left coprime over APW^k_S. Let Λ' be any additive subgroup of $\mathbb{R}^k$ containing $\sigma(A) \cup \sigma(B)$. Under these conditions there exists an $F \in (APW^k_{S\cap\Lambda'})^{m\times p}$ such that*

$$\|F\|_\infty := \sup_{t\in\mathbb{R}^k} \left\|F(t)\right\| \leqslant \gamma, \quad \textit{and} \quad AF = B \tag{5.1}$$

only if

$$T(A)_{S,\Lambda'}\left(T(A)_{S,\Lambda'}\right)^* \geqslant \frac{1}{\gamma^2} T(B)_{S,\Lambda'}\left(T(B)_{S,\Lambda'}\right)^*. \tag{5.2}$$

Conversely, if the operator

$$T(A)_{S,\Lambda'}\left(T(A)_{S,\Lambda'}\right)^* - \frac{1}{\gamma^2} T(B)_{S,\Lambda'}\left(T(B)_{S,\Lambda'}\right)^* \tag{5.3}$$

is positive definite and invertible, then there exists $F \in (APW^k_{S\cap\Lambda'})^{m\times p}$ such that (5.1) *holds. The family of all such matrix functions F can be parameterized by $G \in (APW^k_{\Lambda'\cap S})^{(m-p)\times p}$ with $\|G\|_\infty \leqslant 1$ in terms of a certain fractional linear transformation*

$$F = (\Theta_{11}G + \Theta_{12})(\Theta_{21}G + \Theta_{22})^{-1}. \tag{5.4}$$

The proof of Theorem 5.1 gives a formula (see formula (5.7)) for the function $\Theta = \left[\begin{smallmatrix}\Theta_{11} & \Theta_{12}\\ \Theta_{21} & \Theta_{22}\end{smallmatrix}\right]$ that appears in (5.4).

In connection with Theorem 5.1 notice that $B \in (APW^k)^{p\times p}$ is invertible in $(APW^k)^{p\times p}$ if and only if B is invertible in $(APW^k_\Lambda)^{p\times p}$, where Λ is the additive subgroup of $\mathbb{R}^k$ generated by $\sigma(B)$, if and only if $|\det(B(t))| \geqslant \epsilon > 0$ for all $t \in \mathbb{R}^k$, where ϵ is independent of t. See, e.g., Proposition 2.2 and Corollary 2.7 of [37] for a proof of this statement.

The problem (considered in Theorem 5.1) of solving the functional equation $AF = B$ for given functions A and B under the additional restriction $\|F\| \leqslant \gamma$ for some $\gamma > 0$ is known as the *Toeplitz corona problem*. It has been extensively studied in the literature, in various settings. See, for example, [21,42] for the background on the Toeplitz corona problem in the context of the algebra H^∞ of the upper halfplane or of the unit disk. Results on the Toeplitz corona problem in polydisks, and more generally, in reproducing kernel Hilbert space are found in [2,4]. For algebras of almost periodic functions, the special case of Theorem 5.1, where $B = I$ and $k = 1$, appears as part of Theorem 5.2 in [3]. Theorem 4.1 of [5] is another version of Theorem 5.1, also treating the case $k = 1$.

The proof of the general case of Theorem 5.1 follows using an approach similar to that of [3,5], once we verify the following lemma.

Lemma 5.2. *Let Λ' be a subgroup of $\mathbb{R}^k$. Suppose that the pair of matrix functions (A, B), with $A \in (APW^k_{\Lambda'\cap S})^{p\times m}$, $B \in (APW^k_{\Lambda'\cap S})^{p\times p}$ and B invertible in $(APW^k_{\Lambda'})^{p\times p}$, is a left coprime pair over $APW^k_{\Lambda'\cap S}$. Then*:

(i) *the function $W = B^{-1}A$ has a right coprime factorization $W = CD^{-1}$ over $APW^k_{\Lambda'\cap S}$, with*

$$C \in \left(APW^k_{\Lambda'\cap S}\right)^{p\times m}, \qquad D \in \left(APW^k_{\Lambda'\cap S}\right)^{m\times m},$$

and D invertible in $(APW^k_{\Lambda'})^{m\times m}$.
Moreover,

(ii) *the following kernel-range Toeplitz operator identity holds*:

$$\operatorname{Ker}\begin{bmatrix} T(A)_{S,\Lambda'} & -T(B)_{S,\Lambda'}\end{bmatrix} = \operatorname{Range}\begin{bmatrix} T(D)_{S,\Lambda'} \\ T(C)_{S,\Lambda'}\end{bmatrix}. \tag{5.5}$$

Proof. By assumption, the pair (A, B) is left coprime, and hence $W = B^{-1}A$ is a left coprime factorization. It follows from Theorem 1.3 that $APW^k_{S\cap\Lambda'}$ is a Hermite ring. It has no divisors of zero, as easily follows from the analytic continuation property of Proposition 3.5. By Proposition 2.2, W admits a right coprime factorization. Hence we may write $B^{-1}A = W = CD^{-1}$ where $C \in (APW^k_{\Lambda'\cap S})^{p\times m}$ and $D \in (APW^k_{\Lambda'\cap S})^{m\times m}$ with the determinant of D not identically zero. The rest of the proof goes exactly as that of [5, Lemma 4.2]. $\square$

We also need the following property of canonical factorizations.

Theorem 5.3. *If a Hermitian matrix function admits a canonical factorization, then a factorization can be obtained in the symmetric form $A = B^*DB$, where D is a diagonal matrix with ± 1's on the main diagonal.*

The classical version of this result (for the Wiener–Hopf factorization) goes back to Y.L. Shmulyan [44], the AP version (for $k=1$) is in [45], see also [10, Corollary 9.13]. Finally, the general case is in [38, Theorem 5.1].

Proof of Theorem 5.1. Necessity of the condition (5.2) is straightforward. Indeed, if $F \in (APW^k_{\Lambda' \cap S})^{m\times p}$ satisfies $\|F\|_\infty \leqslant \gamma$ and $AF = B$, then

$$\begin{aligned} T(B)_{S,\Lambda'}\bigl(T(B)_{S,\Lambda'}\bigr)^* &= T(A)_{S,\Lambda'}T(F)_{S,\Lambda'}\bigl(T(F)_{S,\Lambda'}\bigr)^*\bigl(T(A)_{S,\Lambda'}\bigr)^* \\ &\leqslant \gamma^2 T(A)_{S,\Lambda'}\bigl(T(A)_{S,\Lambda'}\bigr)^* \end{aligned}$$

and (5.2) follows.

We next consider the proof of the converse statement. First observe that, by replacing B with $\gamma^{-1}B$ and F with $\gamma^{-1}F$, we may assume without loss of generality that $\gamma = 1$. The assumption that the operator

$$T(A)_{S,\Lambda'}\bigl(T(A)_{S,\Lambda'}\bigr)^* - T(B)_{S,\Lambda'}\bigl(T(B)_{S,\Lambda'}\bigr)^*$$

is positive definite and invertible has the geometric interpretation that the subspace

$$\mathcal{P} := \operatorname{Im}\begin{bmatrix} (T(A)_{S,\Lambda'})^* \\ (T(B)_{S,\Lambda'})^* \end{bmatrix}$$

is a strictly positive subspace in the J_1-inner product on $(B^k_{\Lambda'\cap S})^{(m+p)\times 1}$, where

$$J_1 = \begin{bmatrix} I_m & 0 \\ 0 & -I_p \end{bmatrix}.$$

(Here and in the sequel we use well-known basic properties of geometry of Krein spaces, see, for example, [1].) Consequently the J_1-orthogonal complement $\mathcal{P}^{\perp J_1}$ of $\mathcal{P}$ is a regular subspace of $(B^k_{\Lambda'\cap S})^{(m+p)\times 1}$ in the Krein space with the J_1-inner product. One can easily verify that

$$\mathcal{P}^{\perp J_1} = \operatorname{Ker}\bigl[\, T(A)_{S,\Lambda'} \quad -T(B)_{S,\Lambda'} \,\bigr]. \tag{5.6}$$

Indeed, from the definition of $\mathcal{P}$ it follows that $\mathcal{P}^{\perp J_1}$ consists of exactly such vectors

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \qquad z_1 \in \bigl(B^k_{\Lambda'\cap S}\bigr)^{m\times 1}, \qquad z_2 \in \bigl(B^k_{\Lambda'\cap S}\bigr)^{p\times 1}$$

that

$$0 = \bigl\langle z_1, \bigl(T(A)_{S,\Lambda'}\bigr)^* y\bigr\rangle - \bigl\langle z_2, \bigl(T(B)_{S,\Lambda'}\bigr)^* y\bigr\rangle = \bigl\langle T(A)_{S,\Lambda'}z_1 - T(B)_{S,\Lambda'}z_2, y\bigr\rangle$$

for all $y \in (B^k_{\Lambda'\cap S})^{p\times 1}$. In other words, $z \in \mathcal{P}^{\perp J_1}$ if and only if

$$T(A)_{S,\Lambda'}z_1 - T(B)_{S,\Lambda'}z_2 = 0.$$

This of course is equivalent to (5.6).

By Lemma 5.2 we have the alternative representation

$$\mathcal{P}^{\perp J_1} = \operatorname{Im}\begin{bmatrix} T(D)_{S,\Lambda'} \\ T(C)_{S,\Lambda'} \end{bmatrix}.$$

In terms of this representation, the fact that $\mathcal{P}^{\perp J_1}$ is a regular subspace means that the operator

$$\big(T(D)_{S,\Lambda'}\big)^* T(D)_{S,\Lambda'} - \big(T(C)_{S,\Lambda'}\big)^* T(C)_{S,\Lambda'}$$

is invertible. By Theorem 4.1, the matrix function $(D^*D - C^*C)^{\mathrm{T}}$ admits a canonical factorization. Since all the values of this matrix function are hermitian, its factorization can be chosen in a special form described in Theorem 5.3. Passing to transposed matrices, it can be written in the form

$$D^*D - C^*C = R^* J_0 R$$

for an appropriate signature matrix J_0, where $R^{\pm 1} \in (APW^k_{\Lambda' \cap S})^{m\times m}$. We then let

$$\Theta = \begin{bmatrix} D \\ C \end{bmatrix} R^{-1}. \tag{5.7}$$

One verifies that $\Theta^* J_1 \Theta = J_0$. Thus Θ is (J_0, J_1)-isometric. Decompose $\Theta = \left[\begin{smallmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{smallmatrix}\right]$, with the size of Θ_{11} equal to $m \times (m-p)$ and that of Θ_{22} equal to $p \times p$. Repeating the arguments from the proofs of [3, Theorem 5.2] and [5, Theorem 4.1], we obtain that for every $G \in (APW^k_{\Lambda' \cap S})^{(m-p)\times p}$, the linear fractional parametrization formula (5.4) yields a solution $F \in (APW^k_{\Lambda' \cap S})^{m\times p}$ of the problem

$$AF = B \quad \text{and} \quad \|F\|_\infty \leqslant 1. \tag{5.8}$$

To prove that all solutions $F \in (APW^k_{\Lambda' \cap S})^{m\times p}$ are obtained this way, observe that

$$G = \big(\Theta_{11}^* F - \Theta_{21}^*\big)\big(\Theta_{12}^* F - \Theta_{22}^*\big)^{-1}$$

due to (J_0, J_1)-isometricity of Θ. This concludes the proof of Theorem 5.1. □

Note that the description (5.4) of all $F \in (APW^k_{S\cap\Lambda'})^{m\times p}$ such that (5.1) holds, applies provided the operator (5.3) is positive definite and invertible. If the operator (5.3) is merely positive semidefinite, then there exists an F in the infinity norm matrix Besicovitch space, having its Bohr–Fourier spectrum in $\Lambda' \cap S$ and satisfying (5.1). This statement is completely analogous to the corresponding part of Theorem 5.2 of [3], and can be obtained in the same way. We refer the interested reader to [3] for more details.

Note added in proof

We learned recently about the paper by A. Sasane, The Hermite property of a causal Wiener algebra used in control theory, Complex Analysis and Operator Theory, in press. This paper concerns a certain algebra of functions in the upper half-plane, introduced by Callier and Desoer in 1978. The author derives the Hermite property of this algebra also by first proving the contractibility of its maximal ideal space and then invoking Lin's theorem [30].

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References

[1] T.Y. Azizov, I.S. Iokhvidov, Linear Operators in Spaces with an Indefinite Metric, Wiley, New York, 1989.

[2] J.A. Ball, T.T. Trent, Unitary colligations, reproducing kernel Hilbert spaces, and Nevanlinna–Pick interpolation in several variables, J. Funct. Anal. 157 (1998) 1–61.

[3] J.A. Ball, Y. Karlovich, L. Rodman, I.M. Spitkovsky, Sarason interpolation and Toeplitz corona theorem for almost periodic matrix functions, Integral Equations Operator Theory 32 (1998) 243–281.

[4] J.A. Ball, T.T. Trent, V. Vinnikov, Interpolation and commutant lifting for multipliers on reproducing kernel Hilbert spaces, in: H. Bart, I. Gohberg, A.C.M. Ran (Eds.), Operator Theory and Analysis, Amsterdam, 1997, in: Oper. Theory Adv. Appl., vol. 122, Birkhäuser, Basel, 2001, pp. 89–138.

[5] J.A. Ball, L. Rodman, I.M. Spitkovsky, Toeplitz corona problem for algebras of almost periodic functions, in: Oper. Theory Adv. Appl., vol. 135, Birkhäuser, Basel, 2002, pp. 25–37.

[6] M.A. Bastos, Y.I. Karlovich, A.F. dos Santos, P.M. Tishin, The corona theorem and the existence of canonical factorization of triangular *AP*-matrix functions, J. Math. Anal. Appl. 223 (1998) 494–522.

[7] M.A. Bastos, Y.I. Karlovich, A.F. dos Santos, P.M. Tishin, The corona theorem and the canonical factorization of triangular *AP*-matrix functions—Effective criteria and explicit formulas, J. Math. Anal. Appl. 223 (1998) 523–550.

[8] M.A. Bastos, Y.I. Karlovich, I.M. Spitkovsky, P.M. Tishin, On a new algorithm for almost periodic factorization, in: I. Gohberg, R. Mennicken, C. Tretter (Eds.), Recent Progress in Operator Theory, Regensburg, 1995, in: Oper. Theory Adv. Appl., vol. 103, Birkhäuser, Basel, 1998, pp. 53–74.

[9] A.S. Besicovitch, Almost Periodic Functions, Dover, New York, 1955.

[10] A. Böttcher, Y.I. Karlovich, I.M. Spitkovsky, Convolution Operators and Factorization of Almost Periodic Matrix Functions, Oper. Theory Adv. Appl., vol. 131, Birkhäuser, Basel, 2002.

[11] A. Brudnyi, Contractability of maximal ideal spaces of certain algebras of almost periodic functions, Integral Equations Operator Theory 52 (2005) 595–598.

[12] K.F. Clancey, I. Gohberg, Factorization of Matrix Functions and Singular Integral Operators, Oper. Theory Adv. Appl., vol. 3, Birkhäuser, Basel, 1981.

[13] C. Corduneanu, Almost Periodic Functions, Wiley, New York, 1968.

[14] T. Ehrhardt, C.V.M. van der Mee, Canonical factorization of continuous functions on the d-torus, Proc. Amer. Math. Soc. 131 (3) (2003) 801–813.

[15] T. Ehrhardt, C.V.M. van der Mee, L. Rodman, I.M. Spitkovsky, Factorizations in weighted Wiener algebras on ordered abelian groups, Integral Equations Operator Theory 58 (2007) 65–86.

[16] J. Erdös, On the structure of ordered real vector spaces, Publ. Math. Debrecen 4 (1956) 334–343.

[17] L. Fuchs, Partially Ordered Algebraic Systems, Pergamon, Oxford, 1963.

[18] I.C. Gohberg, I.A. Feldman, Convolution Equations and Projection Methods for Their Solution, Nauka, Moscow, 1971; English translation in: Amer. Math. Soc. Transl. Math. Monogr., vol. 41, Providence, RI, 1974.

[19] I.C. Gohberg, M.G. Krein, Systems of integral equations on a half-line with kernel depending upon the difference of the arguments, Uspekhi Mat. Nauk 13 (2) (1958) 3–72; English translation in: Amer. Math. Soc. Transl. 14 (1960) 217–287.

[20] I.C. Gohberg, M.A. Kaashoek, I.M. Spitkovsky, An overview of matrix factorization theory and operator applications, in: Oper. Theory Adv. Appl., vol. 141, Birkhäuser, Basel, 2003, pp. 1–102.

[21] J.W. Helton, Optimization over spaces of analytic functions and the corona problem, J. Operator Theory 15 (1986) 359–375.

[22] K. Hoffman, Banach Spaces of Analytic Functions, Dover, New York, 1988.

[23] Y.I. Karlovich, Riemann and Haseman boundary value problems with oscillating coefficients, in: Reports of the Extended Sessions of a Seminar of the I.N. Vekua Institute for Applied Mathematics, vol. 5, I.N. Vekua Inst. Appl. Math., Tbilisi, 1990, pp. 86–89.
[24] Y.I. Karlovich, On the Haseman problem, Demonstratio Math. 26 (1993) 581–595.
[25] Y.I. Karlovich, I.M. Spitkovsky, Factorization of almost periodic matrix-valued functions and the Noether theory for certain classes of equations of convolution type, Izv. Akad. Nauk SSSR Ser. Mat. 34 (1990) 281–316.
[26] Y.I. Karlovich, I.M. Spitkovsky, Almost periodic factorization: An analogue of Chebotarev's algorithm, in: Contemp. Math., vol. 189, Amer. Math. Soc., Providence, RI, 1995, pp. 327–352.
[27] Y.I. Karlovich, I.M. Spitkovsky, Factorization of almost periodic matrix functions, J. Math. Anal. Appl. 193 (1995) 209–232.
[28] B.M. Levitan, Almost Periodic Functions, GITTL, Moscow, 1953.
[29] B.M. Levitan, V.V. Zhikov, Almost Periodic Functions and Differential Equations, Cambridge Univ. Press, 1982.
[30] V.Y. Lin, Holomorphic fiberings and multivalued functions of elements of a Banach algebra, Funktsional. Anal. i Prilozhen. 7 (2) (1973) 43–51; English translation in: Functional Anal. Appl. 7 (1973) 122–128.
[31] C.V.M. van der Mee, L. Rodman, I.M. Spitkovsky, H.J. Woerdeman, Factorization of block triangular matrix functions in Wiener algebras on ordered abelian groups, in: Oper. Theory Adv. Appl., vol. 149, Birkhäuser, Basel, 2004, pp. 441–465.
[32] C.V.M. van der Mee, L. Rodman, I.M. Spitkovsky, Factorization of block triangular matrix functions with off diagonal binomials, in: Oper. Theory Adv. Appl., vol. 160, Birkhäuser, Basel, 2005, pp. 423–437.
[33] A.A. Pankov, Bounded and Almost Periodic Solutions of Nonlinear Differential Operator Equations, Kluwer, Dordrecht, 1990.
[34] D. Quint, L. Rodman, I.M. Spitkovsky, New cases of almost periodic factorization of triangular matrix functions, Michigan Math. J. 45 (1998) 73–102.
[35] M. Rakowski, I.M. Spitkovsky, Spectral factorization of measurable rectangular matrix functions and the vector-valued Riemann problem, Rev. Mat. Iberoamericana 12 (3) (1996) 669–696.
[36] L. Rodman, I.M. Spitkovsky, Factorization of matrix functions with subgroup supported Fourier coefficients, J. Math. Anal. Appl. 323 (2006) 604–613.
[37] L. Rodman, I.M. Spitkovsky, H.J. Woerdeman, Carathéodory–Toeplitz and Nehari problems for matrix valued almost periodic functions, Trans. Amer. Math. Soc. 350 (1998) 2185–2227.
[38] L. Rodman, I.M. Spitkovsky, H.J. Woerdeman, Factorization of almost periodic matrix functions of several variables and Toeplitz operators, in: H. Bart, I. Gohberg, A.C.M. Ran (Eds.), Operator Theory and Analysis, Amsterdam, 1997, in: Oper. Theory Adv. Appl., vol. 122, Birkhäuser, Basel, 2001, pp. 385–416.
[39] L. Rodman, I.M. Spitkovsky, H.J. Woerdeman, Abstract band method via factorization, positive and band extensions of multivariable almost periodic matrix functions, and spectral estimation, Mem. Amer. Math. Soc. 160 (762) (2002) viii+71.
[40] L. Rodman, I.M. Spitkovsky, H.J. Woerdeman, Contractive extension problems for matrix valued almost periodic functions of several variables, J. Operator Theory 47 (2002) 3–35.
[41] L. Rodman, I.M. Spitkovsky, H.J. Woerdeman, Noncanonical factorization of almost periodic multivariable matrix functions, in: Singular Integral Operators, Factorization and Applications, in: Oper. Theory Adv. Appl., vol. 142, Birkhäuser, Basel, 2003, pp. 311–344.
[42] M. Rosenblum, A corona theorem for countably many functions, Integral Equations Operator Theory 3 (1980) 125–137.
[43] M. Rosenblum, J. Rovnyak, Hardy Classes and Operator Theory, Dover, Mineola, NY, 1997, corrected reprint of the 1985 original.
[44] Y.L. Shmulyan, The Riemann problem with a hermitian matrix, Uspekhi Mat. Nauk 9 (4) (1954) 243–248.
[45] I.M. Spitkovsky, On the factorization of almost periodic matrix functions, Math. Notes 45 (5–6) (1989) 482–488.
[46] E.L. Stout, The Theory of Uniform Algebras, Bogden & Quigley, Tarrytown-on-Hudson, NY, 1971.
[47] M. Vidyasagar, Control Systems Synthesis. A Factorization Approach, MIT Press, Cambridge, MA, 1985.