# The multivariate Piecing-Together approach revisited

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## ABSTRACT

The univariate Piecing-Together approach (PT) fits a univariate generalized Pareto distribution (GPD) to the upper tail of a given distribution function in a continuous manner. A multivariate extension was established by Aulbach et al. (in press) [2]: the upper tail of a given copula $C$ is cut off and replaced by a multivariate GPD-copula in a continuous manner, yielding a new copula called a PT-copula. Then each margin of this PT-copula is transformed by a given univariate distribution function. This provides a multivariate distribution function with prescribed margins, whose copula is a GPD-copula that coincides in its central part with $C$. In addition to Aulbach et al. (in press) [2], we achieve in the present paper an exact representation of the PT-copula's upper tail, giving further insight into the multivariate PT approach. A variant based on the empirical copula is also added. Furthermore our findings enable us to establish a functional PT version as well.

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## 1. Introduction

As shown by Balkema and de Haan [4] and Pickands [15], the upper tail of a univariate distribution function $F$ can reasonably be approximated only by that of a *generalized Pareto distribution* (GPD), which leads to the Peaks-Over-Threshold (POT) approach: set for a univariate random variable $X$ with distribution function $F$

$$F^{[x_0]}(x) = \mathrm{P}(X \leq x \mid X > x_0) = \frac{F(x) - F(x_0)}{1 - F(x_0)}, \quad x \geq x_0,$$

where we require $F(x_0) < 1$. The univariate POT is the approximation of the upper tail of $F$ by that of a GPD

$$
\begin{aligned}
F(x) =& \ \{1 - F(x_0)\}F^{[x_0]}(x) + F(x_0) \\
\approx&_{\mathrm{POT}} \ \{1 - F(x_0)\}Q_{\gamma,\mu,\sigma}(x) + F(x_0), \quad x \geq x_0,
\end{aligned}
$$

where $\gamma, \mu, \sigma$ are shape, location and scale parameters of the GPD $Q$, respectively. The family of univariate standardized GPD is given by

$$Q_{1,\alpha}(x) = 1 - x^{-\alpha}, \quad x \geq 1,$$

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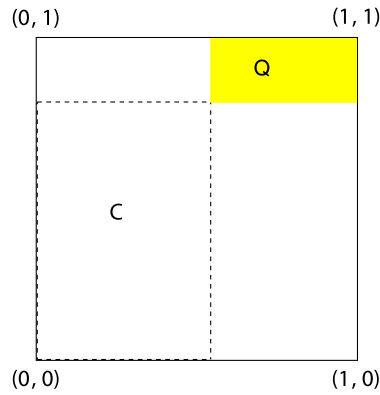
**Fig. 1.** The upper tail of a given copula $C$ is cut off and replaced by GPD-copula $Q$.

$$Q_{2,\alpha}(x) = 1 - (-x)^{\alpha}, \quad -1 \leq x \leq 0,$$
$$Q_3(x) = 1 - \exp(-x), \quad x \geq 0,$$

being the Pareto, beta and exponential GPD. Note that $Q_{2,1}(x) = 1 + x$, $-1 \leq x \leq 0$, is the uniform distribution function on $(-1, 0)$. Multivariate GPD with these margins will play a decisive role in what follows.

The preceding considerations lead to the univariate Piecing-Together approach (PT), by which the underlying distribution function $F$ is replaced by

$$F_{x_0}^*(x) = \begin{cases} F(x), & x < x_0, \\ \{1 - F(x_0)\}Q_{\gamma,\mu,\sigma}(x) + F(x_0), & x \geq x_0, \end{cases} \tag{1}$$

typically in a continuous manner. This approach aims at an investigation of the upper end of $F$ beyond observed data. Replacing $F$ in (1) by the empirical distribution function of the data provides, in particular, a semiparametric approach to the estimation of high quantiles; see, e.g., [16, Section 2.3].

A multivariate extension of the univariate PT approach was developed in [2] and, for illustration, applied to operational loss data. This approach is based on the idea that a multivariate distribution function $F$ can be decomposed into its copula $C$ and its marginal distribution functions. The multivariate PT approach then consists of the two steps.

(i) The upper tail of the given $d$-dimensional copula $C$ is cut off and substituted by the upper tail of a multivariate *GPD-copula* in a continuous manner such that the result is again a copula, called a PT-copula. Fig. 1 illustrates the approach in the bivariate case: the copula $C$ is replaced in the upper right rectangle of the unit square by a GPD-copula $Q$; the lower part of $C$ is kept in the lower left rectangle, whereas the other two rectangles are needed for a continuous transition from $C$ to $Q$.

(ii) Univariate distribution functions $F_1^*, \ldots, F_d^*$ are injected into the resulting copula.

Taken as a whole, this approach provides a multivariate distribution function with prescribed margins $F_i^*$, whose copula coincides in its lower or central part with $C$ and in its upper tail with a GPD-copula.

While in the paper by Aulbach et al. [2] it was merely shown that the generated PT-copula is a GPD-copula, we achieve in the present paper an *exact* characterization, yielding further insight into the multivariate PT approach. A variant based on the empirical copula is also added. Our findings enable us to establish a functional PT version as well.

The present paper is organized as follows. In Section 2, we compile basic definitions, auxiliary results and tools. The multivariate PT result by Aulbach et al. [2] will be revisited and greatly improved in Section 3. In Section 4, we will extend the multivariate PT approach to functional data.

## 2. Auxiliary results and tools

In this section, we compile several auxiliary results and tools from the multivariate extreme value theory (EVT). Precisely, we characterize in Proposition 2.1, Corollaries 2.2 and 2.4 the max-domain of attraction of a multivariate distribution function in terms of its copula. This implies an expansion of the lower tail of a survival copula in Corollary 2.3. Lemma 2.6 provides a characterization of multivariate GPD in terms of random vectors. For recent accounts of basic and advanced topics of EVT, we refer to the monographs by de Haan and Ferreira [7], Resnick [17,18] and Falk et al. [9], among others.

Let $F$ be an arbitrary $d$-dimensional distribution function that is in the *domain of attraction* of a $d$-dimensional extreme value distribution (EVD) $G$ (denoted by $F \in \mathcal{D}(G)$), i.e., there exist norming constants $\boldsymbol{a}_n > \boldsymbol{0} \in \mathbb{R}^d$, $\boldsymbol{b}_n \in \mathbb{R}^d$ such that

$$F^n(\boldsymbol{a}_n \boldsymbol{x} + \boldsymbol{b}_n) \to_{n \to \infty} G(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^d,$$

where all operations on vectors are meant componentwise. The distribution function $G$ is max-stable, i.e., there exist norming constants $\boldsymbol{c}_n > \boldsymbol{0} \in \mathbb{R}^d$, $\boldsymbol{d}_n \in \mathbb{R}^d$ with

$$G^n(\boldsymbol{c}_n \boldsymbol{x} + \boldsymbol{d}_n) = G(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^d.$$

The one-dimensional margins $G_i$ of $G$ are up to scale and location parameters univariate EVD. With shape parameter $\alpha > 0$, the family of (univariate) standardized EVD is

$$G_{1,\alpha}(x) = \exp\left(-x^{-\alpha}\right), \quad x > 0,$$
$$G_{2,\alpha}(x) = \exp\left\{-(-x)^{\alpha}\right\}, \quad x \leq 0,$$
$$G_3(x) = \exp\left(-e^{-x}\right), \quad x \in \mathbb{R},$$

being the Fréchet, (reverse) Weibull and Gumbel EVD, respectively.

The following two results are taken from Aulbach et al. [2].

**Proposition 2.1.** *A distribution function $F$ with copula $C_F$ satisfies $F \in \mathcal{D}(G)$ if, and only if, this is true for the univariate margins of $F$ and if the expansion*

$$C_F(\boldsymbol{u}) = 1 - \|\mathbf{1} - \boldsymbol{u}\|_D + o(\|\mathbf{1} - \boldsymbol{u}\|) \tag{2}$$

*holds uniformly for $\boldsymbol{u} \in [0, 1]^d$, where $\|\cdot\|_D$ is some D-norm.*

A $D$-norm $\|\cdot\|_D$ on $\mathbb{R}^d$ is defined by

$$\|\boldsymbol{x}\|_D := \mathrm{E}\left\{\max_{1 \leq i \leq d}(|x_i|Z_i)\right\}, \quad \boldsymbol{x} \in \mathbb{R}^d,$$

where $\boldsymbol{Z} = (Z_1, \ldots, Z_d)$ is an arbitrary random vector which satisfies $\boldsymbol{Z} \in [0, c]^d$ for some $c > 0$ together with $\mathrm{E}(Z_i) = 1$, $1 \leq i \leq d$. In this case $\boldsymbol{Z}$ is called *generator* of $\|\cdot\|_D$. Note that $\boldsymbol{Z}$ is not uniquely determined.

For example, any random vector of the form $\boldsymbol{Z} = 2(U_1, \ldots, U_d)$, with $(U_1, \ldots, U_d)$ following an arbitrary copula, can be utilized as a generator. This embeds the set of copulas into the set of $D$-norms.

The index $D$ reflects the fact that for $(t_1, \ldots, t_{d-1}) \in [0, 1]^{d-1}$ with $t_1 + \cdots + t_{m-1} \leq 1$,

$$D(t_1, \ldots, t_{d-1}) := \left\|\left(t_1, \ldots, t_{d-1}, 1 - \sum_{i=1}^{d-1} t_i\right)\right\|_D$$

is the *Pickands dependence function*, which provides another way of representing a multivariate EVD $G$ with standard negative exponential margins:

$$G(\boldsymbol{x}) = \exp\left(-\|\boldsymbol{x}\|_D\right) = \exp\left(-\|\boldsymbol{x}\|_1 D\left(\frac{x_1}{\|\boldsymbol{x}\|_1}, \ldots, \frac{x_{d-1}}{\|\boldsymbol{x}\|_1}\right)\right),$$

for $\boldsymbol{x} \leq \boldsymbol{0} \in \mathbb{R}^d$, where $\|\boldsymbol{x}\|_1 = |x_1| + \cdots + |x_d|$ is the usual $p$-norm on $\mathbb{R}^d$ with $p = 1$; for details we refer to Falk et al. [9, Section 4.4].

The following consequence of Proposition 2.1 is obvious. This result is also already contained in [2].

**Corollary 2.2.** *Let $F = C$ be a copula itself. Then $C \in \mathcal{D}(G) \iff$ (2) holds.*

The next result provides an expansion of the lower tail of the *survival copula*

$$\bar{C}(u_1, \ldots, u_d) = \mathrm{P}(1 - U_1 \leq u_1, \ldots, 1 - U_d \leq u_d), \quad \boldsymbol{u} \in [0, 1]^d$$

corresponding to any random vector $\boldsymbol{U}$, whose distribution is a copula $C$ with $C \in \mathcal{D}(G)$. It will be used in the derivation of Proposition 3.2.

**Corollary 2.3.** *Let $(U_1, \ldots, U_d)$ follow a copula $C \in \mathcal{D}(G)$, with corresponding D-norm generated by the random vector $\boldsymbol{Z} = (Z_1, \ldots, Z_d)$. Then for $\boldsymbol{x} \leq \boldsymbol{0} \in \mathbb{R}^d$*

$$\frac{\mathrm{P}(U_1 > 1 + tx_1, \ldots, U_d > 1 + tx_d)}{t} \to_{t \downarrow 0} \mathrm{E}\left\{\min_{1 \leq i \leq d}(|x_i|Z_i)\right\} =: \lambda(\boldsymbol{x}),$$

*where the function $\lambda$ is known as the tail copula [11].*

**Proof.** First note that we have for arbitrary real numbers $a_1, \ldots, a_d$ the equality

$$\min(a_1, \ldots, a_d) = \sum_{\emptyset \neq K \subset \{1, \ldots, d\}} (-1)^{|K|-1} \max(a_k : k \in K),$$

which can be seen by induction. Denote by $\boldsymbol{e}_k$ the $k$-th unit vector in the Euclidean space $\mathbb{R}^d$. The inclusion–exclusion theorem together with Corollary 2.2 then implies for fixed $\boldsymbol{x} \leq \boldsymbol{0} \in \mathbb{R}^d$ and arbitrary $t > 0$

$$\begin{aligned}
\mathrm{P}(U_1 > 1 + tx_1, \ldots, U_d > tx_d) &= 1 - \mathrm{P}\left(\bigcup_{i=1}^{d} \{U_i \leq 1 + tx_i\}\right) \\
&= 1 - \sum_{\emptyset \neq K \subset \{1, \ldots, d\}} (-1)^{|K|-1} \mathrm{P}(U_k \leq 1 + tx_k, \ k \in K) \\
&= 1 - \sum_{\emptyset \neq K \subset \{1, \ldots, d\}} (-1)^{|K|-1} \left(1 - t \left\|\sum_{k \in K} x_k \boldsymbol{e}_k\right\|_D\right) + o(t) \\
&= t \sum_{\emptyset \neq K \subset \{1, \ldots, d\}} (-1)^{|K|-1} \mathrm{E}\left(\max_{k \in K}(|x_k|Z_k)\right) + o(t) \\
&= t\mathrm{E}\left(\min_{1 \leq i \leq d}(|x_i|Z_i)\right) + o(t),
\end{aligned}$$

which yields the assertion. $\square$

A $d$-dimensional distribution function $Q$ is called *multivariate GPD* iff its upper tail equals $1 + \ln(G)$, precisely, iff there exists a $d$-dimensional EVD $G$ and $\boldsymbol{x}_0 \in \mathbb{R}^d$ with $G(\boldsymbol{x}_0) < 1$ such that

$$Q(\boldsymbol{x}) = 1 + \ln\{G(\boldsymbol{x})\}, \quad \boldsymbol{x} \geq \boldsymbol{x}_0. \tag{3}$$

Note that contrary to the univariate case, $H(\boldsymbol{x}) = 1 + \ln\{G(\boldsymbol{x})\}$, defined for each $\boldsymbol{x}$ with $\ln\{G(\boldsymbol{x})\} \geq -1$, does *not* define a distribution function unless $d \in \{1, 2\}$ [13, Theorem 6].

If $G$ has standard negative exponential margins $G_i(x) = \exp(x), x \leq 0$, then $H(\boldsymbol{x}) := 1 + \ln\{G(\boldsymbol{x})\} = 1 - \|\boldsymbol{x}\|_D$, defined for all $\boldsymbol{x} \leq \boldsymbol{0}$ with $\|\boldsymbol{x}\|_D \leq 1$, is a *quasi-copula* [1,10]. Note that $H_i(x) = 1 + x, -1 \leq x \leq 0$. We call $H$ a *GP function*. For each GP function $H$ there exists a distribution function $Q$ with $H(\boldsymbol{x}) = Q(\boldsymbol{x}) = 1 - \|\boldsymbol{x}\|_D, \boldsymbol{x} \geq \boldsymbol{x}_0$; see Corollary 2.2 in [2]. We call $Q$ a *multivariate* GPD with *ultimately uniform margins*. Thus we obtain the following consequence.

**Corollary 2.4.** *A copula $C$ satisfies $C \in \mathcal{D}(G)$ if, and only if, there exists a GPD $Q$ with ultimately uniform margins, i.e., the relation*

$$C(\boldsymbol{u}) = Q(\boldsymbol{u} - \boldsymbol{1}) + o(\|\boldsymbol{u} - \boldsymbol{1}\|)$$

*holds uniformly for $\boldsymbol{u} \in [0, 1]^d$. In this case $Q(\boldsymbol{x}) = 1 + \ln\{G(\boldsymbol{x})\} = 1 - \|\boldsymbol{x}\|_D, \boldsymbol{x}_0 \leq \boldsymbol{x} \leq \boldsymbol{0} \in \mathbb{R}^d$.*

**Example 2.5.** Under suitable conditions, an Archimedean copula $C_A$ is in the domain of attraction of the EVD $G(\boldsymbol{x}) = \exp\left(-\|\boldsymbol{x}\|_{\vartheta}\right), \boldsymbol{x} \leq \boldsymbol{0} \in \mathbb{R}^d$, where $\|\boldsymbol{x}\|_{\vartheta} = \left(\sum_{i=1}^{d} |x_i|^{\vartheta}\right)^{1/\vartheta}, \vartheta \in [1, \infty]$, is the usual $\vartheta$-norm on $\mathbb{R}^d$, with the convention $\|\boldsymbol{x}\|_{\infty} = \max_{1 \leq i \leq d} |x_i|$; see [6,12]. In this case it is reasonable to replace $C_A(\boldsymbol{u})$ for $\boldsymbol{u}$ close to $\boldsymbol{1}$ by $Q(\boldsymbol{u} - \boldsymbol{1}) = 1 - \|\boldsymbol{u} - \boldsymbol{1}\|_{\vartheta}$.

The multivariate PT approach in [2] is formulated in terms of random vectors and based on the following result. Its second part goes back to Buishand et al. [5, Section 2.2], formulated for the bivariate case and for Pareto margins instead of uniform ones.

**Lemma 2.6.** *A distribution function $Q$ is a multivariate GPD with ultimately uniform margins,*

$\iff$ *there exists a D-norm $\|\cdot\|_D$ on $\mathbb{R}^d$ such that $Q(\boldsymbol{x}) = 1 - \|\boldsymbol{x}\|_D, \boldsymbol{x}_0 \leq \boldsymbol{x} \leq \boldsymbol{0} \in \mathbb{R}^d$*
$\iff$ *there exists a generator $\boldsymbol{Z} = (Z_1, \ldots, Z_d)$ such that for $\boldsymbol{x}_0 \leq \boldsymbol{x} \leq \boldsymbol{0} \in \mathbb{R}^d$*

$$Q(\boldsymbol{x}) = \mathrm{P}\left\{-U\left(\frac{1}{Z_1}, \ldots, \frac{1}{Z_d}\right) \leq \boldsymbol{x}\right\},$$

*where the univariate random variable $U$ is uniformly distributed on $(0, 1)$ and independent of $\boldsymbol{Z}$.*

Note that $-U/Z_i$ can be replaced by $\max(M, -U/Z_i), 1 \leq i \leq d$, in the preceding result with some constant $M < 0$ to avoid possible division by zero.

In view of the preceding discussion we call a copula $C$ a *GPD-copula* if there exists $\boldsymbol{u}_0 < \boldsymbol{1} \in \mathbb{R}^d$ such that

$$C(\boldsymbol{u}) = 1 - \|\boldsymbol{u} - \boldsymbol{1}\|_D, \quad \boldsymbol{u}_0 \leq \boldsymbol{u} \leq \boldsymbol{1} \in \mathbb{R}^d,$$

where $\|\cdot\|_D$ is an arbitrary $D$-norm on $\mathbb{R}^d$, i.e., if there exists a generator $\mathbf{Z} = (Z_1, \ldots, Z_d)$ such that for $\mathbf{u}_0 \leq \mathbf{u} \leq \mathbf{1} \in \mathbb{R}^d$

$$C(\mathbf{u}) = P\left\{ -U\left(\frac{1}{Z_1}, \ldots, \frac{1}{Z_d}\right) \leq \mathbf{u} - \mathbf{1} \right\},$$

where the random variable $U$ is uniformly distributed on $(0, 1)$ and independent of $\mathbf{Z}$.

## 3. Multivariate Piecing-Together

Let $\mathbf{U} = (U_1, \ldots, U_d)$ follow an arbitrary copula $C$ and $\mathbf{V} = (V_1, \ldots, V_d)$ follow a GPD-copula with generator $\mathbf{Z}$. We suppose that $\mathbf{U}$ and $\mathbf{V}$ are independent.

Choose a threshold $\mathbf{u} = (u_1, \ldots, u_d) \in (0, 1)^d$ and put for $1 \leq i \leq d$

$$Y_i := U_i \mathbf{1}(U_i \leq u_i) + \{u_i + (1 - u_i)V_i\} \mathbf{1}(U_i > u_i). \tag{4}$$

While it was merely shown in [2] that the random vector $\mathbf{Y} = (Y_1, \ldots, Y_d)$ actually follows a GPD, the following main result of this section provides a precise characterization of the corresponding $D$-norm.

**Theorem 3.1.** *Suppose that* $P(\mathbf{U} > \mathbf{u}) > 0$. *The random vector $\mathbf{Y}$ defined through* (4) *follows a GPD-copula, which coincides with $C$ on* $[\mathbf{0}, \mathbf{u}] \in (0, 1)^d$ *and $D$-norm given by*

$$\|\mathbf{x}\|_D = E\left[ \max_{1 \leq j \leq d} \left\{ |x_j| Z_j \frac{\mathbf{1}(U_j > u_j)}{1 - u_j} \right\} \right],$$

*where $\mathbf{Z}$ and $\mathbf{U}$ are independent.*

Note that $\widetilde{\mathbf{Z}} := (\widetilde{Z}_1, \ldots, \widetilde{Z}_d)$ with $\widetilde{Z}_j := Z_j \mathbf{1}(U_j > u_j)/(1 - u_j)$, is a generator with the characteristic properties of being nonnegative, bounded and satisfying $E(\widetilde{Z}_j) = 1$, $1 \leq j \leq d$, due to the independence of $\mathbf{Z}$ and $\mathbf{U}$. In analogy to a corresponding terminology in point process theory one might call $\widetilde{\mathbf{Z}}$ a *thinned* generator.

**Proof.** Elementary computations yield

$$P(Y_i \leq x) = x, \quad 0 \leq x \leq 1,$$

i.e., $\mathbf{Y}$ follows a copula. We have, moreover, for $\mathbf{0} \leq \mathbf{x} \leq \mathbf{u}$

$$
\begin{aligned}
P(\mathbf{Y} \leq \mathbf{x}) &= \sum_{K \subset \{1, \ldots, d\}} P\left( \mathbf{Y} \leq \mathbf{x}; \, U_k \leq u_k, k \in K; \, U_j > u_j, j \in K^\complement \right) \\
&= \sum_{K \subset \{1, \ldots, d\}} P\Big[ U_i \mathbf{1}(U_i \leq u_i) + \{u_i + (1 - u_i)V_i\} \mathbf{1}(U_i > u_i) \leq x_i, 1 \leq i \leq d; \\
&\qquad\qquad U_k \leq u_k, k \in K; \, U_j > u_j, j \in K^\complement \Big] \\
&= P(U_i \leq x_i, \, 1 \leq i \leq d) \\
&= C(\mathbf{x})
\end{aligned}
$$

and for $\mathbf{u} < \mathbf{x} \leq \mathbf{1}$

$$
\begin{aligned}
P(\mathbf{Y} \leq \mathbf{x}) &= \sum_{K \subset \{1, \ldots, d\}} P\left( \mathbf{Y} \leq \mathbf{x}; \, U_k \leq u_k, k \in K; \, U_j > u_j, j \in K^\complement \right) \\
&= \sum_{K \subset \{1, \ldots, d\}} P\left( U_k \leq u_k, k \in K; \, u_j + (1 - u_j)V_j \leq x_j, U_j > u_j, j \in K^\complement \right) \\
&= \sum_{K \subset \{1, \ldots, d\}} P\left( U_k \leq u_k, k \in K; \, U_j > u_j, j \in K^\complement \right) P\left( V_j \leq \frac{x_j - u_j}{1 - u_j}, j \in K^\complement \right) \\
&= \sum_{K \subset \{1, \ldots, d\}} E\left[ \left\{ \prod_{k \in K} \mathbf{1}(U_k \leq u_k) \right\} \left\{ \prod_{j \in K^\complement} \mathbf{1}(U_j > u_j) \right\} \right] P\left( V_j \leq \frac{x_j - u_j}{1 - u_j}, j \in K^\complement \right).
\end{aligned}
$$

If $\mathbf{x} < \mathbf{1}$ is large enough, then we have for $K^\complement \neq \emptyset$

$$
\begin{aligned}
P\left( V_j \leq \frac{x_j - u_j}{1 - u_j}, j \in K^\complement \right) &= 1 - E\left\{ \max_{j \in K^\complement} \left( \left| \frac{x_j - u_j}{1 - u_j} - 1 \right| Z_j \right) \right\} \\
&= 1 - E\left\{ \max_{j \in K^\complement} \left( \frac{|x_j - 1|}{1 - u_j} Z_j \right) \right\}
\end{aligned}
$$

and, thus,

$$
\begin{aligned}
P(\boldsymbol{Y} \leq \boldsymbol{x}) &= P(U_k \leq u_k, 1 \leq k \leq d) + \sum_{\substack{K \subset \{1,\dots,d\} \\ K^\complement \neq \emptyset}} E\left[ \left\{ \prod_{k \in K} \mathbf{1}(U_k \leq u_k) \right\} \left\{ \prod_{j \in K^\complement} \mathbf{1}(U_j > u_j) \right\} \right] \\
&\quad \times \left[ 1 - E\left\{ \max_{j \in K^\complement}\left( \frac{|x_j - 1|}{1 - u_j} Z_j \right) \right\} \right] \\
&= 1 - \sum_{\substack{K \subset \{1,\dots,d\} \\ K^\complement \neq \emptyset}} E\left[ \left\{ \prod_{k \in K} \mathbf{1}(U_k \leq u_k) \right\} \left\{ \prod_{j \in K^\complement} \mathbf{1}(U_j > u_j) \right\} \max_{j \in K^\complement}\left( \frac{|x_j - 1|}{1 - u_j} Z_j \right) \right] \\
&= 1 - E\left[ \sum_{\substack{K \subset \{1,\dots,d\} \\ K^\complement \neq \emptyset}} \left\{ \prod_{k \in K} \mathbf{1}(U_k \leq u_k) \right\} \left\{ \prod_{j \in K^\complement} \mathbf{1}(U_j > u_j) \right\} \max_{j \in K^\complement}\left( \frac{|x_j - 1|}{1 - u_j} Z_j \right) \right] \\
&= 1 - E\left[ \max_{1 \leq j \leq d} \left\{ |x_j - 1| Z_j \frac{\mathbf{1}(U_j > u_j)}{1 - u_j} \right\} \right] \\
&= 1 - \|\boldsymbol{x} - \mathbf{1}\|_D ,
\end{aligned}
$$

as we can suppose independence of $\boldsymbol{U}$ and the generator $\boldsymbol{Z}$. $\quad\square$

The following result justifies the use of the multivariate PT-approach as it shows that the PT vector $\boldsymbol{Y}$, suitably standardized, approximately follows the distribution of $\boldsymbol{U}$ close to one.

**Proposition 3.2.** *Suppose that $\boldsymbol{U} = (U_1, \dots, U_d)$ follows a copula $C \in \mathcal{D}(G)$ with corresponding D-norm $\|\cdot\|_D$ generated by $\boldsymbol{Z}$. If the random vector $\boldsymbol{V}$ in definition* (4) *of the PT vector $\boldsymbol{Y}$ has this generator $\boldsymbol{Z}$ as well, then we have*

$$
P(\boldsymbol{U} > \boldsymbol{v}) = P\{Y_j > u_j + v_j(1 - u_j), \ 1 \leq j \leq d \mid \boldsymbol{U} > \boldsymbol{u}\} + o(\|\mathbf{1} - \boldsymbol{v}\|)
$$

*uniformly for $\boldsymbol{v} \in [\boldsymbol{u}, \mathbf{1}] \subset \mathbb{R}^d$.*

The term $o(\|\mathbf{1} - \boldsymbol{v}\|)$ can be dropped in the preceding result if $C$ is a GPD-copula itself, precisely, if $C(\boldsymbol{x}) = 1 - \|\boldsymbol{x}\|_D$, $\boldsymbol{x} \geq \boldsymbol{u}$.

**Proof.** Repeating the arguments in the proof of Corollary 2.3 we obtain

$$
P(\boldsymbol{U} > \boldsymbol{v}) = E\left[ \min_{1 \leq j \leq d}\{(1 - v_j)Z_j\} \right] + o(\|\mathbf{1} - \boldsymbol{v}\|)
$$

uniformly for $\boldsymbol{v} \in [0, 1]^d$.

We have, on the other hand, for $\boldsymbol{v}$ close enough to $\mathbf{1}$

$$
P\{Y_j > u_j + v_j(1 - u_j), \ 1 \leq j \leq d \mid \boldsymbol{U} > \boldsymbol{u}\} = P\{U < (1 - v_j)Z_j, \ 1 \leq j \leq d\} = E\left[ \min_{1 \leq j \leq d}\{(1 - v_j)Z_j\} \right],
$$

which completes the proof. $\quad\square$

If the copula $C$ is not known, the preceding PT-approach can be modified as follows, with $C$ replaced by the *empirical copula*. Suppose we are given $n$ copies $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ of a random vector $\boldsymbol{X} = (X^{(1)}, \dots, X^{(d)})$. Set for $1 \leq j \leq d$

$$
F_n^{(j)}(x) := \frac{1}{n + 1} \sum_{i=1}^{n} \mathbf{1}(X_i^{(j)} \leq x), \quad x \in \mathbb{R},
$$

which is essentially the empirical distribution function of the $j$-th components of $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$. Transform each random vector $\boldsymbol{X}_i$ in the sample to the vector of its standardized ranks $\boldsymbol{R}_i := \left( F_n(X_i^{(1)}), \dots, F_n(X_i^{(d)}) \right)$. The empirical copula is then the empirical distribution function corresponding to $\boldsymbol{R}_1, \dots, \boldsymbol{R}_n$:

$$
C_n(\boldsymbol{u}) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}(\boldsymbol{R}_i \leq \boldsymbol{u}), \quad \boldsymbol{u} \in [0, 1]^d.
$$

Properties of the empirical copula are well studied; we refer to Segers [19] and the literature cited therein.

Given the empirical copula $C_n$, let the random vector $\boldsymbol{U}^* = (U_1^*, \dots, U_d^*)$ follow this distribution function $C_n$ and let $\boldsymbol{V} = (V_1, \dots, V_d)$ follow a GPD-copula. Again we suppose that $\boldsymbol{U}$ and $\boldsymbol{V}$ are independent.

Choose a threshold $\boldsymbol{u} = (u_1, \ldots, u_d) \in (0, 1)^d$ and put for $1 \le i \le d$

$$Y_i^* := U_i^* \mathbf{1}(U_i^* \le u_i) + \{u_i^* + (1 - u_i^*)V_i\}\mathbf{1}(U_i^* > u_i), \tag{5}$$

where $u_i^* := P_n(U_i^* \le u_i)$. Recall that the preceding probability is, actually, a conditional one, given the empirical copula $C_n$. To avoid confusion we add the index $n$. The following result can be shown by repeating the arguments in the proof of Theorem 3.1. The minimum $\min(\boldsymbol{u}, \boldsymbol{u}^*)$ is meant to be taken componentwise.

**Proposition 3.3.** *Suppose that the threshold $\boldsymbol{u} \in (0, 1)^d$ satisfies $P_n(\boldsymbol{U}^* > \boldsymbol{u}) > 0$. The random vector $\boldsymbol{Y}^*$, defined componentwise in* (5), *follows a multivariate GPD, which coincides on $[\boldsymbol{0}, \min(\boldsymbol{u}, \boldsymbol{u}^*)]$ with the empirical copula $C_n$ and, for $\boldsymbol{x} < \boldsymbol{1}$ large enough,*

$$P_n(\boldsymbol{Y}^* \le \boldsymbol{x}) = 1 - \|\boldsymbol{x}\|_{D_n},$$

*where the D-norm is given by*

$$\|\boldsymbol{x}\|_{D_n} = E_n\left[\max_{1 \le j \le d}\left\{|x_j|Z_j\frac{\mathbf{1}(U_j^* > u_j)}{1 - u_j^*}\right\}\right],$$

*the generator $\boldsymbol{Z}$ and $\boldsymbol{U}^*$ being independent and $E_n$ denoting the expected value with respect to $P_n$.*

Proposition 3.2 can now be formulated as follows; its proof carries over.

**Proposition 3.4.** *Let $C$ be a copula with $C \in \mathcal{D}(G)$ and corresponding D-norm $\|\cdot\|_D$ generated by $\boldsymbol{Z}$. Let the random vector $\boldsymbol{U}$ follow this copula $C$. Suppose that the random vector $\boldsymbol{V}$ in definition* (5) *of the PT random vector $\boldsymbol{Y}^*$ has this generator $\boldsymbol{Z}$ as well. Then we have*

$$P(\boldsymbol{U} > \boldsymbol{v}) = P_n\{Y_j^* > u_j^* + v_j(1 - u_j^*), \ 1 \le j \le d \mid \boldsymbol{U}^* > \boldsymbol{u}\} + o(\|\boldsymbol{1} - \boldsymbol{v}\|)$$

*uniformly for $\boldsymbol{v} \in [\boldsymbol{u}, \boldsymbol{1}] \in \mathbb{R}^d$, where $\boldsymbol{U}^*$ follows the empirical copula $C_n$.*

The term $o(\|\boldsymbol{1} - \boldsymbol{v}\|)$ can again be dropped in the preceding result if $C$ is a GPD-copula itself, precisely, if $C(\boldsymbol{x}) = 1 - \|\boldsymbol{x} - \boldsymbol{1}\|_D$, $\boldsymbol{x} \ge \boldsymbol{u}$.

## 4. Piecing together: a functional version

In this section, we will extend the PT approach from Section 3 to function spaces. Suppose we are given a stochastic process $\boldsymbol{X} = (X_t)_{t \in [0,1]}$ with corresponding continuous copula process $\boldsymbol{U} = (U_t)_{t \in [0,1]} \in C[0, 1]$, where $C[0, 1]$ denotes the space of continuous functions on $[0, 1]$. A *copula process* $\boldsymbol{U}$ is characterized by the condition that each $U_t$ is uniformly distributed on $(0, 1)$. For a review of the attempts to extend the use of copulas to a dynamic setting, we refer to Ng [14] and the review paper by Andrew Patton in this Special Issue.

Choose a *generator process* $\boldsymbol{Z} = (Z_t)_{t \in [0,1]}$, characterized by the condition

$$0 \le Z_t \le c, \qquad E(Z_t) = 1, \quad 0 \le t \le 1,$$

for some $c \ge 1$. We require that $\boldsymbol{Z} \in C[0, 1]$ as well.

Let $U$ be a uniformly distributed on $(0, 1)$ random variable that is independent of $\boldsymbol{Z}$ and put for some $M < 0$

$$V_t := \max\left(M, -\frac{U}{Z_t}\right), \quad 0 \le t \le 1. \tag{6}$$

The process $\boldsymbol{V} = (V_t)_{t \in [0,1]} \in C[0, 1]$ is called a *standard generalized Pareto process* (GPP) as it has ultimately uniform margins; see below. This functional extension of multivariate GPD goes back to Buishand et al. [5, Section 2.3], again with Pareto margins instead of uniform ones. We incorporate the constant $M$ again to avoid possible division by zero.

Note that for $0 \ge x \ge K := \max(M, -1/c)$

$$P(V_t \le x) = P(U \ge |x|Z_t) = \int_0^c P(U \ge |x|z)\,(P * Z_t)(dz) = 1 + x, \tag{7}$$

i.e., each $V_t$ follows close to zero a uniform distribution.

Denote by $E[0, 1]$ the set of bounded functions $f : [0, 1] \to \mathbb{R}$, which have only a finite number of discontinuities, and put $\bar{E}^-[0, 1] := \{f \in E[0, 1] : f \le 0\}$. Repeating the arguments in the derivation of Eq. (7), we obtain for $f \in \bar{E}^-[0, 1]$ with $\|f\|_\infty \le |K|$

$$P(\boldsymbol{V} \le f) = P\{V_t \le f(t), \ t \in [0, 1]\} = 1 - E\left\{\sup_{t \in [0,1]}(|f(t)|Z_t)\right\}.$$

To improve the readability of this paper, we set stochastic processes such as $\boldsymbol{V}$ in bold font and non stochastic functions such as $f$ in default font. Operations on functions such as $\leq, >$ etc. are meant componentwise.

The process $\boldsymbol{V}$ can easily be modified to obtain a *generalized Pareto copula process* (GPCP) $\boldsymbol{Q} = (Q_t)_{t \in [0,1]}$, i.e., each $Q_t$ follows the uniform distribution on $(0, 1)$ and $(Q_t - 1)_{t \in [0,1]}$ is a GPP. Just put

$$\widetilde{V}_t := \begin{cases} V_t & \text{if } V_t > K \\ \xi & \text{if } V_t \leq K \end{cases}, \quad 0 \leq t \leq 1,$$

where the random variable $\xi$ is uniformly distributed on $(-1, K)$ and independent of the process $\boldsymbol{V}$; we assume that $K > -1$. Note that each $\widetilde{V}_t$ is uniformly distributed on $(-1, 0)$ and that for $f \in \bar{E}^-[0, 1]$ with $\|f\|_\infty < |K|$

$$\begin{aligned} P\left(\widetilde{\boldsymbol{V}} \leq f\right) &= P\left\{\widetilde{V}_t \leq f(t), \ 0 \leq t \leq 1\right\} \\ &= P\{V_t \leq f(t), \ 0 \leq t \leq 1\} = P(\boldsymbol{V} \leq f). \end{aligned} \tag{8}$$

The process $\boldsymbol{Q}$ is now obtained by putting $\boldsymbol{Q} := (\widetilde{V}_t + 1)_{t \in [0,1]}$. It does not have continuous sample paths, but it is continuous in probability, i.e.,

$$P\left(|Q_{t_n} - Q_t| > \varepsilon\right) \rightarrow_{t_n \to t} 0$$

for each $t \in [0, 1]$ and any $\varepsilon > 0$.

Suppose that we are given a copula process $\boldsymbol{U} \in C[0, 1]$. Choose a GPCP $\boldsymbol{Q}$ with generator $\boldsymbol{Z} \in C[0, 1]$, $\boldsymbol{Q}$ independent of $\boldsymbol{U}$, a threshold $u \in (0, 1)$ and put

$$Y_t := U_t \mathbf{1}(U_t \leq u) + \{u + (1 - u)Q_t\}\mathbf{1}(U_t > u), \quad t \in [0, 1]. \tag{9}$$

We call $\boldsymbol{Y} = (Y_t)_{t \in [0,1]}$ a *PT-process*. We require that the processes $\boldsymbol{U}$ and $\boldsymbol{Z}$ are independent. Note that $\boldsymbol{Y}$ is continuous under the condition $\boldsymbol{U} > u$. The following theorem is the main result in this section.

**Theorem 4.1.** *The process $\boldsymbol{Y} = (Y_t)_{t \in [0,1]}$ with $Y_t$ as in (9) is a GPCP, which is continuous in probability, and with D-norm given by*

$$\|f\|_D = \mathrm{E}\left[\sup_{t \in [0,1]} \left\{|f(t)|Z_t \frac{\mathbf{1}(U_t > u)}{1 - u}\right\}\right], \quad f \in E[0, 1].$$

Note that $\mathrm{E}\left[\sup_{t \in [0,1]} \{|f(t)|Z_t \mathbf{1}(U_t > u)/(1 - u)\}\right]$ is well defined, due to the continuity of $\boldsymbol{Z}$ and $\boldsymbol{U}$. The *thinned generator process*

$$\widetilde{\boldsymbol{Z}} = \left\{Z_t \frac{\mathbf{1}(U_t > u)}{1 - u}\right\}_{t \in [0,1]}$$

satisfies

$$0 \leq \widetilde{Z}_t \leq \frac{c}{1 - u}, \qquad \mathrm{E}(\widetilde{Z}_t) = 1, \quad t \in [0, 1],$$

and it is continuous in probability.

**Proof.** Each $Y_t$ is by Theorem 3.1 uniformly distributed on $(0, 1)$. Continuity in probability follows from elementary arguments. Choose $f \in \bar{E}^-[0, 1]$ with $\|f\|_\infty < (1 - u) \min\{|M|, |K|, c^{-1}\}$. We have

$$P(Y_t \leq 1 + f(t), \ t \in [0, 1)) = P\left[\{u + (1 - u)Q_t\}\mathbf{1}(U_t > u) \leq 1 + f(t), \ t \in [0, 1]\right].$$

Note that the term $U_t \mathbf{1}(U_t \leq u)$ can be neglected since $U_t \leq u$ implies $U_t \leq 1 + f(t)$ and, due to the restrictions on $f$, $1 + f(t) > u > 0, t \in [0, 1]$. Analogously, we may rewrite the probability from above as

$$\begin{aligned} &P\left\{(1 - u)Q_t \mathbf{1}(U_t > u) \leq 1 - u + f(t), \ t \in [0, 1]\right\} \\ &= P\left\{(Q_t - 1)\mathbf{1}(U_t > u) \leq 1 - \mathbf{1}(U_t > u) + \frac{f(t)}{1 - u}, \ t \in [0, 1]\right\} \\ &= P\left\{Q_t - 1 - Q_t \mathbf{1}(U_t \leq u) \leq \frac{f(t)}{1 - u}\mathbf{1}(U_t \leq u) + \frac{f(t)}{1 - u}\mathbf{1}(U_t > u), \ t \in [0, 1]\right\} \\ &= P\left\{Q_t - 1 \leq \frac{f(t)}{1 - u}\mathbf{1}(U_t > u), \ t \in [0, 1]\right\} \end{aligned}$$

where the last equality is again a consequence of neglecting the terms corresponding to the case $U_t \leq u$; note that the restrictions on $f$ imply $f(t) \geq u - 1$. This probability has by (6) and (8) the representation

$$
P\left\{V_t \leq \frac{f(t)}{1-u}\mathbf{1}(U_t > u),\ t \in [0,1]\right\} = P\left[U \geq \sup_{t \in [0,1]}\left\{|f(t)|Z_t\frac{\mathbf{1}(U_t > u)}{1-u}\right\}\right]
$$
$$
= 1 - E\left[\sup_{t \in [0,1]}\left\{|f(t)|Z_t\frac{\mathbf{1}(U_t > u)}{1-u}\right\}\right]
$$

which completes the proof. $\square$

In what follows, we justify the functional PT approach by extending Proposition 3.2. We say that a copula process $\boldsymbol{U} \in C[0,1]$ is in the *functional domain of attraction* of a max-stable process $\boldsymbol{\eta} \in C[0,1]$, denoted by $\boldsymbol{U} \in \mathcal{D}(\boldsymbol{\eta})$, if

$$
P\{n(\boldsymbol{U}-1) \leq f\}^n \to_{n\to\infty} P(\boldsymbol{\eta} \leq f), \quad f \in \bar{E}^-[0,1].
$$

The max-stability of $\boldsymbol{\eta}$ is characterized by the equation

$$
P\left(\boldsymbol{\eta} \leq \frac{f}{n}\right)^n = P(\boldsymbol{\eta} \leq f), \quad n \in \mathbb{N},\ f \in \bar{E}^-[0,1].
$$

From Aulbach et al. [3] we know that there exists a generator process $\boldsymbol{Z} = (Z_t)_{t \in [0,1]} \in C[0,1]$ such that for $f \in \bar{E}^-[0,1]$

$$
P(\boldsymbol{\eta} \leq f) = \exp\left[-E\left\{\sup_{t \in [0,1]}(|f(t)|Z_t)\right\}\right] = \exp\left(-\|f\|_D\right),
$$

which shows in particular that the process $\boldsymbol{\eta}$ has standard negative exponential margins. A continuous max-stable process (MSP) with standard negative exponential margins will be called a *standard* MSP. We refer to Aulbach et al. [3] for a detailed investigation of the functional domain of attraction condition, which is weaker than that based on weak convergence developed in [8].

The next result, which justifies the functional PT-approach, is now an immediate consequence of Proposition 3.2. The term $o(\|\mathbf{1}-\boldsymbol{v}\|)$ can again be dropped for $(v_1, \ldots, v_d)$ large enough, if the process $\boldsymbol{U}$ is itself a GPCP.

**Proposition 4.2.** *Suppose that the copula process $\boldsymbol{U} \in C[0,1]$ satisfies $\boldsymbol{U} \in \mathcal{D}(\boldsymbol{\eta})$, where $\boldsymbol{\eta} \in C[0,1]$ is a standard MSP with generator process $\boldsymbol{Z} = (Z_t)_{t \in [0,1]} \in C[0,1]$. Choose a threshold $u \in (0,1)$ and arbitrary indices $0 \leq t_1 < \cdots < t_d \leq 1$, $d \in \mathbb{N}$. If the process $\boldsymbol{V}$ in definition (9) of the PT-process $\boldsymbol{Y}$ has this generator $\boldsymbol{Z}$ as well, then we have*

$$
P\left(U_{t_j} > v_{t_j},\ 1 \leq j \leq d\right) = P\left\{Y_{t_j} > u + (1-u)v_j,\ 1 \leq j \leq d \mid U_{t_j} > u,\ 1 \leq j \leq d\right\} + o(\|\mathbf{1}-\boldsymbol{v}\|),
$$

*uniformly for $\boldsymbol{v} \in [u,1]^d$.*

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