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The Spherical Detonation

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CONTENTS

1. INTRODUCTION

The diverging detonation displays a strong nonlinear interaction among geometry, chemistry, and hydrodynamics. Our analysis of this interaction uses a combination of phase plane analysis, bifurcation theory, and matched asymptotic expansions. The bifurcation and phase plane structures we encounter are highly singular. The stationary manifold of phase space changes its dimensions under perturbation while the bifurcation has infinite co-dimension. These facts place the problem outside of standard theories, and for this reason, an independent bifurcation analysis is

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required. Because the differential equations have to be studied not only locally, in a neighborhood of the phase plane critical points, but also globally, matched asymptotic expansions are required. For these expansions, uniformity with respect to the order ω of the chemical reaction is an additional difficulty which we address. The uniform expansions require at least two terms and contain non-rational exponents as well as logarithms.

From the point of view of physics, the principle effect of the geometrical divergence of the wave front is the decrease of the wave speed due to the slowing of the reaction behind the shock. Our central result is a quantitative theory of this effect. Using curvature dependent theories, the detonation wave has been studied in various geometries [9, 3, 8, 13]. In the *rate stick* experiments the detonation wave propagates along a cylindrical stick of explosive. The detonation waves are diverging due to boundary effects, and the divergence increases as the diameter of the rate stock is decreased. The resulting decrease in speed of the wave is traditionally called the *diameter effect*. The combination [13] of the curvature dependent detonation wave speed analysis with a shock polar analysis at the boundary gives a closed system of equations for the wave front in this geometry. The analysis of [8], which has as its point the relation of curvature dependent detonation wave speed to realistic chemistry, concludes that reaction orders ω greater than one are important. Further discussion of the rate stick problem can be found in [9, pp. 199–229; 3].

Our central result is a bifurcation analysis and an extension of the wave speed analysis of a diverging detonation in two or three dimensions. We will derive an asymptotic approximation for the detonation which is valid for shock radii that are large with respect to the width of the reaction zone. Although this is not a steady state problem, we will show in Section 2 that to leading order in the shock curvature, the detonation may be modeled by a system of autonomous differential equations in space, depending on curvature and wave speed as parameters. This model is specified by Eqs. (2.10)–(2.11) below, see also [9, pp. 207–210]. Our main bifurcation results are stated in Section 3. In order to understand the topological structure of the bifurcation, we propose a normal form in Section 4. This normal form is solved explicitly.

The analysis of the model equations begins in Section 5 with the solution of a free boundary problem for sonic transition surface, which terminates the subsonic region behind the shock. The fluid velocity becomes unbounded along a hypersurface in state space on which the flow velocity relative to the shock equals the sound speed. This surface will be referred to as the *sonic locus*. A smooth transition across the sonic locus is possible only at a critical point of the model system. We will show that an appropriate critical point exists and solves the sonic free boundary problem.

The plane wave sonic critical point is degenerate. In the zero curvature limit, the eigenvalues of the critical point vanish and eigenvectors coincide. In addition, the entire eigenspace becomes stationary. The plane wave sonic critical point is a bifurcation point of infinite codimension. The number of topologically distinct fields that can be obtained by a small perturbation of the plane wave is infinite. These features of the vector field defined by (2.10)–(2.11) can be seen in the normal form of Section 4. A partial derivation of the normal form by means of Poincaré normalizing transformations is given in Section 5. A one-parameter unfolding of the normal form is presented in Section 4 illustrating the effect of curvature on the phase portrait.

For each value of the curvature, the wave speed must be chosen to ensure that the solution passes through the sonic critical point. The evolution of the detonation is determined by this shooting problem connecting the state at the shock interface to the sonic free boundary. The wave speed may be interpreted as a nonlinear eigenvalue, and detonations exhibiting this type of behavior are sometimes termed *eigenvalue detonations* [9]. The shooting problem is solved in Section 7 by an asymptotic expansion in the small curvature limit. The singular behavior of the solution near the sonic critical point introduces nonuniformities into the small curvature expansion which must be resolved by an inner expansion in a thin layer near the sonic-free boundary. The inner solution in this *sonic boundary layer* governs the form of the wave speed expansion. We find that the expansion depends on details of the chemistry. In particular, it depends strongly on the form of the order ω in the chemical rate law.

The leading order terms for the wave speed as given in (7.10) are

$$
\begin{array}{ll}
\text{(i) } \omega \ll 1, & D \approx D_{CJ} + D_1\kappa \\
\text{(ii) } \frac{1}{2} < \omega < \frac{3}{2},\ \omega \neq 1, & D \approx D_{CJ} + D_1\kappa + D_2\kappa^{1/\omega} \\
\text{(iii) } \omega = 1, & D \approx D_{CJ} + D_3\kappa + D_4\kappa \log \kappa \\
\text{(iv) } 1 \ll \omega, & D \approx D_{CJ} + D_2\kappa^{1/\omega}.
\end{array}
$$

The κ interval of validity of these expansions is uniform in ω at the level of formal asymptotics. The terms $D_1\kappa$ and $D_2\kappa^{1/\omega}$ are in resonant competition near $\omega = 1$, and both are required for the expansion to be uniformly valid in ω. The linear term D_1 was calculated by Bdzil and Stewart [4] using a temperature independent rate law and the strong shock limit. They also computed the matching of the inner with the outer solutions for the logarithmic term D_4.

The present work is an extended version of the author's Ph.D. thesis [8]. Portions of these results have been presented in [6, 7].

2. Derivation of the Model

The equations for a spherically symmetric, transport-free, reactive, polytropic gas are

$$\begin{aligned}
\rho_t + (\rho u)_r + \frac{\rho u(d-1)}{r} &= 0 \\
(\rho u)_t + (\rho u^2 + p)_r + \frac{\rho u^2(d-1)}{r} &= 0 \\
E_t + (u(E+p))_r + \frac{u(E+p)(d-1)}{r} &= 0 \\
(\rho\lambda)_t + (\rho\lambda u)_r + \left(\frac{\rho\lambda u(d-1)}{r} - \rho R\right) &= 0.
\end{aligned} \tag{2.1}$$

Here ρ is the density, u is the radial velocity, and p is the pressure. The spatial dimension is $d \in 2,3$ and r is the radial coordinate. The reaction progress parameter, which varies from 0 (all reactant) to 1 (all product), will be denoted by λ. The total energy density is $E = \rho e + \rho u^2/2$. The reaction rate $R(\lambda, T)$ and specific internal energy $e(\lambda, T)$ are assumed to have the form

$$R(\lambda, T) = \begin{cases} k(1-\lambda)^{\omega} g(T), & T_c \leq T \\ 0, & T < T_c \end{cases} \tag{2.2}$$

$$e(\lambda, T) = \frac{T}{\gamma - 1} + (1-\lambda)q,$$

where γ is the polytropic gas constant and q is the heat released per unit mass by the complete reaction. Here $\omega > 0$ and g is a strictly positive dimensionless function of the temperature $T = p/\rho$ (using units in which the ideal gas constant is unity). The rate multiplier k is positive and has dimensions of inverse time. The constant T_c is the critical temperature below which the reaction rate is taken to be identically zero. The role of T_c is to avoid the famous "cold boundary problem." Without such a cutoff temperature there are no solutions for large times because all of the reactant is consumed upstream of the shock. In the case of Arrhenius kinetics, g takes the form

$$g(T) = \exp(-A/T),$$

where $A > 0$ is the activation energy. We will assume that g is differentiable.

We now transform the differential equations (2.1) to a more convenient form for the analysis. Denote the radius of the shock by z. Then $1/z$ is the mean curvature of the shock, $\kappa = (d-1)/z$ is the sum of principle curvatures, and the wave speed is

$$D \equiv \frac{dz}{dt} = -(d-1)\kappa^{-2}\frac{d\kappa}{dt}.$$

Now define

$$x \equiv z - r,$$

and

$$v \equiv D - u.$$

The variable x is just the distance from the shock, oriented inwards. The velocity of the flow relative to the shock is v, again oriented towards the center of the detonation. With this choice of orientation x will be positive behind the shock in the reaction zone. Eliminate the internal energy by substituting (2.2) into (2.1), and change variables from r, t, and u to x, κ, and v, respectively, to obtain

$$\begin{aligned}
-\kappa^2(d-1)^{-1}D\rho_\kappa + v\rho_x + v_x\rho &= -\kappa\frac{\rho(D-v)}{1-(d-1)^{-1}\kappa x} \\
-\kappa^2(d-1)^{-1}Dv_\kappa + vv_x + \frac{p_x}{\rho} &= -\kappa^2(d-1)^{-1}DD_\kappa \\
-\kappa^2(d-1)^{-1}Dp_\kappa + vp_x + \gamma p v_x &= q(\gamma-1)\rho R(\lambda,T) \\
&\quad -\kappa\frac{\gamma p(D-v)}{1-(d-1)^{-1}\kappa x} \\
-\kappa^2(d-1)^{-1}D\lambda_\kappa + v\lambda_x &= R(\lambda,T).
\end{aligned} \tag{2.3}$$

Denote values ahead of the shock by the subscript a. It will be assumed that the ahead state is constant and unreacted ($\lambda_a = 0$), and that the flow velocity is zero ($u_a = 0$, or $v_a = D$). The Rankine–Hugoniot jump conditions are

$$\begin{aligned}
\rho_s v_s &= \rho_a D \equiv m \\
\frac{p_a - p_s}{V_a - V_s} &= -m^2 \\
(V_s - \mu^2 V_a)p_s &= (V_a - \mu^2 V_s)p_a \\
\lambda_s &= \lambda_a,
\end{aligned} \tag{2.4}$$

where $\mu^2 = (\gamma - 1)/(\gamma + 1)$. The s subscript indicates variables evaluated immediately behind the shock.

The Plane Wave Limit

When $\kappa = 0$, we obtain the equations for a one-dimensional steady state detonation studied by Zeldovich, von Neumann, and Doering (ZND). The hydrodynamic equations may be integrated to obtain ρ, v, and p as functions of λ. The chemical rate equation then constitutes an ordinary differential equation for λ as a function of x,

$$\rho v = m \; (= \text{const})$$

$$\frac{p - p_s}{V - V_s} = -m^2$$

$$2\mu^2 q\lambda = (V_s - \mu^2 V)p_s - (V - \mu^2 V_s)p$$

$$\frac{d\lambda}{dx} = \frac{R}{v}, \tag{2.5}$$

where $V \equiv 1/\rho$ is the specific volume. The first of Eqs. (2.5) states that the mass flux is constant in the shock frame. The second equation defines a line of slope $-m^2$ in the p, V plane and is referred to as the *Rayleigh line*. The third of Eqs. (2.5) defines a family of *Hugoniot curves* in the p, V plane. The intersections of the Rayleigh line with the $\lambda = 0$ Hugoniot curve determine the possible shock transitions. The solution terminates at a point where the Rayleigh line intersects the $\lambda = 1$ Hugoniot curve. In general, there is a one-parameter family of solutions, parameterized by the mass flux m, or for a given ambient state ahead of the shock, by the wave speed D. Unlike an inert shock, which may propagate at any speed, a detonation possesses a minimum wave speed at which the Rayleigh line is tangent to the final reaction Hugoniot. This point of tangency is called the *Chapman–Jouguet* point, in honor of the early detonation theorists who discovered it and discussed its significance. The solution terminating at this point will be denoted by a *CJ* subscript. For a steady plane wave which is not supported by a driving force such as a piston, only the Chapman–Jouguet (CJ) solution can be stable. This unique plane solution is taken to be the asymptotic limit of the diverging wave.

The wave speed D_{CJ} and final ($\lambda = 1$) flow velocity v_{CJ} for the plane wave may be determined from Eqs. (2.5) and the condition $v_{CJ} = c_{CJ}$, where c_{CJ} is the sound speed at the CJ point. The result is

$$D_{CJ} = \left[(\gamma^2 - 1)qc_a^{-2} + 1 + \left(\left((\gamma^2 - 1)qc_a^{-2} + 1\right)^2 - 1\right)^{1/2}\right]^{1/2} c_a$$

$$v_{CJ} = \left(\frac{2c_a^2}{\gamma + 1} + \mu^2 D_{CJ}^2 + 2\mu^2 q\right)^{1/2}. \tag{2.6}$$

The Sonic-free Boundary

Let $\mathbf{w}(x, \kappa, D) \equiv (\rho, v, P, \lambda)^{\mathrm{T}}$. The system (2.3) then takes the form

$$-\kappa^2(d - 1)^{-1} D\mathbf{w}_\kappa + \mathbf{O}(\mathbf{w}) \cdot \mathbf{w}_x = \mathbf{h}(\mathbf{w}, x, \kappa, D). \tag{2.7}$$

The eigenvalues of the quasilinear operator $\mathbf{O}$ are v and $v \pm c$, where $c = (\gamma p/\rho)^{1/2}$ is the sound speed. The diverging detonation is weakened by rarefactions produced behind the shock front and terminates below the sonic point on the weak detonation branch of the final reaction Hugoniot. This means that a sonic transition must occur in the reaction zone from the subsonic flow behind the shock to the supersonic flow at termination. Consequently, the desired solution of (2.7) must possess a sonic transition at some point $x = w < \infty$ when $\kappa > 0$. At a sonic transition the quasilinear operator $\mathbf{O}$ becomes singular. This singularity may be interpreted as a turning point of the system. If we attempt to solve for $\mathbf{w}_x$ in (2.7) near the turning point we find that the x derivatives blow up unless certain solvability conditions are satisfied. Specifically $\mathbf{h} + \kappa^2(d - 1)^{-1}D\mathbf{w}_\kappa$ must lie in the range of $\mathbf{O}$. For the plane wave, this reduces to the Chapman–Jouguet condition that a sonic transition $v = c$ may occur only at the termination of the reaction zone $\lambda = 1$. If κ is small but nonzero, the solvability condition requires that $\lambda < 1$ at a sonic transition. We will refer to the problem of determining an admissable sonic transition as the *sonic free boundary problem*. The wave speed D is then determined, at least in principle, by a shooting problem connecting the state at the shock to the sonic free boundary. The solution beyond the sonic boundary is supersonic and has no effect on the detonation wave. We will assume on physical grounds that a smooth transonic solution of (2.3) exists for sufficiently small curvature, and that the $\kappa \to 0$ limit of this solution is an undrive plane wave. We will seek an asymptotic approximation to this solution which is valid for small κ.

The Model Equations

Denote by w the width of the subsonic region between the shock and the sonic transition. For the plane wave, w may be computed by integrating the rate equation.

$$w = \int_0^1 \frac{v(\lambda)}{(1-\lambda)^\omega g(T(\lambda))}\, d\lambda.$$

When $\omega < 1$, w is finite, but for realistic reaction rates with $\omega \geq 1$ the sonic transition is at infinity. This means that the subsonic region expands without bound as $\kappa \to 0$. This will have important consequences for the solution of the shooting problem in Section 7.

We will approximate (2.3) by neglecting terms which we expect to be uniformly small in the subsonic region. We need the following two ansatz:

(i) The width w of the subsonic region is small relative to the radius of curvature of the shock, so that $\kappa w \ll 1$.

(ii) The κ derivatives are bounded or do not blow up too quickly as $\kappa \to 0$, so that $\kappa(d/d\kappa)$ is small and $\kappa^2(d/d\kappa)$ is negligible to leading order.

The first ansatz implies that the lateral divergence of the flow is approximately homogeneous in the subsonic region, and allows us to approximate $(d-1)/r = \kappa/(1-(d-1)^{-1}\kappa x)$ by the total shock curvature κ. The second ansatz permits us to neglect the time derivatives to leading order. We will verify later that the asymptotic approximation is consistent with these assumptions.

Neglecting terms in (2.3), which are uniformly small according to our ansatz, produces

$$\begin{aligned}
v\rho_x + v_x\rho &= -\kappa\rho(D-v)\\
vv_x + \frac{p_x}{\rho} &= 0\\
vp_x + \gamma p v_x &= q(\gamma-1)\rho R(\lambda,T) - \kappa\gamma p(D-v)\\
v\lambda_x &= R(\lambda,T).
\end{aligned} \tag{2.8}$$

The approximations leading to (2.8) are easily understood. First, the derivatives with respect to κ are neglected, so that the system is *quasi-steady*, depending on κ (and therefore on time) only as a parameter. This says that the detonation wave has been propagating for a long time and that all of the initialization transients have had time to die out. Second, the lateral divergence factor $(d-1)/r$ is homogeneous to leading order and equal to the shock curvature κ throughout the subsonic region $[0, w]$. Note that (2.8) is independent of the dimension d.

We now take (2.8) as our model for the expanding detonation and turn our attention to an analysis of this system. Our first objective will be to simplify the model equations. The steady state energy equation is

$$e_x + pV_x = 0.$$

The velocity equation in (2.8) can be written as

$$\tfrac{1}{2}(v^2)_x + Vp_x = 0.$$

These two equations may then be added to obtain

$$\left(\tfrac{1}{2}v^2 + e + Vp\right)_x = 0,$$

which integrates to yield Bernoulli's law:

$$\tfrac{1}{2}v^2 + e + Vp = f(\kappa).$$

For a polytropic equation of state this becomes

$$\frac{1}{2}v^2 + \frac{1}{\gamma - 1}c^2 - \lambda q = \frac{1}{2}D^2 + \frac{c_a^2}{\gamma - 1}. \tag{2.9}$$

Note that we are able to connect across the shock to the ambient state, since Bernoulli's law may be interpreted as one of the Rankine–Hugoniot jump conditions. This fact determines $f(\kappa)$.

Now eliminate p_x between the velocity and pressure equations to obtain

$$v_x = \frac{q(\gamma - 1)k(1 - \lambda)^{\omega}g(T) - (D - v)c^2\kappa}{c^2 - v^2}. \tag{2.10}$$

By (2.9) c^2 is a known function of v and λ, therefore the right-hand side of (2.10) is also a known function of v and λ. This form of the velocity equation may be combined with the rate equation

$$\lambda_x = k\frac{(1 - \lambda)^{\omega}g(T)}{v} \tag{2.11}$$

to obtain a self-contained system of two equations for v and λ.

The denominator in the velocity equation (2.10) vanishes at a sonic transition, so for a smooth sonic transition to occur, the numerator must vanish simultaneously. The sonic transition is just the turning point mentioned previously, and the condition that the numerator vanish is equivalent to leading order to the solvability condition for w_x in (2.7). We may substitute v^2 for c^2 in (2.9) to obtain the *sonic locus*

$$v^2 = \frac{2c_a^2}{\gamma + 1} + \mu^2 D^2 + 2\mu^2 q\lambda. \tag{2.12}$$

A solution of (2.10) and (2.11) which crosses the sonic locus will be called *transonic*. We thus seek a transonic solution which satisfies

$$q(\gamma - 1)k(1 - \lambda)^{\omega} g(T) - \kappa c^2(D - v) = 0 \tag{2.13}$$

at the sonic transition. A solution (v_c, λ_c) of the system (2.12), (2.13) will be referred to as a *sonic critical point*. Note that when $\kappa = 0$, Eq. (2.13) yields $\lambda = 1$ at the sonic critical point, retrieving the aforementioned result that a steady undriven plane detonation terminates at the CJ point.

The system (2.10), (2.11) will be easier to analyze if transformed into a more conventional form. Define the singular change of variable

$$y \equiv \int^x \frac{dx'}{\left(c(x')^2 - v(x')^2\right)v(x')g(T(x'))}. \tag{2.14}$$

Now change variables from x to y to obtain

$$\begin{aligned} v_y &= q(\gamma - 1)k(1 - \lambda)^{\omega} v - \kappa(D - v)vc^2 \frac{1}{g(\gamma^{-1}c^2)} \\ \lambda_y &= k(1 - \lambda)^{\omega}(c^2 - v^2). \end{aligned} \tag{2.15}$$

Observe that the structure of the phase curves in the (v, λ) plane is unaltered by this change of independent variable since the transformed vector field is proportional to the initial vector field. The integral curves have simply been reparameterized to eliminate the singularity in the denominator of the velocity equation. The right-hand side of (2.15) is now bounded in the region of interest, and the sonic critical point defined by (2.12), (2.13) is recognized as a stationary point of the system.

Several observations about (2.15) can be made immediately. Since the first equation is proportional to v, the λ axis is a phase curve of the system. Likewise, the $\lambda = 1$ line is a phase curve, since the second equation has a factor of $(1 - \lambda)^{\omega}$. These two phase curves intersect in a fixed critical point $(0, 1)$. When $\kappa = 0$, the vector field is proportional to $(1 - \lambda)^{\omega}$, so that the entire $\lambda = 1$ line is stationary. The CJ point is thus embedded in a manifold of critical points. We will see that the CJ point is a bifurcation point for the system, i.e., a point in the phase plane where the topology of the phase curves is unstable to small perturbations of the vector field.

The Shooting Problem

For a fixed ambient state ahead of the shock, the Rankine-Hugoniot jump conditions determine a one parameter family of solutions behind the shock, parameterized by the wave speed D. The critical point equations (2.12), (2.13) define v_c and λ_c as functions of κ and D. The desired solution of (2.15) will connect the ambient state (via the jump conditions) to a sonic critical point; this *shooting problem* defines the functional relationship between κ and D.

The method of determining the full dynamic solution of (2.15) is now clear. We first analyze the system allowing κ and D to vary as independent parameters, thus determining the phase plane structure throughout some domain in parameter space. We then may recover the leading order dynamics by solving the shooting problem. The shooting problem will be solved in Section 7 by a matched asymptotic expansion.

After eliminating c_a between (2.6) and (2.12), we obtain

$$c^2 - v^2 = -q(\gamma - 1)(1 - \lambda) - \frac{\gamma + 1}{2}\left(v^2 - v_b^2\right), \qquad (2.16)$$

where

$$v_b \equiv \left(v_{CJ}^2 + \mu^2\left(D^2 - D_{CJ}^2\right)\right)^{1/2}$$

is the flow velocity at the $\kappa = 0$ sonic bifurcation point. With κ and D independent, v_b is now a function of D and D_{CJ}.

We may obtain a single differential equation for $v(\lambda)$ by dividing the first equation in (2.15) by the second to obtain

$$(1 - \lambda)^{\omega}(c^2 - v^2)\frac{dv}{d\lambda} = q(\gamma - 1)(1 - \lambda)^{\omega} v - k^{-1}\kappa(D - v) v c^2 \frac{1}{g\left(c^2/\gamma\right)}. \qquad (2.17)$$

This equation clearly shows the (nonlinear) turning point character of the sonic locus. Let $v = h(\lambda, \kappa, D)$ be a transonic solution of (2.17), so that

$$v_c(\kappa, D) = h(\lambda_c(\kappa, D), \kappa, D).$$

Evaluating h at the shock and using the jump conditions gives an implicit solution

$$v_s(D) = h(0, \kappa, D) \qquad (2.18)$$

to the wave speed shooting problem. If we set $D = -(d - 1)\kappa^{-2}(d\kappa/dt)$

in (2.18), we obtain an ordinary differential equation for the curvature evolution $\kappa(t)$.

When $\kappa = 0$, we may use (2.16) to write Eq. (2.17) in the form

$$\left(1 - \frac{v_b^2 - 2q\mu^2(1-\lambda)}{v^2}\right) dv + \frac{2q\mu^2}{v} d\lambda = 0.$$

The left-hand side of this equation is the differential of the function

$$f(v, \lambda) = v + \frac{v_b^2 - 2q\mu^2(1-\lambda)}{v} + C.$$

Denoting by (v_r, λ_r) any fixed reference point, we have

$$v + \frac{v_b^2 - 2q\mu^2(1-\lambda)}{v} = v_r + \frac{v_b^2 - 2q\mu^2(1-\lambda_r)}{v_r}. \tag{2.19}$$

The choice $v_r = v_b$, $\lambda = 1$, yields the separatrix solution

$$(v - v_b)^2 + 2q\mu^2(\lambda - 1) = 0 \tag{2.20}$$

for the plane wave. This form of the plane wave solution will be useful later on.

3. Statement of Bifurcation Results

One of the main results of this paper is that the velocity of the expanding detonation is equal to the plane wave velocity plus a correction which is to lowest order a function of the shock curvature κ. One consequence of this result is that standard methods of computation of detonation waves [12] which use the experimental values of the planar detonation velocity can be improved in accuracy by these corrections. Moreover, since the correction can be computed from the chemistry, we believe that the correction can be predicted from some phenomenological equation of state and rate law, at least after the latter have been recalibrated to reflect the new requirement that they reproduce both planar speeds and leading order curvature corrections. Such a predictive capability would minimize the amount of experimental calibration necessary to use this new theory in numerical computations.

It is important to verify the existence of an appropriate solution to the model equations (2.15) which terminates at a sonic free boundary. The following theorem is therefore of interest.

THEOREM 3.1. *Let the reaction rate $R(\lambda, T)$ have the first-order Arrhenius form*

$$R(\lambda, T) = k(1 - \lambda)\exp(-A/T)$$

and assume that there is a radius z_1 such that $T_c < T$ whenever $0 < x$ and $z_1 < z$. Then there is a $\kappa_{\max} > 0$ and a neighborhood $(D_{\min}, D_{\max})$ of D_{CJ} such that

(i) *The critical point $v_b \equiv (v_{CJ}^2 + \mu^2(D^2 - D_{CJ}^2))^{1/2}$, $\lambda_b = 1$ in the vector field* (2.15) *bifurcates into a saddle point as κ is increased from zero, for all $D \in (D_{\min}, D_{\max})$.*

(ii) *For $\kappa \in (0, \kappa_{\max})$, and $D \in (D_{\min}, D_{\max})$, the saddle point in i) is the unique sonic critical point of* (2.15).

(iii) *The location of the saddle point in the phase plane is a C^∞ function of $\kappa \in (0, \kappa_{\max})$ and $D \in (D_{\min}, D_{\max})$.*

(iv) *The restriction of the vector field* (3.3) *to the stable separatrix of the saddle point is continuous, uniformly in $D \in (D_{\min}, D_{\max})$.*

(v) *The unique smooth transonic solution of* (2.10), (2.11) *is given by the stable separatrix of the saddle point.*

This theorem is proven in Section 6.

4. THE PROPOSED NORMAL FORM

We propose

$$\begin{pmatrix} \hat{v} \\ \hat{\lambda} \end{pmatrix}_y = \nu \begin{pmatrix} -\alpha\hat{\lambda} \\ (\hat{\lambda} - \eta\kappa)\hat{v} \end{pmatrix} \tag{4.1}$$

as a normal form at the sonic bifurcation point for the vector field (2.15) in the case of first order Arrhenius reaction kinetics. The shock curvature κ is the bifurcation parameter. The coefficients ν, α, and η are positive. The variables $\hat{v}$ and $\hat{\lambda}$ here denote v and λ translated to the transonic critical point, which remains fixed at the origin as κ is varied. In this section we investigate the properties of this proposed normal form. The results here will assist in understanding the properties of the transonic critical point, as well as lay a foundation for an eventual proof of local topological equivalence with (2.16) at the critical point.

If the first of Eqs. (4.1) is divided by the second, a separable ordinary differential equation

$$\hat{v}\,d\hat{v} = -\frac{\alpha\hat{\lambda}}{\hat{\lambda} - \eta\kappa}\,d\hat{\lambda}$$

is obtained for the phase curves $\hat{v}(\hat{\lambda})$. The general solution of this equation is

$$\hat{v}^2 = \hat{v}_r^2 - 2\alpha\left(\hat{\lambda} - \hat{\lambda}_r + \eta\kappa \ln\left(\frac{\hat{\lambda} - \eta\kappa}{\hat{\lambda}_r - \eta\kappa}\right)\right), \tag{4.2}$$

where $(\hat{v}_r, \hat{\lambda}_r)$ denotes any fixed reference point.

For $\kappa > 0$, (4.1) has a unique critical point at $(0, 0)$. The eigenvalues and corresponding eigenvectors are

$$\rho_{\pm} = \pm\nu(\alpha\eta\kappa)^{1/2}, \qquad \mathbf{V}_{\pm} = \begin{pmatrix} 1 \\ \pm(\eta\kappa/\alpha)^{1/2} \end{pmatrix}.$$

Thus the critical point is a saddle.

The phase plane structure for (4.1) is shown in Fig. 4.1. Note that the horizontal line $\hat{\lambda} = \eta\kappa$ is a phase curve for (4.1), as well as a horizontal

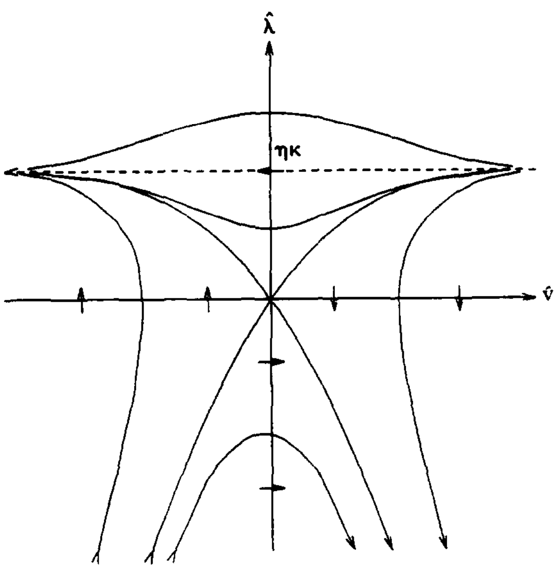

FIGURE 4.1

asymptote for all nearby phase curves. This line corresponds to the $\lambda = 1$ line of the original vector field (2.15) (although the location of the $\lambda = 1$ line is perturbed slightly from $\eta\kappa$). The region above the $\hat{\lambda} = \eta\kappa$ line is non-physical.

We may use (4.2) to eliminate $\hat{v}$ from the $\hat{\lambda}_y$ Eq. in (4.1), obtaining

$$\hat{\lambda}_y = \operatorname{sgn}(\hat{v}_r)\nu(\hat{\lambda} - \eta\kappa)\left[\hat{v}_r^2 - 2\alpha\left(\hat{\lambda} - \hat{\lambda}_r + \eta\kappa \ln\left(\frac{\hat{\lambda} - \eta\kappa}{\hat{\lambda}_r - \eta\kappa}\right)\right)\right]^{1/2}.$$

The solution is

$$y = \operatorname{sgn}(\hat{v}_r)\int_{\hat{\lambda}_r}^{\hat{\lambda}}\left[\hat{v}_r^2 - 2\alpha\left(-\hat{\lambda}_r + \eta\kappa \ln\left(\frac{s - \eta\kappa}{\hat{\lambda}_r - \eta\kappa}\right)\right)\right]^{-1/2} \frac{ds}{\nu(s - \eta\kappa)}. \tag{4.3}$$

The Plane Wave Limit

When $\kappa = 0$ the vector field (4.1) becomes

$$\begin{pmatrix} \hat{v} \\ \hat{\lambda} \end{pmatrix}_y = \nu\hat{\lambda}\begin{pmatrix} -\alpha \\ \hat{v} \end{pmatrix}. \tag{4.4}$$

The factor of $\hat{\lambda}$ in the vector field creates a continuum of critical points on the $\hat{\lambda} = 0$ axis. This corresponds to the $\lambda = 1$ stationary manifold that we observed in Section 3 for the plane wave. Let $v_c \geq 0$. The linear part of (4.4) at $(\pm v_c, 0)$ is

$$\mathbf{A} \equiv \nu\begin{pmatrix} 0 & -\alpha \\ 0 & \pm\hat{v}_c \end{pmatrix}.$$

There is a double zero eigenvalue at the origin, and only one eigenspace (the $\hat{v}$ axis). This is a double zero bifurcation point, about which more will be said later. When $\hat{v} = \hat{v}_c$ (resp. $-\hat{v}_c$), there is one zero eigenvalue, with the $\hat{v}$ axis as corresponding eigenspace, and one positive (resp. negative) eigenvalue with a corresponding unstable (resp. stable) separatrix solution. These are simple zero bifurcation points which vanish for κ positive. The $\hat{v}$ axis is the common center manifold for the bifurcation points.

If the factor of $\hat{\lambda}$ is removed from the vector field (4.4) the resulting modified vector field has the same phase curves as the original field except along $\hat{\lambda} = 0$. This modified field has no critical points and possesses a continuous structurally stable flow. The phase flow of (4.4) thus consists of a line of critical points superimposed over a continuous one-parameter family of phase curves. Setting $\kappa = 0$ in (4.2) yields the phase curve equation

$$\hat{\lambda} = -(2\alpha)^{-1}\hat{v}^2 - \beta, \tag{4.5}$$

where

$$\beta \equiv \hat{v}_r^2 + 2\alpha\hat{\lambda}_r.$$

The phase curves are thus a family of parabolas, symmetric about the $\hat{\lambda}$ axis and concave downward. When $\beta > 0$, the phase curve is a separatrix for the critical points at $(\pm\hat{v}_c, 0)$, where $\hat{v}_c \equiv |\beta|^{1/2}$. When $\beta < 0$, the vertex of the parabola lies below the $\hat{v}$ axis. The $\beta = 0$ phase curve is tangent to the $\hat{v}$ axis at the origin. It is this tangency that produces the second zero eigenvalue at the origin. The phase plane structure of (4.4) is illustrated in Fig. 4.2.

Equation (4.5) may be combined with the first of equations (4.4) to obtain an ordinary differential equation

$$\hat{v}_y = 1/2\nu(\hat{v}^2 - \beta)$$

for $\hat{v}(y)$. This equation may be integrated and the solution substituted

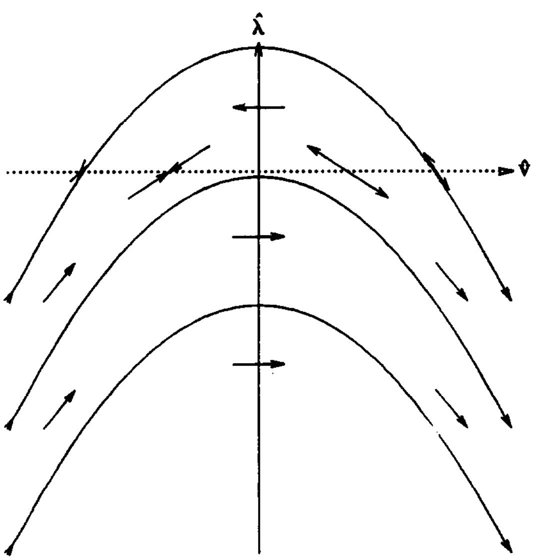

FIGURE 4.2

back into (4.5) to yield $\hat{\lambda}(y)$. The results are

$$\hat{v} = \begin{cases} -\hat{v}_c \tanh(\nu \hat{v}_c y/2 + \phi_+), & \beta > 0, |\hat{v}_r| < \hat{v}_c \\ -\hat{v}_c \coth(\nu \hat{v}_c y/2 + \phi_+), & \beta > 0, |\hat{v}_r| > \hat{v}_c \\ \dfrac{\hat{v}_r}{1 - \hat{v}_r \nu y/2}, & \beta = 0 \\ \hat{v}_c \tan(\nu \hat{v}_c y/2 + \phi_-), & \beta < 0 \end{cases}$$

$$\hat{\lambda} = \begin{cases} \hat{\lambda}_r \left(\dfrac{\cosh(\phi_+)}{\cosh(\nu \hat{v}_c y/2 + \phi_+)} \right)^2, & \beta > 0, |\hat{v}_r| < \hat{v}_c \\ \hat{\lambda}_r \left(\dfrac{\sinh(\phi_+)}{\sinh(\nu \hat{v}_c y/2 + \phi_+)} \right)^2, & \beta > 0, |\hat{v}_r| > \hat{v}_c \\ \dfrac{\hat{\lambda}_r}{(1 - \hat{v}_r \nu y/2)^2}, & \beta = 0 \\ \hat{\lambda}_r \left(\dfrac{\cos(\phi_-)}{\cos(\nu \hat{v}_c y/2 + \phi_-)} \right)^2, & \beta < 0, \end{cases} \tag{4.6}$$

where

$$\phi_+ = \begin{cases} \tanh^{-1}(-\hat{v}_r/\hat{v}_c), & |\hat{v}_r| < \hat{v}_c \\ \coth^{-1}(-\hat{v}_r/\hat{v}_c), & |\hat{v}_r| > \hat{v}_c \end{cases}$$

$$\phi_- = \tan^{-1}(\hat{v}_r/\hat{v}_c).$$

We end this section with some observations about the double zero bifurcation point at the origin of (4.4). There are two choices of resonant terms for **A** in terms of standard basis vectors. They are

$$\begin{pmatrix} \hat{v}^n \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \hat{v}^n \end{pmatrix}$$

and

$$\begin{pmatrix} 0 \\ \hat{v}^n \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \hat{v}^{n-1}\hat{\lambda} \end{pmatrix}.$$

Choosing the latter, we find that the only resonant term in (2.15) at $\kappa = 0$ are $(0, \hat{\lambda}\hat{v})^{\mathrm{T}}$, and $(0, \hat{\lambda}\hat{v}^2)^{\mathrm{T}}$. Only the second degree term is retained in the proposed normal form (4.4). Note that the resonant terms $(0, \hat{v}^n)^{\mathrm{T}}$ are missing from (4.7) for all n. As a consequence of this degeneracy the bifurcation has infinite codimension: the number of distinct topological equivalence classes which may be obtained by a small perturbation of the plane wave vector field is infinite. This exceptional degeneracy is exhibited as the line of critical points. To further emphasize this point, consider the perturbed field

$$\begin{pmatrix} \hat{v} \\ \hat{\lambda} \end{pmatrix}_y = \nu\hat{\lambda}\begin{pmatrix} -\alpha \\ \hat{v} \end{pmatrix} + \begin{pmatrix} 0 \\ f(\hat{v}) \end{pmatrix},$$

with $f(\hat{v}) = \exp(-\hat{v}^{-2})\sin(\hat{v}^{-1})$. The perturbation term is smooth and smaller than any power of $\hat{v}$. The line of critical points breaks up into an infinite sequence of critical points converging on the origin.

A presentation of the nondegenerate case, in which both second degree resonant terms are present, may be found in [10]. The case of second-order degeneracy occurs in models of chemical reactors [14].

5. The Sonic Critical Point

In this section we apply the Poincaré transformations described in the Appendix to facilitate our study of the phase plane structure of (2.15) in a neighborhood of the sonic bifurcation point. We will take $R(\lambda, T)$ to be first-order Arrhenius. We will temporarily ignore the functional relationship between κ and D defined by the shooting problem described in Section 2, and consider κ and D to be independent parameters of the system. It will be seen that this is sufficient to determine the topological structure of the bifurcation point.

In order to carry out the transformations in a way which preserves the correct dependence on κ we employ the standard trick of defining an *augmented* system which consists of (2.15) together with a third equation $d\kappa/dy = 0$. We then expand the augmented system in a Taylor series about the origin $(\hat{v}, \hat{\lambda}, \kappa) = (0, 0, 0)$ and perform the simplifying nonlinear transformations as indicated in Section 3, while allowing only transformations that leave κ invariant.

In what follows we will use Ω to denote a neighborhood of the origin in the $\hat{v}, \hat{\lambda}$ plane. It will be necessary at several points to restrict $(\hat{v}, \hat{\lambda}, \kappa)$ to some cylinder $\Omega_{\max} \times [0, \kappa_{\max}]$ to obtain a desired result. For notational simplicity we will let $\kappa_{\max}$ and $\Omega_{\max}$ denote the minimum over all such restrictions. Let $\Pi_{m,n}$ denote the class of real analytic functions on $U \subset R^m$ which are $O(|\mathbf{w}|^n)$ for $\mathbf{w} \in U$, where U is any neighborhood of the origin.

PROPOSITION 5.1. *For each*

$$\bar{p}_i \in \Pi_{3,2}, \qquad i \in \{1,2\},$$
$$\bar{q}_i \in \Pi_{2,2}, \qquad i \in \{1,2\},$$
$$a_1, a_2, b_1 > 0,$$
$$\bar{a}_j \in R, \qquad j \in \{3,\dots,6\},$$
$$\bar{b}_k \in R, \qquad k \in \{2,3,4\},$$

there exist $p_i \in \Pi_{3,2}$, $q_i \in \Pi_{2,2}$, *and* $b_k \in R$, $k \in \{2,3\}$ *such that the system*

$$\begin{aligned}
\hat{v}_y &= -a_1\hat{\lambda} - a_2\kappa - \bar{a}_3\hat{v}\hat{\lambda} + \bar{a}_4\kappa\hat{v} + \bar{a}_5\kappa\hat{\lambda} + \bar{a}_6\kappa^2 \\
&\quad + \kappa\bar{p}_1(\hat{v},\hat{\lambda},\kappa) + \hat{\lambda}\bar{q}_1(\hat{v},\hat{\lambda}) \\
\hat{\lambda}_y &= b_1\hat{v}\hat{\lambda} + \bar{b}_2\kappa\hat{\lambda} + \bar{b}_3\kappa^2 - \bar{b}_4\hat{\lambda}^2 + \kappa\bar{p}_2(\hat{v},\hat{\lambda},\kappa) + \hat{\lambda}\bar{q}_2(\hat{v},\hat{\lambda}) \\
\kappa_y &= 0
\end{aligned}$$

is smoothly equivalent to the system

$$\begin{aligned}
\hat{v}_y &= -a_1\hat{\lambda} - a_2\kappa + \kappa p_1(\hat{v},\hat{\lambda},\kappa) + \hat{\lambda} q_1(\hat{v},\hat{\lambda}) \\
\hat{\lambda}_y &= b_1\hat{v}\hat{\lambda} + b_2\kappa\hat{\lambda} + b_3\kappa^2 + \kappa p_2(\hat{v},\hat{\lambda},\kappa) + \hat{\lambda} q_2(\hat{v},\hat{\lambda}) \\
\kappa_y &= 0
\end{aligned} \tag{5.1}$$

at the origin.

If the augmented system is expanded in a Taylor series about the point

$$(v_b, \lambda_b, \kappa_b) \equiv \left(\left(v_{CJ}^2 + \mu^2\left(D_{CJ}^2 - D^2\right)\right)^{1/2}, 1, 0\right),$$

it has the form of the first system in Proposition 5.1, providing $0 < v_b < D$. When $D = D_{CJ}$, this condition reads $0 < v_{CJ} < D_{CJ}$. This is satisfied by the plane wave equations (2.5). We may extend this result by continuity to more general D providing D is restricted to a sufficiently small neighborhood $(D_{\min}, D_{\max})$ of D_{CJ}. The values of the coefficients a_i, b_i for our system are

$$\begin{aligned}
a_1 &= (\gamma - 1)kqv_b \\
a_2 &= v_b^3(D - v_b)\exp(A\gamma/v_b^2) \\
b_1 &= 2\gamma k v_b \\
b_2 &= \exp(A\gamma/v_b^2)(2\gamma(D - v_b)[v_b - A(\gamma - 1)]v_b^3) \\
b_3 &= \frac{v_b^2(D - v_b)}{(\gamma - 1)kq}\exp\left(\frac{2A\gamma}{v_b^2}\right) \\
&\quad \times\left(2\gamma(D - v_b)\left[v_b^2 - A(\gamma - 1)\right] + v_b^2(4v_b - 3D)\right).
\end{aligned}$$

The coefficients a_1, a_2, and b_1 are positive. The coefficients b_2 and b_3 may be positive, negative, or zero. We point out that the coefficients depend analytically on $D \in (D_{\min}, D_{\max})$. The transformations leading to (5.1) are analytic, uniformly in $D \in (D_{\min}, D_{\max})$, and thus preserve analytic dependence on D. Thus the p_i, q_i terms must depend analytically on D. Proposition 5.1 is obtained by constructing a sequence of nonlinear transformations of the vector field that remove qualitatively insignificant quadratic terms. These calculations were performed with the aid of Macsyma. The calculations are summarized in the Appendix. The third equation in (5.1) is no longer needed and will be discarded.

A phase plane analysis yields the next result.

PROPOSITION 5.2. *If $p_i = q_i \equiv 0$ in* (5.1), *the system possesses the unique critical point*

$$\begin{aligned}
\hat{v}_c &= \kappa\frac{a_1b_3 - a_2b_2}{a_2b_1} \\
\hat{\lambda}_c &= -\kappa\frac{a_2}{a_1}.
\end{aligned} \tag{5.2}$$

At the critical point, the linear part

$$A_0(\kappa) = \begin{pmatrix} 0 & -a_1 \\ -\dfrac{a_2b_1}{a_1}\kappa & \dfrac{a_1b_3}{a_2}\kappa \end{pmatrix}$$

from (5.1) *has eigenvalues and eigenvectors given by*

$$\rho_{\pm}=\frac{a_1 b_3 \kappa \pm \left((a_1 b_3 \kappa)^2+4a_2^3 b_1 \kappa\right)^{1/2}}{2a_2}$$

$$\mathbf{V}_{\pm}=\begin{pmatrix}-a_1\\ \rho_{\pm}\end{pmatrix}. \tag{5.3}$$

It is clear from (5.3) and the signs of the coefficients a_2 and b_1 that the critical point in Proposition 5.2 is a saddle for all $b_3 \in R$, $D \in (D_{\min}, D_{\max})$. The Hartman–Grobman theorem [10] tells us that all saddle points are topologically equivalent. Consequently for each fixed κ the system (5.1), with p_i and q_i set to zero, is topologically equivalent to the system with b_3 also set to zero. Although the term controlled by b_3 is resonant and cannot be removed by a polynomial change of variables, a more general topological equivalence may indeed remove this term. We thus group the b_3 term with the perturbation terms.

We next prove the existence of a critical point for the system (5.1) for κ sufficiently small, which converges smoothly to the origin as $\kappa \to 0$, uniformly in D (Proposition 5.3). Thus translation of the critical point of 5.1 to the origin constitutes a smooth equivalence transformation. We also show that the saddle point structure persists under the perturbation.

PROPOSITION 5.3. *Let $p_i \in \Pi_{3,2}$, and $q_i \in \Pi_{2,2}$. For $\kappa_{\max}$ sufficiently small the system* (5.1) *possesses a critical point*

$$\left(\hat{v}_c(\kappa, D), \hat{\lambda}_c(\kappa, D)\right) \in C^{\infty}\left([0, \kappa_{\max}] \times (D_{\min}, D_{\max}), R^2\right),$$

corresponding through first order in κ to (5.2). *There is a neighborhood $\Omega_{\max}$ of the origin and a $\kappa_{\max}$ such that, for $0 < \kappa \leq \kappa_{\max}$, this critical point is unique in $\Omega_{\max}$, uniformly in $D \in (D_{\min}, D_{\max})$.*

Proof. The critical points $(\hat{v}_c, \hat{\lambda}_c)$ of (5.1) are defined by setting the right-hand sides of (5.1) to zero. By the Implicit Function Theorem, the first of these equations defines $\hat{\lambda}_c$ as an analytic function of $\hat{v}_c$, D, and κ in a neighborhood of the point $\hat{v} = \hat{\lambda} = \kappa = 0$, $D = D_{CJ}$, provided $a_1 \neq 0$. It is clear from the definition of a_1 that it is strictly positive in the region of interest. Substituting the expansions

$$\hat{\lambda}_c = \hat{\lambda}^0(\hat{v}) + \kappa\hat{\lambda}^1(\hat{v}) + \cdots$$

$$p_1 = p_1^0(\hat{v}, \hat{\lambda}) + \kappa p_1^1(\hat{v}, \hat{\lambda}) + \cdots$$

in powers of κ into the first equation yields

$$\hat{\lambda}_c = \kappa \frac{p_1^0(\hat{v},0) - a_2}{a_1 - q_1(\hat{v},0)} + O(\kappa^2),$$

which is well defined for $\Omega_{\max}$ sufficiently small. Now substitute this result, and a similar expansion for p_2 into the second critical point equation to obtain an implicit equation for $\hat{v}_c(\kappa)$:

$$\kappa\left(\frac{p_1^0(\hat{v}_c,0) - a_2}{a_1 - q_1(\hat{v}_c,0)} b_1\hat{v}_c + p_2^0(\hat{v}_c,0)\right) + O(\kappa^2) = 0.$$

The solutions $\kappa = 0$, $\hat{v}_c$ undetermined correspond to the line of critical points discussed in the previous section. Dividing by the common factor of κ, we obtain an equation with the solution $\hat{v}_c = 0$ at $\kappa = 0$. This is the bifurcation point. Observe that since $p_i^0(\hat{v},0), q_i(\hat{v},0) = O(\hat{v}^2)$, the $\kappa = 0$ equation has the form

$$-\hat{v}_c\left(\frac{a_2 b_1}{a_1} + O(\hat{v}_c)\right) = 0.$$

In a sufficiently small neighborhood of the origin, the second factor is non-zero and the $\hat{v}_c = 0$ solution is unique. By the Implicit Function Theorem $\hat{v}_c$ is a single-valued smooth function of κ in a neighborhood of $\kappa = 0$ providing $-a_2 b_1/a_1$ is nonzero. This follows from (5.1). This solution is unique in some neighborhood $\Omega_{\max}$ of the origin. Solving for the leading coefficients in the Taylor series expansion of $\hat{v}_c(\kappa)$ we obtain agreement through first order in κ to (5.2). This result can be substituted back into the expansion for $\hat{\lambda}_c(\kappa)$ to obtain first-order agreement for $\hat{\lambda}_c$ as well. This completes the proof.

PROPOSITION 5.4. *Let $p_i \in \Pi_{3,2}$ and $q_i \in \Pi_{2,2}$. For $\kappa_{\max}$ sufficiently small the critical point in Proposition* 5.3 *is a saddle point.*

Proof. We showed in the previous proposition that the critical point $(\hat{v}_c, \hat{\lambda}_c)(\kappa)$ is $O(\kappa)$ to lowest order. The perturbation terms p_i are second order in $\hat{v}$, $\hat{\lambda}$, and κ, so when evaluated at the critical point they are $O(\kappa^2)$. The linear part of the perturbed system at the critical point thus has the form

$$\begin{pmatrix} \hat{v} \\ \hat{\lambda} \end{pmatrix}_y = \left(A_0(\kappa) + \kappa^2 B(\kappa)\right) \cdot \begin{pmatrix} \hat{v} \\ \hat{\lambda} \end{pmatrix} \equiv A(\kappa) \cdot \begin{pmatrix} \hat{v} \\ \hat{\lambda} \end{pmatrix}, \tag{5.4}$$

where $B(\kappa)$ is a smooth matrix function. The critical point is a saddle if and only if the determinant of $A(\kappa)$ is negative. This determinant has the form $-\kappa a_2 b_1 + O(\kappa^2)$, which is negative for κ small but positive, proving the proposition.

Denote the critical point of (5.1) by $\mathbf{w}(\kappa)$. Then the translation $\mathbf{T}$ defined by $\mathbf{T}(\mathbf{w}, \kappa) \equiv \mathbf{w} + \mathbf{w}(\kappa)$ is a global diffeomorphism mapping the origin onto the critical point of (5.1). From Propositions 5.2 and 5.3 we know that $\mathbf{T}$ depends smoothly on κ. If we denote the vector field (5.1) by $\mathbf{G}(\mathbf{w}, \kappa)$ and define the translated field $\mathbf{g}(\mathbf{w}, \kappa) \equiv \mathbf{G}(\mathbf{T}(\mathbf{w}, \kappa), \kappa)$, we have the commutation relations

$$\mathbf{T}\left(\phi_y^{\mathbf{G}}(\mathbf{w}), \kappa\right) = \phi_y^{\mathbf{g}}(\mathbf{T}(\mathbf{w}, \kappa))$$

between the translated and untranslated phase flows. We thus have the following result.

THEOREM 5.5. *After translation to the unique critical point in* $\Omega_{\max}$ *given by Proposition* 5.3, *the vector field* (5.1) *has the form*

$$\begin{pmatrix} \hat{v} \\ \hat{\lambda} \end{pmatrix}_y = \nu \begin{pmatrix} -\alpha\hat{\lambda} + \kappa p_1(\hat{v}, \hat{\lambda}, \kappa) + \hat{\lambda} q_1(\hat{v}, \hat{\lambda}) \\ \hat{v}\hat{\lambda} - \eta\kappa\hat{v} + \omega\kappa\hat{\lambda} + \kappa p_2(\hat{v}, \hat{\lambda}, \kappa) + \hat{\lambda} q_2(\hat{v}, \hat{\lambda}) \end{pmatrix}, \quad (5.5)$$

where

$$\nu = b_1, \qquad \alpha = a_1/b_1,$$
$$\eta = a_2/a_1, \qquad \omega = \alpha b_3/b_1,$$
$$q_i \in \Pi_{2,2}, \qquad p_i(\mathbf{w}, \kappa) = O(\kappa|\mathbf{w}| + w^2)$$

The coefficients α, ν, and η are identical to those introduced in Section 4.

Proof. The coefficients are obtained by explicit calculation from the critical point derived in Proposition 5.3. The conditions on p_1 and p_2 follow from the requirement that the critical point of (5.5) be at the origin for all $\kappa \in [0, \kappa_{\max}]$. This proves the theorem.

6. PROOF OF THEOREM 3.1

In Section 4 we have identified the phase plane structure of (2.15) in a neighborhood of the plane wave sonic bifurcation point. In the present section we prove Theorem 3.1. In the course of the proof we will exhibit the topological structure of the solutions for small κ in the physically relevant region of the phase plane. Define

$$\psi(v, \lambda, \kappa, D) \equiv q(\gamma - 1)k(1 - \lambda) - \kappa(D - v)c^2 \exp(A\gamma/c^2), \quad (6.1)$$

where c^2 is given by (2.16). The vector field (2.15) then becomes

$$\begin{aligned} v_y &= v\psi(v, \lambda, \kappa, D) \\ \lambda_y &= k(1 - \lambda)(c^2 - v^2). \end{aligned} \quad (6.2)$$

As always, we assume that q, c_a, k, D, κ, and A are positive and that $\gamma > 1$. The function ψ is analytic providing we avoid a vacuum state $c^2 = 0$. The set of possible vacuum states in the v, λ plane is determined by setting $c^2 = 0$ in Bernoulli's law (2.9), and will be denoted the *vacuum locus*. The result is

$$2q\mu^2(1 - \lambda) = -\mu^2 v^2 + v_b^2.$$

The equation

$$2q\mu^2(1 - \lambda) = -v^2 + v_b^2$$

defining the sonic locus $v^2 = c^2$ is also a parabola. The sonic and vacuum loci possess the same vertex $(0, 1 - v_b^2/2q\mu^2)$ and axis of symmetry (λ axis). From (2.6) we find that $v_{CJ} > 2q\mu^2$, consequently $1 - v_b^2/2q\mu^2 < 0$ for D in a sufficiently small neighborhood $(D_{\min}, D_{\max})$ of D_{CJ}, so that the vertex lies below the v axis. The right intercept with the $\lambda = 1$ line for the sonic locus is v_b. Note that

$$D^2 - v_b^2 = D_{CJ}^2 - v_{CJ}^2 + \frac{2}{\gamma + 1}\left(D^2 - D_{CJ}^2\right).$$

From (2.6) we have

$$\begin{aligned} D_{CJ}^2 - v_{CJ}^2 &= \frac{2}{\gamma + 1}\left(D_{CJ}^2 - c_a^2 - (\gamma - 1)q\right) \\ &= \frac{2}{\gamma + 1}\left(\gamma(\gamma - 1)q + \left(\left((\gamma^2 - 1)qc_a^{-2} + 1\right)^2 - 1\right)^{1/2} c_a^2\right) > 0, \end{aligned}$$

so that $D - v_b > 0$ for D sufficiently close to D_{CJ}. This means that the right branch of the sonic locus up to $\lambda = 1$ lies to the left of the line $v = D$. The vacuum locus is broader than the sonic locus. In fact, from

(2.9) we see that $c^2 > 0$ is equivalent to

$$q(\gamma - 1)\lambda + c_a^2 + \frac{\gamma - 1}{2}(D^2 - v^2) > 0.$$

This inequality holds in a neighborhood of the rectangle

$$\Sigma(D) = \{(v, \lambda) : 0 \le v \le D, 0 \le \lambda \le 1\}.$$

The function ψ is thus analytic on a neighborhood of $\Sigma(D)$, and we will now restrict our attention to this compact set.

The flow velocity $v = D$ of corresponds to a stagnation point $u = 0$ in the original Newtonian frame of reference. The critical point $(D, 1)$ is an artifact of the stagnation point that occurs at the center of the spherical detonation.

Our next objective is to identify all of the critical points of (6.2) in $\Sigma(D)$. This is accomplished by considering the zeros of each of the factors of (6.2), and cataloging the relevant common solutions. When $\kappa = 0$, the $\lambda = 1$ line becomes a manifold of critical points, each possessing at least one zero eigenvalue. There are no other critical points at $\kappa = 0$. As pointed out in Section 2, the λ axis and the line $\lambda = 1$ are fixed phase curves which meet at a fixed critical point $(v, \lambda) = (0, 1)$, which is the unique common zero of the v and $1 - \lambda$ factors of (6.2). The identity $\psi(D, 1, \kappa, D) \equiv 0$ results in a fixed critical point at $(D, 1)$. For $\kappa > 0$, this is the unique common zero of the ψ and $(1 - \lambda)$ factors. For positive κ, the critical point $(0, 1)$ is a sink (two negative eigenvalues), and the critical point $(D, 1)$ is a source. The factors v and $c^2 - v^2$ have no common zeros in $\Sigma(D)$. As shown in the previous section, there is a saddle point on the sonic locus for positive κ, which converges smoothly to the sonic bifurcation point $(v_b, 1)$ as $\kappa \to 0$. The saddle point is at an intersection of the solution of $\psi = 0$ with the sonic locus. We claim that for κ sufficiently small, these three critical points are the only critical points of (6.2) in $\Sigma(D)$. It is clear that any additional critical point must result from another common solution of $\psi = 0$ and $c^2 = v^2$. At $\kappa = 0$, the equation $\psi(v, \lambda, \kappa, D) = 0$ has the unique solution $\lambda = 1$, independent of v and D. More precisely, for each $D_1 \in (D_{\min}, D_{\max})$, and each $v_1 \in [0, D_1]$, we have a solution $\psi(v_1, 1, 0, D_1) = 0$. Since ψ is smooth on $\Sigma(D)$, we may apply the Implicit Function Theorem at each of these solutions to show the existence of a unique smooth local solution $\lambda = f(v, \kappa, D)$ in a neighborhood of $(v_1, 0, D_1)$ satisfying $f(v_1, 0, D_1) = 1$. We have from (6.1),

$$\frac{d\psi}{d\lambda}(v_1, 1, 0, D_1) = -q(\gamma - 1)k,$$

which is strictly nonzero, independent of (and therefore uniformly in) v_1, D_1. This shows that the solution for $\kappa > 0$ is a smooth curve, deformed slightly from $\lambda = 1$. This is the unique solution in a neighborhood

$$B_\delta \equiv \{(v,\lambda): 0 \le v \le D, 1-\delta < \lambda < 1+\delta\}$$

of the $\lambda = 1$ boundary. Define $\Sigma_\delta \equiv \Sigma(D) \setminus B_\delta$. We claim that for κ sufficiently small, no additional branches of $\psi = 0$ are created in $\Sigma(D)$. Specifically, we shall show that for every $\delta > 0$ there is a $\kappa_{\max} > 0$ such that ψ is non-vanishing on the compact set $K(\delta, \kappa_{\max})$ defined by

$$K(\delta, \kappa_{\max}) \equiv \Sigma_\delta \times [0, \kappa_{\max}] \times [D_{\min}, D_{\max}].$$

Fix $\delta > 0$. Since $\lambda = 1$ is the unique solution of $\psi = 0$ at $\kappa = 0$, ψ is non-zero on $K(\delta, 0)$). By the continuity of ψ, there is a neighborhood N_δ of $K(\delta, 0)$ such that ψ is non-zero on N_δ. Since $K(\delta, 0)$ is compact we can choose N_δ to be bounded. Let

$$\kappa_{\max} \equiv 1/2 \inf_{p \in \partial N_\delta, q \in K(\delta, 0)} |p - q|.$$

Because $K(\delta, 0)$ is compact and N_δ is open, the infinum is nonzero. Then $K(\delta, \kappa_{\max}) \subset N_\delta$, so that ψ is non-vanishing on $K(\delta, \kappa_{\max})$.

Differentiating (6.1) implicitly with respect to κ we find

$$f_\kappa(v, 0, D) = -\frac{(D-v)c^2}{q(\gamma-1)k} \exp\left(\frac{A\gamma}{c^2}\right),$$

which is negative for $v < D$ and positive for $v > D$, with a simple zero at $v = D$. Thus the $\psi = 0$ curve enters $\Sigma(D)$ through the $v = 0$ boundary with $\lambda < 1$, and exits through the critical point $v = D$, $\lambda = 1$. Note that any two points in $\Sigma(D)$ on the $\psi = 0$ curve are connected by that curve in $\Sigma(D)$. The same is true for two points on the sonic locus. For κ sufficiently small, the $\psi = 0$ curve must cross the sonic locus at least once. Now the slope f_v may be bounded in an arbitrarily small neighborhood of zero by restricting κ. The slope of the sonic locus in $\Sigma(D)$ is bounded away from zero, since the vertex lies on the negative λ axis, so that the slopes of the $\psi = 0$ curve and the sonic locus in $\Sigma(D)$ are in disjoint intervals, uniformly in κ and D, for sufficiently small κ. Thus the intersection of the two curves is unique in $\Sigma(D)$. This is just an example of the general result that transversality of two smooth curves in the plane is stable under perturbations of the curves.

Note that $\nabla\psi = (0, -q(\gamma - 1)k) + O(\kappa)$, so that $\psi > 0$ below the $\psi = 0$ curve. Now define the sectors $\Sigma_{\pm\pm}$ as follows:

$$\Sigma_{\pm\pm} = \{(v, \lambda) \in \Sigma(D) : \psi > 0, c^2 - v^2 > 0\}$$

$$\Sigma_{+-} = \{(v, \lambda) \in \Sigma(D) : \psi > 0, c^2 - v^2 < 0\}$$

$$\Sigma_{-+} = \{(v, \lambda) \in \Sigma(D) : \psi < 0, c^2 - v^2 > 0\}$$

$$\Sigma_{--} = \{(v, \lambda) \in \Sigma(D) : \psi < 0, c^2 - v^2 < 0\}.$$

The signs of the components of the vector field (7.2) at some point (v, λ) (and therefore the quadrant into which the vector points) are determined by the sector $\Sigma_{\pm\pm}$ containing (v, λ), except possibly at a critical point or at a boundary. We have seen that the only critical point in the closure $\overline{\Sigma}_{++}$ of Σ_{++} is the saddle point at $\psi = c^2 - v^2 = 0$. A phase curve which crosses $\psi = 0$ away from a critical point must do so vertically, i.e., $d\lambda/dv = \pm\infty$. Likewise, a phase curve which crosses the sonic locus at a non-critical point must have zero slope. Further, the sign of the slope of a phase curve which crosses either the $\psi = 0$ curve or the sonic locus at a non-critical point must change, since exactly one of the components of (7.2) reverse sign. Our final observation is that a non-critical intersection of a phase curve with $\psi = 0$ or with the sonic locus in $\Sigma(D)$ is transverse, so that the slope of the phase curve changes sign at the intersection. A non-transverse intersection with the $\psi = 0$ curve can only occur at a point where the curve has a vertical tangent. We have shown that the slope of the $\psi = 0$ curve may be bounded in an arbitrarily small neighborhood of 0 by restricting $\kappa_{\max}$, so that no non-transverse intersections are possible. Likewise, non-transverse intersections with the sonic locus may be excluded because the sonic locus is never horizontal in $\Sigma(D)$. These considerations make possible a classification of the non-critical crossings with the $\psi = 0$ curve and with the sonic locus. Denote by (v_i, λ_i) the coordinates of a non-critical crossing. The possible crossings of a phase curve with the $\psi = 0$ curve are

Case	$\lambda < \lambda_i$	$\lambda > \lambda_i$
$a1$	$v_y < 0, \lambda_y > 0$	$v_y > 0, \lambda_y > 0$
$b1$	$v_y > 0, \lambda_y > 0$	$v_y < 0, \lambda_y > 0$
$c1$	$v_y > 0, \lambda_y < 0$	$v_y < 0, \lambda_y < 0$
$d1$	$v_y < 0, \lambda_y < 0$	$v_y > 0, \lambda_y < 0$

The possible non-critical intersections with the sonic locus are

Case	$v < v_i$	$v > v_i$
$a2$	$v_y > 0, \lambda_y > 0$	$v_y > 0, \lambda_y < 0$
$b2$	$v_y > 0, \lambda_y < 0$	$v_y > 0, \lambda_y > 0$
$c2$	$v_y < 0, \lambda_y < 0$	$v_y < 0, \lambda_y > 0$
$d2$	$v_y < 0, \lambda_y > 0$	$v_y < 0, \lambda_y < 0$

We will show that the stable separatrix of the saddle point has positive slope and intersects the $\lambda = 0$ boundary of $\Sigma(D)$. We will accomplish this in two steps. First we show that the stable eigenspace enters the Σ_{++} sector at the saddle point, and that the unstable eigenspace does not. Then we prove that there are no subsequent intersections of the separatrix with the $\psi = 0$ curve or with the sonic locus. Since Σ_{++} contains no other critical points and since the $v = 0$ boundary of Σ_{++} is a phase curve, the separatrix must then exit through the $\lambda = 0$ boundary.

We claim that for $\kappa > 0$, exactly one separatrix branch enters each of the four sectors. We have shown that the $\psi = 0$ curve is transverse to the sonic locus at the saddle point, so that the sign of the slope of a separatrix which crosses both curves transversely is determined by the sector $\Sigma_{\pm\pm}$ into which the corresponding eigenvector points. Two such eigenvectors which point into the same sector must necessarily have slopes of the same sign. It is therefore sufficient to show that the eigenvectors are not tangent to the sonic locus or to the curve $\psi = 0$, and that one of the eigenvectors has positive slope, and one has negative. Then one separatrix will cross with positive slope from Σ_{++} into Σ_{--}, and the other will cross from Σ_{-+} into Σ_{+-} with negative slope. Let $\mathbf{f}(v, \lambda)$ denote the vector field (6.2), and $\rho, \mathbf{V}$ an eigenvalue and corresponding eigenvector at the saddle point, so that $D\mathbf{f} \cdot \mathbf{V} = \rho \mathbf{V}$. By Proposition 5.3, the slopes of both eigenvectors converge continuously to zero as $\kappa \to 0$, so we may assume that $\mathbf{V}_1 \neq 0$. Since $\mathbf{V}$ is only defined modulo a nonzero factor, we may set $\mathbf{V}_1 = 1$. Eliminating ρ and solving for $\mathbf{V}_2$, we obtain

$$\mathbf{V}_2 = \frac{\mathbf{f}_{2,2} - \mathbf{f}_{1,1} \pm \left((\mathbf{f}_{2,2} - \mathbf{f}_{1,1})^2 + 4\mathbf{f}_{1,2}\mathbf{f}_{2,1}\right)^{1/2}}{2\mathbf{f}_{1,2}}.$$

The two solutions correspond to the two independent eigenvectors. A necessary and sufficient condition for $\mathbf{V}_2$ to have both a positive and a negative solution is $\mathbf{f}_{1,2}\mathbf{f}_{2,1} > 0$. For (6.2) we have

$$\mathbf{f}_{1,2} = -vq(\gamma - 1)k + O(\kappa) < 0,$$

and

$$\mathbf{f}_{2,1} = -(1 - \lambda)(\gamma + 1)/2 < 0,$$

so this condition is satisfied. For $\mathbf{V}$ to be tangent to the sonic locus we must have $\mathbf{V} \cdot \nabla \mathbf{f}_2 = 0$. After eliminating $\mathbf{V}$ this becomes $\mathbf{f}_{1,2}\mathbf{f}_{2,1} - \mathbf{f}_{1,1}\mathbf{f}_{2,2} = 0$, or $\det(D\mathbf{f}) = 0$, which is impossible at a hyperbolic critical point. The same result holds if the sonic locus is replaced by the $\psi = 0$ curve, so the separatrixes are transverse to both curves. Thus the slopes of the separatrixes near the saddle point are given by the sector $\Sigma_{\pm\pm}$ into which the eigenvector points. This means that for each fixed $\kappa > 0$ there is a neighborhood of the saddle point in which one separatrix branch lies in each of the four sectors.

We next show that the separatrix branch in Σ_{++} connects with the $\lambda = 0$ boundary of $\Sigma(D)$, with the slope $d\lambda/dv = \mathbf{f}_2/\mathbf{f}_1$ of the phase curve everywhere positive. We have demonstrated above that the separatrix leaves the saddle point with positive slope. Since $v = 0$ is a phase curve, and since Σ_{++} contains no other critical points, we need only show that the separatrix does not intersect the $\psi = 0$ curve or the sonic locus at a non-critical point in $\Sigma(D)$. We will assume that there is a non-critical intersection of the separatrix with $\psi = 0$, or with the sonic locus, and arrive at a contradiction. Proceeding in the negative y direction from the saddle point into Σ_{++}, there is a first intersection. Assume that the first intersection is with $\psi = 0$. (The argument is identical for the sonic locus.) This intersection must have one of the forms $a1$–$d1$. Since both v_y and λ_y are positive in Σ_{++}, we may eliminate cases $c1$ and $d1$. In case $b1$ the phase curve meets the intersection in the positive y direction from Σ_{++}. Since we must meet the intersection in the negative y direction, we may eliminate this case as well, which leaves us with case $a1$. The slope of the $\psi = 0$ curve may be bounded in an arbitrarily small neighborhood of 0. Since the phase curve $\lambda = \phi(v)$ must intersect with infinite slope, we see from the definition of case $a1$ that there is a neighborhood of the intersection in which $\phi(v) > \psi(v)$ on the Σ_{++} side of the $\psi = 0$ curve. However, $\psi = 0$ is the upper boundary of Σ_{++}. More precisely, let (v_1, λ_1) be a point in the interior of Σ_{++}. The vertical line $v = v_1$ intersects $\psi = 0$ at a unique point $\lambda_2 = \psi(v_1)$, where $\lambda_1 < \lambda_2$. Since the intersecting phase curve is smooth, it is locally approximated by its vertical tangent line, and must satisfy $\phi(v) < \psi(v)$ locally. We have our contradiction.

At this point we will return to a consideration of the vector field (2.10), (2.11), with the original independent variable x. As pointed out in Section 2, this field possesses the same phase curves as the continuous field (2.15). In particular, there is a separatrix solution which increases monotonically from the v axis to the sonic critical point. Since the sonic critical point is unique, all other transonic solutions must cross the sonic locus at a non-critical point. At such points the velocity gradient v_x is unbounded, and the solution is non-smooth. We show now that for fixed κ sufficiently small, the vector field (2.10), (2.11) is in fact continuous when restricted to the separatrix, so that a smooth solution $v(x)$ exists, uniformly in $D \in (D_{\min}, D_{\max})$.

In Proposition 5.3 we proved that the sonic critical point (v_c, λ_c) is a smooth function of κ, and that $1 - \lambda_c = O(\kappa)$, uniformly in D in some neighborhood $(D_{\min}, D_{\max})$ of D_{CJ}. We also have the estimate $1/|\mathbf{V}_2| < O(\kappa^{-1/2})$, uniformly in D, for the reciprocal of the eigenvector slope. The limit of the ratio $v_x/\lambda_x = v_y/\lambda_y$ as one approaches the critical point along a separatrix equals $1/\mathbf{V}_2$. Since λ_x is continuous, this means that v_x is continuous along the separatrix if it is bounded there, since the limits from either side must be equal. We have

$$\begin{aligned}
&\lim_{(v,\lambda)\to(v_c,\lambda_c)} v_x \\
&\quad = \frac{1}{v_c} k(1-\lambda_c)\exp\left(\frac{-A\gamma}{c_c^2}\right) \lim_{(v,\lambda)\to(v_c,\lambda_c)} \frac{v_x}{\lambda_x} \\
&\quad = \frac{1}{v_c} k(1-\lambda_c)\exp\left(\frac{-A\gamma}{c_c^2}\right) \lim_{(v,\lambda)\to(v_c,\lambda_c)} \frac{v_y}{\lambda_y} \\
&\quad \le \kappa^{1/2} C(\kappa, D),
\end{aligned}$$

where $C(\kappa, D)$ is bounded, uniformly in $\kappa \in (0, \kappa_{\max}]$, and in $D \in (D_{\min}, D_{\max})$. Hence v_x is bounded along the separatrix and possesses a removable discontinuity at the sonic critical point. This completes the proof of Theorem 3.1.

7. Solution of the Shooting Problem

The goal of the present section is to obtain the leading order term(s) in the asymptotic expansion of (2.17) for small κ, and to solve the shooting problem for $D(\kappa)$ to leading order. We begin by transforming to dimensionless variables. Define

$$\begin{aligned}
&\sigma \equiv (2q)^{1/2}\frac{\mu}{c_{CJ}}, \qquad \zeta \equiv \frac{2c_a^2}{c_{CJ}^2(\gamma+1)}, \\
&\tilde{v} \equiv \frac{v}{c_{CJ}}, \qquad \tilde{D}(\tilde{\kappa}) \equiv \frac{D}{c_{CJ}}, \\
&\tilde{\kappa} \equiv \frac{\kappa c_{CJ}}{k}, \qquad s \equiv \tilde{v}_c - \tilde{v}, \\
&\phi(\tilde{D}) \equiv \left(1 + \mu^2(\tilde{D}^2 - \tilde{D}_{CJ}^2)\right)^{1/2} \\
&\qquad \approx 1 + \mu^2 \tilde{D}_{CJ}(\tilde{D} - \tilde{D}_{CJ}), \\
&f(s, \tilde{\kappa}) \equiv \sigma^2(\lambda_c - \lambda) - s^2.
\end{aligned}$$

It is easily verified from (2.5), (2.6) that $\zeta, \sigma \in (0,1)$ and $\tilde{D} > 1$. The independent and dependent variables s and f are 0 at the sonic free boundary. This formulation avoids unnecessary singularities and facilitates the expansion. The plane wave (2.20) is now simply

$$s^2 = \sigma^2(1 - \lambda),$$

or $f(s,0) \equiv 0$. Now define

$$F(s) \equiv \left.\frac{2(\tilde{D} - \tilde{v})c^2}{(\gamma + 1)\tilde{v}g(T)c_{CJ}^2}\right]_{\tilde{\kappa}=0}.$$

We may express the sonic free boundary equations (2.12), (2.13) as

$$\sigma^2(1 - \lambda_c) = \phi^2(\tilde{D}) - \tilde{v}_c^2$$
$$\sigma^2(1 - \lambda_c)^{\omega} = \tilde{\kappa}F_0,$$

where

$$F_0 \equiv F(0) = \frac{2\sigma}{(\gamma + 1)g\left(c_{CJ}^2\gamma^{-1}\right)}.$$

Expanding $\tilde{v}_c$ and λ_c to leading order in $\tilde{\kappa}$ and $(\tilde{D} - \tilde{D}_{CJ})$ produces

$$\tilde{v}_c = 1 + \mu^2\tilde{D}_{CJ}\left(\tilde{D} - \tilde{D}_{CJ}\right) - \tfrac{1}{2}\alpha\tilde{\kappa}^{1/\omega}$$
$$\lambda_c = 1 - \sigma^{-2}\alpha\tilde{\kappa}^{1/\omega}, \tag{7.1}$$

where $\alpha \equiv \sigma^{2(1-\omega^{-1})}F_0^{1/\omega}$.

We can use (7.1) to verify the validity of the homogeneous curvature approximation. As discussed in Section 2, this entails demonstrating that the width w of the subsonic region is small relative to the shock radius $z = \kappa^{-1}$ when κ is small, or $\kappa w \to 0$ as $\kappa \to 0$. Write

$$a \equiv \min_{\lambda \in [0,1]} \frac{v_0(\lambda)}{kg(T_0(\lambda))} > 0,$$

$$b \equiv \max_{\lambda \in [0,1]} \frac{v_0(\lambda)}{kg(T_0(\lambda))}.$$

Then

$$\min_{\lambda \in [0,1]} \frac{v(\lambda, \tilde{\kappa})}{kg(T(\lambda, \tilde{\kappa}))} = a + o(1),$$

$$\max_{\lambda \in [0,1]} \frac{v(\lambda, \tilde{\kappa})}{kg(T(\lambda, \tilde{\kappa}))} = b + o(1).$$

The subsonic width w is defined by integrating the rate equation (2.11). For sufficiently small $\tilde{\kappa}$ we have

$$(a + o(1))W(\tilde{\kappa}) < w < (b + o(1))W(\tilde{\kappa}),$$

where

$$W(\tilde{\kappa}) \equiv \int_0^{\lambda_c(\tilde{\kappa})} \frac{d\lambda}{(1-\lambda)^{\omega}}.$$

We can estimate $W(\tilde{\kappa})$ with the use of (7.1) to obtain

$$W(\tilde{\kappa}) \approx \left\{ \begin{array}{cc} (1-\omega)^{-1} & \omega < 1 \\ -\ln(\alpha\sigma^{-2}\tilde{\kappa}) & \omega = 1 \\ (\omega - 1)^{-1}\alpha^{1-\omega}\sigma^{2(\omega-1)}\tilde{\kappa}^{\omega^{-1}-1}, & \omega > 1 \end{array} \right\}.$$

We see that w is finite when $\omega < 1$. For $\omega = 1$, w grows at a logarithmic rate as $\tilde{\kappa} \to 0$, so that $\tilde{\kappa}w \to 0$. The fastest rate of growth for w occurs when $\omega > 1$, in which case $\tilde{\kappa}w = O(\tilde{\kappa}^{1/\omega}) \to 0$. This confirms that for all $\omega > 0$, the flow curvature is nearly homogeneous in the subsonic region, and that this approximation becomes exact in the $\tilde{\kappa} \to 0$ limit.

The plane wave solution (2.20) may be combined with the shock condition (2.4) to produce the identities

$$\begin{aligned} \tilde{D}_{CJ} &= 1 + \sigma, \\ \zeta &= \frac{2(1+\sigma)(1-\gamma\sigma)}{\gamma+1}. \end{aligned} \tag{7.2}$$

The shock conditions are then, to leading order in $\tilde{\kappa}$ and $(\tilde{D} - \tilde{D}_{CJ})$,

$$\begin{aligned} s_s &= \sigma - \frac{\alpha}{2}\tilde{\kappa}^{1/\omega} + \beta(\tilde{D} - \tilde{D}_{CJ}) \\ f_s &= -\alpha(1-\sigma)\tilde{\kappa}^{1/\omega} + 2\sigma\beta(\tilde{D} - \tilde{D}_{CJ}), \end{aligned} \tag{7.3}$$

where

$$\beta \equiv \left[\frac{2(1+\sigma)}{\gamma+1} - \sigma - \frac{2\sigma}{(1+\sigma)^2}\right].$$

For the plane wave, $s_s = \sigma$ at the shock. The parameter σ may be interpreted as a measure of the shock strength. When $\sigma = 0$ we have $q = 0$ and $D = c_{CJ}$, and the shock degenerates into a sound wave. For strong shocks, the pressure ahead of the shock is negligibly small relative to the shock pressure. Then approximately, $c_a = \zeta = 0$. In this case, the identities (7.2) give $\sigma = 1/\gamma$, which is also the maximum possible value for σ. Note that $s_s < 1$ if $\tilde{\kappa}$ and $(\tilde{D} - \tilde{D}_{CJ})$ are sufficiently small. We now have

$$F(s) = 2(\sigma + s)\big(1/(\gamma + 1) + \mu^2 s\big)\frac{1}{g(T_0(s))}. \tag{7.4}$$

Expressing Eq. (2.17) in the dimensionless variables and neglecting terms which are both $o(\tilde{\kappa})$ and $o(\tilde{\kappa}^{1/\omega})$, uniformly in $s \in [0, s_s)$ produces

$$\left(s^2 + f + \alpha\tilde{\kappa}^{1/\omega}\right)^{\omega}\frac{d}{ds}\left(\frac{f}{1-s}\right) = \sigma^{2(\omega-1)}\left(2s + \frac{df}{ds}\right)F(s)\tilde{\kappa}. \tag{7.5}$$

We will now solve the shooting problem for the wave speed in two steps. First, we will use (7.5) to obtain the leading term in the expansion of $f(s, \tilde{\kappa})$. The wave speed is then defined by the equation

$$f\big(s_s(\tilde{\kappa}, \tilde{D}), \tilde{\kappa}\big) = f_s(\tilde{\kappa}, \tilde{D}),$$

with s_s and f_s given by (7.3).

As discussed in Section 2, the curvature of the shock affects the solution in two principle ways. First, the shock curvature determines the local divergence of the flow. Second, the location of the sonic free boundary, and thus the boundary values for the wave speed shooting problem, vary with the curvature. These two competing effects are evident in (7.5). The term on the right-hand side is linear in $\tilde{\kappa}$. This term comes directly from the geometrical source term and represents the flow divergence. The terms proportional to $\tilde{\kappa}^{1/\omega}$ in (7.3) and (7.5) come from the displacement of the sonic free boundary and depend intimately on the order of the reaction. A regular expansion of (7.5) requires powers of $\tilde{\kappa}^{1/\omega}$ as well as of $\tilde{\kappa}$. If omega is rational, there exist relatively prime positive integers m and n such that $\tilde{\kappa}^{m/\omega} = \tilde{\kappa}^n$, and the corresponding terms in the expansion may exhibit resonance. At these resonant values of ω the regular expansion fails. Only the fundamental resonance $\omega = m = n = 1$ appears at leading order in the expansion; this case is of theoretical and practical interest and rates a thorough treatment. When $\omega < 1$, then $\tilde{\kappa} < \tilde{\kappa}^{1/\omega}$ and the leading term in the regular expansion is linear in the curvature. When $\omega > 1$, a $\tilde{\kappa}^{1/\omega}$ term will dominate. When ω is close to but not equal to one, the $\tilde{\kappa}^{1/\omega}$ and $\tilde{\kappa}$ terms are of comparable magnitude. We will see that the range of validity (in $\tilde{\kappa}$) of the leading order expansion shrinks to zero as $\omega \to 1$ unless both of these competing terms are included.

The regular expansion exhibits nonuniformities when s^2 in (7.5) is $O(\kappa^{\omega^{-1}})$. An inner expansion will thus be required at the sonic free boundary to complete the solution.

The Sonic Boundary Layer

Define the stretched inner variable $\tilde{s} = \tilde{\kappa}^{-1/2\omega}s$. Equation (7.5) becomes, to leading order

$$\left[\left(f + (\tilde{s}^2 + \alpha)\tilde{\kappa}^{1/\omega}\right)^{\omega} - \alpha^{\omega}\tilde{\kappa}\right]\frac{df}{d\tilde{s}} = 2\alpha^{\omega}\tilde{s}\tilde{\kappa}^{1/\omega}\tilde{\kappa}. \tag{7.6}$$

We seek the leading term(s) in a regular expansion of f which satisfies the boundary condition $f(0) = 0$. In the light of the above discussion only the $\tilde{\kappa}$ and $\tilde{\kappa}^{1/\omega}$ terms are needed. Inserting $f = f_1\tilde{\kappa} + f_2\tilde{\kappa}^{1/\omega}$ + h.o.t. into (7.6) and expanding yields $f_1 = 0$ and

$$\frac{df_2}{d\tilde{s}}\left(\left(\tilde{s}^2 + f_2 + \alpha\right)^{\omega} - \alpha^{\omega}\right) = 2\alpha^{\omega}\tilde{s}.$$

The solution which satisfies the boundary condition is given implicitly by

$$f_2 = \begin{cases} \alpha(1-\omega)^{-1}\left[\left(1 + \alpha^{-1}(f_2 + \tilde{s}^2)\right)^{1-\omega} - 1\right], & \omega \neq 1 \\ \alpha \log\left(1 + \alpha^{-1}(f_2 + \tilde{s}^2)\right), & \omega = 1. \end{cases} \tag{7.7}$$

The inner solution (7.7) must be matched to an appropriate outer solution valid at the shock. Expressing the inner solution in the outer variable and expanding yields

$$f \approx \begin{cases} \alpha(1-\omega)^{-1}\left[\left(s^2\alpha^{-1}\right)^{1-\omega}\tilde{\kappa} - \tilde{\kappa}^{1/\omega}\right], & \omega \neq 1 \\ \alpha \log(s^2\alpha^{-1})\tilde{\kappa} - \alpha\tilde{\kappa}\log\tilde{\kappa}, & \omega = 1. \end{cases} \tag{7.8}$$

Note that a term proportional to $\tilde{\kappa}\log\tilde{\kappa}$ appears at the $\omega = 1$ resonance. This term results from the competition between the $\tilde{\kappa}$ and $\tilde{\kappa}^{1/\omega}$ terms in the regular expansion, and must be accommodated in the outer expansion.

The Wave Speed Calculation

In the outer (shock) region, (7.5) becomes

$$\frac{d}{ds}\left(\frac{f}{1-s}\right) = 2\tilde{\kappa}\sigma^{2(\omega-1)}s^{1-2\omega}F(s) \tag{7.9}$$

to leading order. The form of the solution depends on ω. A resonant form is needed at $\omega = 1$ which includes a $\tilde{\kappa}\log\tilde{\kappa}$ term. Both $\tilde{\kappa}$ and $\tilde{\kappa}^{1/\omega}$ terms are required for ω close but not equal to one. The two-term expansion breaks down at the resonance $\omega = \frac{3}{2}$ ($I(s)$ fails to converge.). For $\omega \leq \frac{1}{2}$

the $\tilde{\kappa}^{1/\omega}$ term is of the same order as the neglected $\tilde{\kappa}^2$ term and may be dropped. Several overlapping regions are required to obtain an expansion which is valid for all $\omega > 0$. We will label these regions (i) $\omega \ll 1$; (ii) $\frac{1}{2} < \omega < \frac{3}{2}$, $\omega \neq 1$; (iii) $\omega = 1$; and (iv) $1 \ll \omega$. The result is

$$\begin{aligned}
&\text{(i)}\ f \approx f_1\tilde{\kappa}, && \tilde{D} \approx \tilde{D}_{CJ} + \tilde{D}_1\tilde{\kappa}\\
&\text{(ii)}\ f \approx f_1\tilde{\kappa} + f_2\tilde{\kappa}^{1/\omega}, && \tilde{D} \approx \tilde{D}_{CJ} + \tilde{D}_1\tilde{\kappa} + D_2\tilde{\kappa}^{1/\omega}\\
&\text{(iii)}\ f \approx f_3\tilde{\kappa} + f_4\tilde{\kappa}\log\tilde{\kappa}, && \tilde{D} \approx \tilde{D}_{CJ} + \tilde{D}_3\tilde{\kappa} + \tilde{D}_4\tilde{\kappa}\log\tilde{\kappa}\\
&\text{(iv)}\ f \approx f_2\tilde{\kappa}^{1/\omega}, && \tilde{D} \approx \tilde{D}_{CJ} + \tilde{D}_2\tilde{\kappa}^{1/\omega}.
\end{aligned} \tag{7.10}$$

The solutions for f consistent with (7.8) are

$$\begin{aligned}
f_1 &= (1-s)\sigma^{2(\omega-1)}\left[F_0(1-\omega)^{-1}s^{2(1-\omega)} + 2I(s)\right],\\
f_2 &= -\alpha(1-\omega)^{-1}(1-s)\\
f_3 &= (1-s)\left[\alpha\log(s^2/\alpha) + 2I(s)\right],\\
f_4 &= -\alpha(1-s),
\end{aligned} \tag{7.11}$$

where

$$I(s) \equiv \int_0^s t^{2(1-\omega)}G(t)\,dt$$

and

$$G(s) \equiv \frac{F(s) - F_0}{s}.$$

Note that $\lim_{s\to 0} G(s) = F'(0)$ is defined, so $G(s)$ is continuous.

The leading order correction to the wave speed may now be found by evaluating $f(s)$ at the shock and using the shock conditions. The result is

$$\begin{aligned}
\tilde{D}_1 &= 1/2(1-\sigma)\sigma^{-1}\beta^{-1}\left[F_0(1-\omega)^{-1} + 2\sigma^{2(\omega-1)}I(\sigma)\right],\\
\tilde{D}_2 &= -1/2\alpha\omega\sigma^{-1}\beta^{-1}(1-\omega)^{-1}(1-\sigma),\\
\tilde{D}_3 &= (1-\sigma)\sigma^{-1}\beta^{-1}\left[1/2\alpha\left(1 + \log(\sigma^2/\alpha)\right) + I(\sigma)\right],\\
\tilde{D}_4 &= -1/2\alpha\sigma^{-1}\beta^{-1}(1-\sigma).
\end{aligned} \tag{7.12}$$

A straightforward application of L'Hopital's rule reveals that the two-term expansion has a removable discontinuity at the $\omega = 1$ resonance with

$$\lim_{\omega\to 1}\left(f_1(s)\tilde{\kappa} + f_2(s)\tilde{\kappa}^{1/\omega}\right) = f_3(s)\tilde{\kappa} + f_4(s)\tilde{\kappa}\log\tilde{\kappa}$$

and

$$\lim_{\omega \to 1} \left(\tilde{D}_1 \tilde{\kappa} + \tilde{D}_2 \tilde{\kappa}^{1/\omega} \right) = \tilde{D}_3 \tilde{\kappa} + \tilde{D}_4 \tilde{\kappa} \log \tilde{\kappa}.$$

The normal form (4.1) provided the first indication of the qualitative structure of the sonic boundary layer and the resonance at $\omega = 1$. The exact solution (4.2) of the normal form is nearly identical to the inner solution (7.1), but is not valid at the shock and cannot be used to solve the shooting problem.

For the special case of a reaction rate which is independent of the temperature ($g(T) \equiv 1$), the integral $I(\sigma)$ may be calculated in closed form. The result is

$$I(\sigma) = \frac{\sigma^{3-2\omega}}{\gamma + 1} \left[\frac{2(1 + \gamma\sigma)}{3 - 2\omega} + \frac{\gamma\sigma}{2 - \omega} \right].$$

The coefficient $\tilde{D}_1$ (case (i)) was calculated by Bdzil and Stewart [3] for a state independent rate ($g \equiv 1$) in the strong shock limit ($\sigma = \gamma^{-1}$).

Appendix

The method we employ to analyse the sonic bifurcation point is to transform the vector field in a neighborhood of the bifurcation point by a smooth change of variables to a field which is topologically equivalent but simpler in form. The study of these simplified fields (or *normal forms*) was initiated by Poincaré. To find such a field, we will take advantage of the analyticity of the vector field by expanding in powers of the deviations $\hat{v}$, $\hat{\lambda}$ from the critical values $(v_b, 1)$. We then seek a diffeomorphism of the phase space that preserves the linear part of the vector field while eliminating as many nonlinear terms as possible. This procedure is customarily carried out order by order; first quadratic terms are eliminated, then cubic terms and so on. In the present case, only the second degree terms are required. The normal form possesses the same local topological structure as the original field, but is much easier to study. This method may also be applied to an unfolding of the bifurcation, so we may speak of a normal form for the unfolding. Excellent introductions to the theory of normal forms for vector fields are available in Arnold [1] and in Guckenheimer and Holmes [10].

Assume that $\mathbf{A}$ is the matrix of the linear part of a two-dimensional analytic vector field $\mathbf{f}(\mathbf{w})$ with a critical point at the origin, so that $\mathbf{f}$ has the form

$$\frac{d\mathbf{w}}{dy} = \mathbf{f}(\mathbf{w}) = \mathbf{A} \cdot \mathbf{w} + \mathbf{f}^{(2)}(\mathbf{w}),$$

where $\mathbf{f}^{(2)} = O(|\mathbf{w}|^2)$. For each integer $n \geq 2$, the linear operator $\mathbf{A}$ induces a linear operator $\mathbf{L_A}$ on the linear space $\Lambda_{2,n}$ of 2-vectors having entries which are homogeneous nth degree polynomials in $\mathbf{w}_1, \mathbf{w}_2$,

$$(\mathbf{L_A} \cdot \mathbf{h})_i \equiv \sum_{j=1}^{2} \frac{\partial \mathbf{h}_i}{\partial \mathbf{w}_j} (\mathbf{A} \cdot \mathbf{w})_j - (\mathbf{A} \cdot \mathbf{h})_i. \tag{8.1}$$

Note that $\mathbf{L_A} \cdot \mathbf{h}$ is just the Lie bracket $[\mathbf{h}, \mathbf{A} \cdot \mathbf{w}]$ of $\mathbf{h}$ with $\mathbf{A} \cdot \mathbf{w}$. If $\mathbf{L_A}$ were nonsingular, all nth degree terms in the Taylor series for $\mathbf{f}$ could be eliminated by the nonlinear change of variables $\overline{\mathbf{w}} = \mathbf{w} + \mathbf{L_A}^{-1} \cdot \mathbf{h}$, where $\mathbf{h}$ denotes the vector of nth degree terms in the series. In general, only those nth degree terms of $\mathbf{f}$ which lie in the range of $\mathbf{L_A}$ can be eliminated. Those elements of $\Lambda_{2,n}$ which do not lie in the range of $\mathbf{L_A}$ cannot be eliminated by a smooth change of variables and are termed *resonant*. The resonant terms of $\mathbf{f}$ contain the essential nonlinear contributions to the phase plane structure. Applying $\mathbf{L_A}$ to the standard basis $(\mathbf{w}_1^j \mathbf{w}_2^l, 0)^T$ and $(0, \mathbf{w}_1^j \mathbf{w}_2^l)^T$, $j + l = n$, yields a set of vectors which span the range of $\mathbf{L_A}$. A basis for the range may be chosen from this set. We need never include more than codim(range($\mathbf{L_A}$)) in the set of resonant vectors. This is because two resonant vectors which differ only by an element of range($\mathbf{L_A}$) are smoothly equivalent (the element of range($\mathbf{L_A}$) may be transformed away). Consequently we may identify the set of resonant vectors with the non-zero vectors of the quotient space $\Lambda_{2,n}/\text{range}(\mathbf{L_A})$. Choose any basis for this quotient space. These basis vectors are equivalence classes of elements of $\Lambda_{2,n}$. Now choose any particular representatives for these equivalence classes. These representatives define a maximal set of resonant vectors, i.e., $\mathbf{f}$ may be smoothly transformed into a vector field with second-degree terms consisting of a linear combination of this maximal set of resonant vectors. In practice this maximal set is chosen from among the standard basis vectors, if possible, in order to produce the maximum simplification of the vector field.

We summarize here the construction of the Poincaré transformations leading to Eqs. (5.1). The augmented system may be written in the form

$$\begin{aligned}
\hat{v}_y &= -a_1\hat{\lambda} - a_2\kappa - \bar{a}_3\hat{v}\hat{\lambda} + \bar{a}_4\kappa\hat{v} + \bar{a}_5\kappa\hat{\lambda} + \bar{a}_6\kappa^2 \\
&\quad + \kappa\bar{p}_1(\hat{v}, \hat{\lambda}, \kappa) + \hat{\lambda}\bar{q}_1(\hat{v}, \hat{\lambda}) \\
\hat{\lambda}_y &= b_1\hat{v}\hat{\lambda} + \bar{b}_2\kappa\hat{\lambda} + \bar{b}_3\kappa^2 - \bar{b}_4\hat{\lambda}^2 + \kappa\bar{p}_2(\hat{v}, \hat{\lambda}, \kappa) + \hat{\lambda}\bar{q}_2(\hat{v}, \hat{\lambda}) \\
\kappa_y &= 0,
\end{aligned} \tag{8.2}$$

where $\hat{v} \equiv v - v_b$, $\hat{\lambda} \equiv \lambda - 1$, $\underline{p}_i \in \Pi_{3,2}$, and $q_i \in \Pi_{2,2}$. For the augmented system, the coefficients $\bar{b}_2$, $\bar{b}_3$, and $\bar{b}_3$ are zero, as are the terms

$\hat{\lambda}\bar{q}_1$ and $\kappa\bar{p}_2$, but we will do the more general case. The matrix of the linear part of (9.1) is

$$A \equiv \begin{pmatrix} 0 & -a_1 & -a_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{8.3}$$

We seek smooth transformations of the phase space eliminating second-order terms of the vector field while leaving the third variable (here, κ) invariant. The induced linear operator $\mathbf{L_A}$ on $\Lambda_{3,2}$ may be computed from (3.1) by its action on the standard basis. The results are

$\mathbf{h}$	$\mathbf{L_A} \cdot \mathbf{h}$	$\mathbf{h}$	$\mathbf{L_A} \cdot \mathbf{h}$
$\begin{pmatrix} \hat{\upsilon}^2 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -2\hat{\upsilon}(a_1\hat{\lambda} + a_2\kappa) \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} \hat{\upsilon}\hat{\lambda} \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -\hat{\lambda}(a_1\hat{\lambda} + a_2\kappa) \\ 0 \\ 0 \end{pmatrix}$
$\begin{pmatrix} \hat{\upsilon}\kappa \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -\kappa(a_1\hat{\lambda} + a_2\kappa) \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} \hat{\lambda}^2 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
$\begin{pmatrix} \hat{\lambda}\kappa \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} \kappa^2 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
$\begin{pmatrix} 0 \\ \hat{\upsilon}^2 \\ 0 \end{pmatrix}$	$\begin{pmatrix} a_1\hat{\upsilon}^2 \\ -2\hat{\upsilon}(a_1\hat{\lambda} + a_2\kappa) \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ \hat{\upsilon}\hat{\lambda} \\ 0 \end{pmatrix}$	$\begin{pmatrix} a_1\hat{\upsilon}\hat{\lambda} \\ -\hat{\lambda}(a_1\hat{\lambda} + a_2\kappa) \\ 0 \\ 0 \end{pmatrix}$
$\begin{pmatrix} 0 \\ \hat{\upsilon}\kappa \\ 0 \end{pmatrix}$	$\begin{pmatrix} a_1\hat{\upsilon}\kappa \\ -\kappa(a_1\hat{\lambda} + a_2\kappa) \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ \hat{\lambda}^2 \\ 0 \end{pmatrix}$	$\begin{pmatrix} a_1\hat{\lambda}^2 \\ 0 \\ 0 \end{pmatrix}$
$\begin{pmatrix} 0 \\ \hat{\lambda}\kappa \\ 0 \end{pmatrix}$	$\begin{pmatrix} a_1\hat{\lambda}\kappa \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ \kappa^2 \\ 0 \end{pmatrix}$	$\begin{pmatrix} a_1\kappa^2 \\ 0 \\ 0 \end{pmatrix}.$

The remaining basis vectors $\mathbf{h}$ would be used to construct transformations of the bifurcation parameter κ, which are forbidden in Proposition 5.1. We will eliminate the nonresonant terms of 8.2 one term at a time. The order in which the terms are eliminated must be chosen carefully, since a transformation eliminating one term may contribute to another. Five steps are required:

Step 1. Eliminate the $\bar{b}_4$ term using the transformation

$$\begin{pmatrix} \hat{\upsilon} \\ \hat{\lambda} \\ \kappa \end{pmatrix} \to \begin{pmatrix} \hat{\upsilon} \\ \hat{\lambda} \\ \kappa \end{pmatrix} + \frac{\bar{b}_4}{a_1}\begin{pmatrix} 0 \\ \hat{\upsilon}\hat{\lambda} \\ 0 \end{pmatrix}.$$

This transformation contributes to the a_3 and b_2 terms. Specifically $a_3 \to a_3 + a_4$ and $b_2 \to b_2 + a_2 b_4/a_1$. Higher order terms are also created, so that 8.2 becomes

$$\begin{aligned}
\hat{v}_y &= -a_1\hat{\lambda} - a_2\kappa - a_3\hat{v}\hat{\lambda} + a_4\kappa\hat{v} + a_5\kappa\hat{\lambda} + a_6\kappa^2 \\
&\quad + \kappa p_1(\hat{v},\hat{\lambda},\kappa) + \hat{\lambda}q_1(\hat{v},\hat{\lambda}) \\
\hat{\lambda}_y &= b_1\hat{v}\hat{\lambda} + b_2\kappa\hat{\lambda} + \kappa p_2(\hat{v},\hat{\lambda},\kappa) + \hat{\lambda}q_2(\hat{v},\hat{\lambda}) \\
\kappa_y &= 0,
\end{aligned}$$

where $p_i \in \Pi_{3,2}$ and $q_i \in \Pi_{2,2}$. Since we will perform several transformations, we will hereafter drop the bars on the coefficients and remainders and not bother to adopt distinctive notation for the values at each step of the transformation.

Step 2. Eliminate the a_3 term using the transformation

$$\begin{pmatrix} \hat{v} \\ \hat{\lambda} \\ \kappa \end{pmatrix} \to \begin{pmatrix} \hat{v} \\ \hat{\lambda} \\ \kappa \end{pmatrix} + \frac{a_3}{2a_1}\begin{pmatrix} \hat{v}^2 \\ 0 \\ 0 \end{pmatrix}.$$

The a_4 term is replaced by $a_4 + a_2 a_3/a_1$.

Step 3. Eliminate the a_4 term using the transformation

$$\begin{pmatrix} \hat{v} \\ \hat{\lambda} \\ \kappa \end{pmatrix} \to \begin{pmatrix} \hat{v} \\ \hat{\lambda} \\ \kappa \end{pmatrix} + \frac{a_4}{a_1}\begin{pmatrix} 0 \\ \hat{v}\kappa \\ 0 \end{pmatrix}.$$

This transformation contributes to the b_2 and b_3 terms, which are replaced by $b_2 + a_4$ and $b_3 + a_2 a_4/a_1$, respectively.

Step 4. Eliminate the a_5 term using the transformation

$$\begin{pmatrix} \hat{v} \\ \hat{\lambda} \\ \kappa \end{pmatrix} \to \begin{pmatrix} \hat{v} \\ \hat{\lambda} \\ \kappa \end{pmatrix} + \frac{a_5}{a_1}\begin{pmatrix} 0 \\ \kappa\hat{\lambda} \\ 0 \end{pmatrix}.$$

Step 5. Eliminate the a_6 term using the transformation

$$\begin{pmatrix} \hat{v} \\ \hat{\lambda} \\ \kappa \end{pmatrix} \to \begin{pmatrix} \hat{v} \\ \hat{\lambda} \\ \kappa \end{pmatrix} + \frac{a_6}{a_1}\begin{pmatrix} 0 \\ \kappa^2 \\ 0 \end{pmatrix}.$$

We have arrived at (5.1).

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