Asymptotic Convergence in Finite and Boundary Element Methods: Part 2: The LBB Constant for Rigid and Elastic Scattering Problems

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Abstract—The paper is a continuation of [1] and contains the evaluation of the (exact) LBB constant, in terms of the wave number, for typical problems (all with spherical geometry) in elastic scattering. Solutions to the problem of scattering of a plane wave on an elastic spherical shell, for different wave numbers, illustrate the dramatic effect of the magnitude of the LBB constant on the convergence.

Keywords—Finite and boundary element methods, Approximation theory, Elastic scattering, Vibrations.

1. INTRODUCTION

The paper is a continuation of [1] and it is devoted to the evaluation of the exact LBB constant for a number of classical problems in elastic scattering (all with spherical geometry) including:

- Helmholtz integral equation for rigid scattering,
- hypersingular integral equation for rigid scattering,
- Burton-Miller integral equation for rigid scattering,
- vibrations of an elastic submerged shell.

As shown in the example concluding [1], the effect of the radiation damping on the LBB constant varies (as expected) with the physical data, and the task of determining the effect, in context of typical data for a steel shell submerged in the water, is undertaken in the last example.

Finally, using the technique described in [2], the classical problem of elastic scattering of a plane wave on the elastic shell is solved for three different wave numbers, illustrating the dramatic effect of the LBB constant on the convergence.

2. RIGID SCATTERING PROBLEMS

Given a sphere S with radius R, we investigate the four classical integral operators associated with the Helmholtz equation in $\mathbb{R}^3$ (see [3]):
• single layer operator $A : H^{-1/2+\epsilon}(S) \to H^{1/2+\epsilon}(S)$

$$(Ap)(R_0) = \int_S g(R_0, R) p(R) dS_R$$  \hspace{1cm} (2.1)

• double layer operator $C : H^{1/2+\epsilon}(S) \to H^{-1/2+\epsilon}(S)$

$$(Cp)(R_0) = \int_S \frac{\partial g}{\partial n_R}(R_0, R) p(R) dS_R$$  \hspace{1cm} (2.2)

• adjoint of double layer operator $B : H^{-1/2+\epsilon}(S) \to H^{-1/2+\epsilon}(S)$

$$(Bp)(R_0) = \int_S \frac{\partial g}{\partial n_{R_0}}(R_0, R) p(R) dS_R$$  \hspace{1cm} (2.3)

• hypersingular operator $D : H^{1/2+\epsilon}(S) \to H^{-1/2+\epsilon}(S)$

$$(Dp)(R_0) = \int_S \frac{\partial^2 g}{\partial n_{R_0} \partial n_R}(R_0, R) p(R) dS_R,$$  \hspace{1cm} (2.4)

where

- $\epsilon \in [-1/2, 1/2]$,
- $g$ denotes the free space Green function for the Helmholtz operator

$$g(R_0, R) = g(r) = -\frac{e^{ikr}}{4\pi r},$$  \hspace{1cm} (2.5)

with $r = R - R_0$, $r = ||r||$,  

$$\frac{\partial g}{\partial n_R} = g'(r) \frac{r \circ n_R}{r},$$  \hspace{1cm} (2.6)

$$\frac{\partial g}{\partial n_{R_0}} = g'(r) \frac{r \circ n_{R_0}}{r},$$  \hspace{1cm} (2.7)

$$\frac{\partial^2 g}{\partial n_{R_0} \partial n_R} = g''(r) \frac{r \circ n_{R_0}}{r} \frac{r_0 \circ n_R}{r} + g'(r) u_{R_0} \circ u_R,$$  \hspace{1cm} (2.8)

and the last integral is understood in the Hadamard finite part sense. Operators $A$, $C$ and $B$ are classical integral operators with $L^2$-kernels and they may be defined, in particular, on the whole $L^2(S)$ space. As usual, the actual $L^2$-adjoint $C^*$ of operator $C$ is equal to $\overline{B}$, where $(\cdot)$ denotes the complex conjugate, i.e.,

$$(C^*p)(R_0) = \int_S \overline{\frac{\partial g}{\partial n_{R_0}}(R_0, R)} p(R) dS_R.$$  \hspace{1cm} (2.9)

The domain of the hypersingular operator must be restricted to $H^1(S)$ in order to guarantee that values of the operator are in $L^2(S)$.

All four integral operators are normal, and therefore, admit a spectral decomposition with, in fact, the same $L^2(S)$-orthogonal eigenfunctions

$$p^M_N(\theta, \phi) = P^M_N(\cos \theta) \cos(M \phi), \quad M \leq N,$$  \hspace{1cm} (2.10)

where

- $\theta, \phi$ are the spherical coordinates,
- $P^M_N(\eta)$ are the Legendre polynomials.
We shall also need the hypersingular operator corresponding to the Laplace operator in $\mathbb{R}^3, L : H^{1/2+r}(S) \to H^{-1/2+r}(S), \|r\| \leq 1/2$

\[
(Lp)(\mathbf{R}_0) = \int_S \frac{\partial^2 h}{\partial n_{\mathbf{R}_0} \partial n_{\mathbf{R}}} (\mathbf{R}_0, \mathbf{R}) p(\mathbf{R}) \ dS_{\mathbf{R}},
\]

where $h$ is the free space Green function for the Laplace operator

\[
h(\mathbf{R}_0, \mathbf{R}) = h(r) = \frac{1}{4\pi r}, \quad r = \mathbf{R} - \mathbf{R}_0, \quad r = \|r\|,
\]

and

\[
\frac{\partial^2 h}{\partial n_{\mathbf{R}_0} \partial n_{\mathbf{R}}} = h''(r) \frac{\mathbf{R}_0 \circ \mathbf{n}_{\mathbf{R}_0}}{r} + h'(r) \mathbf{n}_{\mathbf{R}_0} \circ \mathbf{n}_{\mathbf{R}}.
\]

The operator is self-adjoint and semipositive definite with the eigenvectors again given by (2.10).

Evaluation of values of the described operators on eigenvectors (2.10) is done in two steps:

**Step 1.** We consider first $\mathbf{R}_0$ outside of the sphere and use the classical expansion formulas for both free space Green functions (see [4])

\[
g(\mathbf{R}_0, \mathbf{R}) = -\frac{i k}{4\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{\varepsilon_m (n - m)!}{(n + m)!} (2n + 1) \cos m(\varphi - \varphi_0)
\]

\[
\times P_n^m(\cos \theta) P_{m}^{n}(\cos \theta_0) j_n(kR) h_n(kR_0),
\]

\[
\frac{\partial g}{\partial n_{\mathbf{R}}} (\mathbf{R}_0, \mathbf{R}) = \frac{\partial g}{\partial R_0} (\mathbf{R}_0, \mathbf{R}) = -\frac{i k}{4\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \cdots j_n'(kR) k h_n(kR_0),
\]

\[
\frac{\partial g}{\partial R_0} (\mathbf{R}_0, \mathbf{R}) = \frac{\partial^2 g}{\partial R_0 \partial R} (\mathbf{R}_0, \mathbf{R}) = -\frac{i k}{4\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \cdots j_n'(kR) k h_n'(kR_0) k,
\]

\[
\frac{\partial^2 g}{\partial n_{\mathbf{R}_0} \partial n_{\mathbf{R}}} (\mathbf{R}_0, \mathbf{R}) = \frac{\partial^2 g}{\partial R_0 \partial R} (\mathbf{R}_0, \mathbf{R}) = -\frac{i k}{4\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \cdots j_n'(kR) k h_n'(kR_0) k^2,
\]

\[
h(\mathbf{R}_0, \mathbf{R}) = \frac{1}{4\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{\varepsilon_m (n - m)!}{(n + m)!} \cos m(\varphi - \varphi_0)
\]

\[
\times P_n^m(\cos \theta) P_{m}^{n}(\cos \theta_0) \frac{R^n}{R_0^{n+1}},
\]

\[
\frac{\partial^2 h}{\partial n_{\mathbf{R}_0} \partial n_{\mathbf{R}}} (\mathbf{R}_0, \mathbf{R}) = \frac{\partial^2 h}{\partial R_0 \partial R} (\mathbf{R}_0, \mathbf{R}) = \frac{1}{4\pi} \sum_{n=1}^{\infty} \sum_{m=0}^{n} \cdots n(n + 1) \frac{R^{n-1}}{R_0^{n+2}}.
\]

with the assumption that $R < R_0$ and $j_n, h_n$ denoting the usual Bessel and Hankel functions, respectively.

**Step 2.** We evaluate the value of the corresponding operators for (2.10) and move point $\mathbf{R}_0$ to the sphere, using the limiting properties of the operators

\[
(Ap)(\mathbf{R}_0) = \lim_{\mathbf{R}_o \to \mathbf{R}_0} (Ap)(\mathbf{R}_0),
\]

\[
(Cp)(\mathbf{R}_0) = \lim_{\mathbf{R}_o \to \mathbf{R}_0} (Cp)(\mathbf{R}_0) - 1/2p(\mathbf{R}_0),
\]

\[
(Bp)(\mathbf{R}_0) = \lim_{\mathbf{R}_o \to \mathbf{R}_0} (Bp)(\mathbf{R}_0) + 1/2p(\mathbf{R}_0),
\]

\[
(Dp)(\mathbf{R}_0) = \lim_{\mathbf{R}_o \to \mathbf{R}_0} (Dp)(\mathbf{R}_0),
\]

\[
(Lp)(\mathbf{R}_0) = \lim_{\mathbf{R}_o \to \mathbf{R}_0} (Lp)(\mathbf{R}_0).
\]

(2.20)
We end up with the the following formulas:

\[(AP)(\theta, \varphi) = -ikR^2j_n(kR)h_n(kR) p(\theta, \varphi),\]

\[(CP)(\theta, \varphi) = \left\{ -i(kR)^2j'_n(kR)h_n(kR) - \frac{1}{2} \right\} p(\theta, \varphi),\]

\[(BP)(\theta, \varphi) = \left\{ -i(kR)^2j'_n(kR)h'_n(kR) + \frac{1}{2} \right\} p(\theta, \varphi),\]

\[(DP)(\theta, \varphi) = ik(kR)^2j'_n(kR)h'_n(kR) p(\theta, \varphi),\]

\[(LP)(\theta, \varphi) = \frac{n(n + 1)}{2n + 1} \frac{1}{R} p(\theta, \varphi).\]  

We are now ready to determine the LBB constants for the problems of interest.

**Helmholtz Formulation**

The operator

\[\frac{1}{2} I - C\]  

is a compact perturbation of \((1/2) I\) and the LBB constant \(\lambda\) is determined (see [1]) by solving the eigenvalue problem

\[\frac{1}{2} I q = \left( \frac{1}{2} I - C \right) p, \quad \lambda\frac{1}{2} I p = \left( \frac{1}{2} I - B \right) q,\]  

where \((p, q)\) is the corresponding eigensolution.

Substituting (2.10) for \(p\), we use (2.21) and end up with the following formulas for the \(n^{th}\) eigenvalue \(\lambda_n\)

\[\lambda_n = 4\left[ -i(kH)^2j_n(kH)h_n'(kH) \right] \left[ 1 + i(kH)^2j'_n(kR)h_n(kR) \right].\]  

\[\lambda_n = \frac{n(n + 1)}{2n + 1} \frac{1}{R} p(\theta, \varphi).\]  

\[\lambda_n = 4\left[ -i(kH)^2j_n(kH)h_n'(kH) \right] \left[ 1 + i(kH)^2j'_n(kR)h_n(kR) \right].\]
The LBB Constant for Rigid and Elastic Scattering Problems

Values of the first 20 eigenvalues for $0 < k < 10 (R = 1)$ are displayed in Figure 1. The pointwise infimum of the curves represents the actual LBB constant (the minimum eigenvalue). We note that values of Bessel and Hankel functions are evaluated using the classical recursion formulas (see [3] for details). The algorithm breaks down for small wave numbers which is the reason for the discontinued lines for small wave numbers. As expected, the LBB constant approaches unity for $k \to 0$ and goes down to zero for the forbidden (fictitious) frequencies, identified as the eigenvalues of the interior Dirichlet problem for the Laplace operator.

**Hypersingular Formulation**

The operator $D$ is a compact perturbation of the corresponding hypersingular operator $L$ for the Laplace equation. As operator $L$ is only semi-positive definite, we augment it with the identity operator (premultiplied by $R^{-1}$ for scaling purposes) and end up with the following eigenvalue problem

$$\left(\frac{1}{R} I + L\right) q = D p, \quad \lambda \left(\frac{1}{R} I + L\right) p = D q,$$

where, as previously, $(p, q)$ is the corresponding eigensolution (see also the subsequent discussion in the next section of the case of a self-adjoint operator with zero eigenvalue).

Substituting (2.10) for $p$, we use (2.21) and obtain the following formula for the $n$th eigenvalue $\lambda_n$,

$$\lambda_n = \frac{k^6 R^6 (j_n'(kR))^2 h_n'(kR) h_n(kR)}{\left(\frac{n(n+1)}{2n+1} + 1\right)^2}.$$

Values for the first 20 eigenvalues for $0 < k < 10 (R = 1)$ are displayed in Figure 2. Again, the pointwise infimum of the curves represents the actual LBB constant. The zero values of the LBB constant correspond to the forbidden frequencies identified this time as the eigenvalues to the interior Neumann problem for the Laplace operator (including the zero wave number!).
Burton-Miller Formulation

Following [5], we define the Burton-Miller operator as

\[ \left( \frac{1}{2} I - C \right) + \frac{i}{k} D \]  

(2.27)

with the adjoint operator equal to

\[ \left( \frac{1}{2} I - B \right) - \frac{i}{k} D. \]  

(2.28)

The eigenvalue problem takes the form

\[ \left( \frac{1}{R} I + L \right) q = \left[ \left( \frac{1}{2} I - C \right) + \frac{i}{k} D \right] p, \quad \lambda \left( \frac{1}{R} I + L \right) p = \left[ \left( \frac{1}{2} I - B \right) - \frac{i}{k} D \right] q, \]  

(2.29)

and the corresponding formula for the \( n \)th eigenvalue is

\[ \lambda_n = \left( \frac{n(n + 1)}{2n + 1} + 1 \right)^{-2} R^2 \left[ -i(kR)^2 j_n(kR)h'_n(kR) + (kR)^2 j'_n(kR)h''_n(kR) \right] \]
\[ \times \left[ 1 + i(kR)^2 j'_n(kR)h_n(kR) + (kR)^2 j''_n(kR)h'_n(kR) \right]. \]  

(2.30)

The first 20 eigenvalues are displayed in Figure 3. The \( 1/k \) scaling factor in front of operator \( D \), advocated in [1], indeed has produced a uniformly stable formulation, except for wave number \( k < 0 \) where, due to the \( 1/k \) factor, the hypersingular operator dominates the Helmholtz one. A simple remedy to this problem is to replace \( 1/k \) factor in the formulation with 1 for \( k < 1 \). The resulting eigenvalues are then shown in Figure 4.

![Figure 3. Burton-Miller integral operator. Pointwise infimum of the curves shown represents dependence of LBB constant \( \gamma \) upon the wave number \( k \).](image)
3. ELASTIC SCATTERING PROBLEMS

In this section, we investigate the LBB constant for the operator governing vibrations of an elastic spherical shell in fluid. We do not investigate the full coupled problem, consisting of the elasticity equations and the Helmholtz integral equation (or similarly the Burton-Miller integral equation) with the velocity components and pressure as unknowns. Rather, following the standard idea (see [4,6]), we solve the Helmholtz (Burton-Miller) equation for pressure in terms of the normal velocity on the boundary and substitute it into the elasticity problem. The procedure results, in general, in a nonlocal boundary condition for the elasticity equations. For the sphere problem, however, due to the same spectral representation (eigenfunctions) for both elasticity and Burton-Miller operators, a full spectral decoupling for both the original and the adjoint problems is possible, and the determination of the LBB constant reduces, as in the previous section, to the solution of simple scalar equations for each of the modes separately.

Before we turn into the discussion of the shell problem, we would like to point out an extra technical detail connected with the fact that the shell is freely floating in the fluid, i.e., the spectrum of the elasticity operator includes $\lambda = 0$. In both examples in [1], the elastic body was supported which had eliminated the zero eigenvalue and, consequently, allowed to use the energy norm to evaluate the discrete LBB constant $\gamma_h$. More precisely, we had for the vibrating string problem

$$\gamma_h = \inf_{\|u_h\| = 1} \sup_{\|v_h\| = 1} |b(u_h, v_h)|$$

$$= \sum_{i=1}^{N_h} \frac{\lambda_i^h}{(\lambda_i^h - k^2)} \frac{\lambda_i^h}{(\lambda_i^h - k^2)}$$

$$= \min_{i=1,...,N_h} |\lambda_i^h - k^2| (\lambda_i^h)^{-1}, \quad (3.1)$$

where $\lambda_i^h$, $i = 1, \ldots, N_h$ are the discrete eigenvalues.
When selecting the norm for the case of the string “flying freely in the air” (traction boundary conditions are applied only), we have to supplement the energy with an extra (e.g., $L^2$-) term to make it a norm

$$\|u\|^2 - C\|u\|^2_0 + \|u\|^2_E,$$

(3.2)

where $C > 0$ is an arbitrary, positive constant. Using the discrete spectral decomposition, we have then

$$\|u_h\|^2 = \sum_{i=0}^{N_h} (C + \lambda_i^h) (u_i^h)^2,$$

(3.3)

where $\lambda_0^h = 0$.

Consequently, the determination of the discrete LBB constant reduces to the saddle point problem

$$\gamma_h = \inf_{\|u_h\|=1} \sup_{\|v_h\|=1} |b(u_h, v_h)|$$

$$= \inf_{\sum_{i=0}^{N_h} (C + \lambda_i^h) (u_i^h)^2 = 1} \sup_{\sum_{j=0}^{N_h} (C + \lambda_j^h) (v_j^h)^2 = 1} \sum_{i=0}^{N_h} (\lambda_i^h - k^2) u_i^h v_i^h$$

$$= \min \left\{ \frac{k^2}{C}, \frac{\|\lambda_i - k^2\|}{C + \lambda_i}, i = 1, \ldots, N_h \right\}.$$  

(3.4)

Thus, the essential difference between the supported and free body cases is the presence of the $k^2/C$ term in the final formula. In fact, exactly the same situation has already been encountered in the previous section for the hypersingular formulation.

We recall now (see [4]) the equations for axisymmetric vibrations of a spherical shell subjected to an external pressure

$$L_{uvv} + L_{uvw} + \Omega^2 u = 0,$$

$$L_{uwv} + L_{uwv} + \Omega^2 w = -p\frac{a^2 (1 - \nu^2)}{E h},$$

(3.5)

where the operators $L_{uvv}, L_{uvw}, L_{uwv}$, and $L_{uwv}$ are given by

$$L_{uv} = (1 + \beta^2) \left\{ (1 - \eta^2)^{1/2} \frac{d^2}{d\eta^2} \left( 1 - \eta^2 \right)^{1/2} + (1 - \nu) \right\},$$

$$L_{uw} = (1 - \eta^2)^{1/2} \left\{ [\beta^2(1 - \nu) - (1 + \nu)] \frac{d}{d\eta} + \beta^2 \frac{d}{d\eta} \right\},$$

$$L_{uwv} = -\left\{ [\beta^2(1 - \nu) - (1 + \nu)] \frac{d}{d\eta} \left( 1 - \eta^2 \right)^{1/2} + \beta^2 \frac{d}{d\eta} \left( 1 - \eta^2 \right)^{1/2} \right\},$$

$$L_{uwv} = -\beta^2 \frac{d}{d\eta} \left( 1 - \eta^2 \right)^{1/2} - \beta^2(1 - \nu) \frac{d}{d\eta} - 2(1 + \nu),$$

(3.6)

with $\beta = (1/\sqrt{12}) (h/a)$ and

$$\nabla^2_\eta = \frac{d}{d\eta} \left( 1 - \eta^2 \right) \frac{d}{d\eta}. $$

(3.7)

The following notation has been used

- $a$–the radius of the middle surface of the shell,
- $E, \nu$–the Young modulus and Poisson ratio,
- $h$–the thickness of the shell,
- $p$ –the pressure,
- $\eta = \cos \theta$,
- $\Omega$ - dimensionless frequency of the shell

$$\Omega = \frac{\omega a}{c_p} = \left( \frac{c}{c_p} \right) k a$$  

(3.8)
The equations admit a spectral decomposition using the usual eigenfunction representation

\[
\psi(\eta) = \sum_{n=0}^{\infty} V_n (1 - \eta^2)^{1/2} P_n(\eta), \quad \omega(\eta) = \sum_{n=0}^{\infty} W_n P_n(\eta),
\]

where \(P_n(\eta)\) are the Legendre polynomials of order \(n\) and \(V_n, W_n\) are unknown modal components.

Equations (3.5) are now accompanied by the formula for pressure in terms of specific acoustic impedance and the unknown components \(W_n\) (see [4])

\[
p_n(\eta) = -\sum_{n=0}^{\infty} (-i\omega W_n r_n - \omega^2 W_n m_n) \ P_n(\eta),
\]

where modal resistance \(r_n\) and modal accession to inertia \(m_n\) are given by the formulas

\[
r_n = \rho c \Re \left[ \frac{i h_n(ka)}{h_n'(ka)} \right], \quad m_n = -\frac{\rho c}{\omega} \Im \left[ \frac{i h_n(ka)}{h_n'(ka)} \right].
\]

Formula for the pressure for the adjoint problem will simply take the form

\[
p_n(\eta) = -\sum_{n=0}^{\infty} (+i\omega W_n r_n - \omega^2 W_n m_n) \ P_n(\eta).
\]

When selecting the norm for the evaluation of the LBB constant, we choose, as in the discussion of the free string problem, the energy norm augmented with an extra \(L^2\)-term. The equations for the \(n^{th}\) eigenvalue will now look as follows

\[
\left( \begin{array}{cc} A_n & B_n \\ B_n & \lambda_n A_n \end{array} \right) \left( \begin{array}{c} u_n \\ u_n^* \end{array} \right) = 0
\]

with \(u_n = (V_n, W_n)^T, u_n^* = (V_n^*, W_n^*)^T\)-modal components of the eigensolution and the following matrices:

- matrix \(L_n\) corresponding to the free vibrations in vacuum

\[
L_n = \begin{pmatrix}
\Omega^2 & -1 \beta^2 \nu + \kappa_n 1 \\
\beta^2 \nu + \kappa_n & \Omega^2 - 2(1 + \nu) - \beta^2 \kappa_n (\nu + \kappa_n - 1)
\end{pmatrix}
\]

with \(\kappa_n = n(n + 1)\),

- \(B_n\) corresponding to the free vibrations in fluid

\[
B_n = L_n + \begin{pmatrix}
0 & 0 \\
0 & \Omega^2 \frac{m_n}{\rho s c_p} + i\Omega \frac{a}{\rho s c_p} \frac{r_n}{\rho s c_p}
\end{pmatrix}
\]

- matrix \(A_n\) corresponding to the choice of norm

\[
A_n = L_n + \begin{pmatrix}
C & 0 \\
0 & C
\end{pmatrix}
\]

\((C = 1\) in calculations)
Figure 5. Vibrations of a submerged shell. Pointwise infimum of the curves shown represents dependence of LBB constant $\gamma$ upon the wave number $k$.

The characteristic equation corresponding to (3.13) is solved numerically with the help of the following algebraic identities:

\[
\begin{vmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} \\
  a_{21} & a_{22} & a_{23} & a_{24} \\
  a_{31} & a_{32} & \lambda a_{33} & \lambda a_{34} \\
  a_{41} & a_{42} & \lambda a_{43} & \lambda a_{44}
\end{vmatrix}
\]

\[
= (\lambda - 1)^2 \begin{vmatrix}
  a_{11} & a_{12} & 0 & 0 \\
  a_{21} & a_{22} & 0 & 0 \\
  a_{31} & a_{32} & a_{33} & a_{34} \\
  a_{41} & a_{42} & a_{43} & a_{44}
\end{vmatrix}
\]

\[
+ (\lambda - 1) \begin{vmatrix}
  a_{11} & a_{12} & 0 & a_{14} \\
  a_{21} & a_{22} & 0 & a_{24} \\
  a_{31} & a_{32} & a_{33} & a_{34} \\
  a_{41} & a_{42} & a_{43} & a_{44}
\end{vmatrix}
\]

\[
+ \begin{vmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} \\
  a_{21} & a_{22} & a_{23} & a_{24} \\
  a_{31} & a_{32} & a_{33} & a_{34} \\
  a_{41} & a_{42} & a_{43} & a_{44}
\end{vmatrix}
\]

(3.17)

Finally, the actual LBB constant is evaluated taking again the pointwise (in terms of wave number $k$) infimum of $\lambda_n = \lambda_n(k)$

\[
\gamma = \inf_{n=1,2,...} \lambda_n(k).
\]

(3.18)

Figure 5 displays results of the calculations for the first 50 modes using the standard data for a steel shell imbedded in the water

\[
a = 1 \text{ m}, \quad E = 2 \times 10^{11} \text{ N/m}^2, \quad \nu = 0.3, \quad h = 1 \text{ cm}, \quad c = 1,460 \text{ m/sec},
\]
and the wave number $k$ ranging from 0 to 10. A zoom for $k$ between 0 and 2, presented in Figure 6 allows for a careful examination of the LBB constant around the first eigenfrequencies of the problem. While the value of $\gamma$ is equal around $2 \cdot 10^{-4}$ for the first eigenvalue, it drops down quickly to $10^{-7}$ for the third one, and beginning with the fifth eigenvalue, it reaches the machine zero (around $10^{-15}$). Thus, except for the first couple of eigenvalues, the radiation dumping for this problem is practically negligible.
4. A "PRACTICAL" VERIFICATION: CONCLUSIONS

In order to verify the theoretical investigations, the classical problem of scattering of a plane wave on an elastic spherical shell was solved, considering three different wave numbers:

- $k = 1.13$ (near the first resonant frequency of the submerged shell),
- $k = 1.156353$ (the first local minimum for the LBB constant),
- $k = 4.15$ (away from resonant frequencies, see Figure 14).

With the same physical data as in the previous section, the problem was solved using a BE/FE approximation based on the Burton-Miller integral equation coupled with the standard 3-D elasticity formulation described in [3].

Figures 7, 8 and 9 display the real part of the pressure along a cross section of the sphere compared with the exact pressure distribution, obtained using a series representation (see [4]). Three uniform meshes of quadratic meshes were considered, with 2, 4 and 8 elements per meridian. An excellent convergence is observed. We note that the LBB constant for this wave number case is around $10^{-3}$.

The next four figures, Figures 10-13, present results for the same problem on meshes with 2, 4, 8 and a maximum of 12 elements per meridian (this was about the maximum for the workstation being used) but for the wave number yielding the first local minimum of the LBB constant with the value around $2 \cdot 10^{-4}$. The results are rather depressing. While the first, coarse mesh approximation seems to be quite good, the next ones are completely wrong and the method evidently diverges. The presented theory provides a perfect explanation of the observed behavior. On the coarse mesh, the discrete LBB constant $\gamma_h$ supposedly is still far away from the exact, minimal one, and the approximation is stable. With more degrees of freedom $\gamma_h$ gets closer to $\gamma$, and it evidently reaches a threshold value for the discrete LBB constant, above which the approximation becomes unstable. We note that the standard Gaussian elimination with no pivoting was used to solve the resulting system of linear equations, see [1] for the details on the solver.
We mention, at this point, that exactly the same unstable behavior of the solution was observed for wave numbers corresponding to the second and third minimum of $\gamma$.

Finally, Figures 15 and 16 present two solutions of the same problem, obtained on uniform meshes with 10 and 12 elements per meridian, for wave number $k = 4.15$. Based on the results from the previous section, the wave number $k$ was selected this time in such a way as to yield roughly a local maximum of the LBB constant (see Figure 14).
As for the first wave number considered, the method converges, although, due to a more complicated pattern of the solution, certainly more degrees of freedom are needed. The following conclusions suggest themselves.

1. Magnitude of the wave number plays a secondary role in solving the problem. Obviously, for larger wave numbers one needs more degrees of freedom.
2. In the absence of the structural dumping (how to model it?) magnitude of the LBB constant depends upon the distance from the nearest resonant frequency (in fluid, with the shift due to the accession to inertia terms taken into account) and plays the absolutely deciding role in the possibility of solving the problem. For small LBB constants (around $10^{-4}$ with the present implementation on a 15 digits machine) the problem is simply not solvable.

3. Without a strict control of the discrete LBB constant during the solution process, the results may be completely unreliable!

We emphasize that all these conclusions do not apply to the rigid scattering problems where the Burton-Miller formulation provides means for a uniformly stable approximation and eventually allows for the use of $hp$-approximations in achieving high convergence rates and superior quality of the solution (see [7]). For the elastic scattering however, the a posteriori control of the discrete LBB constant $\gamma_h$ seems to be absolutely crucial, and for small $\gamma_h$, the use of higher orders of approximation may be restricted. In any event, the use of all possible techniques to minimize the effects of the round-off error (pivoting, preconditioning, etc.) seems to be inevitable.
Figure 15. Elastic scattering of a plane wave on a spherical shell for $k = 4.15$. Comparison of the exact and approximate solutions on a uniform mesh of quadratic elements with 10 elements per meridian.

Figure 16. Elastic scattering of a plane wave on a spherical shell for $k = 4.15$. Comparison of the exact and approximate solutions on a uniform mesh of quadratic elements with 12 elements per meridian.

REFERENCES


