Uniqueness of the Inverse Conductive Scattering Problem

G. YAN AND P. Y. H. PANG
Department of Mathematics, National University of Singapore
Singapore 119260

(Received and accepted April 1998)

Abstract—In this paper, we show that for the inverse obstacle scattering problem with conductive boundary condition, the conductive boundary is uniquely determined by the far field pattern. © 1998 Elsevier Science Ltd. All rights reserved.

Keywords—Inverse obstacle scattering, Helmholtz equation, Conductive boundary value problem.

1. INTRODUCTION

During the last two decades or so, inverse scattering problems for the Helmholtz equation have enjoyed a remarkable degree of popularity, both in pure and applied contexts (see the monograph [1] and the references therein). One of the most important theoretical considerations in inverse scattering problems is uniqueness. Different approaches have been proposed [2-10]. Many of them are based on the finite dimensionality of the eigenspaces of the negative Laplacian in bounded domains. In [5], Isakov proposed a variational approach which was later extended and simplified by Kirsch and Kress [7] by means of boundary integral equation methods. The purpose of this paper is to adapt the method of [7] to the inverse conductive problem [8,9,11,12]. This problem, which generalizes the more classical transmission problem [13], arises in geophysical models in which an obstacle is covered by a thin layer of high conductivity [14].

The scattering of acoustic time-harmonic waves by a penetrable bounded conductive obstacle \( D \), which is assumed to be an open and bounded region in \( \mathbb{R}^3 \) with \( \mathbb{R}^3 \setminus \overline{D} \) connected, can be modelled by a boundary value problem as follows. We look for a pair of functions \( u \in C^2(\mathbb{R}^3 \setminus \overline{D}) \cap C^1(\mathbb{R}^3 \setminus D) \) and \( v \in C^2(D) \cap C^1(\partial D) \) satisfying the Helmholtz equations

\[
\begin{align*}
\Delta u + k^2 u &= 0, & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\
\Delta v + k^2 v &= 0, & \text{in } D,
\end{align*}
\]

with the boundary conditions

\[
\begin{align*}
&u - \mu v = f, & \text{in } \partial D, \\
&\frac{\partial u}{\partial n} - \frac{\partial v}{\partial n} = \lambda u + g, & \text{on } \partial D,
\end{align*}
\]

where \( u \) is the superposition of the given incident plane wave \( u^i(x) = e^{ikx \cdot d} \) and scattered wave \( u^s \), i.e., \( u(x) = u^i(x) + u^s(x) \), and the scattered wave \( u^s \) is required to satisfy the Sommerfeld radiation condition

\[
\lim_{|x| \to \infty} |x| \left( \frac{\partial u^s}{\partial n} -iku^s \right) = 0
\]
uniformly in all directions \( \hat{x} = x/|x| \). This condition ensures the uniqueness for the exterior boundary value problem and leads to an asymptotic behavior of the form

\[
u_{\infty}(x) = \frac{e^{ik|x|}}{|x|} \left\{ u_{\infty}(\hat{x}) + O\left(\frac{1}{|x|}\right) \right\}, \quad |x| \to \infty
\]  

uniformly in all directions. The function \( u_{\infty}(\hat{x}) \), defined on the unit sphere \( \Omega \subset \mathbb{R}^3 \), is called the far field pattern or scattering amplitude of the scattered wave. It is a consequence of Rellich's Lemma [15] that the far field pattern uniquely determines the scattered wave.

In (1.1) and (1.2), \( k, k_0, \) and \( \mu \) are positive constants with \( \mu \neq 1 \), and \( f \in C^{1,\alpha}(\partial D) \) and \( g \in C^{0,\alpha}(\partial D) \) are given functions in the specified Hölder spaces with exponent \( 0 < \alpha < 1 \). We assume that the boundary \( \partial D \) is connected and of class \( C^2 \) and we denote by \( \nu \) the outward unit normal of \( \partial D \). For the sake of simplicity, in the subsequent analysis, we will assume that \( \lambda \) is a nonzero constant, although our result remains valid for more general \( \lambda \). Our main result is the following.

**Theorem 1.1.** For the problem (1.1), (1.2) with \( \mu(\neq 1), \lambda, f, \) and \( g \) as above, suppose that \( D_1 \) and \( D_2 \) are two scatterers such that the far field patterns coincide for an infinite number of incident plane waves with distinct directions \( d \) and fixed wave-numbers \( k \) and \( k_0 \). Then \( D_1 = D_2 \).

## 2. BASIC LEMMALS

The following boundary integral operators \([1,15]\) will be used:

\[
(S\varphi)(x) = 2\int_{\partial D} \Phi(x,y)\varphi(y)\,ds(y), \quad (2.1)
\]

\[
(K\varphi)(x) = 2\int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\varphi(y)\,ds(y), \quad (2.2)
\]

\[
(K'\varphi)(x) = 2\frac{\partial}{\partial \nu(x)} \int_{\partial D} \Phi(x,y)\varphi(y)\,ds(y), \quad (2.3)
\]

\[
(T\varphi)(x) = 2\frac{\partial}{\partial \nu(x)} \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\varphi(y)\,ds(y), \quad (2.4)
\]

where

\[
\Phi_k(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}, \quad x \neq y
\]

is the fundamental solution to the Helmholtz equation

\[
\Delta u + k^2 u = 0 \quad (2.6)
\]

in \( \mathbb{R}^3 \). When only one \( k \) is present and no confusion arises, we will drop the subscript \( k \) of \( \Phi_k \) in (2.5).

**Lemma 2.1.** Let \( \partial D \) be of class \( C^2 \). Then the operators \( S, K, \) and \( K' \) are compact from \( C(\partial D) \) into \( C(\partial D) \) and from \( L^2(\partial D) \) into \( L^2(\partial D) \). \( T \) is bounded from \( C^{1,\alpha}(\partial D) \) into \( C^{0,\alpha}(\partial D)(\alpha \in (0,1)) \). The operator \( S \) is self-adjoint. \( K \) and \( K' \) are adjoint with respect to the standard \( L^2 \) inner product.

The following lemma follows immediately from Lemma 2.1 \([1,7]\).

**Lemma 2.2.** Let \( D \) be an open bounded domain with \( C^2 \) boundary \( \partial D \) such that \( \mathbb{R}^3 \setminus \overline{D} \) is connected. If \( u \in C^2(D) \cap C^1(\overline{D}) \) is a solution to the Helmholtz equation (2.6) in \( D \), then there exists a sequence \( \{u_n\} \) in the span of \( \{u^i(\cdot; d), d \in \Omega\} \) such that

\[
u_n \to u, \quad \text{grad } u_n \to \text{grad } u, \quad \text{as } n \to \infty.
\]
For the two scatterers $D_1$ and $D_2$, let $G$ be the unbounded component of the complement of $\overline{D_1} \cup \overline{D_2}$. We have already mentioned that if the far field patterns for the two scatterers coincide for an incident plane waves $u^i(x;d)$, then the scattered waves coincide in $G$. It is also known the same result holds for point sources in $R^3\setminus(\overline{D_1} \cup \overline{D_2})$.

The scattering problem for a point source is also modelled by a boundary value problem. Let $x_0 \in G$ and consider radiating solutions of the following conductive boundary value problem (we still use $u^j$ to denote scattered waves corresponding to point sources):

$$\begin{align*}
\Delta u^j + k^2 u^j &= 0, & \text{in } R^3\setminus D_j, \\
\Delta v^j + k^2 v^j &= 0, & \text{in } D_j,
\end{align*}$$

(2.7)

with the conductive boundary conditions

$$\frac{\partial u^j}{\partial n} + \Phi(\cdot; x_0) = \frac{\partial v^j}{\partial n} + g,$$

(2.8)

where $j = 1, 2$ and $\Phi$ is given in (2.5).

Radiating solutions of (2.7),(2.8) also have the representation (1.4) where $u_{\infty,j}$ are the far field patterns.

**Lemma 2.3.** (See [7].) Let $x_0 \in G$. For the above problem (2.7),(2.8), suppose $u_{\infty,1} = u_{\infty,2}$, then $u^1 = u^2$ in $G$.

### 3. PROOF OF MAIN RESULT

Let $D$ be a bounded domain and $x_0 \in \partial D$ which is of class $C^2$. Consider the Banach space

$$C_0(\partial D) = \left\{ \phi \in C(\partial D \setminus \{x_0\}); \lim_{x \to x_0} |x - x_0| \phi \text{ exists} \right\},$$

with the weighted supremum norm

$$\|\phi\|_{\infty,0} = \sup_{x \neq x_0} |(x - x_0)\phi(x)|.$$

**Lemma 3.1.** (See [7].) Let $p$ be a weakly singular kernel on $\partial D$ which is continuous for $x \neq y$ and satisfies

$$\|p(x,y)\| \leq \frac{M}{|x - y|}, \quad x \neq y,$$

for some constant $M$. An operator $A : C_0(\partial D) \to C_0(\partial D)$ defined by

$$(A\phi)(x) := \int_{\partial D} p(x,y)\phi(y) \, ds(y)$$

is compact.

**Lemma 3.2.** Let $D$ and $x_0$ be as above, $\Gamma \subset \partial D$ be compact with nonempty interior such that $x_0 \notin \Gamma$, and suppose that $f$ and $g$ belong to $C_0(\partial D) \cap C^{1,\alpha}(\partial D)$, i.e., on the boundary

$$\|f\|_{\infty,0} + \|f\|_{1,\alpha} + \|g\|_{\infty,0} + \|g\|_{1,\alpha} < \infty.$$

Let $u \in C^2(R^3\setminus \overline{D})\cap C^{1,\alpha}(R^3\setminus \overline{D})$ be a solution of the Helmholtz equation (2.6) in $R^3\setminus \overline{D}$ satisfying the radiation condition, and let $v \in C^2(D)\cap C^{1,\alpha}(\overline{D})$ be a solution of the Helmholtz equation (2.6) in $D$. Then there exists a constant $C = C(D, \Gamma)$ such that

$$\|u\|_{\infty,\Gamma} + \left\|\frac{\partial v}{\partial n}\right\|_{\infty,\Gamma} \leq C \left( \|\psi_1\|_{\infty,0} + \|\psi_2\|_{\infty,0} + \|\psi_1\|_{1,\alpha,\Gamma} + \|\psi_2\|_{1,\alpha,\Gamma} \right),$$

where $\psi_1$ and $\psi_2$ are functions related to the far field patterns of $u$ and $v$, respectively.
where  
\[ \psi_1 = u - \mu v, \quad \psi_2 = \frac{\partial u}{\partial v} - \frac{\partial v}{\partial v} - \lambda u \]

are defined on \( \partial D \) and the subscript \( \Gamma \) denotes restriction to \( \Gamma \).

**Proof.** Using a combination of single-layer and double-layer potentials, we construct the unique solution of the conductive boundary problem in the following form:

\[
\begin{align*}
\psi &= \int_{\partial D} \left\{ \psi(y) \frac{\partial \Phi_k(x,y)}{\partial y} + \varphi(y) \Phi_k(x,y) \right\} ds(y), & \quad x \in R^3 \setminus \overline{D}, \\
v &= \int_{\partial D} \left\{ \psi(y) \frac{\partial \Phi_{k_0}(x,y)}{\partial y} + \varphi(y) \Phi_{k_0}(x,y) \right\} ds(y), & \quad x \in D,
\end{align*}
\]

where \( \psi \in C^{1,\alpha}(\partial D) \) and \( \varphi \in C^{0,\alpha}(\partial D) \).

By the jump conditions [1, Theorem 3.1], we have, on \( \partial D \),

\[
(1 + \mu)\psi + (K_k - \mu K_{k_0}) \psi + (S_k - \mu S_{k_0}) \varphi = 2\psi_1, \\
(1 + \mu)\varphi - (K_k + T_k - T_{k_0}) \psi - (S_k + K_k' - \mu K_{k_0}') \varphi = -2\psi_2.
\]

Define

\[
A = \frac{1}{1 + \mu} \begin{pmatrix} K_k - \mu K_{k_0} & S_k - \mu S_{k_0} \\ T_{k_0} - K_k - T_k & \mu K_{k_0}' - S_k - K_k' \end{pmatrix}.
\]

By Lemmas 2.1 and 3.1, we see that

\[ A : C_0(\partial D) \times C_0(\partial D) \to C_0(\partial D) \times C_0(\partial D) \]

is compact.

Now, the operator \( I + A \) has a trivial null-space in \( C^{1,\alpha}(\partial D) \times C^{0,\alpha}(\partial D) \) [1, Chapter 3]. Using the Fredholm alternative theorem and Lemma 2.2, we observe that \( I + A \) also has a trivial null-space in \( C_0(\partial D) \times C_0(\partial D) \). So \( (I + A)^{-1} : C_0(\partial D) \times C_0(\partial D) \to C_0(\partial D) \times C_0(\partial D) \) is bounded by the Riesz-Fredholm theory, i.e., there exists a constant \( C = C(D) \) such that

\[
\|\psi\|_{C^{1,\alpha}} + \|\varphi\|_{C^{0,\alpha}} \leq C (\|\psi_1\|_{C^{1,\alpha}} + \|\psi_2\|_{C^{0,\alpha}}).
\]

Then we can use the technique of [7, Lemma 4.3] to prove that

\[
\|\psi\|_{C^{1,\alpha,\Gamma}} + \|\varphi\|_{C^{0,\alpha,\Gamma}} \leq C (\|\psi_1\|_{C^{1,\alpha}} + \|\psi_2\|_{C^{0,\alpha}} + \|\psi_1\|_{C^{1,\alpha,\Gamma}} + \|\psi_2\|_{C^{0,\alpha,\Gamma}}).
\]

Our result now follows immediately by using the mapping properties of the single- and double-layer operators [1, Theorem 3.3].

**Proof of Theorem 1.1.** Let \( D_1 \) and \( D_2 \) be two obstacles which satisfy the assumptions in Theorem 1.1. By (2.7), (2.8), and Lemma 2.3, we have \( u_1 = u_2 \) in \( G = R^3 \setminus (\overline{D_1} \cup \overline{D_2}) \). Suppose \( D_1 \neq D_2 \). Then there exists a point \( x_0 \in \partial D_1 \) and \( x_0 \notin \partial D_2 \). (See Figure 1.)

We construct a sequence \( \{x_n\} \) such that

\[
x_n = x_0 + \frac{h}{n} \nu(x_0), \quad (h > 0)
\]

is contained in \( G \). We denote by \( u_{n,j}^* \) and \( u_{n,j} \) the solutions of (2.7), (2.8) with \( x_0 \) replaced by \( x_n \).

Then, \( u_{n,1}^* = u_{n,2}^* \) in \( G \) and we shall use a single symbol \( u_n^* \) to denote \( u_{n,1}^* \) and \( u_{n,2}^* \) in \( G \). Consider the obstacle \( D_2 \). The point \( x_n \) is an interior point of the domain \( R^3 \setminus \overline{D_2} \). Thus,

\[
\|u_n^*\|_{C^{1,\alpha}} + \|\frac{\partial u_n^*}{\partial v}\|_{C^{0,\alpha}} < \infty,
\]

and

\[
\|u_n^*\|_{C^{1,\alpha,\Gamma}} + \|\frac{\partial u_n^*}{\partial v}\|_{C^{0,\alpha,\Gamma}} < \infty.
\]

Using the fact that \( u_n^* \) is continuous in \( x_n \), we have

\[
\left| \frac{\partial u_n^*}{\partial v}(x_n) \right| < \infty.
\]

Then, we can use the technique of [7, Lemma 4.3] to prove that

\[
\|u_n^*\|_{C^{1,\alpha}} + \|\varphi\|_{C^{0,\alpha}} \leq C (\|\psi_1\|_{C^{1,\alpha}} + \|\psi_2\|_{C^{0,\alpha}}).
\]

This completes the proof of Theorem 1.1.
for $n = 1, 2, \ldots$, and some sufficiently small ball $B$ centred at $x_0$. It follows that

$$
\left\| \frac{\partial}{\partial \nu} \{ v_{n,1} - \Phi_{k_0} (\cdot, x_n) \} \right\|_{\infty, \partial D_1 \cap B} < \infty,
$$

(3.8)

since $\| \text{grad } \Phi_{k_0} (\cdot, x_n) - \text{grad } \Phi_k (\cdot, x_n) \|_{\infty, \partial D_1}$ is bounded uniformly. By Lemma 3.2, we have

$$
\| v_{n,1} - \Phi_{k_0} (\cdot, x_n) \|_{\infty, \Gamma} + \left\| \frac{\partial}{\partial \nu} \{ v_{n,1} - \Phi_{k_0} (\cdot, x_n) \} \right\|_{\infty, \Gamma} < \infty.
$$

(3.9)

Now, combining (3.11), (3.12), and [7, Lemma 4.4], we get

$$
\| v_{n,1} - \Phi_{k_0} (\cdot, x_n) \|_{\infty, \partial D_1} < \infty,
$$

(3.10)

for $n = 1, 2, \ldots$. By (2.8), (3.7), and (3.10), we have

$$
\| \Phi_k (\cdot, x_n) - \mu \Phi_{k_0} (\cdot, x_n) \|_{\infty, \partial D_1 \cap B} < \infty,
$$

(3.11)

for $n = 1, 2, \ldots$. However, as $\mu \neq 1$, (3.11) is impossible. The contradiction completes the proof.

REFERENCES