A circuit axiomatisation of Lagrangian matroids

Richard F. Booth\(^{a, *}\), Maria Leonor Moreira\(^{b, 1}\), Maria Rosário Pinto\(^{b, 1}\)

\(^{a}\)Department of Mathematics, UMIST, P.O. Box 88, Manchester M60 1QD, UK
\(^{b}\)Centro de Matematica da, Universidade do Porto, Departamento de Matematica Pura, Rua do Campo Alegre 687, 4169-007 Porto, Portugal

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Abstract

Characterisations of symplectic and orthogonal Lagrangian matroids in terms of basis exchange are well known. We define circuits of Lagrangian matroids in a natural way and characterise Lagrangian matroids in terms of circuit axioms. We go on to characterise orthogonal Lagrangian matroids in terms of circuits, and prove a result that may be useful in establishing orientations in terms of circuits.

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1. Lagrangian matroids

The theory of Coxeter matroids is an extremely beautiful and natural algebraic generalisation of classical matroid theory, and contains many of the previous generalisations of matroid theory as special cases [3]. Lagrangian matroids are a particularly interesting special case; they are equivalent to the \(^{A}\)-matroids and symmetric matroids of Bouchet, and exhibit interesting analogues of many classical matroid properties.

Let \(I = \{1, \ldots, n\}\), \(I^* = \{1^*, \ldots, n^*\}\), and \(J = I \sqcup I^*\), where \(\sqcup\) denotes the disjoint union. We define the involution \(*\) on \(J\) by setting \((i^*)^* = i\) for \(i^* \in I^*\) and extend it to sets in the obvious way. A set \(A \subseteq J\) is said to be admissible if \(A \cap A^* = \emptyset\), and

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\(^{*}\) Corresponding author.

E-mail address: richard.booth@umist.ac.uk (R.F. Booth).

URL: \texttt{http://www.ma.umist.ac.uk/rb/}

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we write \( J_k \) for the collection of admissible \( k \)-subsets of \( J \). The \textit{symmetric difference} of two sets \( A \) and \( B \) is written and defined by
\[
A \Delta B = (A \cup B) - (A \cap B).
\]

There are many equivalent definitions of Lagrangian matroids; for the purposes of this paper, we shall use the definitions in terms of basis exchange properties. This definition is due to Bouchet, who calls the objects thus defined symmetric matroids [4]. The proof that symmetric matroids are equivalent to Lagrangian matroids as usually defined [2] follows from the characterisation of symmetric matroids in terms of a greedy algorithm in [4].

**Definition 1.** A collection of admissible \( n \)-sets \( \mathcal{B} \subseteq J_n \) is the collection of bases of a Lagrangian matroid \( \mathcal{M} \) if and only if:

For all \( A, B \in \mathcal{B} \) and \( a \in A \Delta B \), there exists \( b \in A \Delta B \) such that \( A \Delta \{a, b, a^*, b^*\} \in \mathcal{B} \).

This is called the \textit{symmetric exchange property}.

Note that the symmetric exchange property is equivalent to:

For all \( A, B \in \mathcal{B} \) and \( a \in A - B \), there exists \( b \in B - A \) such that \( A \Delta \{a, b, a^*, b^*\} \in \mathcal{B} \).

This is because for every \( i \in I \) exactly one of \( \{i, i^*\} \) is in any given basis of \( \mathcal{M} \). Note also that there is no requirement that \( |\{a, b, a^*, b^*\}| = 4 \).

Although non-Lagrangian symplectic and orthogonal matroids will play no part in this paper, we mention that, in general, a Lagrangian matroid is a symplectic matroid [2] of rank \( n \). Since all orthogonal matroids are also symplectic matroids [7], a particular Lagrangian matroid may also be an orthogonal matroid. This case is conveniently characterised by the following definition:

**Definition 2.** A Lagrangian matroid is an \textit{orthogonal Lagrangian matroid} if it is a Lagrangian matroid and the parity of \( B \cap I \) is the same for every basis \( B \) in \( \mathcal{B} \) [7].

The \textit{dual} Lagrangian matroid \( \mathcal{M}^* \) is the Lagrangian matroid whose bases are given by \( \mathcal{B}^* = \{B^* \mid B \in \mathcal{B}\} \).

Consider now an ordinary matroid [6,8,9]. It has bases which are \( j \)-element subsets of \( J \). Make each basis \( B \) into an element of \( J_n \) by sending \( B \) to \( B \sqcup (I - B)^* \). The resulting subset of \( J_n \) is clearly an orthogonal Lagrangian matroid, and this construction commutes with taking duals. (Relatively few Lagrangian matroids can be obtained in this way, or as relabellings of Lagrangian matroids obtained in this way. Those which are can be characterised by the existence of an admissible \( n \)-set \( T \) such that for every basis \( B \in \mathcal{B} \) the intersection \( T \cap B \) is of the same size; in particular, this means that all Lagrangian matroids so obtained are orthogonal.)
The notion of duality introduced here has a natural topological interpretation. Some Lagrangian matroids may be constructed from topological maps on surfaces [1,5], and here the dual Lagrangian matroid can be constructed from the dual map.

2. Circuit properties

Throughout this section \( M \) is a Lagrangian matroid and \( \mathcal{B} \) the set of its bases.

**Definition 3.** A set \( A \subset J \) is independent if it is a subset of some basis in \( \mathcal{B} \) (so \( A \subset B \in \mathcal{B} \) for some \( B \)). In particular, independent sets are admissible. A dependent set is one which is not independent. A loop is an element which does not belong to any basis.

A circuit is a minimal admissible dependent subset of \( J \). We write \( \mathcal{C} \) for the set of all circuits of \( M \).

A cocircuit of \( M \) is an element of \( \mathcal{C}^* = \{ C^* | C \in \mathcal{C} \} \). Note that the cocircuits are exactly the circuits of the dual Lagrangian matroid \( M^* \).

Note that in the case described at the end of the last section, where \( M \) is obtained from an ordinary matroid \( M \), the set of circuits of \( M \) is exactly

\[ \mathcal{C} = \{ C | C \text{ a circuit of } M \} \cup \{ C^* | C \text{ a cocircuit of } M \} \]

This follows without much difficulty from Lemma 4 below.

**Lemma 4.** Take \( B \in \mathcal{B} \) and some \( x \notin B \). Then either \( B \Delta \{ x, x^* \} \) is a basis, or \( B \cup \{ x \} \) contains a unique circuit \( C \). Furthermore, \( C \) is given by

\[ C = \{ x \} \cup \{ b \in B | B \Delta \{ x, b, x^*, b^* \} \in \mathcal{B} \} \]

**Proof.** If \( B \Delta \{ x, x^* \} \) is a basis, there is nothing to prove. If \( x \) is a loop then \( C = \{ x \} \), and the proof is finished. Otherwise, write

\[ D = \{ b \in B | B \Delta \{ x, b, x^*, b^* \} \in \mathcal{B} \} \]

and set \( C = D \cup \{ x \} \). Since \( x \) is not a loop it is in some basis; since it is not in \( B \), we have \( x \in A - B \) for some \( A \in \mathcal{B} \). Thus, from the exchange property for Lagrangian matroids (interchanging the roles of \( B \) and \( A \)), \( D \) is non-empty. Furthermore, since \( B \Delta \{ x, x^* \} \) is not a basis, \( x^* \notin D \). Thus, since \( D \subset B \) which is admissible, we see that \( C = \{ x \} \cup D \) is also admissible.

We show next that \( C \) is dependent. If not, then \( C \subseteq Y \), for some basis \( Y \). Now, since \( C - \{ x \} \subseteq B \), we see that

\[ B \Delta Y = \{ x, y_1, \ldots, y_j, x^*, y_1^*, \ldots, y_j^* \} \]

where \( y_1, \ldots, y_j \in B - C \). Since \( B \) and \( Y \) are both bases, we can do an exchange, and choose \( x \) as an element to exchange. Since \( B \Delta \{ x, x^* \} \) is not a basis, we see that there
is some \( y_i \in B - Y \) such that \( B \Delta \{ x, y_i, x^*, y_i^* \} \) is a basis. But then \( y_i \in C \), which is a contradiction.

To show minimality, we show that \( C - \{ y \} \) is independent for each \( y \in C \). If \( y = x \), then \( C - \{ y \} = D \subseteq B \in \mathcal{B} \), and we are done. Otherwise, \( C - \{ y \} \subseteq B \Delta \{ x, x^*, y, y^* \} \), which is a basis by definition of \( D \).

Finally, we show that \( C \) is the unique circuit inside \( \{ x \} \cup B \). Suppose then that \( Z \subseteq \{ x \} \cup B \) is a circuit. Certainly \( x \in Z \), as otherwise we have \( Z \subseteq B \). Now given \( y \in B - Z \), we see that

\[
Z \subseteq (B \cup \{ x, y^* \}) - \{ x^*, y \} = B \Delta \{ x, x^*, y, y^* \}.
\]

Thus \( B \Delta \{ x, x^*, y, y^* \} \) is not a basis, and so \( y \notin D \). Thus \( C \subseteq Z \), and so by minimality of circuits \( C = Z \).

**Definition 5.** Given \( B \in \mathcal{B} \) and \( x \notin B \) such that \( B \Delta \{ x, x^* \} \notin \mathcal{B} \), the unique circuit

\[
\{ x \} \cup \{ b \in B \mid B \Delta \{ x, b, x^*, b^* \} \in \mathcal{B} \}
\]

contained in \( B \cup \{ x \} \) is called the fundamental circuit for \( B \) and \( x \).

Notice that if \( \mathcal{M} \) is orthogonal, Lemma 4 states that there exists a fundamental circuit for every \( B \in \mathcal{B} \) and \( x \notin B \).

The next lemma is very similar to [8, Theorem 1.9.2]. Our proof begins with a section by contradiction, in order to establish the equivalent of the weak circuit elimination axiom. After that, the proofs are identical except for notation and phrasing; in particular, that given here is phrased as an induction rather than a contradiction.

**Lemma 6.** Let \( C_1 \) and \( C_2 \) be two distinct circuits of \( \mathcal{M}, C_1 \cup C_2 \) admissible and \( x \in C_1 \cap C_2 \). Then for every \( c \in C_1 \Delta C_2 \) there is some circuit \( C_c \in \mathcal{C} \) with

\[
c \subseteq C_c \subseteq (C_1 \cup C_2) - \{ x \}.
\]

**Proof.** Note that \( x \) is not a loop, as otherwise \( C_1 = C_2 = \{ x \} \).

Let us now suppose that \( (C_1 \cup C_2) - \{ x \} \) is independent, for contradiction. Let \( B \) be a basis such that \( (C_1 \cup C_2) - \{ x \} \subseteq B \). Clearly \( x \notin B \), as otherwise \( B \) contains a circuit. Since circuits are admissible we have

\[
C_1, C_2 \subseteq C_1 \cup C_2 \subseteq B \Delta \{ x, x^* \}.
\]

Now, \( B \Delta \{ x, x^* \} \) cannot be a basis, since it contains the circuits \( C_1 \) and \( C_2 \). By Lemma 4, any circuit contained in \( B \Delta \{ x, x^* \} \) must be the unique fundamental circuit, and so \( C_1 = C_2 \), a contradiction.

Thus \( (C_1 \cup C_2) - \{ x \} \) is dependent and admissible, and so certainly contains a circuit. We must now show that we can find such a circuit containing any \( c \in C_1 \Delta C_2 \); we proceed by induction on \( |C_1 \cup C_2| \).
For the basis of induction, the smallest possible case is clearly $C_1 = \{c_1, x\}$ and $C_2 = \{c_2, x\}$. Now $C = \{c_1, c_2\}$ must be a circuit, and can play the role of both $C_{c_1}$ and $C_{c_2}$.

Now fix $c \in C_1 \cup C_2$, and without loss of generality take $c \in C_2 - C_1$. We have shown that there exists a circuit $C \subseteq (C_1 \cup C_2) - \{x\}$; suppose that it does not contain $c$, as otherwise we are done. Since $C \not\subseteq C_2$ (by minimality of circuits), there exists some $y \in (C \cap C_1) - C_2$. Notice that $x \in C_1 - C$. Since $c \notin C \cup C_1$, we have $C \cup C_1 \subseteq C_1 \cup C_2$.

Thus we can apply induction to $C$, $C_1$, $x$ and $y$ to find a circuit $C_3$ with

$$x \in C_3 \subseteq (C \cup C_1) - \{y\}.$$

Now, $C_3 \cup C_2 \subseteq C_1 \cup C_2$, since $y \notin C_2, C_3$. Furthermore, $x \in C_3 \cap C_2$ and $c \notin C_2 - C_3$, so by induction we obtain some circuit $C_c$ with

$$c \in C_c \subseteq (C_3 \cup C_2) - \{x\} \subseteq (C_1 \cup C_2) - \{x\},$$

which completes the proof. □

The following result is known in the theory of classical matroids as orthogonality of circuits and cocircuits.

**Lemma 7.** Let $C$ be a circuit and $K$ a cocircuit of $\mathcal{M}$. Then $|C \cap K| \neq 1$.

**Proof.** Let $C$ will be a circuit for which the lemma fails. Thus, there exist a cocircuit $K$ and basis $B$ of $\mathcal{M}$ such that

$$|C \cap K| = 1 \quad \text{and} \quad C - K \subseteq B.$$ Choose such a pair $(K, B)$ for which $|K \cap B|$ is minimal. Write $x$ for the unique element of $C \cap K$.

1. There exists some $b \in B \cap K$. Otherwise, $K$ and $B$ are disjoint, and so the basis $B$ contains the circuit $K^*$, a contradiction.
2. $b \neq x, x^*$. Since, if $x = b$, $x \in B$; but then $C \subseteq B \cup \{x\} = B$. If $b = x^*$, then $x, x^* = x, b \in K$, contradicting admissibility.
3. There exists a (unique) cocircuit $K_b \subseteq B^* \cup \{b\}$. Otherwise, by Lemma 4, $B_0 = B \Delta \{b, b^*\}$ is a basis. But clearly $C - \{x\} \subseteq B_0$ and $K \cap B_0 \subseteq K \cap B$, contradicting the choice of $(K, B)$.
4. $x \notin K_b$. Otherwise, $x^* \in K_b^*$, and so by Lemma 4 we obtain a basis

$$B_1 = B \Delta \{x, b, x^*, b^*\} \in \mathcal{B}.$$ But now, since $b \notin C$ and $C \subseteq B \cup \{x\}$, we have $C \subseteq B_1$.
5. There exists some $d \in B$ such that

$$\{d, d^*\} \subseteq (K_b \cup K) - \{b\}.$$
Since \( B \) is a basis of a Lagrangian matroid, it contains one of \( d, d^* \) for any \( d \). Suppose for contradiction that \((K_b \cup K) - \{b\}\) is admissible. Since \( b \in K_b \cap K \), so \( K_b \cup K \) is admissible, and by Lemma 6 there is a cocircuit
\[
K_x \subseteq (K_b \cup K) - \{b\} \quad \text{with} \quad x \in K_x.
\]
Since \( K_b \subseteq B^* \cup \{b\} \) and \( C - \{x\} \subseteq B \), we obtain \( K_b \cap C \subseteq \{b\} \). Since \( K \cap C = \{x\} \), we have \( K_b \cap C = \{x\} \). Since \( K_b \cup B = \{b\} \), we have \( K_b \cap B \subseteq (K \cap B) - \{b\} \subseteq K \cap B \), contradicting the choice of \((K, B)\).

But now \( d \in B \) and \( B \cap K_b = \{b\} \), so \( d \notin K_b \). Since \( \{d, d^*\} \subseteq (K_b \cup K) \), we have \( d \in K \) and so \( d^* \in K_b \). Since \( K_b^* \subseteq B \cup \{b\} \), by Lemma 4 there is a basis
\[
B_2 = B \Delta \{b, d, b^*, d^*\} \in \mathcal{B}.
\]
Since \( b, d \in K \) we have \( b, d \notin C - \{x\} \). Thus
\[
C - \{x\} \subseteq B - \{b, d\} \subseteq B_2.
\]
However, \( K \cap B_2 = (K \cap B) - \{b, d\} \subset K \cap B \), which contradicts the choice of \((K, B)\).

Earlier versions of these results, restricted to orthogonal Lagrangian matroids, and with a different (and not quite true) statement of Lemma 6 were obtained in early 2000 by the first author and Alexander Kelmans, whom we thank. We also wish to thank Neil White for suggesting the current form of Lemma 6, and providing a proof of this new statement (less elementary than the proof given here).

3. Circuit axiomatisations

We now proceed to give an alternative definition of Lagrangian matroids in terms of circuits. Let \( \mathcal{C} \) be a collection of admissible subsets of \( J \) such that:

\begin{enumerate}
    \item[(C0)] \( \emptyset \notin \mathcal{C} \).
    \item[(C1)] If \( C_1, C_2 \in \mathcal{C} \) with \( C_1 \subseteq C_2 \), then \( C_1 = C_2 \).
    \item[(C2)] If \( C_1, C_2 \in \mathcal{C} \) with \( C_1 \neq C_2 \), \( x \in C_1 \cap C_2 \) and \( C_1 \cup C_2 \) is admissible, then there exists some \( C \in \mathcal{C} \) with \( C \subseteq (C_1 \cup C_2) - \{x\} \).
    \item[(C3)] For every \( C \in \mathcal{C} \) and \( K \in \mathcal{C}^* \), \( |C \cap K| \neq 1 \).
\end{enumerate}

**Theorem 8.** Let \( \mathcal{B}, \mathcal{C} \) be collections of admissible subsets of \( J \) such that:

\begin{enumerate}
    \item \( \mathcal{B} \) is the collection of maximal admissible subsets not containing members of \( \mathcal{C} \); equivalently,
    \item \( \mathcal{C} \) is the collection of minimal admissible subsets not contained in any member of \( \mathcal{B} \).
\end{enumerate}

Then \( \mathcal{B} \) is the collection of bases of a Lagrangian matroid if and only if \( \mathcal{C} \) satisfies Axioms (C0) to (C3) above.
Note that (C0)–(C2) are the axioms defining ordinary matroids in terms of circuits (in the rather different setting where admissibility is not an issue). Axiom (C3) is also a well-known property of ordinary matroids, but in that setting is not required for a characterisation. To see that it is required here, consider

\[ C = \{1^*, 3^*, 12, 2^*3\} \text{ and thus } \mathcal{B} = \{12^*, 23, 13\}. \]

This \( C \) trivially satisfies all but Axiom (C3), but \( \mathcal{B} \) is not the set of bases of a Lagrangian matroid (or even a symplectic matroid).

The proof of this theorem constitutes the rest of this section. From the previous section, we see at once that axioms (C0)–(C3) hold for the collection of circuits of a Lagrangian matroid. For the remainder of the section, then, we assume that \( C \) is a collection satisfying axioms (C0)–(C3) and \( \mathcal{B} \) a set obtained from it as above. We shall prove (in Lemma 11) that the symmetric exchange property hold on \( \mathcal{B} \), and so that it is the collection of bases of a Lagrangian matroid.

**Lemma 9.** The members of \( \mathcal{B} \) are of size \( n \).

**Proof.** Suppose not; then there is some \( B \in \mathcal{B} \) with \( a, a^* \notin B \). Now \( B \cup \{a\}, B \cup \{a^*\} \) are admissible but not contained in members of \( \mathcal{B} \). Thus there exist \( C_1, C_2 \in \mathcal{C} \) with \( a \in C_1, a^* \in C_2 \), and \( C_1 - \{a\}, C_2 - \{a^*\} \subseteq B \). But then \( C_1 \cap C_2^* = \{a\} \), contradicting Axiom (C3). \( \Box \)

The following lemma corresponds exactly with Lemma 4.

**Lemma 10.** Take \( x \in J \) with \( \{x\} \notin \mathcal{C} \) and \( B \in \mathcal{B} \) with \( x \notin B \). Then either \( B \Delta \{x, x^*\} \) \( \mathcal{C} \) or \( B \cup \{x\} \) contains a unique \( C \in \mathcal{C} \). Furthermore, \( C \) is given by

\[ C = \{x\} \cup \{b \in B \mid B \Delta \{x, b, x^*, b^*\} \in \mathcal{B}\}. \]

**Proof.** If \( B \Delta \{x, x^*\} \in \mathcal{B} \) there is nothing to prove, so assume not. Thus, there is some \( D \in \mathcal{C} \) with \( D \subseteq B \Delta \{x, x^*\} \). We shall take \( C \) as in the last line of the lemma, and show that \( C = D \).

First we must show that \( D \subseteq C \). Certainly \( x \in D \) to avoid \( D \subseteq B \). Let \( y \in D - \{x\} \), and take \( A = B \Delta \{x, y, x^*, y^*\} \). Suppose, for contradiction, that \( A \) contains some \( E \in \mathcal{C} \); then

\[ D^* \cap E \subseteq \left( B^* \cup \{x^*\} \right) \cap A = \{x, y^*\}. \]

We shall show that \( D^* \cap E = \{y^*\} \), contradicting Axiom (C3).

Observe that \( x \notin D^* \), since \( x \in D \) which is admissible. Since \( y \in D \), we have \( y^* \in D^* \); we need only show that \( y^* \in E \). Suppose not; then \( y^* \notin E \) and so \( E \subseteq B \cup \{x\} \), and so since \( E \in \mathcal{C} \) and \( B \in \mathcal{B} \) we obtain \( x \in E \). So \( x \in D \cap E \), and \( D \cup E \) is admissible since \( x^* \notin D \cup E \subseteq B \cup \{x\} \). Thus, by Axiom (C2), there exists a member of \( \mathcal{C} \) inside \( (D \cup E) - \{x\} \subseteq B \in \mathcal{B} \), which contradicts the definition of \( \mathcal{B} \).

This contradiction establishes \( A \in \mathcal{B} \), as required, and so indeed \( D \subseteq C \). Certainly, then, \( C \notin \mathcal{B} \). To show that \( C \in \mathcal{C} \), we must show that \( C - \{y\} \) lies inside some
member of $\mathcal{B}$ for all $y \in C$. This proceeds exactly as in Lemma 4: if $y = x$, then $C - \{y\} \subseteq B \in \mathcal{B}$, and we are done. Otherwise, $C - \{y\} \subseteq B \Delta \{x, x^*, y, y^*\}$, which is a member of $\mathcal{B}$ by definition of $C$.

So we have $C, D \in \mathcal{C}$ with $D \subseteq C$, and applying Axiom (C1) completes the proof. □

**Lemma 11.** Let $A, B \in \mathcal{B}$ with $a \in A - B$. Then there exists $b \in B - A$ such that $B \Delta \{a, a^*, b, b^*\} \in \mathcal{B}$.

**Proof.** If $B \Delta \{a, a^*\} \in \mathcal{B}$ then we can take $b = a^*$. Otherwise, Lemma 10 applies and we have

$$C - \{a\} = \{b \in B \mid B \Delta \{a, b, a^*, b^*\} \in \mathcal{B}\}.$$

Thus, we need only show that there is some $b \in C - A$; but if not then $C \subseteq A$, a contradiction. □

This completes the proof of Theorem 8.

### 3.1. Orthogonal Lagrangian matroids

We can characterise orthogonal Lagrangian matroids by the addition of one more circuit axiom (C4), as follows.

**Theorem 12.** Let $\mathcal{B}, \mathcal{C}$ be collections of admissible subsets of $J$ such that:

1. $\mathcal{B}$ is the collection of maximal admissible subsets not containing members of $\mathcal{C}$; equivalently,
2. $\mathcal{C}$ is the collection of minimal admissible subsets not contained in any member of $\mathcal{B}$.

Then $\mathcal{B}$ is the collection of bases of an orthogonal Lagrangian matroid if and only if $\mathcal{C}$ satisfies:

- (C0) $\emptyset \notin \mathcal{C}$.
- (C1) If $C_1, C_2 \in \mathcal{C}$ with $C_1 \subseteq C_2$, then $C_1 = C_2$.
- (C2) If $C_1, C_2 \in \mathcal{C}$ with $C_1 \neq C_2$, $x \in C_1 \cap C_2$ and $C_1 \cup C_2$ is admissible, then there exists some $C \in \mathcal{C}$ with $C \subseteq (C_1 \cup C_2) - \{x\}$.
- (C3) For every $C \in \mathcal{C}$ and $K \in \mathcal{C}^*$, $|C \cap K| \neq 1$.
- (C4) Every set $X \subseteq J$ of size $n + 1$ with $|X \cap X^*| = 2$ contains a circuit.

**Proof.** Suppose $\mathcal{B}$ is the collection of bases of an orthogonal Lagrangian matroid. Then axioms (C0)–(C3) hold, by Theorem 8. If (C4) does not hold, then there is some $X$ as described containing no circuit. Let $X \cap X^* = \{j, j^*\}$. Then $A = X - \{j\}$ and $B = X - \{j^*\}$ are admissible $n$-sets. Since they are contained in $X$, which contains no circuit, they are bases. But now $B = A \Delta \{j, j^*\}$, which is a contradiction.
Conversely, suppose that \( \mathcal{C} \) satisfies the axioms given. Then, by Theorem 8, \( \mathcal{B} \) is the set of bases of a Lagrangian matroid. Suppose it is not orthogonal; then there are two bases whose symmetric difference is of the form

\[
\{ j_1, j_2, \ldots, j_k, j^*_1, j^*_2, \ldots, j^*_k \},
\]

with \( k \) odd. Now, by the exchange property (Definition 1), we can find intermediate bases by exchanging either two or four elements; thus, we can find two bases \( A, B \in \mathcal{B} \) such that \( A \Delta B = \{ j, j^* \} \) for some \( j \in J \). But then \( X = A \cup B \) is an \((n+1)\)-set, \( X \cap X^* = \{ j, j^* \} \), and \( X \) contains no circuit, contradicting axiom (C4). \( \square \)

The theory of orientations of classical matroids makes use of the fact that, for any circuit \( C \) and distinct pair of elements \( x, y \in C \), there is a cocircuit \( K \) with \( C \cap K = \{ x, y \} \).

This motivates the following:

**Theorem 13.** Let \( \mathcal{M} \) be an orthogonal Lagrangian matroid with set of circuits \( \mathcal{C} \). Then for every circuit \( C \in \mathcal{C} \) and pair of distinct elements \( x, y \in C \), there exists some cocircuit \( K \in \mathcal{C}^* \) with \( C \cap K = \{ x, y \} \).

**Proof.** Take a general circuit \( C \) and distinct \( x, y \in C \). Now, there exists some basis \( B \) containing the independent set \( C - \{ x \} \). Thus \( x \notin B \), so \( y \in B \) and \( x^* \in B \). By Lemma 4 and the orthogonality of \( \mathcal{M}, B \cup \{ y^* \} \) contains a unique circuit \( K^* \). Now, since \( y^* \in K^* \), we have \( y \in C \cap K \). Thus,

\[
\{ y \} \subseteq C \cap K \subseteq (B \cup \{ x \}) \cap (B^* \cup \{ y \}) = \{ x, y \}.
\]

Since, by Lemma 7, \( |C \cap K| \neq 1 \), we see that \( C \cap K = \{ x, y \} \), as required. \( \square \)

This property does not suffice, along with Axioms (C0)–(C3), to characterise Lagrangian orthogonal matroids; the Lagrangian matroid with

\[
\mathcal{B} = \{ 123, 123^*, 1^* 2^* 3, 1^* 2^* 3^* \}
\]

satisfies the conclusion of the theorem, but is not orthogonal.

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