Sufficient Conditions for the Existence of Nonnegative Solutions of a Nonlocal Boundary Value Problem

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Abstract—In this paper, we provide sufficient conditions for the existence of nonnegative solutions of a nonlocal boundary value problem for a second-order ordinary differential equation. By applying Krasnoeleskij's fixed-point theorem in a cone, first we prove the existence of solutions of an auxiliary BVP formulated by truncating the response function. Then the Arzela-Ascoli Theorem is used to take $C^1$ limits of sequences of such solutions. © 2002 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

We establish sufficient conditions ensuring that a second-order ordinary differential equation admits a nonnegative solution, whose slope at the end of times depends on its values on the whole time interval. To be more precise, consider the following ordinary differential equation:

$$x''(t) + q(t)f(x(t), x'(t)) = 0, \quad \text{a.a. } t \in [0, 1],$$

associated with the boundary conditions

$$x(0) = 0 \quad (c_1)$$

and

$$x'(1) = \int_0^1 x'(s) dg(s), \quad (c_2)$$

where $f : I \times \mathbb{R}^2 \to \mathbb{R}$, $q, g : I \to [0, \infty)$ are given functions and in $(c_2)$ the integral is meant in the Riemann-Stieljes sense.

Nonlocal boundary value problems of this form were considered in the early 1960s by Bitsadze [1] and later on by Bitsadze and Samarskii [2] and Il'in and Moiseev [3]. This class of problems includes, as special cases, multipoint boundary value problems considered by many authors (see, e.g., [4–7] and the references therein). Nowadays, the problem of the existence of positive (or of nonnegative) solutions for various types of boundary value problems is the subject

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of many papers. Among others, we refer to the papers [4,8–13] and to the recent book by Agarwal and O'Regan [14]. A very usual technique to get such results is based on fixed-point theorems in cones and especially on the following well-known fixed-point theorem due to Krasnoselskii [15].

**Theorem 1.1.** Let $B$ be a Banach space and let $K$ be a cone in $B$. Assume $\Omega_1, \Omega_2$ are open subsets of $E$, with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$, and let

$$A : K \cap (\Omega_2 \setminus \overline{\Omega_1}) \rightarrow K$$

be a completely continuous operator such that either

$$\|Au\| \leq \|u\|, \quad u \in K \cap \partial \Omega_1,$$

or

$$\|Au\| \geq \|u\|, \quad u \in K \cap \partial \Omega_1,$$

and

$$\|Au\| \leq \|u\|, \quad u \in K \cap \partial \Omega_2.$$

Then $A$ has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega_1})$.

The most common "secret" in applying Theorem 1.1 is the knowledge of the behavior of the response function at 0 and at $+\infty$ relatively to some linear functions, whose slopes satisfy known conditions.

In this paper, we assume that the response function is positive at 0 and that it satisfies an integral condition at $+\infty$. So, our method goes as follows: first we consider truncations of the response function and formulate an (infinite) sequence of BVPs. Then the above fixed-point theorem is used to prove the existence of nonnegative solutions for each of these problems. By applying the classical Arzela-Ascoli Theorem, we conclude that an accumulation point of the family of these solutions exists. By continuous dependence arguments, we conclude that such a point is a solution of the boundary value problem under investigation.

## 2. THE ASSUMPTIONS AND THE BASIC NOTATIONS

In the sequel, we shall denote by $\mathbb{R}$ the real line, by $\mathbb{R}^+$ the interval $[0, \infty)$, and by $I$ the interval $[0,1]$. Let also $C^0_0(I)$ be the space of all functions $x : I \rightarrow \mathbb{R}$, whose first derivative $x'$ is absolutely continuous on $I$ and $x(0) = 0$. The set $C^0_0(I)$ is a Banach space when it is furnished with the norm $\| \|$ defined by

$$\|x\| := \sup \{|x'(t)| : t \in I\}.$$  

Finally, we denote by $L^+_1(I)$ the space of all functions $x : I \rightarrow \mathbb{R}^+$ which are Lebesgue integrable on $I$, endowed with the usual norm $\| \cdot \|_1$.  

Consider equation (e) associated with the conditions (c1), (c2). It is clear that, without loss of generality, we can assume that $g(0) = 0$. By a solution of this boundary value problem we mean a function $x \in C^0_0(I)$ satisfying condition (c2), as well as equation (e) for almost all $t \in I$. Searching for the existence of solutions, we shall first reformulate the problem to an operator equation of the form $x = Ax$, where $A$ is a suitable operator. To find $A$, consider an equation of the form

$$x'' = -z, \quad \text{a.e. on } I,$$

subject to conditions (c1), (c2). By integration we get

$$x'(t) = x'(1) + \int_t^1 z(s) \, ds, \quad t \in I. \quad (2.1)$$

Then from (c2) it follows that

$$x'(1) = \gamma \int_0^1 \int_t^1 z(s) \, ds \, dg(t),$$
where
\[
\gamma = \frac{1}{1 - g(1)}
\]
provided that \(g(1) \neq 1\). From (2.1) and (c₁), we finally obtain
\[
x(t) = \gamma t \int_0^1 \int_s^1 z(r) \, dr \, dg(s) + \int_0^t \int_s^1 z(r) \, dr \, ds, \quad t \in I.
\]
This process shows that solving the boundary value problem (e),(c₁),(c₂) is equivalent to solving the operator equation \(x = Ax \in C₀^1(I)\), where \(A\) is the operator defined by
\[
(Ax)(t) := \gamma t \int_0^1 \int_s^1 q(r)Z(x)(r) \, dr \, dg(s) + \int_0^t \int_s^1 q(r)Z(x)(r) \, dr \, ds,
\]
for \(x \in C₀^1(I)\) and \(t \in I\), where \(Z(x)(t) = f(x(t), x'(t))\). It is clear that \(A\) is a completely continuous operator.

For our convenience, we also define
\[
\Phi(\psi) := \gamma \int_0^1 \int_s^1 q(r)Z \left( \left( \int_0^s \psi \right)(r) \right) \, dr \, dg(s) + \int_0^t \int_s^1 q(r)Z \left( \int_0^t \psi \right)(r) \, dr,
\]
for \(\psi \in AC(I)\), the space of absolutely continuous real-valued functions defined on \(I\), endowed with the sup-norm \(\|\cdot\|_{AC}\), i.e., \(\|\psi\|_{AC} := \sup_{s \in I} |\psi(s)|\), \(\psi \in AC(I)\). Then for every \(x \in C₀^1(I)\) we have
\[
(Ax)'(0) = \Phi(x').
\]

Before presenting our results, we give the notation and the list of our assumptions, which we use in this paper.

Let
\[
M(L) := \max\{f(u, v) : u, v \in [0, L]\}, \quad L > 0,
\]
\[
\sigma := \gamma \int_0^1 \int_s^1 q(r) \, dr \, dg(s) + \|q\|_1 = \gamma \|q\|_1 + \|q\|_1,
\]
and
\[
K_+ := \{x \in C₀^1(I) : x \geq 0, x \text{ is nondecreasing and } x' \text{ is nonincreasing}\},
\]
which is a cone in \(C₀^1(I)\).

(H₁) \(f\) is a real-valued continuous function defined at least on \(I \times \mathbb{R}^2\), satisfying the inequality \(f(u, v) \geq 0\) when \(u > 0, \ v > 0\). Also, \(q \in L_1^1(I)\) and \(g : I \to \mathbb{R}\) is a nondecreasing function with \(0 = g(0) \leq g(1) < 1\) and \(\sigma > 0\).

It is easy to see that, under condition (H₁), the operator \(A\) maps the cone \(K_+\) into itself.

(H₂) It holds that \(f(0, 0) > 0\).

(H₃) There exists a nondecreasing function \(\omega : \mathbb{R}^+ \to (0, +\infty)\) such that
\[
f(u, v) \leq \omega(v), \quad \text{for all } u, v \in \mathbb{R}^+,
\]
and moreover,
\[
\lim_{\tau \to +\infty} \inf_{\gamma \|q\|_1, \omega(\tau)} \frac{dr}{\omega(\tau)} > \|q\|_1.
\]

(H₄) It holds that \(\inf_{L > 0} M(L)/L < 1/\sigma\).
REMARK. If we set 
\[ m(L) := \min\{f(u, v) : u, v \in [0, L]\}, \quad L > 0, \]
then Assumption (H₂) is equivalent to the following:

(H₂₇) It holds that \( \sup_{L>0} m(L)/L = +\infty. \)

Indeed, assume (H₂). Then \( m(T) > 0 \), for some \( T > 0 \). Consider a positive real number \( c \) and set \( N := \min\{T, m(T)/c\}(>0) \). Then observe that \( m(N) \geq m(T) \geq cN \). (Notice that \( m(\cdot) \) is a nonincreasing function.) Hence, \( m(N)/N > c \).

Conversely, if (H₂₇) holds, then there exists a certain \( T > 0 \) such that \( m(T) \geq T \) and so \( f(0, 0) \geq m(T) \geq T > 0. \)

3. THE MAIN RESULTS

Before presenting our main results we give a lemma.

**Lemma 3.1.** Consider the functions \( f, q, \) and \( g \) satisfying Assumptions (H₁) and (H₂). Then there exists \( m > 0 \) such that for any \( x \in K_+ \) with \( ||x|| = m \), we have \( ||Ax|| \geq ||x||. \)

**Proof.** We assume the contrary. Then for every positive integer \( n \), there exists a function \( x_n \in K_+ \), with \( ||x_n|| = n^{-1} \) and \( ||Ax_n|| < ||x_n|| \). Let \( \psi_n := x'_n \). Then for all \( n \) and every \( s \in [0, 1] \) we have

\[ 0 \leq \psi_n(s) \leq \psi_n(0) = ||x_n||, \]

which implies that \( \psi_n \rightarrow 0 \) in \( AC(I) \). So, we must have

\[ 0 \geq \lim_{n \to \infty} \psi_n(0) = \lim_{n \to \infty} \Phi(\psi_n) = \Phi(0) \]

\[ = f(0, 0) \left[ \gamma \int_0^1 \int_s^1 q(r) \, dr \, dg(s) + \|q\|_1 \right] \]

\[ = \sigma f(0, 0), \]

which, due to (H₂), is a contradiction.

Now we are ready to give our first main result.

**Theorem 3.2.** Consider the functions \( f, q, \) and \( g \) satisfying (H₁), (H₂), and (H₃). Then the boundary value problem \((e),(c₁),(c₂)\) has at least one nonnegative solution.

**Proof.** For each positive integer \( n \), define the function

\[ f_n(u, v) := \min\{f(u, v), n\} \]

and consider the problem \((e_n),(c₁),(c₂)\), where \((e_n)\) stands for the equation

\[ x''(t) + q(t)f_n(x(t), x'(t)) = 0, \quad \text{a.a. } t \in [0, 1]. \]

(eₙ)

From (H₃), we have \( f_n(u, v) \leq \omega(v) \) for all \( u, v \in \mathbb{R}^+ \), \( n = 1, 2, \ldots. \) Moreover, let \( A_n \) and \( \Phi_n \) be the operators corresponding to \( A \) and \( \Phi \) given by the relations (2.2) and (2.3), respectively.

Since the function \( f_n \) satisfies Assumption (H₁), by Lemma 3.1, there exists a positive real number \( m_n \) such that for every \( x \in K_+ \) with \( ||x|| = m_n \), it holds that \( ||A_n x|| \geq ||x||. \) Moreover, if \( x \in K_+ \) is such that

\[ ||x|| = n\sigma =: M_n, \]

then it is not hard to see that

\[ ||A_n x|| = \Phi_n x' \leq M_n = ||x||. \]
holds. Hence, by Theorem 1.1, there exists a solution $x_n \in C_0^1(I)$ of the problem (e_n),(c_1),(c_2), such that $m_n \leq \|x_n\| \leq M_n$.

Now we claim that the set $\{x_n : n = 1, 2, \ldots\}$ is a precompact subset of $C_0^1(I)$. To prove the claim, we shall use the classical Arzela-Ascoli Theorem. Thus, it is enough to show that the sets $\{x'_n : n = 1, 2, \ldots\}$ and $\{x''_n : n = 1, 2, \ldots\}$ are bounded. Keep also in mind that $x_n(0) = 0$ for all $n = 1, 2, \ldots$.

Let $n$ be a fixed index and define

$$y_n := x'_n.$$ 

Then observe that $y_n \geq 0 \geq y'_n$, and for every $t \in I$ we have

$$0 \leq -y'_n(t) \leq q(t)\omega(y_n(t)).$$  \tag{3.1}

This implies that

$$\int_{y_n(1)}^{y_n(0)} \frac{dr}{\omega(r)} \leq \|q\|_1. \tag{3.2}$$

On the other hand, from condition (c_2) and the fact that $g(0) = 0$, we get

$$y_n(1) = \int_0^1 y_n(s) dg(s) = y_n(1)g(1) - \int_0^1 y'_n(s)g(s) \, ds$$

$$= y_n(1)g(1) + \int_0^1 q(s)f_n(x_n(s), x'_n(s))g(s) \, ds$$

$$\leq y_n(1)g(1) + \int_0^1 q(s)\omega(y_n(s))g(s) \, ds.$$ 

Thus,

$$y_n(1) \leq \gamma \omega(y_n(0))\|qg\|_1.$$ 

Taking into account (3.2), we obtain

$$\int_{\gamma \omega(y_n(0))\|qg\|_1}^{y_n(0)} \frac{dr}{\omega(r)} \leq \|q\|_1.$$ 

Now, if the sequence $(y_n(0))$ is not bounded, by taking a subsequence, if necessary, we can assume that $y_n(0) \to +\infty$. This fact implies that

$$\liminf_{r \to +\infty} \int_{\gamma \omega(r)\|qg\|_1}^{r} \frac{dr}{\omega(r)} \leq \|q\|_1,$$

contrary to (2.4). Thus, the sequence $(y_n(0))$ is bounded and, by (3.1), also the sequence $(y'_n)$ is bounded. Our claim is proved.

Consequently, we can assume that the sequence $(x_n)$ converges in $C_0^1(I)$ to a certain $x$. This is equivalent to saying that $x_n \to x$ and $x'_n \to x'$ uniformly on $I$. Then, from the equation (e_n), by using continuous dependence arguments, we can easily obtain that $x$ is a nonnegative solution of the problem (e),(c_1),(c_2).

**LEMMA 3.3.** Consider the functions $f$, $q$, and $g$ satisfying Assumptions (H_1) and (H_4). Then, there exists a certain $T > 0$ such that for every $x \in K_+$, with $\|x\| = T$, we have $\|Ax\| \leq \|x\|$.

**PROOF.** Because of (H_4), there exists $T > 0$ such that

$$\frac{M(T)}{T} \leq \frac{1}{\sigma}.$$
Let $x$ be a point in the cone $K_+$, with $\|x\| = x'(0) = T$. Then for every $r \in I$, the numbers $x(r), x'(r)$ belong to the interval $[0, T]$, and hence, we obtain

$$\|Ax\| = (Ax)'(0) = \gamma \int_0^1 \int_s^1 q(r)f(x(r), x'(r)) \, dr \, dg(s) + \int_0^1 q(r)f(x(r), x'(r)) \, dr \leq M(T) \left[ \gamma \int_0^1 \int_s^1 q(r) \, dr \, dg(s) + \|q\| \right] = M(T) \sigma \leq T = \|x\|.$$ 

Here we give the following result.

**Theorem 3.4.** Consider the functions $f$, $q$, and $g$ satisfying Assumptions (H₁), (H₂), and (H₄). Then the boundary value problem (e₁),(c₁),(c₂) has at least one nonnegative solution.

**Proof.** This follows by applying Theorem 1.1, when we take into account Lemmas 3.1 and 3.3 and set $\Omega_1 := \{x \in C^1_0(I) : \|x\| < r_1\}$, $\Omega_2 := \{x \in C^1_0(I) : \|x\| < r_2\}$, with $r_1 := \min\{m, T\}$ and $r_2 := \max\{m, T\}$.

**Example 1.** Consider the following boundary value problem:

$$x''(t) + \frac{1}{2} \cos^2 x(t) + \beta x'(t) = 0, \quad t \in [0, 1],$$
$$x(0) = 0,$$
$$x'(1) = \frac{1}{2} x'\left(\frac{1}{2}\right).$$

We observe that for

$$g(t) = \begin{cases} 
0, & \text{if } 0 \leq t < \frac{1}{2}, \\
\frac{1}{2}, & \text{if } \frac{1}{2} \leq t \leq 1,
\end{cases}$$

the boundary condition (c₂) reduces to the boundary condition (c). Moreover, if we set

$$\omega(v) := \frac{1}{2} + \beta v,$$

we can see that (2.4) is satisfied if and only if

$$\beta e^\theta < 2.$$ 

So, for every $\beta$ such that $\beta e^\theta < 2$ (hence, for every $\beta \leq 0.85259$), we have that Assumption (H₄) is fulfilled. Since, obviously, Assumptions (H₁) and (H₂) are also satisfied, by Theorem 3.2, the boundary value problem (e₁),(c₁),(c) has at least one nonnegative solution, provided that $\beta \leq 0.85259$.

For the same boundary value problem, we can also use Assumption (H₄). To this end, we set

$$M(L) := \frac{1}{2} + \beta L, \quad L > 0.$$ 

Then, since $\sigma = 3/2$ and $\inf_{L \in (0, \infty)} M(L)/L = \beta$, by Theorem 3.4 we conclude that for every $\beta < 2/3$, the boundary value problem (e₁),(c₁),(c) has at least one nonnegative solution. However, as one can see, Theorem 3.3 provides a better upper bound for $\beta$. On the other hand, in the following example we see that Theorem 3.4 gives existence results, but Theorem 3.3 cannot be applied at all.

**Example 2.** Consider the boundary value problem (e₂),(c₁),(c), where (e₂) stands for the equation

$$x''(t) + \frac{1}{2} \cos^2 x'(t) + \beta x(t) = 0, \quad t \in [0, 1].$$

(e₂)
It is clear that for this boundary value problem, Assumptions (H₁), (H₂) are satisfied. Also, as in the previous example, we have \( M(L) = 1/2 + \beta L \) and \( \sigma = 3/2 \). So, for every \( \beta < 2/3 \), Assumption (H₄) is satisfied and, so, the boundary value problem \((e₂),(c₁),(c)\) has at least one nonnegative solution.

It is clear that Assumption (H₃) cannot be applied for the boundary value problem \((e₂),(c₁),(c)\).

**REFERENCES**