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# Properties of four partial orders on standard Young tableaux ☆

Müge Taşkin

*School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA*

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**Abstract**

Let $\mathrm{SYT}_n$ be the set of all standard Young tableaux with $n$ cells. After recalling the definitions of four partial orders, the weak, KL, geometric and chain orders on $\mathrm{SYT}_n$ and some of their crucial properties, we prove three main results:

- Intervals in any of these four orders essentially describe the product in a Hopf algebra of tableaux defined by Poirier and Reutenauer.
- The map sending a tableau to its descent set induces a homotopy equivalence of the proper parts of all of these orders on tableaux with that of the Boolean algebra $2^{[n-1]}$. In particular, the Möbius function of these orders on tableaux is $(-1)^{n-3}$.
- For two of the four orders, one can define a more general order on skew tableaux having fixed inner boundary, and similarly analyze their homotopy type and Möbius function.



*Keywords:* Robinson–Schensted algorithm (RSK); Standard Young tableaux; Skew standard tableaux; Partial orders; Möbius function and poset homotopy type

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*E-mail address:* taskin@math.umn.edu.

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## 1. Introduction

This paper is about four partial orders on the set $\mathrm{SYT}_n$ of all standard Young tableaux of size $n$ satisfying:

$$\text{weak order} \subsetneq \text{Kazhdan–Lusztig (KL) order} \subseteq \text{geometric order} \subsetneq \text{chain order}.$$

Here $P \subset Q$ means that $u \leqslant v$ in $P$ implies $u \leqslant v$ in $Q$, in which case we say $Q$ is *stronger* than $P$ (or $P$ is *weaker* then $Q$).

All four of these orders have appeared in the work of Melnikov [30–32], who refers to what we are calling the weak order as the *induced Duflo order*. Roughly speaking,

- the weak order is induced from the weak Bruhat order on the symmetric group $\mathfrak{S}_n$ via the Robinson–Schensted insertion map,
- the KL order is induced by the Kazhdan–Lusztig preorder on $\mathfrak{S}_n$ arising in the theory of Kazhdan–Lusztig (right) cells,
- geometric order describes inclusions of certain algebraic varieties indexed by tableaux (*orbital varieties*), and
- chain order is induced by the dominance order on partitions; for each interval of values $[i, j]$, one restricts the tableau to these values and compares the insertion shapes in dominance order.

All four of these orders on $\mathrm{SYT}_n$ coincide for $n \leqslant 5$, and are depicted in Fig. 1. For $n \geqslant 6$, they differ (see Examples 3.6 and 3.8). After reviewing their definitions in Section 2, we recall some of their known properties in Section 3.

We then prove three main new results. The first result, proven in Section 4, relates to a Hopf algebra defined by Poirier and Reutenauer [34] whose basis elements are indexed by standard Young tableaux $T$ of all sizes. The multiplication in this Hopf algebra is somewhat nontrivial to describe, but turns out to be described essentially by any of our four partial orders.[1]

**Theorem 1.1.** *For any of the four partial order $\leqslant$ above, one has*

$$T * S = \sum_{\substack{R \in \mathrm{SYT}_n: \\ T/S \leqslant R \leqslant T\backslash S}} R,$$

*where $T/S$ and $T\backslash S$ are obtained by sliding $S$ over $T$ from the left and from the bottom, respectively.*

The second result is about the Möbius function and homotopy type of these orders. The weak Bruhat order on $\mathfrak{S}_n$ is well-known to have each interval homotopy equivalent to either a sphere or a point, and hence have Möbius function values all in $\{\pm 1, 0\}$. Although it is not true in general for the intervals in the weak, KL, geometric and chain orders on $\mathrm{SYT}_n$ (see Fig. 2 for some examples) the interval from bottom to top is homotopy equivalent to either a sphere or a point. This result is proven in Section 5, by associating descent sets to tableaux and thereby obtaining a poset map to a Boolean algebra.

[1] This result for the weak order was asserted without proof in [16, middle of p. 579].

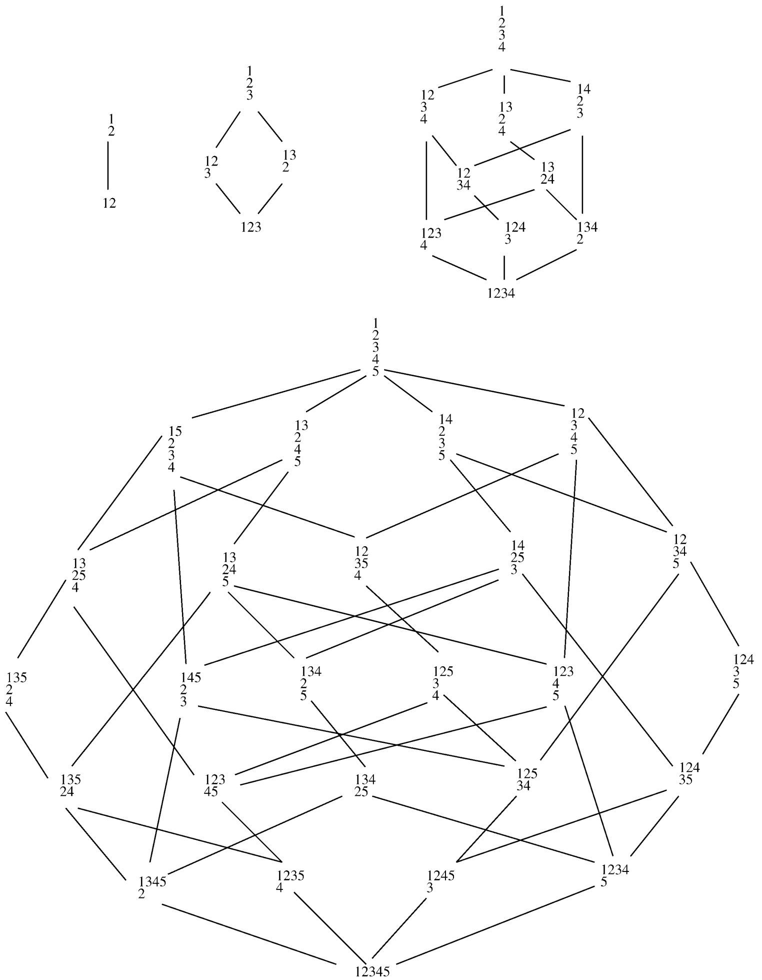



Fig. 1. Chain, the weak, KL and geometric order on $\mathrm{SYT}_n$, which coincide for $n = 2, 3, 4, 5$ (but not in general).

**Theorem 1.2.** *Let $\leqslant$ be any of the four partial orders. Then the map $\mathrm{SYT}_n \mapsto 2^{[n-1]}$ sending a tableau to its descent set is order-preserving, and induces a homotopy equivalence of the proper parts.*

*In particular, for any such order $\mu(\hat{0}, \hat{1}) = (-1)^{n-3}$.*

The third result, proven in Section 7 deals with a generalization of the above orders to skew tableaux with fixed inner boundary. The most crucial step in the proof is the application of Rambau's Suspension Lemma [35] which makes the proof (compared to the standard methods in topological combinatorics) much shorter and comprehensible. Given a partition $\mu$, let $\mathrm{SYT}_n^{\mu}$ denote the set of all skew standard tableaux of having $n$ cells which are "skewed by $\mu$," that is, whose shape is $\lambda/\mu$ for some $\lambda$. It turns out that two of the four orders (KL, geometric) have a property (the *inner translation* property; see Theorem 6.4) which allows us to generalize them

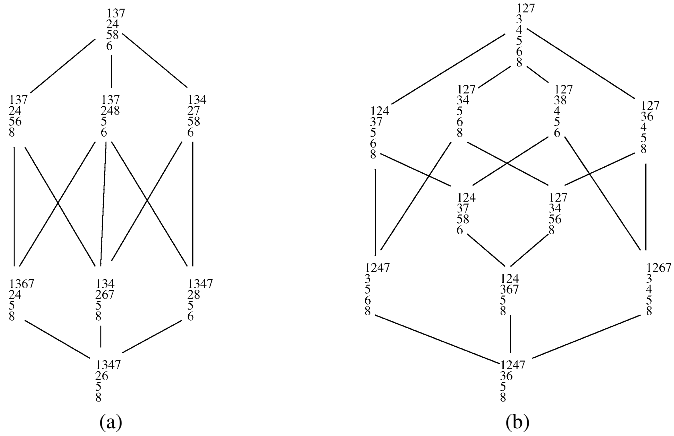



Fig. 2. (a) An interval in $(\mathrm{SYT}_8, \leqslant_{\mathrm{chain}})$ and (b) an interval in $\mathrm{SYT}_8$ ordered with $\leqslant_{\mathrm{weak}}$, $\leqslant_{\mathrm{KL}}^{\mathrm{op}}$ and $\leqslant_{\mathrm{geom}}$ having Möbius function 2 and $-2$, respectively.

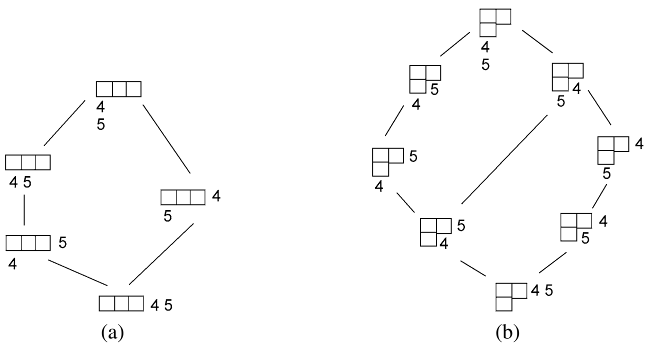



Fig. 3. An illustration of the skew orders on $\mathrm{SYT}_n^{\mu}$ for $n = 2$. When $\mu$ is (a) rectangular and (b) nonrectangular, $\mathrm{SYT}_2^{\mu}$ has its proper part homotopy equivalent to a 0-dimensional sphere and a point, respectively.

on $\mathrm{SYT}_n^{\mu}$. Each of these skew orders has a top element $\hat{1}$ and a bottom element $\hat{0}$, so that one can speak of the homotopy of their *proper parts* obtained by removing $\hat{0}, \hat{1}$.

**Theorem 1.3.** *Let $\leqslant$ be* KL *or geometric orders on* $\mathrm{SYT}_n$. *Then the associated order $\leqslant$ on* $\mathrm{SYT}_n^{\mu}$ *has the homotopy type of its proper part equal to that of*

$$\begin{cases} \text{an } (n-2)\text{-dimensional sphere} & \text{if } \mu \text{ is rectangular,} \\ \text{a point} & \text{otherwise.} \end{cases}$$

*In particular, for any such order either $\mu(\hat{0}, \hat{1}) = (-1)^{n-2}$ or $\mu(\hat{0}, \hat{1}) = 0$, depending on $\mu$.*

Figure 3 provides an illustration for Theorem 1.3 where both posets are considered with KL or geometric orders. In fact, this theorem follows from a more general statement (Proposition 7.1) about the homotopy types of certain intervals, which applies to any order between the weak and chain orders (including the weak order itself).

We close this section with some context and motivation for Theorems 1.1 and 1.2, stemming from two commutative diagrams that appear in the work of Loday and Ronco [26]

$$\begin{tikzcd} \mathfrak{S}_n \arrow[r] \arrow[dr] & Y_n \arrow[d] \\ & 2^{[n-1]} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{Z}\mathfrak{S} & \mathbb{Z}Y \arrow[l] \\ & \Sigma \arrow[u] \arrow[ul] \end{tikzcd} \tag{1.1}$$

In the left diagram of (1.1), $Y_n$ denotes the set of planar binary trees with $n$ vertices. The horizontal map sends a permutation $w$ to a certain tree $T(w)$, and has been considered in many contexts (see e.g. [43, §1.3], [7, §9]). The southeast map $\mathfrak{S}_n \to 2^{[n-1]}$ sends a permutation $w$ to its descent set $\mathrm{Des}_L(w)$. These maps of sets become order-preserving if one orders $\mathfrak{S}_n$ by weak order, $Y_n$ by the *Tamari order* (see [7, §9]), and $2^{[n-1]}$ by inclusion.

Indeed, the order preserving maps of the first diagram induce the inclusions of Hopf algebras in the second diagram of (1.1), in which $\mathbb{Z}\mathfrak{S}$ is the Malvenuto–Reutenauer algebra, $\mathbb{Z}Y$ is a subalgebra isomorphic to Loday and Ronco's free *dendriform algebra* on one generator [25], and $\Sigma$ is a subalgebra known as the algebra of *noncommutative symmetric functions*. In [26], Loday and Ronco proved a description of the product structure for each of these three algebras very much analogous to Theorem 1.1, which should be viewed as the analogue replacing $\mathbb{Z}Y$ by $\mathbb{Z}\mathrm{SYT}$; see Theorem 4.1 below for their description of the product in $\mathbb{Z}\mathfrak{S}$. The analogy between the standard Young tableaux $\mathrm{SYT}_n$ and the planar binary trees $Y_n$ is tightened further by recent work of Hivert et al. [16]. They show that the planar binary trees $Y_n$ can be interpreted as the plactic monoid structure given by a Knuth-like relation, similar to the interpretation of the set of standard Young tableaux as Knuth/plactic classes.

We were further motivated in proving Theorem 1.1 by the results of Aguiar and Sottile in [1,2] where the Möbius functions of the weak order on $\mathfrak{S}_n$ and Tamari order on $Y_n$ have key roles in understanding the structures of the Hopf algebras of permutations and planar binary trees.

In [7, Remark 9.12], Björner and Wachs (essentially) show that the triangle on the left induces a diagram of homotopy equivalences on the proper parts of the posets involved. Theorem 1.2 gives the analogue of this statement in which one replaces $(Y_n, \leqslant_{\mathrm{Tamari}})$ by $(\mathrm{SYT}_n, \leqslant)$ where $\leqslant$ is any order between the weak and chain orders.

## 2. Definitions

### *2.1. Chain order*

The first partial order on $\mathrm{SYT}_n$ that will be discussed is the strongest one: *chain order*.

Given $T \in \mathrm{SYT}_n$, we denote by $\mathrm{sh}(T)$ the partition corresponding to the shape of $T$. For $1 \leqslant i < j \leqslant n$, let $T_{[i,j]}$ be the skew subtableau obtained by restricting $T$ to the segment $[i, j]$. Let $\mathrm{std}(T_{[i,j]})$ be the tableau obtained by lowering all entries of $T_{[i,j]}$ by $i-1$ and sliding it into normal shape by jeu-de-taquin [39].

The definition of chain order also involves the dominance order. We denote by $(\mathrm{Par}_n, \leqslant_{\mathrm{dom}}^{\mathrm{op}})$ the set of all partitions of the number $n$ ordered by the *opposite* (*or dual*) *dominance order*, that is, $\lambda \leqslant_{\mathrm{dom}}^{\mathrm{op}} \mu$ if

$$\lambda_1 + \cdots + \lambda_k \geqslant \mu_1 + \cdots + \mu_k \quad \text{for all } k.$$

**Definition 2.1.** Let $S, T \in \mathrm{SYT}_n$ and We say $S$ is less that $T$ in chain order ($S \leqslant_{\text{chain}} T$) if for every $1 \leqslant i < j \leqslant n$,

$$\mathrm{sh}\big(\mathrm{std}(S_{[i,j]})\big) \leqslant^{\mathrm{op}}_{\mathrm{dom}} \mathrm{sh}\big(\mathrm{std}(T_{[i,j]})\big).$$

*2.2. Weak order*

Before giving the definition of the weak order it is necessary to recall the Robinson–Schensted (RSK) correspondence; see [36, §3] for more details and references on RSK. The RSK correspondence is a bijection between $\mathfrak{S}_n$ and $\{(P, Q)\colon\ P, Q \in \mathrm{SYT}_n$ of same shape$\}$. Here $P$ and $Q$ are called the *insertion* and *recording tableau* respectively. Knuth [21] defined an equivalence relation $\sim_K$ on $\mathfrak{S}_n$ with the property that $u \sim_K w$ if and only if they have the same insertion tableaux $P(u) = P(w)$. We will denote the corresponding equivalence classes in $\mathfrak{S}_n$ by $\{C_T\}_{T \in \mathrm{SYT}_n}$.

We now recall the (*right*) *weak Bruhat order*, $\leqslant_{\text{weak}}$, on $\mathfrak{S}_n$. It is the transitive closure of the relation $u \leqslant_{\text{weak}} w$ if $w = u \cdot s_i$ for some $i$ with $u_i < u_{i+1}$, and where $s_i$ is the adjacent transposition $(i\ i+1)$. The weak order has an alternative characterization [5, Proposition 3.1] in terms of (*left*) *inversion sets*

$$\mathrm{Inv}_L(u) := \big\{(i, j)\colon\ 1 \leqslant i < j \leqslant n \text{ and } u^{-1}(i) > u^{-1}(j)\big\},$$

namely $u \leqslant_{\text{weak}} w$ if and only if $\mathrm{Inv}_L(u) \subset \mathrm{Inv}_L(w)$.

For $1 \leqslant i < j \leqslant n$ let $[i, j]$ be a segment of the alphabet $[n]$ and $u_{[i,j]}$ be the subword of $u$ obtained by restricting to the alphabets $[i, j]$ and $\mathrm{std}(u_{[i,j]})$ in $\mathfrak{S}_{j-i+1}$ be the word obtained from $u_{[i,j]}$ by subtracting $i-1$ from each letter.

In fact $\mathrm{Inv}_L(u) \subset \mathrm{Inv}_L(w)$ gives $\mathrm{Inv}_L(u_{[i,j]}) \subset \mathrm{Inv}_L(w_{[i,j]})$ for all $1 \leqslant i < j \leqslant n$ and hence

$$u \leqslant_{\text{weak}} w \quad \text{implies} \quad u_{[i,j]} \leqslant_{\text{weak}} w_{[i,j]} \quad \text{for all } 1 \leqslant i < j \leqslant n. \tag{2.1}$$

The following basic fact about RSK, Knuth equivalence, and jeu-de-taquin are essentially due to Knuth and Schützenberger; see Knuth [20, Section 5.1.4] for detailed explanations.

**Lemma 2.2.** *Given $u \in \mathfrak{S}_n$, let $P(u)$ be the insertion tableau of $u$. Then for $1 \leqslant i < j \leqslant n$,*

$$\mathrm{std}\big(P(u)_{[i,j]}\big) = P\big(\mathrm{std}(u_{[i,j]})\big).$$

Furthermore one can use Greene's theorem [14] for the following fact:

$$\text{If} \quad u \leqslant_{\text{weak}} w \quad \text{then} \quad \mathrm{sh}\big(\mathrm{std}\big(P(u)_{[i,j]}\big)\big) \leqslant^{\mathrm{op}}_{\mathrm{dom}} \mathrm{sh}\big(\mathrm{std}\big(P(u)_{[i,j]}\big)\big) \quad \text{for all } 1 \leqslant i < j \leqslant n. \tag{2.2}$$

Now (2.1) and (2.2) shows that the following order is weaker than chain order on $\mathrm{SYT}_n$ and hence it is well defined.

**Definition 2.3.** The *weak order* $(\mathrm{SYT}_n, \leqslant_{\text{weak}})$, first introduced by Melnikov [30] under the name *induced Duflo order*, is the partial order induced by taking transitive closure of the following rule. Denoting the Knuth class of $T$ by $C_T$,

$$S \leqslant_{\text{weak}} T \quad \text{if there exist } \sigma \in C_S,\ \tau \in C_T \quad \text{such that } \sigma \leqslant_{\text{weak}} \tau.$$

The necessity of taking the transitive closure in the definition of the weak order is illustrated by the following example (cf. Melnikov [30, Example 4.3.1]).

**Example 2.4.** Let $R = \begin{smallmatrix} 1 & 2 & 5 \\ 3 & 4 \end{smallmatrix}$, $S = \begin{smallmatrix} 1 & 4 & 5 \\ 2 \\ 3 \end{smallmatrix}$, $T = \begin{smallmatrix} 1 & 4 \\ 2 & 5 \\ 3 \end{smallmatrix}$ with

$$C_R = \{31425, 34125, 31452, 34152, 34512\},$$
$$C_S = \{32145, 32415, 32451, 34215, 34251, 34521\},$$
$$C_T = \{32154, 32514, 35214, 32541, 35241\}.$$

Here $R <_{\text{weak}} S$ since $34125 <_{\text{weak}} 34215 = 34125 \cdot s_3$, and $S <_{\text{weak}} T$ since $32145 <_{\text{weak}} 32154 = 32145 \cdot s_4$. Therefore $R <_{\text{weak}} T$.

On the other hand, for every $\rho \in C_R$ one has $(2,4) \in \mathrm{Inv}_L(\rho)$, whereas for every $\tau \in C_T$ one has $(2,4) \notin \mathrm{Inv}_L(\tau)$. This shows that there is no $\rho \in C_R$ and $\tau \in C_T$ such that $\rho <_{\text{weak}} \tau$.

### 2.3. *Kazhdan–Lusztig order*

It turns out that RSK is closely related to *Kazhdan–Lusztig* preorders on $\mathfrak{S}_n$. Recall that a *preorder* on a set $X$ is a binary relation $\leqslant$ which is reflexive ($x \leqslant x$) and transitive ($x \leqslant y$, $y \leqslant z$ implies $x \leqslant z$). It need not be antisymmetric, that is, the equivalence relation $x \sim y$ defined by $x \leqslant y$, $y \leqslant x$ need not have singleton equivalence classes. Note that a preorder induces a *partial order* on the set $X/\!\sim$ of equivalence classes.

Kazhdan and Lusztig [19] introduced two preorders (the left and right KL preorders) on Coxeter groups whose equivalence classes are called the left and right cells respectively. The theory of left (or right) cells provides a decomposition of the regular representation of the Hecke algebras of Coxeter groups (cf. [8, Chapter 6]) such that, in case the Coxeter group is $\mathfrak{S}_n$, each summand is irreducible.

In this paper we will denote by $\leqslant_{\mathrm{KL}}^{\mathrm{op}}$ the *opposite* of the usual KL right preorder on $\mathfrak{S}_n$. For example, with our convention, the identity element 1 and the longest element $w_0$ satisfy $1 \leqslant_{\mathrm{KL}}^{\mathrm{op}} w_0$. It turns out [19] (and explicitly in [12, p. 54]) that the associated equivalence relation for this KL preorder is the Knuth equivalence $\sim_K$. Hence an equivalence class (usually called either a *Knuth class* or *plactic class* or a *Kazhdan–Lusztig right cell* in $\mathfrak{S}_n$) corresponds to a tableau $T$ in $\mathrm{SYT}_n$.

**Definition 2.5.** KL *order* on $\mathrm{SYT}_n$ is defined by the rule

$$S \leqslant_{\mathrm{KL}}^{\mathrm{op}} T \quad \text{if } C_S \leqslant_{\mathrm{KL}}^{\mathrm{op}} C_T$$

where $C_S$ is the Knuth class (or KL right cell) in $\mathfrak{S}_n$ corresponding to $S \in \mathrm{SYT}_n$.

For later use, we now recall the basic construction of the KL right preorder on $\mathfrak{S}_n$. Recall that the *right descent set* $D_R(u)$ and the *left descent set* $D_L(u)$ of a permutation $u \in \mathfrak{S}_n$, are defined by

$$\mathrm{Des}_R(u) := \{(i, i+1)\colon\ 1 \leqslant i \leqslant n-1 \text{ and } u(i) > u(i+1)\},$$
$$\mathrm{Des}_L(u) := \{(i, i+1)\colon\ 1 \leqslant i \leqslant n-1 \text{ and } u^{-1}(i) > u^{-1}(i+1)\} = \mathrm{Inv}_L(u) \cap S,$$

where $S = \{(i, i+1)\colon\ 1 \leqslant i \leqslant n-1\}$. In what follows, we will often identify the set $S$ of adjacent transpositions with the numbers $[n-1] := \{1, 2, \ldots, n-1\}$ via the obvious map $(i, i+1) \mapsto i$.

In [19], Kazhdan and Lusztig prove the existence of unique polynomials $\{P_{u,w}(q)\} \subseteq \mathbb{Z}[q]$ indexed by permutations in $\mathfrak{S}_n$. Denoting by $\leqslant$ the Bruhat order on $\mathfrak{S}_n$, $l(u)$ the length of the permutation $u$ and $l(u,w) = l(w) - l(u)$, these polynomials satisfy:

$$P_{u,w}(q) = 1 \quad \text{if } u = w,$$
$$P_{u,w}(q) = 0 \quad \text{if } u \nleqslant w,$$
$$\deg\bigl(P_{u,w}(q)\bigr) \leqslant \frac{1}{2}\bigl(l(u,w) - 1\bigr). \tag{2.3}$$

Let $[q^k]P_{u,w}(q)$ denote the coefficient of $q^k$ in $P_{u,w}(q)$ and define

$$\bar{\mu}(u,w) := \begin{cases} [q^{\frac{l(u,w)-1}{2}}]P_{u,sw}(q) & \text{if } l(u,w) \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

Then a recursive formula for these polynomials is given in the following way: For $u \leqslant w$ and $s \in D_L(w)$,

$$P_{u,w}(q) = q^{1-c}P_{su,sw}(q) + q^c P_{u,sw}(q) - \sum_{\{v:\ s \in D_L(v)\}} q^{l(v,w)/2}\bar{\mu}(v,sw)P_{u,v}(q) \tag{2.5}$$

where $c = 1$ if $s \in D_L(u)$ and $c = 0$ otherwise. Moreover the dual of right $KL$ preorder on $\mathfrak{S}_n$ is given by taking the transitive closure of the following relation:

$$u \leqslant^{\mathrm{op}}_{\mathrm{KL}} w \quad \text{if} \begin{cases} D_R(w) - D_R(u) \neq \emptyset, \\ \text{and} \\ \bar{\mu}(u,w) \neq 0 \text{ or } \bar{\mu}(w,u) \neq 0. \end{cases} \tag{2.6}$$

*2.4. Geometric order*

The final order on $\mathrm{SYT}_n$ to be discussed in this paper relates to the preorder on $\mathfrak{S}_n$ induced from geometric order on the orbital varieties associated to the Lie algebra $\mathfrak{sl}_n$.

The theory of orbital varieties arise from the work of Spaltenstein [41,42] and Steinberg [44,45] on the unipotent variety of a connected complex semi-simple group $G$. They have a key role in the studies of primitive ideals (i.e. annihilators of irreducible representations) in the enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of Lie algebra $\mathfrak{g}$ corresponding to $G$ (cf. [11,24,33]). They also play an important role in Springer's Weyl group representations.

Let $\mathfrak{g}$ be the Lie algebra of $G$ and $B$ be the Borel subgroup of $G$ given with respect to some triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ such that $\mathfrak{h}$ is a Cartan subalgebra and $\mathfrak{n}$ is the corresponding nilradical.

For given $\eta \in \mathfrak{n}$, we denote by $\mathcal{O}_\eta$ the nilpotent orbit determined by the adjoint action of $G$ on $\eta$. Therefore $\bar{\mathcal{O}}_\eta$ is an irreducible variety. Now an *orbital variety* $\mathcal{V}$ associated to $\mathcal{O}_\eta$ is defined to be an irreducible component of the intersection $\mathcal{O}_\eta \cap \mathfrak{n}$. Given orbital varieties $\mathcal{V}$ and $\mathcal{W}$, the *geometric order* is defined by

$$\mathcal{V} \leqslant_{\mathrm{geom}} \mathcal{W} \quad \text{if } \mathcal{W} \subseteq \bar{\mathcal{V}}$$

where $\bar{\mathcal{V}}$ denotes the Zariski closure of $\mathcal{V}$ inside $\mathfrak{n}$. The only general description of orbital varieties provided below is due Steinberg [44].

Given a positive root system $R^+ \subset \mathfrak{h}^*$, recall that $\mathfrak{n} = \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha$ where $\mathfrak{g}_\alpha$ is the root space corresponding to $\alpha$. Let $W$ be the Weyl group of $\mathfrak{g}$ generated by simple roots in $R^+$, and for $w \in W$ let

$$\mathfrak{n} \cap^w \mathfrak{n} := \bigoplus_{\alpha \in\ w(R^+) \cap R^+} \mathfrak{g}_\alpha.$$

Since $B$ is an irreducible closed subgroup of $G$, the action of $B$ on $\mathfrak{n} \cap^w \mathfrak{n}$ gives an irreducible locally closed subvariety $B(\mathfrak{n} \cap^w \mathfrak{n})$ which, therefore, lies in a unique nilpotent orbit $\mathcal{O}_\eta$ for some $\eta \in \mathfrak{n}$ and $\overline{G(\mathfrak{n} \cap^w \mathfrak{n})} = \bar{\mathcal{O}}_\eta$. By the result of Steinberg

$$\mathcal{V}_w := \overline{B(\mathfrak{n} \cap^w \mathfrak{n})} \cap \mathcal{O}_\eta \tag{2.7}$$

is an orbital variety and the map $w \mapsto \mathcal{V}_w$ is a surjection. Moreover geometric order induces a preorder on $W$ such that, for $u, w \in W$

$$u \leqslant_{\text{geom}} w \quad \text{if } \mathcal{V}_w \subset \bar{\mathcal{V}}_u \text{ or equivalently } \overline{B(\mathfrak{n} \cap^w \mathfrak{n})} \subseteq \overline{B(\mathfrak{n} \ \cap^u \mathfrak{n})}. \tag{2.8}$$

According to Steinberg [44], the fibers of the map $w \mapsto \mathcal{V}_w$ for $\mathfrak{g} = \mathfrak{sl}_n$ are the Knuth classes of $\mathfrak{S}_n$ and therefore each orbital variety $\mathcal{V}$ in $\mathfrak{sl}_n$ can be identified with some $T \in \mathrm{SYT}_n$ i.e., $\mathcal{V} = \mathcal{V}_T$. This leads to the following definition.

**Definition 2.6.** The *geometric order* on $\mathrm{SYT}_n$, $(\mathrm{SYT}_n, \leqslant_{\text{geom}})$, is given by the following rule:

$$S \leqslant_{\text{geom}} T \quad \text{if } \mathcal{V}_T \subset \bar{\mathcal{V}}_S.$$

When $\mathfrak{g} = \mathfrak{sl}_n$, an explicit description of orbital varieties can be given in the following way. Let $B$ be the Borel subgroup of invertible upper triangular $n \times n$ matrices given by the Cartan decomposition of $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$ of trace 0 diagonal matrices and nilradicals $\mathfrak{n}$ and $\mathfrak{n}^-$, whose elements are strictly upper and strictly lower triangular matrices, respectively. Then the set of matrices $\{E_{ij}\}_{i<j}$ (respectively $\{E_{ij}\}_{i>j}$), where $E_{i,j}$ has 1 at the position $(i, j)$ and 0 elsewhere, provides a basis for $\mathfrak{n}$ (respectively $\mathfrak{n}^-$).

The action of the Weyl group $\mathfrak{S}_n$ on $E_{i,j}$ can be described by

$$w \cdot E_{i,j} = p_w E_{i,j}\ p_w^{-1} = E_{w(i),w(j)}$$

where $p_w$ is the permutation matrix of $w \in \mathfrak{S}_n$ and this leads to the following characterization

$$\mathfrak{n} \cap^w \mathfrak{n} = \operatorname{span}\bigl\{E_{i,j} \mid (i, j) \notin \mathrm{Inv}_L(w)\bigr\}. \tag{2.9}$$

On the other hand, the adjoint action of $B$ on $E_{i,j}$ sweeps the corner at $(i, j)$ to the northeast direction. In other words $B \cdot E_{i,j}$ consists of all matrices of rank 1, having a nonzero entry at $(i, j)$ and with all other nonzero entries are located at some positions to the northeast of $(i, j)$. Therefore all matrices in $\overline{B(\mathfrak{n} \cap^w \mathfrak{n})}$ have their nonzero entries in some boundary provided by $\{B \cdot E_{i,j} \mid (i, j) \notin \mathrm{Inv}_L(w)\}$, and $\mathcal{V}_w = \overline{B(\mathfrak{n} \cap^w \mathfrak{n})} \cap \mathcal{O}_\eta$ consists of all those matrices in $\overline{B(\mathfrak{n} \cap^w \mathfrak{n})}$ whose Jordan form is the same as that of $\eta$. Recall that $\eta$ is uniquely determined by the condition $\overline{G(\mathfrak{n} \cap^w \mathfrak{n})} = \bar{\mathcal{O}}_\eta$. Actually one can show that the partition determined by the Jordan form of $\eta$ and the partition obtained from $w$ through the RSK correspondence are the same.

There is also a bijection, revealed by Steinberg [45], between the orbital varieties determined by $\eta$ and Springer fiber $\mathcal{F}_\eta$ of the complete flag variety $\mathcal{F}$. Moreover geometric order results in an ordering between the irreducible components of $\mathcal{F}_\eta$. We next discuss this connection.

Let $\lambda = J(\eta)$ be the Jordan form of $\eta$, $\mathcal{O}_\lambda = \{\eta \mid J(\eta) = \lambda\}$ be the $\mathrm{GL}(V)$-orbit of $\eta$ and

$$\tilde{\mathcal{O}}_\lambda := \bigl\{(\eta, f) \mid \eta \in \mathcal{O}_\lambda,\ f \in \mathcal{F}\colon\ \eta(f) \subset f\bigr\}.$$

Here $\mathrm{GL}(V)$ acts on $\mathcal{O}_\lambda$ and $\mathcal{F}$ by conjugation and left translation respectively; therefore it acts on $\tilde{\mathcal{O}}_\lambda$, and the projections onto $\mathcal{O}_\lambda$ and $\mathcal{F}$ are equivariant maps. We have the following diagram:

$$\begin{tikzcd} & \tilde{\mathcal{O}}_\lambda \arrow[dl] \arrow[dr] & \\ \mathcal{O}_\lambda & & \mathcal{F} \end{tikzcd}$$

In this diagram, the fiber of any $\eta \in \mathcal{O}_\lambda$ is equal to $\mathcal{F}_\eta := \{(\eta, f)\colon\ f \in \mathcal{F},\ \eta(f) \subset f\}$. Since $\mathrm{GL}(V)$ is irreducible and its action on $\mathcal{O}_\lambda$ is transitive, the irreducible components of this *Springer fiber* $\mathcal{F}_\eta$ are in bijection with the irreducible components of $\tilde{\mathcal{O}}_\lambda$. On the other hand, for any $f \in \mathcal{F}$, let $B$ be the Borel subgroup of $\mathrm{GL}(V)$ which fixes $f$ and let $\mathfrak{n}$ be nilradical of the corresponding Borel algebra $\mathfrak{b}$. Then the fiber of $f$ is equal to $\{(\eta, f)\colon\ \eta \in \mathcal{O}_\lambda \cap \mathfrak{n}\}$ and again the transitivity of the action of $\mathrm{GL}(V)$ on $\mathcal{F}$ implies that the irreducible components of $\mathcal{O}_\lambda \cap \mathfrak{n}$ are in bijection with the irreducible components of $\tilde{\mathcal{O}}_\lambda$. These two bijections determine the correspondence between the orbital varieties and the irreducible components of Springer fibers in the flag variety. The geometric order describes the inclusions among (the closures of) these components as one varies $\lambda$, in either context.

## 3. Known properties

In this section we recall some of the main properties of these four orders which we need later in proving our main results. These properties also can be found in or deduced from the works of Melnikov [30,31,33] and Barbash and Vogan [3]. In order to make these posets more understandable we provide the proofs of those which are combinatorially approachable, while for those which need theoretical approaches the reader is directed to the references.

### *3.1. Restriction to segments*

For $u \in \mathfrak{S}_n$ and $T \in \mathrm{SYT}_n$ recall the definitions of $\mathrm{std}(u_{[i,j]})$ and $\mathrm{std}(T_{[i,j]})$ from Sections 2.2 and 2.1, respectively. Say that a family of preorders $\leqslant$ on $\mathfrak{S}_n$ *restricts to segments* if

$$u \leqslant w \quad \text{implies} \quad \mathrm{std}(u_{[i,j]}) \leqslant \mathrm{std}(w_{[i,j]}) \quad \text{for all } 1 \leqslant i < j \leqslant n.$$

Melnikov shows in [30, p. 45] the preorder $\leqslant_{\mathrm{geom}}$ on the Weyl group $W$ of any reductive Lie algebra restricts to $W_I$, where $I$ is any subset of simple roots generating $W_I$. Therefore geometric order on $\mathfrak{S}_n$ restricts to segments. The same fact about KL preorder was first shown by Barbash and Vogan [3] for arbitrary finite Weyl groups (see also work by Lusztig [28]) whereas the generalization to Coxeter groups is due to Geck [13, Corollary 3.4]. On the other hand, this result for the weak order on $\mathfrak{S}_n$ follows from an easy analysis on the (left) inversion sets.

We say the order $\leqslant$ on $\mathrm{SYT}_n$ *restricts to segments* if

$$S \leqslant T \quad \text{implies} \quad \mathrm{std}(S_{[i,j]}) \leqslant \mathrm{std}(T_{[i,j]}) \quad \text{for all } 1 \leqslant i < j \leqslant n.$$

The following result for the weak, KL and geometric order on $\mathrm{SYT}_n$ is an easy consequence of the above discussion together with Lemma 2.2, whereas for chain order it follows directly from its definition.

**Corollary 3.1.** *On* $\mathrm{SYT}_n$ *all of the four orders restrict to segments of standard Young tableaux.*

In fact any order $\leqslant$ on $\mathrm{SYT}_n$ which is stronger than the weak order and which restricts to segments shares a crucial property that we describe now.

Recall that (*left*) *descent set* of a permutation $\tau$ is defined by

$$\mathrm{Des}_L(\tau) := \{i\colon\ 1 \leqslant i \leqslant n-1 \text{ and } \tau^{-1}(i) > \tau^{-1}(i+1)\}.$$

As a consequence of a well-known properties of RSK, the left descent set $\mathrm{Des}_L(-)$ is constant on Knuth classes $C_T$; the *descent set* of the standard Young tableau $T$ is described intrinsically by

$$\mathrm{Des}(T) := \{(i, i+1)\colon\ 1 \leqslant i \leqslant n-1 \text{ and } i+1 \text{ appears in a row below } i \text{ in } T\}.$$

We let $(2^{[n-1]}, \subseteq)$ be the Boolean algebra of all subsets of $[n-1]$ ordered by inclusion.

**Lemma 3.2.** *Let $\leqslant$ be any order on $\mathrm{SYT}_n$ which is stronger than the weak order and restricts to segments. Then the map*

$$(\mathrm{SYT}_n, \leqslant) \mapsto \left(2^{[n-1]}, \subseteq\right)$$

*sending any tableau $T$ to its descent set $\mathrm{Des}(T)$ is order preserving.*

**Proof.** For $n = 2$, such an order is isomorphic to weak order on $\mathrm{SYT}_2$ and the statement follows directly by examination of Fig. 1. For $n > 2$, one can use the fact that

$$\mathrm{Des}_L(T) = \mathrm{Des}_L(T_{[1,n-1]}) \cup \mathrm{Des}_L(T_{[2,n]})$$

to get the desired result by induction. $\square$

*3.2. Poset morphisms*

For the record, we note here some symmetries and order-preserving maps of $\leqslant_{\mathrm{chain}}$, $\leqslant_{\mathrm{weak}}$, $\leqslant^{\mathrm{op}}_{\mathrm{KL}}$ and $\leqslant_{\mathrm{geom}}$ on $\mathrm{SYT}_n$, to other posets.

**Proposition 3.3.** *Let $\leqslant$ represent to any of the orders $\leqslant_{\mathrm{weak}}$, $\leqslant^{\mathrm{op}}_{\mathrm{KL}}$ $\leqslant_{\mathrm{geom}}$ or $\leqslant_{\mathrm{chain}}$ on $\mathrm{SYT}_n$. Then the following maps are order preserving*:

(i) *The map*

$$(\mathrm{SYT}_n, \leqslant) \to \left(2^{[n-1]}, \subseteq\right)$$

*sending a tableau $T$ to its descent set $\mathrm{Des}(T)$.*

(ii) *The map*

$$(\mathrm{SYT}_n, \leqslant) \to \left(\mathrm{Par}_n, \leqslant^{\mathrm{op}}_{\mathrm{dom}}\right)$$

*sending $T$ to its shape $\lambda(T)$.*

*On the other hand, for $\leqslant$ equal to any of the orders $\leqslant_{\mathrm{weak}}$, $\leqslant^{\mathrm{op}}_{\mathrm{KL}}$, $\leqslant_{\mathrm{geom}}$ or $\leqslant_{\mathrm{chain}}$.*

(iii) *The Schützenberger's evacuation map*

$$(\mathrm{SYT}_n, \leqslant) \to (\mathrm{SYT}_n, \leqslant)$$

*sending $T$ to its evacuation tableau $T^{\mathrm{evac}}$ is an poset automorphism,*

*whereas for $\leqslant$ equal to $\leqslant_{\mathrm{weak}}$, $\leqslant^{\mathrm{op}}_{\mathrm{KL}}$ or $\leqslant_{\mathrm{chain}}$.*

(iv) *The map*

$$(\mathrm{SYT}_n, \leqslant) \to (\mathrm{SYT}_n, \leqslant)$$

*sending $T$ to its transpose $T^t$ is a poset anti-automorphism.*

**Proof.** The first assertion follows from Lemma 3.2, since all of the four orders are stronger than the weak order and restrict to segments.

Second assertion for $\leqslant_{\mathrm{chain}}$ follows from its definition. For $\leqslant_{\mathrm{weak}}$, as it mentioned earlier, one can apply Greene's theorem [14]. If $S \leqslant_{\mathrm{geom}} T$ then there are orbital varieties given by $\mathcal{V}_S$ and $\mathcal{V}_T$ such that $\mathcal{V}_T \subseteq \bar{\mathcal{V}}_S$. Now the nilpotent orbits that these orbital varieties belong to can be characterized by the partition given by $\mathrm{sh}(S)$ and $\mathrm{sh}(T)$. Moreover we have $\mathcal{O}_{\mathrm{sh}(T)} \subseteq \bar{\mathcal{O}}_{\mathrm{sh}(S)}$. By the result of Gerstenhaber, see [11, Chapter 6] for example, last inclusions implies $\mathrm{sh}(T) \leqslant_{\mathrm{dom}} \mathrm{sh}(S)$, proving the statement for geometric order. For KL order the proof based on the theory that relates the Kazhdan–Lusztig cells to the primitive ideals: let $\mathfrak{g}$ be a semisimple algebra with universal enveloping Lie algebra $U(\mathfrak{g})$ and Weyl group $W$. As it is shown in [3] and [9], for any primitive ideal $I$ of $U(\mathfrak{g})$, the set of the form $\{w \in W \mid I_w = I\}$ can be characterized by a Kazhdan–Lusztig left cell. Moreover $v <_{\mathrm{KL}}^{\mathrm{op}} w$ (right dual KL order) if and only if $I_{v^{-1}} \subseteq I_{w^{-1}}$, whence the associated variety of the primitive ideal $I_{w^{-1}}$ is contained in that of $I_{v^{-1}}$. On the other hand, by the result of Borho and Brylinski [9] and Joseph [17] associated variety of a primitive ideal is the closure of a nilpotent orbit in $\mathfrak{g}^*$. In our case $\mathfrak{g} = \mathfrak{sl}_n$, $W = \mathfrak{S}_n$ and the nilpotent orbits are characterized by partitions of $n$, therefore the result of Gerstenhaber reveals the desired property on the shapes of the corresponding tableaux of $v$ and $w$.

The assertions about transposition and evacuation for $\leqslant_{\mathrm{KL}}^{\mathrm{op}}$ and $\leqslant_{\mathrm{weak}}$, follow from the fact that the involutive maps

$$w \mapsto w_0 w \quad \text{and} \quad w \mapsto w w_0$$

are antiautomorphisms of both $(\mathfrak{S}_n, \leqslant_{\mathrm{KL}}^{\mathrm{op}})$ [12] and $(\mathfrak{S}_n, \leqslant_{\mathrm{weak}})$. Hence $w \mapsto w_0 w w_0$ is an automorphism of both. On the other hand, $P(ww_0)$ is just the transpose tableau of $P(w)$ [37] and $P(w_0 w w_0)$ is nothing but the evacuation of $P(w)$ [38].

Indeed $w_0 w$ and $w w_0$ correspond reversing the value and the order of numbers in $w$, respectively. Therefore by Greene's theorem they reverse the dominance order on the RSK insertion shapes which then gives the desired property for $(\mathrm{SYT}_n, \leqslant_{\mathrm{chain}})$.

The assertion that Schützenberger's evacuation map gives a poset automorphism of $(\mathrm{SYT}_n, \leqslant_{\mathrm{geom}})$ follows from Melnikov's work [31, pp. 17–18]. □

**Question 3.4.** (See discussion by Van Leeuwen [23, §8].) Is the map which sends a tableau to its transpose an anti-automorphism of the geometric order?

By part (ii) of the Proposition 3.3, if $S \leqslant T$ under the weak, KL, geometric or chain orders then $\mathrm{sh}(S) \leqslant_{\mathrm{dom}}^{\mathrm{op}} \mathrm{sh}(T)$. Actually we have a stronger condition for the first three orders which is given in Proposition 3.5 below. On the other hand, Example 3.6 shows that this property is not satisfied by chain order.

**Proposition 3.5.** *Let $\leqslant$ be any of $\leqslant_{\mathrm{weak}}$, $\leqslant_{\mathrm{KL}}^{\mathrm{op}}$ or $\leqslant_{\mathrm{geom}}$ on $\mathrm{SYT}_n$. Then*

$$S \lneqq T \;\Rightarrow\; \mathrm{sh}(S) \lneqq_{\mathrm{dom}}^{\mathrm{op}} \mathrm{sh}(T)$$

*e.g., under these orders the shape of the tableaux is not fixed.*

**Proof.** For $\leqslant_{\mathrm{KL}}^{\mathrm{op}}$, this property can be induced from the work of Lusztig [27] which result in the conclusion that, for $\mathfrak{S}_n$ right cells given by the tableaux of the same shape form an antichain in the KL order.

For $\leqslant_{\mathrm{geom}}$, Gerstenhaber's result mentioned in the proof of Proposition 3.3(ii) gives the required property; if $\mathrm{sh}(S) = \mathrm{sh}(T) = \lambda$, the orbital varieties $\mathcal{V}_T$ and $\mathcal{V}_S$ lie in the same nilpotent orbit $\mathcal{O}_\lambda$. As being the irreducible components of $\mathcal{O}_\lambda \cap \mathfrak{n}$ they satisfy neither $\mathcal{V}_T \subseteq \bar{\mathcal{V}}_S$ nor $\mathcal{V}_S \subseteq \bar{\mathcal{V}}_T$. Therefore $T$ and $S$ are not comparable under $\leqslant_{\mathrm{geom}}$ and this proves the hypothesis.

Now $\leqslant_{\mathrm{weak}}$ satisfy the hypothesis since it is weaker then KL and geometric orders. □

**Example 3.6.** The following tableaux have $T \gneqq_{\mathrm{chain}} S$ although they have the same shape.

$$T = \begin{matrix} 1 & 3 & 6 \\ 2 & 4 & \\ 5 & & \end{matrix} \quad \text{and} \quad S = \begin{matrix} 1 & 3 & 4 \\ 2 & 6 & \\ 5 & & \end{matrix}\ .$$

## 3.3. Embedding

It is known that the (right) weak order on $\mathfrak{S}_n$ is weaker than the (right) KL preorder on $\mathfrak{S}_n$ [19, p. 171]. As it is described, for instance in [30, p. 9], the weak order is also weaker than geometric order on $\mathfrak{S}_n$. Therefore by the virtue of its definition $(\mathrm{SYT}_n, \leqslant_{\mathrm{weak}})$ embeds in $(\mathrm{SYT}_n, \leqslant_{\mathrm{KL}})$ and $(\mathrm{SYT}_n, \leqslant_{\mathrm{geom}})$.

On the other hand, by Corollary 3.1 and by Proposition 3.3(ii) the weak, KL and geometric orders on $\mathrm{SYT}_n$ are weaker then chain order.

The following important result, which reveals that KL order embeds in geometric order on $\mathrm{SYT}_n$, can be deduced from the work of Melnikov [32, Corollary 1.2], Borho and Brylinski [10, 6.3] and Vogan [46].

**Theorem 3.7.** *On $\mathfrak{S}_n$,* KL *order is weaker than geometric order. Therefore for all $S, T \in \mathrm{SYT}_n$,*

$$S \leqslant_{\mathrm{KL}}^{\mathrm{op}} T \ \Rightarrow\ S \leqslant_{\mathrm{geom}} T.$$

It happens that all these four orders coincide for $n \leqslant 5$, but they start to differ for $n = 6$. Proposition 3.5 and the Example 3.6 provided above show that $(\mathrm{SYT}_n, \leqslant_{\mathrm{chain}})$ differs from all the other orders for $n = 6$. The following examples reveals the same fact for $(\mathrm{SYT}_n, \leqslant_{\mathrm{weak}})$ (cf. Melnikov [30, Example 4.1.6]).

**Example 3.8.** Let $S = \begin{smallmatrix} 1&2&3 \\ 4&5&6 \end{smallmatrix}$, $T_1 = \begin{smallmatrix} 1&2&5 \\ 3&6& \\ 4&& \end{smallmatrix}$ and $T_2 = \begin{smallmatrix} 1&3&6 \\ 2&4& \\ 5&& \end{smallmatrix}$ .

Computer calculations show that $S \leqslant_{\mathrm{KL}}^{\mathrm{op}} T_1, T_2$, but $S \nleqslant_{\mathrm{weak}} T_1, T_2$. By using the anti-automorphism of $\leqslant_{\mathrm{KL}}^{\mathrm{op}}$ and $\leqslant_{\mathrm{weak}}$ that transposes a standard Young tableau (see Proposition 3.3) one obtains two more examples of pairs of tableaux which are comparable in $\leqslant_{\mathrm{KL}}^{\mathrm{op}}$, but not in $\leqslant_{\mathrm{weak}}$. These are the *only* such examples in $\mathrm{SYT}_6$.

To summarize we have the following diagram:

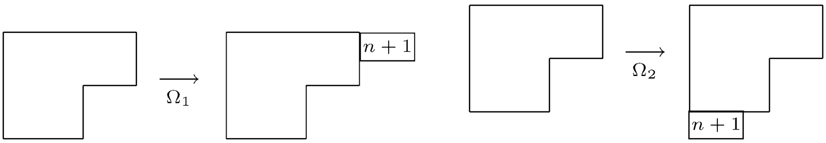
$$(\mathrm{SYT}_n, \leqslant_{\mathrm{weak}}) \subsetneqq \left(\mathrm{SYT}_n, \leqslant_{\mathrm{KL}}^{\mathrm{op}}\right) \subseteq (\mathrm{SYT}_n, \leqslant_{\mathrm{geom}}) \subsetneqq (\mathrm{SYT}_n, \leqslant_{\mathrm{chain}}).$$

**Question 3.9.** Do $(\mathrm{SYT}_n, \leqslant_{\mathrm{KL}}^{\mathrm{op}})$ and $(\mathrm{SYT}_n, \leqslant_{\mathrm{geom}})$ coincide?

### *3.4. Extension from segments*

In this section we discuss two order preserving maps which embed $\mathrm{SYT}_n$ into $\mathrm{SYT}_{n+1}$ under any of the four orders.

Denoted by $\Omega_1$ and $\Omega_2$, these maps are given by the following rule: For each $T \in \mathrm{SYT}_n$, $\Omega_1 : \mathrm{SYT}_n \mapsto \mathrm{SYT}_{n+1}$ concatenates $n+1$ to the first row of $T$ from the right whereas $\Omega_2 : \mathrm{SYT}_n \mapsto \mathrm{SYT}_{n+1}$ concatenates $n+1$ to the first column of $T$ from the bottom i.e.,




**Definition 3.10.** Any partial order $\leqslant$ on $\mathrm{SYT}_n$ is said to have the property of *extension from segments* if the maps $\Omega_1, \Omega_2 : \mathrm{SYT}_n \mapsto \mathrm{SYT}_{n+1}$ are order preserving.

In what follows we will prove that all of the four orders have the extension from segments property.

**Lemma 3.11.** *The maps $\Omega_1$ and $\Omega_2$ are order preserving under the weak,* KL*, the geometric and chain orders.*

**Proof.** For any $T \in \mathrm{SYT}_n$ and $\tau \in C_T$, let $\tau(n+1)$ and $(n+1)\tau$ be the words obtained by concatenating $n+1$ to $\tau$ from the right and respectively from the left. The RSK insertion algorithm yields that

$$P\big(\tau(n+1)\big) = \Omega_1(T) \quad \text{and} \quad P\big((n+1)\tau\big) = \Omega_2(T).$$

Conventionally, we use the following notation:

$$\Omega_1(\tau) := \tau(n+1) \quad \text{and} \quad \Omega_2(\tau) := (n+1)\tau.$$

*Chain order*: Let $S \leqslant_{\mathrm{chain}} T$ in $\mathrm{SYT}_n$, i.e., for any $1 \leqslant i < j \leqslant n$ one has

$$\mathrm{sh}\big(\mathrm{std}(S_{[i,j]})\big) \leqslant_{\mathrm{dom}}^{\mathrm{op}} \mathrm{sh}\big(\mathrm{std}(T_{[i,j]})\big).$$

Now concatenating $n+1$ to the first row of $S$ and $T$ from the right (after applying jeu de taquin slides) obviously does not affect $\mathrm{sh}(\mathrm{std}(S_{[i,j]}))$ and $\mathrm{sh}(\mathrm{std}(T_{[i,j]}))$ if $j < n+1$, and both have $n+1$ added to first row if $j = n+1$. Therefore

$$\Omega_1(S) \leqslant_{\mathrm{chain}} \Omega_1(T).$$

On the other hand, by Proposition 3.3(iv) one has:

$$S \leqslant_{\mathrm{chain}} T \;\Rightarrow\; S^t \geqslant_{\mathrm{chain}} T^t \;\Rightarrow\; \Omega_1\big(S^t\big)^t \leqslant_{\mathrm{chain}} \Omega_1\big(T^t\big)^t$$

and since $\Omega_1(S^t)^t = \Omega_2(S)$ for any tableau $S$, now $\Omega_2$ is also order preserving.

*Weak order*: For this it is enough to consider the covering relations of $(\mathrm{SYT}_n, \leqslant_{\mathrm{weak}})$. If $S$ is covered by $T$ then there exist two permutations $\sigma \in C_S$ and $\tau \in C_T$ such that $\sigma \leqslant_{\mathrm{weak}} \tau$. Equivalently $\mathrm{Inv}_L(\sigma) \subset \mathrm{Inv}_L(\tau)$. On the other hand, the last assertion implies

$$\mathrm{Inv}_L\big(\Omega_1(\sigma)\big) \subset \mathrm{Inv}_L\big(\Omega_1(\tau)\big) \quad \text{and} \quad \mathrm{Inv}_L\big(\Omega_2(\sigma)\big) \subset \mathrm{Inv}_L\big(\Omega_2(\tau)\big).$$

Therefore in either case the weak order relation is preserved and we have

$$\Omega_1(S) \leqslant_{\text{weak}} \Omega_1(T) \quad \text{and} \quad \Omega_2(S) \leqslant_{\text{weak}} \Omega(T).$$

KL *order*: This fact for KL order can be deduced easily by considering $\mathfrak{S}_n$ as a parabolic subgroup of $\mathfrak{S}_{n+1}$: any two permutations $v, w \in \mathfrak{S}_n$ satisfying $v \leqslant^{\text{op}}_{\text{KL}} w$ in the parabolic subgroup $\mathfrak{S}_n$ still have the same relation in $\mathfrak{S}_{n+1}$.

If $S \leqslant^{\text{op}}_{\text{KL}} T$ then there exist $\sigma \in C_S$ and $\tau \in C_T$ satisfying $\sigma \leqslant^{\text{op}}_{\text{KL}} \tau$ in $\mathfrak{S}_n$. Then concatenating $n+1$ to the right side of both words still yields $\Omega_1(\sigma) \leqslant^{\text{op}}_{\text{KL}} \Omega_1(\tau)$ in $\mathfrak{S}_{n+1}$. Hence $\Omega_1(S) \leqslant^{\text{op}}_{\text{KL}} \Omega_1(T)$ and by Proposition 3.3(iv) $\Omega_2(S) \leqslant^{\text{op}}_{\text{KL}} \Omega_2(T)$.

*Geometric order*: This fact follows from the result of Melnikov [33, Proposition 6.6]. □

### 3.5. *Extension by* RSK *insertions*

In [30], Melnikov indicates another extension property of the weak and geometric order on $\text{SYT}_n$ which also generalize the property of extension from segments. Let $\leqslant$ be any order on $\text{SYT}_n$, $i \leqslant n$ and $\acute{S}$ and $\acute{T}$ are some tableaux on $[n] - \{i\}$. Suppose $S$ and $T$ are the tableaux in $\text{SYT}_{n-1}$ obtained by standardizing $\acute{S}$ and $\acute{T}$, respectively. Define an order on $\acute{S}$ and $\acute{T}$ in the following way

$$\acute{S} \leqslant \acute{T} \quad \text{if} \quad S \leqslant T.$$

Then $\leqslant$ is said to have the property of *extension by* RSK *insertions* if the RSK insertion of the element $i$ into both tableaux $\acute{S}$ and $\acute{T}$ from above (or from the left) still preserves the order, in other words, denoting the resulting tableaux by $\acute{S}^{\downarrow i}$ and $\acute{T}^{\downarrow i}$, if one has

$$\acute{S} \leqslant \acute{T} \;\Rightarrow\; \acute{S}^{\downarrow i} \leqslant \acute{T}^{\downarrow i}.$$

The property of extension by RSK insertions for the weak order and geometric order was first proven by Melnikov in [30,33], respectively. The same fact for KL order can be deduced from the work of Barbash and Vogan [3, 2.34, 3.7] by using the theory that relates Kazhdan–Lusztig (left) cells to primitive ideals. Below, independently from this theory, we provide a proof that shows KL order has the property of extension by RSK insertions. On the other hand, the following example shows that chain order does not have this property:

$$\acute{T} = \begin{matrix} 1 & 3 & 7 \\ 2 & 5 & \\ 6 & & \end{matrix} \;\geqslant_{\text{chain}}\; \begin{matrix} 1 & 3 & 5 \\ 2 & 7 & \\ 6 & & \end{matrix} = \acute{S} \quad \text{but} \quad \acute{T}^{\downarrow 4} = \begin{matrix} 1 & 3 & 4 \\ 2 & 5 & 7 \\ 6 & & \end{matrix} \;\ngeqslant_{\text{chain}}\; \begin{matrix} 1 & 3 & 4 \\ 2 & 5 & \\ 6 & 7 & \end{matrix} = \acute{S}^{\downarrow 4}.$$

**Lemma 3.12.** KL *order on* $\text{SYT}_n$ *has the extension by* RSK *insertions property.*

**Proof.** Let $\acute{S}$ and $\acute{T}$ be two tableaux on $[n] - \{i\}$ such that $\acute{S} \leqslant^{\text{op}}_{\text{KL}} \acute{T}$. In other words for $S$ and $T$ which are obtained by standardizing $\acute{S}$ and $\acute{T}$, respectively, we have $S \leqslant^{\text{op}}_{\text{KL}} T$. We may assume that $S$ is covered by $T$. Then there exist $\sigma$ and $\tau$ in the Knuth classes of $S$ and $T$, respectively such that $\sigma <^{\text{op}}_{\text{KL}} \tau$ in $\mathfrak{S}_{n-1}$. Since $\mathfrak{S}_{n-1}$ is a parabolic subgroup of $\mathfrak{S}_n$, as Lemma 3.11 for the KL order shows, concatenating $n$ to the right side of both permutations yields $\sigma n <^{\text{op}}_{\text{KL}} \tau n$ in $\mathfrak{S}_n$. Therefore we have

$$D_R(\tau n) - D_R(\sigma n) \neq \emptyset \quad \text{and} \quad \begin{cases} \text{either} & \sigma n \leqslant \tau n \text{ and } \bar{\mu}(\sigma n, \tau n) \neq 0, \\ \text{or} & \tau n \leqslant \sigma n \text{ and } \bar{\mu}(\tau n, \sigma n) \neq 0, \end{cases}$$

where $\leqslant$ denotes Bruhat order. Without lost of generality we assume $\sigma n \leqslant \tau n$ and $\bar{\mu}(\sigma n, \tau n) \neq 0$.

Consider the permutations $s_i s_{i+1} \dots s_{n-1}(\sigma n)$ and $s_i s_{i+1} \dots s_{n-1}(\tau n)$ which are obtained by multiplying $\sigma n$ and $\tau n$ from the left by the transpositions $s_{n-1}, s_{n-2}, \dots, s_{i+1}, s_i$ in this order. It is easy to check that the RSK insertion tableaux of $s_i s_{i+1} \dots s_{n-1}(\sigma n)$ and $s_i s_{i+1} \dots s_{n-1}(\tau n)$ are nothing but $\acute{S}^{\downarrow i}$ and respectively $\acute{T}^{\downarrow i}$. Then $\acute{S}^{\downarrow i} \leqslant_{\mathrm{KL}}^{\mathrm{op}} \acute{T}^{\downarrow i}$ follows, once it is shown that

$$s_i s_{i+1} \dots s_{n-1}(\sigma n) \leqslant_{\mathrm{KL}}^{\mathrm{op}} s_i s_{i+1} \dots s_{n-1}(\tau n). \tag{3.1}$$

Let

$$u_n = \sigma n \quad \text{and} \quad w_n = \tau n$$

and for each $k$ such that $i \leqslant k \leqslant n-1$, let

$$u_k = s_k \dots s_{n-1}(\sigma n) \quad \text{and} \quad w_k = s_k \dots s_{n-1}(\tau n).$$

Obviously for each $i \leqslant k \leqslant n$, analysis on the (left) inversion sets yields

$$l(u_k, w_k) = l(\sigma n, \tau n) \tag{3.2}$$

and one can check that

$$u_k \leqslant w_k \tag{3.3}$$

by using a basic characterization of Bruhat order. That is: $u \leqslant w$ in $\mathfrak{S}_n$ if and only if for each $j \leqslant n$, the sets of the form $\{u_1, \dots, u_j\}$ and $\{w_1, \dots, w_j\}$ can be compared in the manner that after ordering their elements from the smallest to the biggest, the $i$th element of the first set is always smaller than or equal to the $i$th element of the second set for each $i \leqslant j$.

On the other hand, multiplying $\sigma n$ and $\tau n$ by $s_k \dots s_{n-2} s_{n-1}$ from the left does not change the right descents of these permutations on the first $n-1$ positions. In other words, when restricted to the first $n-1$ positions $\sigma n$ and $u_k$ (similarly $\tau n$ and $w_k$) share the same right descents. Therefore

$$D_R(\tau n) - D_R(\sigma n) \neq \emptyset \Rightarrow D_R(w_k) - D_R(u_k) \neq \emptyset. \tag{3.4}$$

Now we will show that

$$P_{u_k, w_k}(q) = P_{\sigma n, \tau n}(q).$$

Obviously $P_{u_n, w_n}(q) = P_{\sigma n, \tau n}(q)$ and therefore it is enough to prove that $P_{u_k, w_k}(q) = P_{u_{k+1}, w_{k+1}}(q)$, since then the required equality follows by induction. Observe that $u_k = s_k u_{k+1}$, $w_k = s_k w_{k+1}$ i.e., both of them are permutations in $\mathfrak{S}_n$ ending with the number $k$. So $s_k$ lies both in $D_L(u_k)$ and $D_L(w_k)$ and by (2.5)

$$P_{u_k, w_k}(q) = P_{u_{k+1}, w_{k+1}}(q) + q P_{u_k, w_{k+1}}(q) - \sum_{\{v:\ s_k \in D_L(v)\}} q^{l(v,w)/2} \bar{\mu}(v, w_{k+1}) P_{u_k, v}(q).$$

Since $u_k$ ends with $k$ and $w_{k+1}$ ends with $k+1$, from the characterization of the Bruhat order it follows that $u_k \nleqslant w_{k+1}$ and furthermore there exist no permutation $v$ satisfying $u_k \leqslant v \leqslant w_{k+1}$. Then by (2.3), all the summation terms on the right-hand side, except $P_{u_{k+1}, w_{k+1}}(q)$, are equal to 0. Henceforth

$$P_{u_k, w_k}(q) = P_{u_{k+1}, w_{k+1}}(q)$$

and $P_{u_k, w_k}(q) = P_{\sigma n, \tau n}(q)$ follows by induction. This result together with (2.4) and (3.2) imply that

$$\bar{\mu}(u_k, w_k) = \bar{\mu}(\sigma n, \tau n) \neq 0. \tag{3.5}$$

Therefore by (2.6), (3.3), (3.4) and (3.5) we have $u_k \leqslant_{\mathrm{KL}}^{\mathrm{op}} w_k$ for each $i \leqslant k \leqslant n-1$ and so (3.1) is true. Hence $\acute{S}^{\downarrow i} \leqslant_{\mathrm{KL}}^{\mathrm{op}} \acute{T}^{\downarrow i}$. $\square$

## 4. Proof of Theorem 1.1

Malvenuto and Reutenauer, in [29] construct two graded Hopf algebra structures on the $\mathbb{Z}$ module of all permutations $\mathbb{Z}S = \bigoplus_{n\geqslant 0}\mathbb{Z}S_n$ which are dual to each other, and shown to be free as associative algebras by Poirier and Reutenauer in [34]. The product structure of the one that concerns us here is given by

$$u * w := \mathrm{shf}(u, \bar{w})$$

where $\bar{w}$ is obtained by increasing the indices of $w$ by the length of $u$ and shf denotes the shuffle product.

Poirier and Reutenauer also show that $\mathbb{Z}$ module of all plactic classes $\{\mathrm{PC}_T\}_{T\in\mathrm{SYT}}$, where $\mathrm{PC}_T = \sum_{P(u)=T} u$ becomes a Hopf subalgebra of permutations whose product (also shown in [22,40]) is given by the formula

$$\mathrm{PC}_T * \mathrm{PC}_{T'} = \sum_{\substack{P(u)=T\\ P(w)=T'}} \mathrm{shf}(u, \bar{w}). \tag{4.1}$$

Then the bijection sending each plactic class to its defining tableau gives us a Hopf algebra structure on the $\mathbb{Z}$ module of all standard Young tableaux, $\mathbb{Z}\mathrm{SYT} = \bigoplus_{n\geqslant 0}\mathbb{Z}\mathrm{SYT}_n$.

For example,

$$\begin{aligned}
\mathrm{PC}_{\substack{1\;2\\3\phantom{\;2}}} * \mathrm{PC}_{\substack{1\\2}} &= \mathrm{shf}(312, 54) + \mathrm{shf}(132, 54)\\
&= 31254 + 31524 + 35124 + 53124 + 31542 + 35142 + 35412\\
&\quad + 53142 + 53412 + 54312 + 13254 + 13524 + 15324 + 51324\\
&\quad + 13542 + 15342 + 15432 + 51342 + 51432 + 54132\\
&= \mathrm{PC}_{\substack{1\;2\;4\\3\;5\phantom{\;4}}} + \mathrm{PC}_{\substack{1\;2\;4\\3\phantom{\;2\;4}\\5\phantom{\;2\;4}}} + \mathrm{PC}_{\substack{1\;2\\3\;4\\5\phantom{\;4}}} + \mathrm{PC}_{\substack{1\;2\\3\phantom{\;2}\\4\phantom{\;2}\\5\phantom{\;2}}}.
\end{aligned} \tag{4.2}$$

Another approach to calculate the product of two tableaux is given in [34] where Poirier and Reutenauer explain this product using jeu de taquin slides. Our goal is to show that it can also be described by a formula using partial orders, analogous to a result of Loday and Ronco [26, Theorem 4.1]. To state their result, given $\sigma \in \mathfrak{S}_k$ and $\tau \in \mathfrak{S}_\ell$, with $n := k + \ell$, let $\bar{\tau}$ be obtained from $\tau$ by adding $k$ to each letter. Then let $\sigma/\tau$ and $\sigma\backslash\tau$ denote the concatenations of $\sigma, \bar{\tau}$ and of $\bar{\tau}, \sigma$, respectively.

**Theorem 4.1.** *For $\tau \in \mathfrak{S}_k$ and $\sigma \in \mathfrak{S}_\ell$, with $n := k + \ell$, one has in the Malvenuto–Reutenauer Hopf algebra*

$$\sigma * \tau = \sum_{\substack{\rho\in\mathfrak{S}_n:\\ \sigma/\tau\leqslant\rho\leqslant\sigma\backslash\tau}} \rho.$$

*Equivalently, the shuffles* $\mathrm{shf}(\sigma, \bar{\tau})$ *are the interval* $[\sigma/\tau, \sigma\backslash\tau]_{\leqslant_{\mathrm{weak}}}$.

The following facts are crucial for transporting the Loday and Ronco result to $\mathrm{SYT}_n$.

Let $\sigma \in \mathfrak{S}_k, \tau \in \mathfrak{S}_\ell$. When $P(\sigma) = S$ and $P(\tau) = T$, let $\bar{T}$ denote the result of adding $k$ to every entry of $T$. It is easily seen that

$$P(\sigma/\tau) = S/T \quad \text{and} \quad P(\sigma\backslash\tau) = S\backslash T, \tag{4.3}$$

where $S/T$ (respectively $S\backslash T$) is the tableaux whose columns (respectively rows) are obtained by concatenating the columns (respectively rows) of $S$ and $\bar{T}$. Note also that Lemma 2.2 shows for $I = [k]$

$$\begin{aligned} &(S/T)_I = S, \quad \operatorname{std}\big((S/T)_{I^c}\big) = T, \\ &(S\backslash T)_I = S, \quad \operatorname{std}\big((S\backslash T)_{I^c}\big) = T. \end{aligned} \tag{4.4}$$

The following theorem is a consequence of Lemma 2.2, Corollary 3.1 and Theorem 4.1.

**Theorem 4.2.** *Let $\leqslant$ be a partial order on $\mathrm{SYT}_n$, for all $n > 0$, that*

(a) *is stronger than $\leqslant_{\text{weak}}$ and*
(b) *restricts to segments.*

*Then in the Poirier–Reutenauer Hopf algebra,*

$$S * T = \sum_{\substack{R \in \mathrm{SYT}_n: \\ S/T \leqslant R \leqslant S\backslash T}} R.$$

**Proof.** Let $\leqslant$ be a partial order on $\mathrm{STY}_n$ satisfying hypothesis. From (4.1), (4.3) and Theorem 4.1 it follows that any tableau $R$ appearing in the product $S * T$ satisfies: $S/T \leqslant_{\text{weak}} R \leqslant_{\text{weak}} S\backslash T$. Therefore we have $S/T \leqslant R \leqslant S\backslash T$ and this proves one direction.

Let $R$ be any tableau such that $S/T \leqslant R \leqslant S\backslash T$. Also let $I = [k]$ where $k$ is the size of the tableau $S$. By hypothesis

$$\begin{aligned} &S = (S/T)_I \leqslant R_I \leqslant (S\backslash T)_I = S, \\ &T = \operatorname{std}\big((S/T)_{I^c}\big) \leqslant \operatorname{std}\big(R_{I^c}\big) \leqslant \operatorname{std}\big((S\backslash T)_{I^c}\big) = T \end{aligned}$$

i.e., $R_I = S$ and $\operatorname{std}(R_{I^c}) = T$ and this shows that $R$ can be found by shuffling $S$ and $T$ in a certain way. Therefore $R$ lies in the product $S * T$. $\square$

**Proof of Theorem 1.1.** All four orders $\leqslant_{\text{chain}}$, $\leqslant_{\text{weak}}$, $\leqslant_{\text{KL}}^{\text{op}}$ and $\leqslant_{\text{geom}}$ on $\mathrm{SYT}_n$ satisfy the hypotheses of Theorem 4.2 by Corollary 3.1. Therefore the result follows. $\square$

**Example 4.3.** Let $T = \begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix}$ and $S = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$. Then $T/S = \begin{smallmatrix} 1 & 2 & 4 \\ 3 & 5 & \end{smallmatrix}$, $T\backslash S = \begin{smallmatrix} 1 & 2 \\ 3 & \\ 4 & \\ 5 & \end{smallmatrix}$ and (4.2) gives

$$T * S = \begin{matrix} 1 & 2 \\ 3 & \end{matrix} * \begin{matrix} 1 \\ 2 \end{matrix} = \begin{matrix} 1 & 2 & 4 \\ 3 & 5 & \end{matrix} + \begin{matrix} 1 & 2 & 4 \\ 3 & & \\ 5 & & \end{matrix} + \begin{matrix} 1 & 2 \\ 3 & 4 \\ 5 & \end{matrix} + \begin{matrix} 1 & 2 \\ 3 & \\ 4 & \\ 5 & \end{matrix}.$$

On the other hand, when considered with any of the four orders, the Hasse diagram of $\mathrm{SYT}_5$ in Fig. 1 shows that the product above is equal to the sum of all tableaux in the interval $[T/S, T\backslash S]$.

## 5. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We will view the commutative diagram

$$\begin{tikzcd} S_n \arrow[r] \arrow[dr] & \mathrm{SYT}_n \arrow[d] \\ & 2^{[n-1]} \end{tikzcd} \tag{5.1}$$

as an instance of the following set-up, involving closure relations, equivalence relations, order-preserving maps, and the topology of posets. For background on poset topology, see [4].

Let $P$ be a partially ordered set $(P, \leqslant_P)$ and $p \mapsto \bar{p}$ a closure relation on $P$, that is,

$$\bar{\bar{p}} = \bar{p}, \quad p \leqslant_P \bar{p} \quad \text{and} \quad p \leqslant_P q \quad \text{implies} \quad \bar{p} \leqslant_P \bar{q}.$$

It is well known [4, Corollary 10.12] that in this instance, the order-preserving closure map $P \to \bar{P}$ has the property that its associated simplicial map of order complexes $\Delta(P) \to \Delta(\bar{P})$ is a strong deformation retraction.

Now assume $\sim$ is an equivalence relation on $P$ such that, as maps of sets, the closure map $P \to \bar{P}$ factors through the quotient map $P \to P/\sim$. Equivalently, the vertical map below is well-defined, and makes the diagram commute:

$$\begin{tikzcd} P \arrow[r] \arrow[dr] & P/\sim \arrow[d] \\ & \bar{P} \end{tikzcd} \tag{5.2}$$

**Proposition 5.1.** *In the above situation, partially order $\bar{P}$ by the restriction of $\leqslant_P$, and assume that $P/\sim$ has been given a partial order $\leqslant$ in such a way that the horizontal and vertical maps in the* (5.2) *are also order-preserving. Then the commutative diagram of associated simplicial maps of order complexes are all homotopy equivalences.*

**Proof.** Obviously one can define a closure relation on $P/\sim$ such that $\overline{P/\sim} = \bar{P}$, and the result follows. $\square$

**Lemma 5.2.** *Given any subset $D \subset [n-1]$, there exists a maximum element $\tau(D)$ in $(\mathfrak{S}_n, \leqslant_{\text{weak}})$ for the descent class*

$$\mathrm{Des}_L^{-1}(D) := \{\sigma \in \mathfrak{S}_n \colon \mathrm{Des}_L(\sigma) = D\}.$$

*Consequently, the map $\mathfrak{S}_n \to \mathfrak{S}_n$ defined by $\sigma \mapsto \tau(\mathrm{Des}_L(\sigma))$ is a closure relation which also restricts to the proper parts and its image is isomorphic to $(2^{[n-1]}, \subseteq)$.*

**Proof.** It is known that [5, pp. 98–100]

$$\mathrm{Des}_L^{-1}(D) := \{\sigma \in \mathfrak{S}_n \colon \mathrm{Des}_L(\sigma) = D\}$$

is actually an interval of the weak Bruhat order on $S_n$. Therefore the map $\sigma \mapsto \tau(\mathrm{Des}_L(\sigma))$ is a closure relation and since $\mathrm{Des}_L^{-1}(\emptyset)$ and $\mathrm{Des}_L^{-1}([n-1])$ consist of respectively $\hat{0}$ and $\hat{1}$ in $(\mathfrak{S}_n, \leqslant_{\text{weak}})$, it restricts to the proper parts. Now it is easy to see that its image is isomorphic to $(2^{[n-1]}, \subseteq)$. $\square$

**Corollary 5.3.** *Order* $\mathfrak{S}_n$ *by* $\leqslant_{\text{weak}}$ *and* $2^{[n-1]}$ *by* $\subseteq$. *Let* $\leqslant$ *be any order on* $\mathrm{SYT}_n$ *such that the commuting diagram* (5.1) *has all the maps order-preserving. Then these restrict to a commuting diagram of order-preserving maps on the proper parts, each of which induces a homotopy equivalence of order complexes. Consequently,* $\mu(\hat{0}, \hat{1}) = (-1)^{n-3}$.

**Proof.** The fact that the maps restrict to the proper parts follows because we know the maps explicitly as maps of sets, and the images of $\hat{0}, \hat{1}$ in $(\mathfrak{S}_n, \leqslant_{\text{weak}})$ must be exactly the $\hat{0}, \hat{1}$ in $(\mathrm{SYT}_n, \leqslant)$ (namely the single-row and single-column tableaux) because the horizontal map is order-preserving.

The fact that they induce homotopy equivalences follows from Proposition 5.1 applied to the three proper parts, using the closure relation in Lemma 5.2 and letting $\sim$ be Knuth equivalence $\sim_K$. One must observe that $\mathrm{Des}_L(\sigma)$ depends only on the Knuth class of $\sigma$.

The fact that $\mu(\hat{0}, \hat{1}) = (-1)^{n-3}$ for the Boolean algebra $(2^{[n-1]}, \subseteq)$ is well known [43, Proposition 3.8.4]. $\square$

**Proof of Theorem 1.2.** By Proposition 3.3 all four orders on $\mathrm{SYT}_n$ satisfy the hypotheses of Corollary 5.3. $\square$

**Example 5.4.** In Fig. 2, the first interval in $(\mathrm{SYT}_8, \leqslant_{\text{chain}})$ has Möbius function value 2, whereas Möbius function value of the second interval which is found in $(\mathrm{SYT}_8, \leqslant_{\text{weak}})$, $(\mathrm{SYT}_8, \leqslant_{\text{KL}}^{\text{op}})$ and $(\mathrm{SYT}_8, \leqslant_{\text{geom}})$ is $-2$. Therefore the Möbius function values of the proper intervals in $\leqslant_{\text{chain}}$, $\leqslant_{\text{weak}}$, $\leqslant_{\text{KL}}^{\text{op}}$ and $\leqslant_{\text{geom}}$ on $\mathrm{SYT}_n$ need not all lie in $\{\pm 1, 0\}$ as they do in $(\mathfrak{S}_n, \leqslant_{\text{weak}})$.

## 6. Inner translation and skew orders

In this section we describe the *inner translation property* of KL and geometric order on $\mathrm{SYT}_n$ which enable us to generalize these orders to the skew standard Young tableaux $\mathrm{SYT}_n^{\mu}$ of size $n$ with some fixed inner boundary $\mu$.

To do this first we need to recall the notion of *dual Knuth* relations on $\mathfrak{S}_n$: permutations $\sigma, \tau \in \mathfrak{S}_n$ are said to be *differ by a single dual Knuth relation* if for some $i \in [n-2]$, $i \in \mathrm{Des}_L(\sigma)$ and $i+1 \notin \mathrm{Des}_L(\sigma)$ whereas $i+1 \in \mathrm{Des}_L(\tau)$ and $i \notin \mathrm{Des}_L(\tau)$. In this case

$$\begin{array}{llll} \text{either} & \sigma = \ldots i+1 \ldots i \ldots i+2 \ldots & \text{and} & \tau = \ldots i+2 \ldots i \ldots i+1 \ldots \\ \text{or} & \sigma = \ldots i+1 \ldots i+2 \ldots i \ldots & \text{and} & \tau = \ldots i \ldots i+2 \ldots i+1 \ldots . \end{array}$$

We say $\sigma, \tau$ are *Knuth equivalent* written as $\sigma \sim_{K^*} \tau$, if $\tau$ can be generated from $\sigma$ by a sequence of single dual Knuth relations. Observe that $\sigma \sim_{K^*} \tau$ if and only if $\sigma^{-1} \sim_K \tau^{-1}$.

Since left descent sets are all equal for the permutations in a Knuth class $C_T$, $T \in \mathrm{SYT}_n$, a single dual Knuth relation gives the following action on tableaux: Let $r_T(i)$ be the row number of $i$ in $T$ from the top.

**Case 1.** If $i+1 \in \mathrm{Des}(T)$ and $i \notin \mathrm{Des}(T)$ then

$$\begin{array}{ll} \text{either} & r_T(i+2) > r_T(i) \geqslant r_T(i+1) \\ \text{or} & r_T(i) \geqslant r_T(i+2) > r_T(i+1). \end{array}$$

The resulting tableau is found by interchanging $i+2$ and $i+1$ in the first case and interchanging $i$ and $i+1$ in the second case.

**Case 2.** If $i \in \mathrm{Des}(T)$ and $i+1 \notin \mathrm{Des}(T)$ then

$$\begin{array}{ll} \text{either} & r_T(i+1) > r_T(i) \geqslant r_T(i+2) \\ \text{or} & r_T(i+1) \geqslant r_T(i+2) > r_T(i). \end{array}$$

This time interchanging $i+2$ and $i+1$ in the first case and interchanging $i$ and $i+1$ in the second case gives us the resulting tableau under the action of the single dual Knuth relation given with the triple $\{i, i+1, i+2\}$.

We say $T \sim_{K^*} T'$ if $T'$ can be obtained from $T$ by applying a sequence of single dual Knuth relations as described above. The following theorem, see [36, Proposition 3.8.1] for example, is a nice characterization of this relation.

**Theorem 6.1.** *Let $S, T \in \mathrm{SYT}_n$. Then $S \sim_{K^*} T$ if and only if $\mathrm{sh}(S) = \mathrm{sh}(T)$.*

Let $\{\alpha, \beta\} = \{i, i+1\}$ and $\mathrm{SYT}_n^{[\alpha,\beta]}$ be a subset of $\mathrm{SYT}_n$ given by

$$\mathrm{SYT}_n^{[\alpha,\beta]} := \bigl\{T \in \mathrm{SYT}_n \mid \alpha \in \mathrm{Des}(T),\ \beta \notin \mathrm{Des}(T)\bigr\}.$$

As we described above we can apply a single dual Knuth relation determined with the triple $\{i, i+1, i+2\}$ on each $T \in \mathrm{SYT}_n^{[\alpha,\beta]}$ and this gives us the following *inner translation map*:

$$\mathcal{V}_{[\alpha,\beta]} : \mathrm{SYT}_n^{[\alpha,\beta]} \mapsto \mathrm{SYT}_n^{[\beta,\alpha]},$$

where $\mathcal{V}_{[\beta,\alpha]} \circ \mathcal{V}_{[\alpha,\beta]}$ and $\mathcal{V}_{[\alpha,\beta]} \circ \mathcal{V}_{[\beta,\alpha]}$ are just identity maps on their domains.

**Definition 6.2.** Any order $\leqslant$ on $\mathrm{SYT}_n$ is said to have the *inner translation property* if the inner translation map

$$\mathcal{V}_{[\alpha,\beta]} : \bigl(\mathrm{SYT}_n^{[\alpha,\beta]}, \leqslant\bigr) \mapsto \bigl(\mathrm{SYT}_n^{[\beta,\alpha]}, \leqslant\bigr)$$

is order preserving.

Now we give the following corollary which is crucial in the sense that it provides the sufficient tool for generalizing any partial order on standard Young tableaux to the skew standard tableaux.

For $1 \leqslant k < n$, let $R$ be a tableau in $\mathrm{SYT}_k$ and

$$\mathrm{SYT}_n^R := \{T \in \mathrm{SYT}_n \mid T_{[1,k]} = R\}.$$

**Corollary 6.3.** *Suppose $S, T \in \mathrm{SYT}_n$ and $R, R' \in \mathrm{SYT}_k$ satisfy*

$$S_{[1,k]} = T_{[1,k]} = R$$
$$\mathrm{sh}(R) = \mathrm{sh}(R').$$

*Moreover suppose $S'$ and $T'$ are the tableaux in $\mathrm{SYT}_n$ obtained by replacing $R$ by $R'$ in $S$ and $T$, respectively.*

*Then for $\leqslant$ having the inner translation property on $\mathrm{SYT}_n$, one has*

$$S \leqslant T \quad \text{if and only if } S' \leqslant T'.$$

*In particular $(\mathrm{SYT}_n^R, \leqslant)$ and $(\mathrm{SYT}_n^{R'}, \leqslant)$ are isomorphic subposets of $(\mathrm{SYT}_n, \leqslant)$.*

**Proof.** As a consequence of Theorem 6.1, by applying to $S$ and $T$ the same sequence of dual Knuth relations on their subtableau $R$, one can generate $S'$ and $T'$ respectively. On the other hand, since $\leqslant$ has inner translation property at each step the order is preserved. □

**Theorem 6.4.** KL *and geometric order on* $\mathrm{SYT}_n$ *have the inner translation property. Therefore for any* $R, R' \in \mathrm{SYT}_k$ *such that* $\mathrm{sh}(R) = \mathrm{sh}(R')$ *and* $k < n$,

$$\left(\mathrm{SYT}_n^R, \leqslant\right) \quad \textit{and} \quad \left(\mathrm{SYT}_n^{R'}, \leqslant\right)$$

*are isomorphic subposet of* $\mathrm{SYT}_n$ *in* KL *and geometric orders.*

**Proof.** This map is first introduced by Vogan in [46] for $KL$ order, where he also shows the desired property. For geometric order this result is due to Melnikov [33, Proposition 6.6]. □

The example given below shows that chain and the weak order do not satisfy the inner translation property (see also Remark 9.3).

**Example 6.5.**

$$\begin{array}{lll} 1 & 3 & \mathbf{6} \\ 2 & \mathbf{4} & \\ \mathbf{5} & & \end{array} \geqslant_{\text{chain}} \begin{array}{lll} 1 & 3 & \mathbf{4} \\ 2 & \mathbf{6} & \\ \mathbf{5} & & \end{array} \quad \text{but} \quad \begin{array}{lll} 1 & 3 & \mathbf{5} \\ 2 & \mathbf{4} & \\ \mathbf{6} & & \end{array} \ngeqslant_{\text{chain}} \begin{array}{lll} 1 & 3 & \mathbf{5} \\ 2 & \mathbf{6} & \\ \mathbf{4} & & \end{array}$$

where the latter pair is obtained from the former by applying a single dual Knuth relation on the triple $\{4, 5, 6\}$.

$$\begin{array}{lll} 1 & 2 & \mathbf{4} \\ \mathbf{3} & \mathbf{5} & 6 \end{array} \leqslant_{\text{weak}} \begin{array}{lll} 1 & 2 & \mathbf{4} \\ \mathbf{3} & 6 & \\ \mathbf{5} & & \end{array} \quad \text{but} \quad \begin{array}{lll} 1 & 2 & \mathbf{3} \\ \mathbf{4} & \mathbf{5} & 6 \end{array} \nleqslant_{\text{weak}} \begin{array}{lll} 1 & 2 & \mathbf{5} \\ \mathbf{3} & 6 & \\ \mathbf{4} & & \end{array}$$

where the latter pair is obtained from the former by applying a single dual Knuth relation on the triple $\{3, 4, 5\}$.

### *6.1. The definition of the skew orders*

Let $m = k + n$, $\lambda \models m$ and $\mu \models k$ such that $\mu \subset \lambda$. For $T \in \mathrm{SYT}_m$ of shape $\lambda$, define

$$T_{\lambda/\mu}$$

to be the skew standard tableau on $[n]$ of shape $\lambda/\mu$ obtained by standardizing the skew segment of $T$ having shape $\lambda/\mu$.

**Definition 6.6.** Let $\leqslant$ be partial order on $\mathrm{SYT}_n$ having inner translation property. For $U$ and $V$ be two skew standard tableaux in $\mathrm{SYT}_n^{\mu}$, we set

$$U \leqslant V$$

if there exist two tableaux $S$ and $T$ in $\mathrm{SYT}_m$ of shape $\lambda$ and $\lambda'$, respectively, which satisfy:

$$S_\mu = T_\mu = R \quad \text{for some } R \in \mathrm{SYT}_k \text{ of shape } \mu,$$
$$S_{\lambda/\mu} = U \quad \text{and} \quad T_{\lambda'/\mu} = V,$$
$$S \leqslant T.$$

**Remark 6.7.** As a consequence of Theorem 6.4, the skew orders, $\leqslant_{\mathrm{KL}}^{\mathrm{op}}$ and $\leqslant_{\mathrm{geom}}$ on $\mathrm{SYT}_n^{\mu}$ becomes well defined.

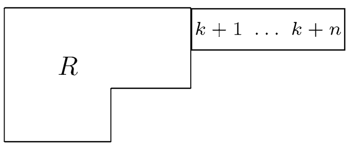
## 7. Proof of Theorem 1.3

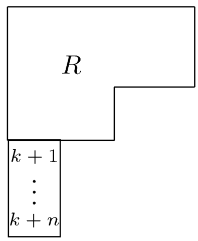
In what follows we first prove a result, namely Proposition 7.1 below, which is about the Möbius function of the subposet $\mathrm{SYT}_m^R$ of $\mathrm{SYT}_n$ ordered by any order that is stronger than $\leqslant_{\mathrm{weak}}$, restricts to segments and has the property of extension from the segments. Consequently Theorem 1.3 follows from this results together with Theorems 3.11 and 6.4 and Definition 6.6.

Let $R$ be a tableau in $\mathrm{SYT}_k$ and $m = k + n$. Recall that

$$\mathrm{SYT}_m^R := \{T \in \mathrm{SYT}_m \mid T_{[1,k]} = R\}.$$

Since the weak order restricts to segment, it can be induced on $\mathrm{SYT}_m^R$. Moreover the analysis made by comparing the left inversion sets yields that any tableau $T \in \mathrm{SYT}_m^R$, under the weak order, lies between two tableaux $\hat{0}_{R,n}$ and $\hat{1}_{R,n}$ given below.

$\hat{0}_{R,n}=$ 


$\hat{1}_{R,n}=$ 


Therefore $(\mathrm{SYT}_m^R, \leqslant_{\mathrm{weak}}) = [\hat{0}_{R,n}, \hat{1}_{R,n}]_{\leqslant_{\mathrm{weak}}}$ and for any order $\leqslant$ which is stronger than the weak order and which restricts to segments we have

$$\left(\mathrm{SYT}_m^R, \leqslant\right) = [\hat{0}_{R,n}, \hat{1}_{R,n}]_{\leqslant}$$

**Proposition 7.1.** *Let $\leqslant$ be any order on $\mathrm{SYT}_m$ with the following properties*

(i) *$\leqslant$ is stronger than $\leqslant_{\mathrm{weak}}$,*
(ii) *$\leqslant$ restricts to segments,*
(iii) *$\leqslant$ extends from segments.*

*Then for $\hat{0}_{R,n}$ and $\hat{1}_{R,n}$ as above one has*

$$\mathrm{SYT}_m^R = [\hat{0}_{R,n}, \hat{1}_{R,n}],$$

*and the proper part of $\mathrm{SYT}_m^R$ is homotopy equivalent to*

$$\begin{cases} \text{an } (n-2)\text{-dimensional sphere} & \text{if } R \text{ is rectangular,} \\ \text{a point} & \text{otherwise.} \end{cases}$$

Below we recall Rambau's Suspension Lemma about bounded posets [35], which will be used to prove Proposition 7.1.

**Lemma 7.2** *(Rambau's Suspension Lemma). Let $\mathcal{P}$ and $\mathcal{Q}$ be two bounded posets such that $\hat{0}_{\mathcal{Q}} \neq \hat{1}_{\mathcal{Q}}$. Assume $\mathcal{P}$ is the disjoint union of its two subsets $\mathcal{I}$ and $\mathcal{J}$ where $\mathcal{I}$ forms an order ideal and $\mathcal{J}$ forms an order filter of $\mathcal{P}$. Assume further that there are order preserving maps*

$$f : \mathcal{P} \mapsto \mathcal{Q} \quad \textit{and} \quad i, j : \mathcal{Q} \mapsto \mathcal{P}$$

*satisfying the following properties*:

(i) *The image of $i$ lies in $\mathcal{I}$ and the image of $j$ lies in $\mathcal{J}$.*
(ii) *The maps $f \circ i$ and $f \circ j$ are identity on $\mathcal{Q}$.*
(iii) *For every $p \in \mathcal{P}$, $i \circ f(p) \leqslant p \leqslant j \circ f(p)$.*
(iv) *The fiber $f^{-1}(\hat{0}_{\mathcal{Q}})$ lies in $\mathcal{J}$ except for $\hat{0}_{\mathcal{P}}$ and the fiber $f^{-1}(\hat{1}_{\mathcal{Q}})$ lies in $\mathcal{I}$ except for $\hat{1}_{\mathcal{P}}$.*

*Then the proper part $\mathcal{P} - \{\hat{0}_{\mathcal{P}}, \hat{1}_{\mathcal{P}}\}$ of $\mathcal{P}$ is homotopy equivalent to the suspension of the proper part of $\mathcal{Q}$.*

**Proof of Proposition 7.1.** For $n \geqslant 2$, let

$$\mathcal{P} = [\hat{0}_{R,n}, \hat{1}_{R,n}] \quad \text{and} \quad \mathcal{Q} = [\hat{0}_{R,n-1}, \hat{1}_{R,n-1}]$$

together with the subposets of $\mathcal{P}$ given as

$$\mathcal{I} = \{T \in \mathcal{P}\colon\ m-1 \notin \operatorname{Des}(T)\},$$
$$\mathcal{J} = \{T \in \mathcal{P}\colon\ m-1 \in \operatorname{Des}(T)\}.$$

Moreover let

$$f : \mathcal{P} \mapsto \mathcal{Q} \quad \text{and} \quad i, j : \mathcal{Q} \mapsto \mathcal{P}$$

where the map $f$ restricts any $T \in \mathcal{P}$ to its initial segment $T_{[1,m-1]}$ and the map $i$ concatenates $m$ to the first row of any $S \in \mathcal{Q}$ from right whereas $j$ concatenates $m$ to the first column of $S$ from the bottom.

First we will show that $\mathcal{I}$ is an order ideal of $\mathcal{P}$. Let $T \in \mathcal{I}$ and $T' < T$. Then by Lemma 3.2, $\operatorname{Des}(T') \subseteq \operatorname{Des}(T)$ and therefore $m-1$ does not belong to $\operatorname{Des}(T')$. This shows that $T' \in \mathcal{I}$ and $\mathcal{I}$ is an order ideal. A similar argument also shows that $\mathcal{J}$ is an order filter of $\mathcal{P}$. On the other hand, it can be easily seen that $\mathcal{P}$ is the disjoint union of $\mathcal{I}$ and $\mathcal{J}$.

Since the tableau $R$ is common for both $\mathcal{P}$ and $\mathcal{Q}$ and $\leqslant$ restricts to the initial segments, the map $f : \mathcal{P} \mapsto \mathcal{Q}$ is well defined and order preserving. By virtue of their definitions the maps $i, j : \mathcal{Q} \mapsto \mathcal{P}$ are also well defined. On the other hand, since $\leqslant$ has the property of extension from segments therefore they both are order preserving.

Now part (i) follows from the fact that the map $i$ concatenates $m$ to the right of the first row of $S \in \mathcal{Q}$, which provides no possibility that $m$ appears below $m-1$ in $i(S)$. Therefore $m-1 \notin \operatorname{Des}(i(S))$ and $i(S) \in \mathcal{I}$. On the other hand, in $j(S)$, $m$ always appears below $m-1$ and this shows that $j(S) \in \mathcal{J}$.

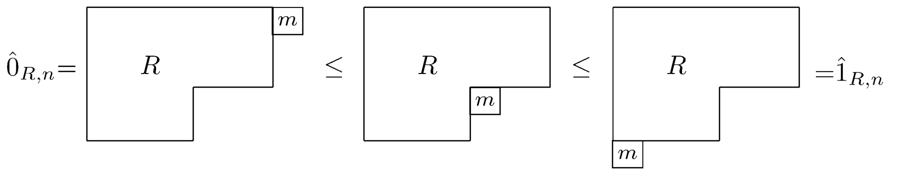
For part (iii), let $\rho_T = a_1 \ldots a_{l-1} m a_{l+1} \ldots a_m$ be the row word of $T \in \mathcal{P}$. The analysis on the (left) inversion sets gives:

$$a_1 \ldots a_{l-1} a_{l+1} \ldots a_m m \leqslant_{\text{weak}} a_1 \ldots a_{l-1} m a_{l+1} \ldots a_m \leqslant_{\text{weak}} m a_1 \ldots a_{l-1}\, a_{l+1} \ldots a_m$$

and by RSK correspondence $i \circ f(T) \leqslant_{\text{weak}} T \leqslant_{\text{weak}} j \circ f(T)$ and hence $i \circ f(T) \leqslant T \leqslant j \circ f(T)$.

One can check the hypotheses (ii) and (iv) easily. Therefore by Lemma 7.2, the proper part of $\mathcal{P}$ is homotopy equivalent to the suspension of the proper part of $\mathcal{Q}$.

In the rest we proceed by induction: Let $n = 1$. Then all tableaux in the poset $\mathcal{P} = [\hat{0}_{R,1}, \hat{1}_{R,1}]$ are obtained by placing $m$ in some outer corner of $R$, i.e., in an empty cell along the boundary of $R$ whose addition to $R$ still gives a Young tableau shape. Moreover it can be easily checked, for example by comparing the left inversion sets of their row words, that these tableaux form a saturated chain in $(\mathrm{SYT}_m, \leqslant_{\mathrm{weak}})$. On the other hand, since $\leqslant$ is stronger then the $\leqslant_{\mathrm{weak}}$ and restricts to segments this chain remains saturated in $(\mathrm{SYT}_m, \leqslant)$. The following diagram illustrates the case when $R$ has three outer corners.




Now if $R$ has rectangular shape then it has two outer corners and the poset $\mathcal{P} = [\hat{0}_{R,1}, \hat{1}_{R,1}]$ consists of two tableaux. It has the Möbius function from the bottom to the top elements to be $-1$ and moreover the proper part of $\mathcal{P}$ is homotopy equivalent to the empty set i.e., $(-1)$-dimensional sphere.

If $R$ is non rectangular then as in the above diagram $\mathcal{P}$ is a saturated chain having more than two elements. Hence its Möbius function is 0 from the bottom to the top elements and it is homotopic to a point.

Now assume that for $n = r$ the poset $\mathcal{Q} = [\hat{0}_{R,r}, \hat{1}_{R,r}]$ satisfies the hypothesis i.e., the proper part of $\mathcal{Q}$ is homotopic to a $(r-2)$-sphere in case $R$ is rectangular and it is homotopic to a point otherwise.

On the other hand, we already see that the proper part of $\mathcal{P} = [\hat{0}_{R,r+1}, \hat{1}_{R,r+1}]$ is homotopy equivalent to the suspension of the proper part of $\mathcal{Q}$, so that the former becomes homotopy equivalent to a $(r-1)$-sphere if $R$ is rectangular and to a point otherwise. Therefore the assertion of Proposition 7.1 follows. □

**Proof of Theorem 1.3.** By Remark 6.7, KL and geometric orders are well defined on $\mathrm{SYT}_n^{\mu}$. On the other hand, they restrict segments and have the property of embedding from initial segments by Lemma 3.11. So the required statement follows from Proposition 7.1. □

## 8. Shortest and longest chains

By observing Figure 1, one can see that the posets of $\mathrm{SYT}_n$ with all these orders are not lattices and not ranked. On the other hand, we can still say something about the size of their shortest or longest chains, where by convention $c_1 < c_2 < \cdots < c_i$ has size $i$.

**Proposition 8.1.**

(i) *The size of a shortest saturated chain in* $(\mathrm{SYT}_n, \leqslant_{\mathrm{weak}})$ *is* $n$.

(ii) *The size of a longest chain in* $(\mathrm{SYT}_n, \leqslant_{\mathrm{weak}})$, $(\mathrm{SYT}_n, \leqslant_{\mathrm{KL}}^{\mathrm{op}})$ *and* $(\mathrm{SYT}_n, \leqslant_{\mathrm{geom}})$ *is equal to the size of the longest chain in* $(\mathrm{Par}_n, \leqslant_{\mathrm{dom}})$, *which is asymptotically* $(\sqrt{8}n^{3/2})/3$.

**Proof.** Observe that if $\sigma$ is covered by $\tau$ in $(\mathfrak{S}_n, \leqslant_{\text{weak}})$ then the size of the (left) descent set $\mathrm{Des}_L(\tau)$ of $\tau$ is at most one bigger than the size of $\mathrm{Des}_L(\sigma)$. This fact is also true for $(\mathrm{SYT}_n, \leqslant_{\text{weak}})$: If $S$ is covered by $T$ in $(\mathrm{SYT}_n, \leqslant_{\text{weak}})$ then

$$0 \leqslant \left|\mathrm{Des}(T)\backslash\mathrm{Des}(S)\right| < 2. \tag{8.1}$$

This shows that the size of a shortest saturated chain must be at least $n$. On the other hand, it can be seen by an easy induction that there exist a saturated chain in $(\mathrm{SYT}_n, \leqslant_{\text{weak}})$ of size $n$ with the following form:

$$\begin{matrix} 1 & 2 & 3 & \ldots & n \end{matrix} \leqslant \begin{matrix} 1 & 2 & 3 & \ldots & n-1 \\ n & & & & \end{matrix} \leqslant \cdots \leqslant \begin{matrix} 1 & 2 \\ 3 & \\ \vdots & \\ n & \end{matrix} \leqslant \begin{matrix} 1 \\ 2 \\ 3 \\ \vdots \\ n \end{matrix}. \tag{8.2}$$

Therefore the statement about shortest chains in $(\mathrm{SYT}_n, \leqslant_{\text{weak}})$ follows.

For longest chains the proof is based on two facts: a result of Greene and Kleitman [15, p. 9] which calculates the size of longest chain in the lattice of integer partitions ordered by the dominance order and the result of Melnikov [30, Proposition 4.1.8] which shows that for any tableau $S$ of shape $\mu$ in $\mathrm{SYT}_n$ and for any partition $\lambda \models n$ such that $\mu <^{\text{op}}_{\text{dom}} \lambda$, there is a tableau $T \in \mathrm{SYT}_n$ such that $\mathrm{sh}(T) = \lambda$ and $S <_{\text{weak}} T$. These two facts enable us to calculate the longest chain in $\mathrm{SYT}_n$ ordered by the weak order. Since $\leqslant^{\text{op}}_{\text{KL}}$ and $\leqslant_{\text{geom}}$ also change the shapes of the tableaux, the longest chain of $(\mathrm{SYT}_n, \leqslant_{\text{weak}})$ still remains saturated in $(\mathrm{SYT}_n, \leqslant^{\text{op}}_{\text{KL}})$ and $(\mathrm{SYT}_n, \leqslant_{\text{geom}})$. $\square$

**Remark 8.2.** By an easy induction one can see that chain in (8.2) still remains saturated in $\mathrm{SYT}_n$ for KL, geometric and chain orders. Therefore if it were known (8.1) is satisfied by these three orders, we could deduce the same conclusion about their shortest chains.

## 9. Remarks and questions

**Remark 9.1.** Theorem 1.2 also follows from Proposition 7.1 by taking $R = 1$. The original proof is kept here for indicating different approaches to the subject.

**Remark 9.2.** The order complex of the proper part of $(\mathrm{SYT}_n, \leqslant)$ under any of the four orders is not homeomorphic to a sphere. One can observe $\mathrm{SYT}_4$ in Fig. 1 to see the smallest example. Moreover since these posets are not ranked for $n \geqslant 4$, the order complex of their proper parts are not pseudomanifolds.

**Remark 9.3.** Although the weak order on $\mathrm{SYT}_n$ does not have the inner translation property, it might still satisfy Corollary 6.3 without this property, which would then make it possible to define weak order on skew standard tableaux.

For chain order, two pairs of tableaux given below where the inner tableau $\begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix}$ common to the first pair is replaced by $\begin{smallmatrix} 1 & 3 \\ 2 & \end{smallmatrix}$ in the second pair, show that Corollary 6.3 is not satisfied by chain order:

$$\begin{matrix} \mathbf{1} & \mathbf{2} & 5 \\ \mathbf{3} & 6 & \\ 4 & & \end{matrix} \geqslant_{\text{chain}} \begin{matrix} \mathbf{1} & \mathbf{2} & 5 \\ \mathbf{3} & 4 & \\ 6 & & \end{matrix} \quad \text{but} \quad \begin{matrix} \mathbf{1} & \mathbf{3} & 5 \\ \mathbf{2} & 6 & \\ 4 & & \end{matrix} \ngeqslant_{\text{chain}} \begin{matrix} \mathbf{1} & \mathbf{3} & 5 \\ \mathbf{2} & 4 & \\ 6 & & \end{matrix}.$$

**Question 9.4.** One might ask to what extent the definitions and results in this paper apply to other Lexicographic Coxeter systems $(W, S)$. The weak order on $W$ is well-defined, as are KL and the geometric order, where the former still remains weaker than the latter ([19]; see [12, Fact 7]). Definition 2.3 makes sense and remains valid, and so does Proposition 3.3(i) for KL order ([19]; see [12, Fact 7]). For geometric order the same property follows from [10, Theorem 6.11] or [18, Theorem 9.9].

For the analysis of Möbius function and homotopy types, the crucial Lemma 5.2 was proven by Björner and Wachs [6, Theorem 6.1] for all *finite* Coxeter groups $W$. Hence Corollary 5.3 and Theorem 1.2 are valid also in this generality, with the same proof.

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