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The Baum–Connes conjecture via localisation of categories☆

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Abstract

We redefine the Baum–Connes assembly map using simplicial approximation in the equivariant Kasparov category. This new interpretation is ideal for studying functorial properties and gives analogues of the Baum–Connes assembly map for other equivariant homology theories. We extend many of the known techniques for proving the Baum–Connes conjecture to this more general setting.

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Contents

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1. Introduction

Let G be a second countable locally compact group. Let A be a separable C^*-algebra with a strongly continuous action of G and let $G \ltimes_{\mathrm{r}} A$ be the reduced crossed product, which is another separable C^*-algebra. The aim of the Baum–Connes conjecture (with coefficients) is to compute the K-theory of $G \ltimes_{\mathrm{r}} A$. For the trivial action of G on $\mathbb{C}$ (or $\mathbb{R}$), this specialises to $\mathrm{K}_*(C^*_{\mathrm{r}}(G))$, the K-theory of the reduced C^*-algebra of G. One defines a certain graded Abelian group $\mathrm{K}^{\mathrm{top}}_*(G, A)$, called the *topological* K*-theory of G with coefficients A*, and a homomorphism

$$\mu_A\colon \mathrm{K}^{\mathrm{top}}_*(G, A) \to \mathrm{K}_*(G \ltimes_{\mathrm{r}} A), \tag{1}$$

which is called the *Baum–Connes assembly map*. The *Baum–Connes conjecture for G with coefficients A* asserts that this map is an isomorphism. It has important applications in topology and ring theory. The conjecture is known to hold in many cases, for instance, for amenable groups [23]. A recent survey article on the Baum–Connes conjecture is [22].

Despite its evident success, the usual definition of the Baum–Connes assembly map has some important shortcomings. At first sight $\mathrm{K}^{\mathrm{top}}_*(G, A)$ may seem even harder to compute than $\mathrm{K}_*(G \ltimes_{\mathrm{r}} A)$. Experience shows that this is not the case. Nevertheless, there are situations where $\mathrm{K}^{\mathrm{top}}_*(G, A)$ creates more trouble than $\mathrm{K}_*(G \ltimes_{\mathrm{r}} A)$. For instance, most of the work required to prove the permanence properties of the

Baum–Connes conjecture is needed to extend evident properties of $\mathrm{K}_*(G \ltimes_{\mathrm{r}} A)$ to $\mathrm{K}_*^{\mathrm{top}}(G, A)$. The meaning of the Baum–Connes conjecture is rather mysterious: it is not *a priori* clear that $\mathrm{K}_*^{\mathrm{top}}(G, A)$ should have anything to do with $\mathrm{K}_*(G \ltimes_{\mathrm{r}} A)$. A related problem is that the Baum–Connes assembly map only makes sense for K-theory and not for other interesting equivariant homology theories. For instance, in connection with the Chern character it would be desirable to have a Baum–Connes assembly map for local cyclic homology as well.

Our alternative description of the assembly map addresses these shortcomings. It applies to any equivariant homology theory, that is, any functor defined on the equivariant Kasparov category KK^G. For instance, we can also apply K-homology and local cyclic homology to the crossed product. Actually, this is nothing so new. Gennadi Kasparov did this using his Dirac dual Dirac method—for all groups to which his method applies (see [28,29]). In his approach, the topological side of the Baum–Connes conjecture appears as the γ-part of $\mathrm{K}_*(G \ltimes_{\mathrm{r}} A)$, and this γ-part makes sense for any functor defined on KK^G. Indeed, our approach is very close to Kasparov's. We show that one half of Kasparov's method, namely, the Dirac morphism, exists in complete generality, and we observe that this suffices to construct the assembly map. From the technical point of view, this is the main innovation in this article.

Our approach is very suitable to state and prove general functorial properties of the assembly map. The various known permanence results of the Baum–Connes conjecture become rather transparent in our setup. Such permanence results have been investigated by several authors. There is a series of papers by Chabert et al. [10–12,15,16]. Both authors of this article have been quite familiar with their work, and it has greatly influenced this article. We also reprove a permanence result for unions of groups by Baum et al. [7] and a result relating the real and complex versions of the Baum–Connes conjecture by Baum and Karoubi [6] and independently by Schick [39]. In addition, we use results of [39] to prove that the existence of a γ-element for a group G for real and complex coefficients is equivalent.

A good blueprint for our approach towards the Baum–Connes conjecture is the work of Davis and Lück in [18]. As kindly pointed out by the referee, the approach of Balmer and Matthey in [3–5] is even closer. However, these are only formal analogies, as we shall explain below.

Davis and Lück only consider discrete groups and reinterpret the Baum–Connes assembly map for $\mathrm{K}_*(G \ltimes_{\mathrm{r}} C_0(X))$ as follows. A proper G-CW-complex $\tilde{X}$ with a G-equivariant continuous map $\tilde{X} \to X$ is called a *proper G-CW-approximation* for X if it has the following universal property: any map from a proper G-CW-complex to X factors through $\tilde{X}$, and this factorisation is unique up to equivariant homotopy. Such approximations always exist and are unique up to equivariant homotopy equivalence. Given a functor F on the category of G-spaces, one defines its localisation by $\mathbb{L}F(X) := F(\tilde{X})$ (up to isomorphism). It comes equipped with a map $\mathbb{L}F(X) \to F(X)$. For suitable F, this is the Baum–Connes assembly map.

We replace the homotopy category of G-spaces by the G-equivariant Kasparov category KK^G, whose objects are the separable G-C^*-algebras and whose morphism spaces are the bivariant groups $\mathrm{KK}_0^G(A, B)$ defined by Kasparov. We need some extra structure, of course, in order to do algebraic topology. For our purposes, it is enough to turn KK^G into a triangulated category (see [38,46]). The basic examples of triangulated categories are the derived categories in homological algebra and the stable homotopy category in algebraic topology. They have enough structure to localise and to do rudimentary homological algebra. According to our knowledge, Andreas Thom's thesis [43] is the first work on C^*-algebras where triangulated categories are explicitly used. Since this structure is crucial for us and not well-known among operator algebraists, we discuss it in an operator algebraic context in Section 2. We also devote an appendix to a detailed proof that KK^G is a triangulated category. This verification of axioms is not very illuminating. The reason for including it is that we could not find a good reference.

We call $A \in \mathrm{KK}^G$ *compactly induced* if it is KK^G-equivalent to $\mathrm{Ind}_H^G A'$ for some compact subgroup $H \subseteq G$ and some H-C^*-algebra A'. We let $\mathscr{CI} \subseteq \mathrm{KK}^G$ be the full subcategory of compactly induced objects and $\langle\mathscr{CI}\rangle$ the localising subcategory generated by it. The objects of $\langle\mathscr{CI}\rangle$ are our substitute for proper G-CW-complexes. The objects of $\mathscr{CI}$ behave like the cells out of which proper G-CW-complexes are built. We define a *$\mathscr{CI}$-simplicial approximation* of $A \in \mathrm{KK}^G$ as a morphism $\tilde{A} \to A$ in KK^G with $\tilde{A} \in \langle\mathscr{CI}\rangle$ such that $\mathrm{KK}^G(P, \tilde{A}) \cong \mathrm{KK}^G(P, A)$ for all $P \in \langle\mathscr{CI}\rangle$. We show that *$\mathscr{CI}$-simplicial approximations* always exist, are unique, functorial, and have good exactness properties. Therefore, if $F\colon \mathrm{KK}^G \to \mathfrak{C}$ is any homological functor into an Abelian category, then its *localisation* $\mathbb{L}F(A) := F(\tilde{A})$ is again a homological functor $\mathrm{KK}^G \to \mathfrak{C}$. It comes equipped with a natural transformation $\mathbb{L}F(A) \to F(A)$. For the functor $F(A) := \mathrm{K}_*(G \ltimes_{\mathrm{r}} A)$, this map is naturally isomorphic to the Baum–Connes assembly map. In particular, $\mathrm{K}_*^{\mathrm{top}}(G, A) \cong \mathbb{L}F(A)$. Thus we have redefined the Baum–Connes assembly map as a localisation.

Of course, we do not expect the map $\mathbb{L}F(A) \to F(A)$ to be an isomorphism for all functors F. For instance, consider the K-theory of the full and reduced crossed products. We will show that both functors have the same localisation. However, the full and reduced group C^*-algebras may have different K-theory.

A variant of Green's Imprimitivity Theorem [20] for reduced crossed products says that $G \ltimes_{\mathrm{r}} \mathrm{Ind}_H^G A$ for a compact subgroup $H \subseteq G$ is Morita–Rieffel equivalent to $H \ltimes A$. Combining this with the Green–Julg Theorem [25], we get

$$\mathrm{K}_*(G \ltimes_{\mathrm{r}} \mathrm{Ind}_H^G A) \cong \mathrm{K}_*(H \ltimes A) \cong \mathrm{K}_*^H(A).$$

Hence $\mathrm{K}_*(G \ltimes_{\mathrm{r}} B)$ is comparatively easy to compute for $B \in \mathscr{CI}$. For an object of $\langle\mathscr{CI}\rangle$, we can, in principle, compute its K-theory by decomposing it into building blocks from $\mathscr{CI}$. In a forthcoming article, we will discuss a spectral sequence that organises this computation. As a result, $\mathrm{K}_*(G \ltimes_{\mathrm{r}} \tilde{A})$ is quite tractable for $\tilde{A} \in \langle\mathscr{CI}\rangle$. The $\mathscr{CI}$-simplicial approximation replaces an arbitrary coefficient algebra A by the best approximation to A in this tractable subcategory in the hope that $\mathrm{K}_*(G \ltimes_{\mathrm{r}} \tilde{A}) \cong \mathrm{K}_*^{\mathrm{top}}(G, A)$ is then a good approximation to $\mathrm{K}_*(G \ltimes_{\mathrm{r}} A)$.

Above we have related the Baum–Connes assembly map to simplicial approximation in homotopy theory. Alternatively, we can use an analogy to homological algebra. In this picture, the category KK^G corresponds to the homotopy category of chain complexes over an Abelian category. The latter has chain complexes as objects and homotopy classes of chain maps as morphisms. To do homological algebra, we also need exact chain complexes and quasi-isomorphisms. In our context, these have the following analogues.

A G-C^*-algebra is called *weakly contractible* if it is KK^H-equivalent to 0 for all compact subgroups $H \subseteq G$. We let $\mathscr{CC} \subseteq \mathrm{KK}^G$ be the full subcategory of weakly contractible objects. This is a localising subcategory of KK^G. We call $f \in \mathrm{KK}^G(A, B)$ a *weak equivalence* if it is invertible in $\mathrm{KK}^H(A, B)$ for all compact subgroups $H \subseteq G$. The weakly contractible objects and the weak equivalences determine each other: a morphism is a weak equivalence if and only if its "mapping cone" is weakly contractible, and A is weakly contractible if and only if the zero map $0 \to A$ is a weak equivalence.

The subcategories $\mathscr{CC}$ and $\langle\mathscr{CI}\rangle$ are "orthogonal complements" in the sense that $B \in \mathscr{CC}$ if and only if $\mathrm{KK}^G(A, B) = 0$ for all $A \in \langle\mathscr{CI}\rangle$, and $A \in \langle\mathscr{CI}\rangle$ if and only if $\mathrm{KK}^G(A, B) = 0$ for all $B \in \mathscr{CC}$. Hence $f \in \mathrm{KK}^G(B, B')$ is a weak equivalence if and only if the induced map $\mathrm{KK}^G(A, B) \to \mathrm{KK}^G(A, B')$ is an isomorphism for all $A \in \langle\mathscr{CI}\rangle$. Therefore, a $\mathscr{CI}$-simplicial approximation for A is the same as a weak equivalence $f \in \mathrm{KK}^G(\tilde{A}, A)$ with $\tilde{A} \in \langle\mathscr{CI}\rangle$.

We now return to our analogy with homological algebra. The weakly contractible objects play the role of the exact chain complexes and the weak equivalences play the role of the quasi-isomorphisms. Objects of $\langle \mathscr{CI} \rangle$ correspond to projective chain complexes as defined in [31]. Hence $\mathscr{CI}$-simplicial approximations correspond to projective resolutions. In homological algebra, we can compute the total left derived functor of a functor F by applying F to a projective resolution. Thus $\mathbb{L}F$ as defined above corresponds to the total left derived functor of F. In particular, $\mathrm{K}_*^{\mathrm{top}}(G, A)$ appears as the total left derived functor of $\mathrm{K}_*(G \ltimes_{\mathrm{r}} A)$.

Bernhard Keller's presentation of homological algebra in [31] is quite close to our constructions because it relies very much on triangulated categories. This is unusual because most authors prefer to use the finer structure of Abelian categories. However, nothing in our setup corresponds to the underlying Abelian category. Hence we only get an analogue of the total derived functor, not of the satellite functors that are usually called derived functors. A more serious difference is that there are almost no interesting exact functors in homological algebra. In contrast, the Baum–Connes conjecture asserts that the functor $\mathrm{K}_*(G \ltimes_{\mathrm{r}} A)$ agrees with its total derived functor, which is equivalent to exactness in classical homological algebra. Hence the analogy to homological algebra is somewhat misleading.

Using weak equivalences, we can also formulate the Baum–Connes conjecture with coefficients as a *rigidity* statement. The assembly map $\mathbb{L}F(A) \to F(A)$ is an isomorphism for all A if and only if F maps all weak equivalences to isomorphisms. If F satisfies some exactness property, this is equivalent to $F(A) = 0$ for all $A \in \mathscr{CC}$. If $A \in \mathscr{CC}$, then A is KK^G-equivalent to a G-C^*-algebra that is H-equivariantly contractible for any compact subgroup $H \subseteq G$ (replace A by the universal algebra $q_s A$ defined in [34]). Thus the Baum–Connes conjecture with coefficients is equivalent to the statement that $\mathrm{K}_*(G \ltimes_{\mathrm{r}} A) = 0$ if A is H-equivariantly contractible for all compact subgroups $H \subseteq G$. Another equivalent formulation that we obtain in Section 9 is the following. The Baum–Connes conjecture with coefficients is equivalent to the statement that $\mathrm{K}_*(G \ltimes_{\mathrm{r}} A) = 0$ if $\mathrm{K}_*(H \ltimes A) = 0$ for all compact subgroups $H \subseteq G$. Both reformulations of the Baum–Connes conjecture with coefficients are as elementary as possible: they involve nothing more than compact subgroups, K-theory and reduced crossed products.

The localisation of the homotopy category of chain complexes over an Abelian category at the subcategory of exact chain complexes is its derived category, which is the category of primary interest in homological algebra. In our context, it corresponds to the localisation $\mathrm{KK}^G/\mathscr{CC}$. We describe $\mathrm{KK}^G/\mathscr{CC}$ in more classical terms, using the universal proper G-space $\mathscr{E}G$. We identify the space of morphisms $A \to B$ in $\mathrm{KK}^G/\mathscr{CC}$ with the group $\mathrm{RKK}^G(\mathscr{E}G; A, B)$ as defined by Kasparov [28]. The canonical functor $\mathrm{KK}^G \to \mathrm{KK}^G/\mathscr{CC}$ is the obvious one,

$$p_{\mathscr{E}G}^* \colon \mathrm{KK}^G(A, B) \to \mathrm{RKK}^G(\mathscr{E}G; A, B).$$

As a consequence, if A is weakly contractible, then $p_Y^*(A) \cong 0$ for any proper G-space Y. This means that the homogeneous spaces G/H for $H \subseteq G$ compact, which are implicitly used in the definition of weak contractibility, already generate all proper G-spaces. Another consequence is that proper G-C^*-algebras in the sense of Kasparov belong to $\langle \mathscr{CI} \rangle$. Conversely, for many groups any object of $\langle \mathscr{CI} \rangle$ is KK^G-equivalent to a proper G-C^*-algebra (see the end of Section 7).

Let $\star \in \mathrm{KK}^G$ be the real or complex numbers, depending on the category we work with. We have a tensor product operation in KK^G, which is nicely compatible with the subcategories $\mathscr{CC}$ and $\mathscr{CI}$. Therefore, if $\mathsf{D} \in \mathrm{KK}^G(\mathsf{P}, \star)$ is a $\mathscr{CI}$-simplicial approximation for $\star$, then $\mathsf{D} \otimes \mathrm{id}_A \in \mathrm{KK}^G(\mathsf{P} \otimes A, A)$ is a $\mathscr{CI}$-simplicial approximation for $A \in \mathrm{KK}^G$. Thus we can describe the localisation of a functor more explicitly as $\mathbb{L}F(A) := F(\mathsf{P} \otimes A)$. We call D a *Dirac morphism* for G. Its existence is equivalent to the representability of a certain functor. Eventually, this is deduced from a generalisation of Brown's Representability Theorem to triangulated categories.

The following example of a Dirac morphism motivates our terminology. Suppose that $\mathscr{E}G$ is a smooth manifold. Replacing it by a suspension of $T^*\mathscr{E}G$, we achieve that $\mathscr{E}G$ has a G-invariant spin structure and that $8 \mid \dim \mathscr{E}G$. Then the Dirac operator on $\mathscr{E}G$ defines an element of $\mathrm{KK}_0^G(C_0(\mathscr{E}G), \star)$; this is a Dirac morphism for G. We can also describe it as the element $p! \in \mathrm{KK}_0^G(C_0(\mathscr{E}G), \star)$ associated to the constant map $p: \mathscr{E}G \to \star$ by wrong way functoriality. Unfortunately, wrong way functoriality only works for manifolds. Extending it to non-Hausdorff manifolds as in [29], one can construct explicit Dirac morphisms also for groups acting properly and simplicially on finite dimensional simplicial complexes. However, it is unclear how to adapt this to infinite dimensional situations.

Since we work in the Kasparov category, Bott periodicity is an integral part of our setup. The above example of a Dirac morphism shows that wrong way functoriality and hence Bott periodicity indeed play significant roles. This distinguishes our approach from [3–5,18]. The bad news is that we cannot treat homology theories such as algebraic K-theory that do not satisfy periodicity. The good news is that the Dirac dual Dirac method, which is one of the main proof techniques in connection with the Baum–Connes conjecture, is already part of our setup. In examples, this method usually arises as an equivariant version of Bott periodicity.

A *dual Dirac morphism* is an element $\eta \in \mathrm{KK}^G(\star, \mathsf{P})$ that is a left-inverse to the Dirac morphism $\mathsf{D} \in \mathrm{KK}^G(\mathsf{P}, \star)$, that is, $\eta\mathsf{D} = \mathrm{id}_{\mathsf{P}}$. Suppose that it exists. Then $\gamma = \mathsf{D}\eta$ is an idempotent in $\mathrm{KK}^G(\star, \star)$. By exterior product, we get idempotents $\gamma_A \in \mathrm{KK}^G(A, A)$ for all $A \in \mathrm{KK}^G$. We have $A \in \mathscr{CC}$ if and only if $\gamma_A = 0$, and $A \in \langle \mathscr{CI} \rangle$ if and only if $\gamma_A = 1$. The category KK^G is equivalent to the direct product $\mathrm{KK}^G \cong \mathscr{CC} \times \langle \mathscr{CI} \rangle$. Therefore, the assembly map is split injective for any covariant functor. For groups with the Haagerup property and, in particular, for amenable groups, a dual Dirac morphism exists and we have $\gamma = 1$. This important theorem is due to Higson and Kasparov [23]. In this case, weak equivalences are already isomorphisms in KK^G. Hence $\mathbb{L}F = F$ for any functor F.

When we compose two functors in homological algebra, it frequently happens that $\mathbb{L}(F' \circ F) \cong \mathbb{L}F' \circ \mathbb{L}F$. This holds, for instance, if F maps projectives to projectives. We check that the restriction and induction functors preserve the subcategories $\mathscr{CC}$ and $\langle \mathscr{CI} \rangle$. The same holds for the complexification functor from real to complex KK-theory and many others. The ensuing identities of localised functors imply permanence properties of the Baum–Connes conjecture.

Another useful idea that our new approach allows is the following. Instead of deriving the functor $A \mapsto \mathrm{K}_*(G \ltimes_{\mathrm{r}} A)$, we may also derive the crossed product functor $A \mapsto G \ltimes_{\mathrm{r}} A$ itself. Its localisation $G \ltimes_{\mathrm{r}}^{\mathbb{L}} A$ is a triangulated functor from KK^G to KK. It can be described explicitly as $G \ltimes_{\mathrm{r}}^{\mathbb{L}} A = G \ltimes_{\mathrm{r}} (\mathsf{P} \otimes A)$ if $\mathsf{D} \in \mathrm{KK}^G(\mathsf{P}, \star)$ is a Dirac morphism. The Baum–Connes conjecture asks for $\mathsf{D}_* \in \mathrm{KK}(G \ltimes_{\mathrm{r}}^{\mathbb{L}} A, G \ltimes_{\mathrm{r}} A)$ to induce an isomorphism on K-theory. Instead, we can ask it to be a KK-equivalence. Then the Baum–Connes conjecture holds for $F(G \ltimes_{\mathrm{r}} A)$ for any split exact, stable homotopy functor F on C^*-algebras because such functors descend to the category KK. For instance, this covers local cyclic (co)homology and K-homology.

This stronger conjecture is known to be false in some cases where the Baum–Connes conjecture holds. Nevertheless, it holds in many examples. For groups with the Haagerup property, we have $\gamma = 1$, so that $\mathbb{L}F = F$ for any functor, anyway. If both $G \ltimes_{\mathrm{r}} A$ and $G \ltimes_{\mathrm{r}}^{\mathbb{L}} A$ satisfy the Universal Coefficient Theorem (UCT) in KK, then an isomorphism on K-theory is automatically a KK-equivalence. Since $G \ltimes_{\mathrm{r}}^{\mathbb{L}} \star$ always satisfies the UCT, the strong Baum–Connes conjecture with trivial coefficients holds if and only if the usual Baum–Connes conjecture holds and $C_{\mathrm{r}}^*(G)$ satisfies the UCT. This is known to be the case for almost connected groups and linear algebraic groups over p-adic number fields, see [14,16].

This article is the first step in a programme to extend the Baum–Connes conjecture to quantum group crossed products. It does not seem a good idea to extend the usual construction in the group case because it is not clear whether the resulting analogue of $\mathrm{K}_*^{\mathrm{top}}(G, A)$ can be computed. Even if we had a good notion of a proper action of a quantum group, these actions would certainly occur on very non-commutative spaces, so that we have to "quantise" the algebraic topology needed to compute $\mathrm{K}_*^{\mathrm{top}}(G, A)$. The framework of triangulated categories and localisation of functors is ideal for this purpose. In the group case, the homogeneous spaces G/H for compact subgroups $H \subseteq G$ generate all proper actions. Thus we expect that we can formulate the Baum–Connes conjecture for quantum groups using quantum homogeneous spaces instead of proper actions. However, we still need some further algebraic structure: restriction and induction functors and tensor products of coactions. We plan to treat this additional structure and to construct a Baum–Connes assembly map for quantum groups in a sequel to this paper. Here we only consider the classical case of group actions.

1.1. Some general conventions

Let $\mathfrak{C}$ be a category. We write $A \in \mathfrak{C}$ to denote that A is an object of $\mathfrak{C}$, and $\mathfrak{C}(A, B)$ for the space of morphisms $A \to B$ in $\mathfrak{C}$.

It makes no difference whether we work with real, "real", or complex C^*-algebras. Except for Section 10.6, we do not distinguish between these cases in our notation. Of course, standard C^*-algebras like $C_0(X)$ and $C_{\mathrm{r}}^*(G)$ have to be taken in the appropriate category. We denote the one-point space by $\star$ and also write $\star = C(\star)$. Thus $\star$ denotes the complex or real numbers depending on the category we use.

Locally compact groups and spaces are tacitly assumed to be second countable, and C^*-algebras are tacitly assumed to be separable. Let G be a locally compact group and let X be a locally compact G-space. A *G-C^*-algebra* is a C^*-algebra equipped with a strongly continuous action of G by automorphisms. A *$G \ltimes X$-C^*-algebra* is a G-C^*-algebra equipped with a G-equivariant essential $*$-homomorphism from $C_0(X)$ to the centre of its multiplier algebra. Kasparov defines bivariant K-theory groups $\mathscr{R}\mathrm{KK}_*^G(X; A, B)$ involving these data in [28, Definition 2.19]. The notation $\mathscr{R}\mathrm{KK}_*^G(X; A, B)$ should be distinguished from $\mathrm{RKK}_*^G(X; A, B)$. The latter is defined for two G-C^*-algebras A and B by

$$\mathrm{RKK}_*^G(X; A, B) := \mathscr{R}\mathrm{KK}_*^G(X; C_0(X, A), C_0(X, B)). \tag{2}$$

Since $\mathscr{R}\mathrm{KK}_*^G(X; A, B)$ agrees with the bivariant K-groups for the groupoid $G \ltimes X$ as defined in [32], we denote it by $\mathrm{KK}_*^{G \ltimes X}(A, B)$. For several purposes, it is useful to generalise from groups to groupoids. However, we do not treat arbitrary groupoids because it is not so clear what should correspond to the compact subgroups in this case. We work with transformation groups throughout because this generalisation is not more difficult than the group case and useful for several applications.

We write $\mathrm{K}_*(A)$ for the graded Abelian group $n \mapsto \mathrm{K}_n(A)$, $n \in \mathbb{Z}$, and similarly for $\mathrm{KK}_*^{G \ltimes X}(A, B)$. We usually omit the subscript 0, that is, $\mathrm{K}(A) := \mathrm{K}_0(A)$, etc.

The *$G \ltimes X$-equivariant Kasparov category* is the additive category whose objects are the $G \ltimes X$-C^*-algebras and whose group of morphisms $A \to B$ is $\mathrm{KK}_0^{G \ltimes X}(A, B)$; the composition is the Kasparov product. We denote this category by $\mathrm{KK}^{G \ltimes X}$.

The notion of equivalence for $G \ltimes X$-C^*-algebras that we encounter most frequently is KK-equivalence, that is, isomorphism in $\mathrm{KK}^{G\ltimes X}$, which we simply denote by "$\cong$". Sometimes we may want to stress that two $G\ltimes X$-C^*-algebras are more than just KK-equivalent. We write $A \approx B$ if A and B are isomorphic as $G\ltimes X$-C^*-algebras and $A\sim_M B$ if A and B are $G\ltimes X$-equivariantly Morita–Rieffel equivalent. Both relations imply $A\cong B$.

2. Triangulated categories of operator algebras

In this section, we explain triangulated categories in the context of equivariant Kasparov theory. The purpose is to introduce operator algebraists to the language of triangulated categories, which we are using throughout this article. We hope that it allows them to understand this article without having to read the specialised literature on triangulated categories (like [38,46]). Thus we translate various known results of non-commutative topology into the language of triangulated categories. In addition, we sketch how to prove basic facts about localisation of triangulated categories in the special case where there are enough projectives. There is nothing essentially new in this section. The only small exception is the rather satisfactory treatment of inductive limits of C^*-algebras in Section 2.4.

The two motivating examples of triangulated categories are the stable homotopy category from algebraic topology and the derived categories of Abelian categories from homological algebra. The definition of a triangulated category formalises some important structure that is present in these categories. The additional structure consists of a *translation automorphism* and a class of *exact triangles* (often called *distinguished triangles*). We first explain what these are for $\mathrm{KK}^{G\ltimes X}$.

2.1. Suspensions and mapping cones

Let $\Sigma\colon \mathrm{KK}^{G\ltimes X} \to \mathrm{KK}^{G\ltimes X}$ be the suspension functor $\Sigma A := C_0(\mathbb{R}) \otimes A$. This is supposed to be the translation automorphism in our case. However, it is only an equivalence and not an isomorphism of categories. This defect is repaired by the following trick. We replace $\mathrm{KK}^{G\ltimes X}$ by the category $\widetilde{\mathrm{KK}}^{G\ltimes X}$ whose objects are pairs (A, n) with $A \in \mathrm{KK}^{G\ltimes X}$, $n \in \mathbb{Z}$, with morphisms

$$\widetilde{\mathrm{KK}}^{G\ltimes X}((A,n),(B,m)) := \varinjlim_{p\in\mathbb{N}} \mathrm{KK}^{G\ltimes X}(\Sigma^{p+n}A, \Sigma^{p+m}B).$$

Actually, since the maps $\mathrm{KK}^{G\ltimes X}(A,B) \to \mathrm{KK}^{G\ltimes X}(\Sigma A, \Sigma B)$ are isomorphisms by Bott periodicity, we can omit the direct limit over p. Morphisms in $\widetilde{\mathrm{KK}}^{G\ltimes X}$ are composed in the obvious fashion. We define the *translation* or *suspension* automorphism on $\widetilde{\mathrm{KK}}^{G\ltimes X}$ by $\Sigma(A,n) := (A, n+1)$. The evident functor $\mathrm{KK}^{G\ltimes X} \to \widetilde{\mathrm{KK}}^{G\ltimes X}$, $A \mapsto (A,0)$, identifies $\mathrm{KK}^{G\ltimes X}$ with a full subcategory of $\widetilde{\mathrm{KK}}^{G\ltimes X}$. Any object of $\widetilde{\mathrm{KK}}^{G\ltimes X}$ is isomorphic to one from this subcategory because Bott periodicity yields $(A,n)\cong(\Sigma^{n \bmod 8}A, 0)$ for all $n \in \mathbb{Z}$, $A \in \mathrm{KK}^{G\ltimes X}$. Thus the categories $\widetilde{\mathrm{KK}}^{G\ltimes X}$ and $\mathrm{KK}^{G\ltimes X}$ are equivalent. It is not necessary to distinguish between $\widetilde{\mathrm{KK}}^{G\ltimes X}$ and $\mathrm{KK}^{G\ltimes X}$ except in very formal arguments and definitions. Most of the time, we ignore the difference between these two categories.

Let $f\colon A \to B$ be an equivariant $*$-homomorphism. Then its *mapping cone*

$$\mathrm{cone}(f) := \{(a,b) \in A \times C_0(]0,1], B) \mid f(a) = b(1)\} \tag{3}$$

is again a $G \ltimes X$-C^*-algebra and there are natural equivariant $*$-homomorphisms

$$\Sigma B \xrightarrow{\iota} \operatorname{cone}(f) \xrightarrow{\varepsilon} A \xrightarrow{f} B. \tag{4}$$

Such diagrams are called *mapping cone triangles*. A diagram $\Sigma B' \to C' \to A' \to B'$ in $\widetilde{\mathrm{KK}}^{G \ltimes X}$ is called an *exact triangle* if it is isomorphic to a mapping cone triangle. That is, there is an equivariant $*$-homomorphism $f: A \to B$ and a commutative diagram

$$\begin{array}{ccccccc}
\Sigma B & \longrightarrow & \operatorname{cone}(f) & \longrightarrow & A & \longrightarrow & B \\
\cong\downarrow \Sigma\beta & & \cong\downarrow \gamma & & \cong\downarrow \alpha & & \cong\downarrow \beta \\
\Sigma B' & \longrightarrow & C' & \longrightarrow & A' & \longrightarrow & B'
\end{array}$$

where α, β, γ are isomorphisms in $\widetilde{\mathrm{KK}}^{G \ltimes X}$ and $\Sigma\beta$ is the suspension of β.

Proposition 2.1. *The category $\widetilde{\mathrm{KK}}^{G \ltimes X}$ with Σ^{-1} as translation functor and with the exact triangles as described above is a triangulated category. It has countable direct sums: they are the usual C^*-direct sums.*

It is proven in the appendix that $\widetilde{\mathrm{KK}}^{G \ltimes X}$ is triangulated. It is shown in [28] that $\mathrm{KK}^{G \ltimes X}(\bigoplus A_n, B) \cong \prod \mathrm{KK}^{G \ltimes X}(A_n, B)$. This means that the usual C^*-direct sum is a direct sum operation also in $\mathrm{KK}^{G \ltimes X}$.

Notice that the translation functor is the inverse of the suspension Σ. The reason for this is as follows. The axioms of a triangulated category are modelled after the stable homotopy category, and the functor from spaces to C^*-algebras is contravariant. Hence we ought to work with the opposite category of $\mathrm{KK}^{G \ltimes X}$. The opposite category of a triangulated category becomes again triangulated if we use “the same” exact triangles and replace the translation functor by its inverse. Since we want to work with $\mathrm{KK}^{G \ltimes X}$ and not its opposite and retain the usual constructions from the stable homotopy category, we sometimes deviate in our conventions from the usual ones for a triangulated category. For instance, we always write exact triangles in the form $\Sigma B \to C \to A \to B$.

One of the axioms of a triangulated category requires any $f \in \mathrm{KK}^{G \ltimes X}(A, B)$ to be part of an exact triangle $\Sigma B \to C \to A \xrightarrow{f} B$. We call this triangle a *mapping cone triangle* for f and C a *mapping cone* for f. We can use the mapping cone triangle (4) if f is an equivariant $*$-homomorphism. In general, we replace f by an equivariant $*$-isomorphism $f': q_s A \to q_s B$ and then take a mapping cone triangle of f' as in (4). The universal C^*-algebra $q_s A$ is defined in [34]. It is important that $q_s A$ is isomorphic to A in $\mathrm{KK}^{G \ltimes X}$. We warn the reader that the above construction only works for ungraded C^*-algebras. For this reason, the Kasparov category of graded C^*-algebras is *not* triangulated. We can represent elements of $\mathrm{KK}(A, B)$ by equivariant, grading preserving $*$-homomorphisms $\chi A \to \mathbb{K} \otimes B$ as in [34]. However, χA is no longer KK-equivalent to A.

The mapping cone triangle has the weak functoriality property that for any commutative diagram

$$\begin{array}{ccccccc}
\Sigma B & \longrightarrow & C & \longrightarrow & A & \longrightarrow & B \\
\downarrow \Sigma\beta & & & & \downarrow \alpha & & \downarrow \beta \\
\Sigma B' & \longrightarrow & C' & \longrightarrow & A' & \longrightarrow & B'
\end{array}$$

whose rows are exact triangles there is a morphism $\gamma: C \to C'$ making the diagram commute. The triple (γ, α, β) is called a *morphism of triangles*. We do not have a functor essentially because γ is not unique.

At least, the axioms of a triangulated category guarantee that γ is an isomorphism if α and β are. Thus the mapping cone and the mapping cone triangle are unique up to a non-canonical isomorphism.

The following facts are proven in [38].

Lemma 2.2. *Let $\Sigma B \to C \to A \xrightarrow{f} B$ be an exact triangle. Then $C = 0$ if and only if f is an isomorphism. That is, a morphism f is an isomorphism if and only if its mapping cone vanishes.*

If the map $\Sigma B \to C$ vanishes, then there is an isomorphism $A \cong C \oplus B$ such that the maps $C \to A \to B$ become the obvious ones. That is, the triangle is isomorphic to a "direct sum" triangle. Conversely, direct sum triangles are exact.

2.2. *Long exact sequences*

Let $\mathscr{T}$ be a triangulated category, for instance, $\mathrm{KK}^{G\ltimes X}$, let Ab be the category of Abelian groups (or any Abelian category). We call a covariant functor $F\colon \mathscr{T} \to \mathsf{Ab}$ *homological* if $F(C) \to F(A) \to F(B)$ is exact for any exact triangle $\Sigma B \to C \to A \to B$. We define $F_n(A) := F(\Sigma^n A)$ for $n \in \mathbb{Z}$. Similarly, we call a contravariant functor $F\colon \mathscr{T} \to \mathsf{Ab}$ *cohomological* if $F(B) \to F(A) \to F(C)$ is exact for any exact triangle, and we define $F^n(A) := F(\Sigma^n A)$. The functor $A \mapsto \mathscr{T}(A, B)$ is cohomological for any fixed B and the functor $B \mapsto \mathscr{T}(A, B)$ is homological for any fixed A. This follows from the axioms of a triangulated category. Since we can rotate exact triangles, we obtain a long exact sequence (infinite in both directions)

$$\cdots \to F_n(C) \to F_n(A) \to F_n(B) \to F_{n-1}(C) \to F_{n-1}(A) \to F_{n-1}(B) \to \cdots$$

if F is homological, and a dual long exact sequence for cohomological F. The maps in this sequence are induced by the maps of the exact triangle, of course.

2.3. *Extension triangles*

Since any exact triangle in $\mathrm{KK}^{G\ltimes X}$ is isomorphic to a mapping cone triangle, Section 2.2 only yields long exact sequences for mapping cone triangles. As in [17], this suffices to get long exact sequences for suitable extensions. Let $K \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} Q$ be an extension of $G\ltimes X$-C^*-algebras. There is a canonical equivariant $*$-homomorphism $K \to \mathrm{cone}(p)$ that makes the diagram

$$\begin{array}{ccccccc} \Sigma Q & \dashrightarrow & K & \xrightarrow{i} & E & \xrightarrow{p} & Q \\ \| & & \downarrow & & \| & & \| \\ \Sigma Q & \xrightarrow{\iota} & \mathrm{cone}(p) & \xrightarrow{\epsilon} & E & \xrightarrow{p} & Q \end{array} \tag{5}$$

commute. The bottom row is the mapping cone triangle, of course. In the non-equivariant case, there is a canonical isomorphism $\mathrm{KK}(\Sigma Q, K) \cong \mathrm{Ext}(Q, K)$. There also exist similar results in the equivariant case [44]. If the extension has a completely positive, contractive, equivariant cross section, then it defines an element of $\mathrm{Ext}^{G\ltimes X}(Q, K) \cong \mathrm{KK}^{G\ltimes X}(\Sigma Q, K)$. This provides the dotted arrow in (5). Furthermore, the vertical map $K \to \mathrm{cone}(p)$ is invertible in $\mathrm{KK}^{G\ltimes X}$ in this case. This can be proven directly and then used to prove excision for the given extension. Conversely, it follows from excision and the Puppe sequence using the Five Lemma.

Definition 2.3. We call the extension *admissible* if the map $K \to \mathrm{cone}(p)$ in (5) is invertible in $\mathrm{KK}^{G\ltimes X}$. Then there is a unique map $\Sigma Q \to K$ that makes (5) commute. The triangle $\Sigma Q \to K \to E \to Q$ is called an *extension triangle*.

If an extension is admissible, then the vertical maps in (5) form an isomorphism of triangles. Hence extension triangles are exact. Not every extension is admissible. As we remarked above, extensions with an equivariant, contractive, completely positive section are admissible. If we replace $\mathrm{KK}^{G\ltimes X}$ by $E^{G\ltimes X}$, then every extension becomes admissible.

Let $f: A \to B$ be an equivariant $*$-homomorphism. We claim that the mapping cone triangle for f is the extension triangle for an appropriate extension. For this we need the *mapping cylinder*

$$\mathrm{cyl}(f) := \{(a, b) \in A \times C([0,1], B) \mid f(a) = b(1)\}. \tag{6}$$

Given $b \in B$, let $\mathrm{const}\, b \in C([0,1], B)$ be the constant function with value b. Define natural $*$-homomorphisms

$$\begin{aligned} p_A&: \mathrm{cyl}(f) \to A, \quad (a,b) \mapsto a,\\ j_A&: A \to \mathrm{cyl}(f), \quad a \mapsto (a, \mathrm{const}\, f(a)),\\ \tilde{f}&: \mathrm{cyl}(f) \to B, \quad (a,b) \mapsto b(0). \end{aligned}$$

Then $p_A j_A = \mathrm{id}_A$, $\tilde{f} j_A = f$, and $j_A p_A$ is homotopic to the identity map in a natural way. Thus $\mathrm{cyl}(f)$ is homotopy equivalent to A and this homotopy equivalence identifies the maps $\tilde{f}$ and f. We have a natural C^*-extension

$$0 \to \mathrm{cone}(f) \to \mathrm{cyl}(f) \xrightarrow{\tilde{f}} B \to 0. \tag{7}$$

Build the diagram (5) for this extension. One checks easily that the resulting map $\mathrm{cone}(f) \to \mathrm{cone}(\tilde{f})$ is a homotopy equivalence, so that the extension (7) is admissible. The composition $\Sigma B \to \mathrm{cone}(\tilde{f}) \xleftarrow{\sim} \mathrm{cone}(f)$ is naturally homotopy equivalent to the inclusion map $\Sigma B \to \mathrm{cone}(f)$. Thus the extension triangle for the admissible extension (7) is isomorphic to the mapping cone triangle for f. It follows that any exact triangle in $\mathrm{KK}^{G\ltimes X}$ is isomorphic to an extension triangle for some admissible extension.

2.4. Homotopy limits

Let (A_n, α_m^n) be a countable inductive system in $\mathrm{KK}^{G\ltimes X}$, with structure maps $\alpha_m^n: A_m \to A_n$ for $m \leqslant n$. (Of course, it suffices to give the maps α_m^{m+1}.) Roughly speaking, its homotopy direct limit is the correct substitute for the inductive limit for homological computations. Homotopy direct limits play an important role in the proof of the Brown Representability Theorem 6.1. They also occur in connection with the behaviour of the Baum–Connes conjecture for unions of open subgroups in Section 10.3.

The *homotopy direct limit* $\text{ho-}\varinjlim A_m$ is defined to fit in an exact triangle

$$\Sigma\, \text{ho-}\varinjlim A_m \to \bigoplus A_m \xrightarrow{\mathrm{id}-S} \bigoplus A_m \to \text{ho-}\varinjlim A_m. \tag{8}$$

Here S is the shift map that maps the summand A_m to the summand A_{m+1} via α_m^{m+1}. Thus the homotopy direct limit is $\Sigma^{-1}\mathrm{cone}(\mathrm{id} - S)$; it is well-defined up to non-canonical isomorphism and has the same weak kind of functoriality as mapping cones. The (de)suspensions are due to the passage to opposite categories that is implicit in our conventions. This also means that homotopy direct limits in KK behave like homotopy inverse limits of spaces. The map $\bigoplus A_m \to \text{ho-}\varinjlim A_m$ in (8) is equivalent to maps $\alpha_m^\infty: A_m \to \text{ho-}\varinjlim A_m$ with $\alpha_n^\infty \circ \alpha_m^n = \alpha_m^\infty$ for $m \leqslant n$.

To formulate the characteristic properties of the homotopy limit, we consider (co)homological functors to the category of Abelian groups that are *compatible with direct sums*. This means $F(\bigoplus A_m) \cong \bigoplus F(A_m)$ in the covariant case and $F(\bigoplus A_m) \cong \prod F(A_m)$ in the contravariant case. The functor $B \mapsto \mathscr{T}(A, B)$ is not always compatible with direct sums. We call A *compact* if it is. The functor $A \mapsto \mathscr{T}(A, B)$ is always compatible with direct sums: this is just the universal property of direct sums. Hence the following lemma applies to $F(A) := \mathscr{T}(A, B)$ for any B.

Lemma 2.4 (*Neeman [36]*). *If F is homological and compatible with direct sums, then the maps $\alpha_m^\infty: A_m \to \text{ho-}\varinjlim A_m$ yield an isomorphism $\varinjlim F_n(A_m) \xrightarrow{\cong} F_n(\text{ho-}\varinjlim A_m)$. If F is cohomological and compatible with direct sums, then there is a short exact sequence*

$$0 \to \varprojlim{}^1 F^{n-1}(A_m) \to F^n(\text{ho-}\varinjlim A_m) \to \varprojlim F^n(A_m) \to 0.$$

The map $F^n(\text{ho-}\varinjlim A_m) \to \varprojlim F^n(A_m)$ is induced by $(\alpha_m^\infty)_{m\in\mathbb{N}}$.

Proof. Consider the homological case first. Apply the long exact homology sequence to (8) and cut the result into short exact sequences of the form

$$\operatorname{coker}\left(\mathrm{id} - S: \bigoplus F_n(A_m) \to \bigoplus F_n(A_m)\right) \rightarrowtail F_n(\text{ho-}\varinjlim A_m) \twoheadrightarrow \ker\left(\mathrm{id} - S: \bigoplus F_{n-1}(A_m) \to \bigoplus F_{n-1}(A_m)\right).$$

The kernel of $\mathrm{id} - S$ vanishes and its cokernel is, by definition, $\varinjlim F_n(A_m)$. Whence the assertion. In the cohomological case, we get a short exact sequence

$$\operatorname{coker}\left(\mathrm{id} - S: \prod F^{n-1}(A_m) \to \prod F^{n-1}(A_m)\right) \rightarrowtail F^n(\text{ho-}\varinjlim A_m) \twoheadrightarrow \ker\left(\mathrm{id} - S: \prod F^n(A_m) \to \prod F^n(A_m)\right).$$

By definition, the kernel is $\varprojlim F^n(A_m)$ and the cokernel is $\varprojlim{}^1 F^{n-1}(A_m)$. □

We now specialise to the category $\mathrm{KK}^{G\ltimes X}$ and relate homotopy direct limits to ordinary direct limits via mapping telescopes. This is used in our discussion of unions of groups in Section 10.3. Any inductive system in $\mathrm{KK}^{G\ltimes X}$ is isomorphic to the image of a direct system of $G\ltimes X$-C^*-algebras. That is, the maps α_m^n are equivariant $*$-homomorphisms and satisfy $\alpha_m^n \circ \alpha_l^m = \alpha_l^n$ as such. To get this, replace the A_m by the universal algebra $q_s(A_m)$ as in the appendix.

The following discussion follows the treatment of inductive limits in [40]. Let (A_m, α_m^n) be an inductive system of $G\ltimes X$-C^*-algebras. We let $A_\infty := \varinjlim A_m$ and denote the natural maps $A_m \to A_\infty$ by α_m^∞.

The *mapping telescope* of the system is defined as the G-C^*-algebra

$$T(A_m,\alpha_m^n) := \left\{ (f_m) \in \bigoplus_{m\in\mathbb{N}} C([m,m+1],A_m) \middle| \; f_0(0)=0 \text{ and } f_{m+1}(m+1) = \alpha_m^{m+1}(f_m(m+1)) \right\}.$$

In the special case where the homomorphisms α_m^n are injective, $T(A_m,\alpha_m^n)$ is the space of all $f \in C_0(]0,\infty[,A_\infty)$ with $f(t)\in A_m$ for $t\leqslant m+1$. In particular, for the constant inductive system (A_∞,id) we obtain just the suspension ΣA_∞. Since the mapping telescope construction is functorial, there is a natural equivariant $*$-homomorphism $T(A_n,\alpha_m^n)\to\Sigma A_\infty$.

Definition 2.5. An inductive system (A_m,α_m^n) is called *admissible* if the map $T(A_n,\alpha_m^n)\to\Sigma A_\infty$ is invertible in $\mathrm{KK}^{G\ltimes X}$.

Proposition 2.6. *We have* $\varinjlim(A_m,\alpha_m^n)\cong\text{ho-}\varinjlim(A_m,\alpha_m^n)$ *for admissible inductive systems.*

Proof. Evaluation at positive integers defines a natural, surjective, equivariant $*$-homomorphism $\pi: T(A_m,\alpha_m^n)\to\bigoplus A_m$. Its kernel is naturally isomorphic to $\bigoplus\Sigma A_m$. Thus we obtain a natural extension

$$0\longrightarrow\bigoplus\Sigma A_m\xrightarrow{\iota}T(A_m,\alpha_m^n)\xrightarrow{\pi}\bigoplus A_m\longrightarrow 0.$$

Build the diagram (5) for this extension. The map $\bigoplus\Sigma A_n\to\mathrm{cone}(\pi)$ is a homotopy equivalence in a natural and hence equivariant fashion. Hence the extension is admissible. Moreover, one easily identifies the map $\Sigma\left(\bigoplus A_m\right)\to\bigoplus\Sigma A_m$ with $S-\mathrm{id}$, where S is the shift map defined above. Rotating the extension triangle, we obtain an exact triangle

$$T(A_m,\alpha_m^n)\xrightarrow{-\pi}\bigoplus A_m\xrightarrow{\mathrm{id}-S}\bigoplus A_m\xrightarrow{\Sigma^{-1}\iota}\Sigma^{-1}T(A_m,\alpha_m^n).$$

This implies $\Sigma^{-1}T(A_m,\alpha_m^n)\cong\text{ho-}\varinjlim(A_m,\alpha_m^n)$ and hence the assertion. □

To obtain a concrete criterion for admissibility, we let $\tilde T(A_m,\alpha_m^n)$ be the variant of $T(A_m,\alpha_m^n)$ where we require $\lim_{t\to\infty}\alpha_m^\infty(f_m(t))$ to exist in A_∞ instead of $\lim f_m(t)=0$. The algebra $\tilde T(A_m,\alpha_m^n)$ is equivariantly contractible in a natural way. The contracting homotopy is obtained by making sense of the formula $H_s f(t) := \alpha_{[st]}^{[t]} f(st)$ for $0\leqslant s\leqslant 1$. There is a natural commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T(A_m,\alpha_m^n) & \xrightarrow{\subseteq} & \tilde T(A_m,\alpha_m^n) & \xrightarrow{\mathrm{ev}_\infty} & A_\infty & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & T(A_\infty,\mathrm{id}) & \xrightarrow{\subseteq} & \tilde T(A_\infty,\mathrm{id}) & \xrightarrow{\mathrm{ev}_\infty} & A_\infty & \longrightarrow & 0
\end{array}$$

whose rows are short exact sequences. The bottom extension is evidently admissible. By definition, the vertical map on $T(\ldots)$ is invertible in $\mathrm{KK}^{G\ltimes X}$ if and only if the inductive system is admissible. The other vertical maps are invertible in any case because $\tilde{T}(\ldots) \cong 0$ in $\mathrm{KK}^{G\ltimes X}$. Therefore, if the inductive system is admissible, then the top row is an admissible extension whose extension triangle is isomorphic to the one for the bottom row. Conversely, if the top row is an admissible extension, then the vertical map on $T(\ldots)$ is invertible in $\mathrm{KK}^{G\ltimes X}$ by the uniqueness of mapping cones. As a result, the inductive system is admissible if and only if the extension in the top row above is admissible. In $E^{G\ltimes X}$, all inductive systems are admissible because all extensions are admissible.

Lemma 2.7. *An inductive system* (A_m, α_m^n) *is admissible if there exist equivariant completely positive contractions* $\phi_m: A_\infty \to A_m$ *such that* $\alpha_m^\infty \circ \phi_m: A_\infty \to A_\infty$ *converges in the point norm topology towards the identity*.

Proof. By the above discussion, the inductive system is admissible if there is an equivariant, contractive, completely positive cross section $A_\infty \to \tilde{T}(A_m, \alpha_m^n)$. It is not hard to see that such a cross section exists if and only if there are maps ϕ_m as in the statement of the lemma. □

2.5. *Triangulated functors and subcategories*

A *triangulated subcategory* of a triangulated category $\mathscr{T}$ is a full subcategory $\mathscr{T}' \subseteq \mathscr{T}$ that is closed under suspensions and has the exactness property that if $\Sigma B \to C \to A \to B$ is an exact triangle with $A, B \in \mathscr{T}'$, then $C \in \mathscr{T}'$ as well. In particular, $\mathscr{T}'$ is closed under isomorphisms and finite direct sums. A triangulated subcategory is called *thick* if all retracts (direct summands) of objects of $\mathscr{T}'$ belong to $\mathscr{T}'$. A triangulated subcategory is indeed a triangulated category in its own right. Given any class of objects $\mathscr{G}$, there is a smallest (thick) triangulated subcategory containing $\mathscr{G}$. This is called the (*thick*) *triangulated subcategory generated by* $\mathscr{G}$. Since a full subcategory is determined by its class of objects, we do not distinguish between full subcategories and classes of objects.

Let $\aleph$ be some infinite regular cardinal number. In our applications we only use the countable cardinal number $\aleph_0$. We suppose that direct sums of cardinality $\aleph$ exist in $\mathscr{T}$. A subcategory of $\mathscr{T}$ is called ($\aleph$-)*localising* if it is triangulated and closed under direct sums of cardinality $\aleph$. We can define the *localising subcategory generated by some class* $\mathscr{G}$ *of objects* as above. We denote it by $\langle\mathscr{G}\rangle$ or $\langle\mathscr{G}\rangle^{\aleph}$. Notice that a triangulated subcategory that is closed under direct sums is also closed under homotopy direct limits. Localising subcategories are automatically thick (see [38]).

It is easy to see that an $\aleph_0$-localising subcategory of $\mathrm{KK}^{G\ltimes X}$ amounts to a class $\mathscr{N}$ of $G\ltimes X$-C^*-algebras with the following properties:

(1) if A and B are $\mathrm{KK}^{G\ltimes X}$-equivalent and $A \in \mathscr{N}$, then $B \in \mathscr{N}$;
(2) $\mathscr{N}$ is closed under suspension;
(3) if $f: A \to B$ is an equivariant $*$-homomorphism with $A, B \in \mathscr{N}$, then also $\mathrm{cone}(f) \in \mathscr{N}$;
(4) if $A_n \in \mathscr{N}$ for all $n \in \mathbb{N}$, then also $\bigoplus_{n\in\mathbb{N}} A_n \in \mathscr{N}$.

We can replace (3) and (4) by the equivalent conditions

(3′) if $K \rightarrowtail E \twoheadrightarrow Q$ is an admissible extension and two of K, E, Q belong to $\mathscr{N}$, so does the third;
(4′) if (A_n, α_m^n) is an admissible inductive system with $A_n \in \mathscr{N}$ for all $n \in \mathbb{N}$, then $\varinjlim A_n \in \mathscr{N}$ as well.

Thus the localising subcategory generated by a class $\mathscr{G}$ of $G\ltimes X$-C^*-algebras is the smallest class of $G\ltimes X$-C^*-algebras containing $\mathscr{G}$ with the above four properties. For example, the localising subcategory of KK generated by $\star$ is exactly the *bootstrap category* (see [9]). The proof uses that extensions and inductive systems of nuclear C^*-algebras are automatically admissible. Another example of a localising subcategory of KK is the class of C^*-algebras with vanishing K-theory. We discuss these two subcategories further in Section 6 to give an easy application of the Brown Representability Theorem.

Let $\mathscr{T}$ and $\mathscr{T}'$ be triangulated categories. A functor $F:\mathscr{T}\to\mathscr{T}'$ is called *triangulated* if it is additive, intertwines the translation automorphisms, and maps exact triangles to exact triangles. Although the latter condition may look like an exactness condition, it is almost empty. Since any exact triangle in $\mathrm{KK}^{G\ltimes X}$ is isomorphic to a mapping cone triangle, a functor is triangulated once it commutes with suspensions and preserves mapping cone triangles. For instance, the functor $A\mapsto A\otimes_{\min}B$ has this property regardless of whether B is exact. Similarly, the full and reduced crossed product functors $\mathrm{KK}^{G\ltimes X}\to\mathrm{KK}$ are triangulated. An analogous situation occurs in homological algebra: any additive functor between Abelian categories gives rise to a triangulated functor between the homotopy categories of chain complexes. The exactness of the functor only becomes relevant for the derived category.

Let $F:\mathscr{T}\to\mathscr{T}'$ be a triangulated functor. Its *kernel* is the class $\ker F$ of all objects X of $\mathscr{T}$ with $F(X)\cong 0$. It is easy to see that $\ker F$ is a thick triangulated subcategory of $\mathscr{T}$. If F commutes with direct sums of cardinality $\aleph$, then $\ker F$ is $\aleph$-localising.

2.6. Localisation of categories and functors

A basic (and not quite correct) result on triangulated categories asserts that any thick triangulated subcategory $\mathscr{N}\subseteq\mathscr{T}$ arises as the kernel of a triangulated functor. Even more, there exists a universal triangulated functor $\mathscr{T}\to\mathscr{T}/\mathscr{N}$ with kernel $\mathscr{N}$, called *localisation functor*, such that any other functor whose kernel contains $\mathscr{N}$ factorises uniquely through $\mathscr{T}/\mathscr{N}$ (see [38]). Its construction is quite involved and may fail to work in general because the morphism spaces in $\mathscr{T}/\mathscr{N}$ may turn out to be classes and not sets.

There are two basic examples of localisations, which have motivated the whole theory of triangulated categories. They come from homological algebra and homotopy theory, respectively. In homological algebra, the ambient category $\mathscr{T}$ is the homotopy category of chain complexes over an Abelian category. The subcategory $\mathscr{N}\subseteq\mathscr{T}$ consists of the exact complexes, that is, complexes with vanishing homology. A chain map is called a *quasi-isomorphism* if it induces an isomorphism on homology. The localisation $\mathscr{T}/\mathscr{N}$ is, by definition, the derived category of the underlying Abelian category. One of the motivations for developing the theory of triangulated categories was to understand what additional structure of the homotopy category of chain complexes is inherited by the derived category.

In homotopy theory there are several important instances of localisations. We only discuss one very elementary situation which provides a good analogy for our treatment of the Baum–Connes assembly map. Let $\mathscr{T}$ be the stable homotopy category of all topological spaces. We call an object of $\mathscr{T}$ *weakly contractible* if its stable homotopy groups vanish. A map is called a *weak homotopy equivalence* if it induces an isomorphism on stable homotopy groups. Let $\mathscr{N}\subseteq\mathscr{T}$ be the subcategory of weakly contractible objects. In homotopy theory one often wants to disregard objects of $\mathscr{N}$, that is, work in the localisation $\mathscr{T}/\mathscr{N}$.

The concepts of a weak equivalence in homotopy theory and of a quasi-isomorphism in homological algebra become equivalent once formulated in terms of triangulated categories: we call a morphism

$f \in \mathscr{T}(A, B)$ an *$\mathscr{N}$-weak equivalence* or an *$\mathscr{N}$-quasi-isomorphisms* if $\mathrm{cone}(f) \in \mathscr{N}$. Since $N \in \mathscr{N}$ if and only if $0 \to N$ is an $\mathscr{N}$-weak equivalence, the weak equivalences and $\mathscr{N}$ determine each other uniquely.

A morphism is a weak equivalence if and only if its image in the localisation $\mathscr{T}/\mathscr{N}$ is an isomorphism. This implies several cancellation assertions about weak equivalences. For instance, if f and g are composable and two of the three morphisms f, g and $f \circ g$ are weak equivalences, so is the third. The localisation has the universal property that any functor out of $\mathscr{T}$, triangulated or not, that maps $\mathscr{N}$-weak equivalences to isomorphisms, factorises uniquely through the localisation.

In many examples of localisation, some more structure is present. This is formalised in the following definition. In the simplicial approximation example, let $\mathscr{P} \subseteq \mathscr{T}$ be the subcategory of all objects that have the stable homotopy type of a CW-complex, that is, are isomorphic in $\mathscr{T}$ to a CW-complex. The Whitehead Lemma implies that $f: X \to Y$ is a weak homotopy equivalence if and only if $f_*: \mathscr{T}(P, X) \to \mathscr{T}(P, Y)$ is an isomorphism for all $P \in \mathscr{P}$. Similarly, $N \in \mathscr{N}$ if and only $\mathscr{T}(P, N) = 0$ for all $P \in \mathscr{P}$. Another important fact is that any space S has a *simplicial approximation*. This is just a weak equivalence $\tilde{X} \to X$ with $\tilde{X} \in \mathscr{P}$. In homological algebra, a similar situation arises if there are "enough projectives". Then one lets $\mathscr{P}$ be the subcategory of projective chain complexes (see [31]).

Definition 2.8. Let $\mathscr{T}$ be a triangulated category and let $\mathscr{P}$ and $\mathscr{N}$ be thick triangulated subcategories of $\mathscr{T}$. We call the pair $(\mathscr{P}, \mathscr{N})$ *complementary* if $\mathscr{T}(P, N) = 0$ for all $P \in \mathscr{P}$, $N \in \mathscr{N}$ and if for any $A \in \mathscr{T}$ there is an exact triangle $\Sigma N \to P \to A \to N$ with $P \in \mathscr{P}$, $N \in \mathscr{N}$.

We shall only need localisations in the situation of complementary subcategories. In this case, the construction of $\mathscr{T}/\mathscr{N}$ is easier and there is some important (and well-known) additional structure (see [36]). We prove some basic results because they are important for our treatment of the Baum–Connes assembly map.

Proposition 2.9. *Let $\mathscr{T}$ be a triangulated category and let $(\mathscr{P}, \mathscr{N})$ be complementary thick triangulated subcategories of $\mathscr{T}$.*

2.9.1. *We have $N \in \mathscr{N}$ if and only if $\mathscr{T}(P, N) = 0$ for all $P \in \mathscr{P}$, and $P \in \mathscr{P}$ if and only if $\mathscr{T}(P, N) = 0$ for all $N \in \mathscr{N}$; thus $\mathscr{P}$ and $\mathscr{N}$ determine each other.*
2.9.2. *The exact triangle $\Sigma N \to P \to A \to N$ with $P \in \mathscr{P}$ and $N \in \mathscr{N}$ is uniquely determined up to isomorphism and depends functorially on A. In particular, its entries define functors $P: \mathscr{T} \to \mathscr{P}$ and $N: \mathscr{T} \to \mathscr{N}$.*
2.9.3. *The functors $P, N: \mathscr{T} \to \mathscr{T}$ are triangulated.*
2.9.4. *The localisations $\mathscr{T}/\mathscr{N}$ and $\mathscr{T}/\mathscr{P}$ exist.*
2.9.5. *The compositions $\mathscr{P} \to \mathscr{T} \to \mathscr{T}/\mathscr{N}$ and $\mathscr{N} \to \mathscr{T} \to \mathscr{T}/\mathscr{P}$ are equivalences of triangulated categories.*
2.9.6. *The functors $P, N: \mathscr{T} \to \mathscr{T}$ descend to triangulated functors $P: \mathscr{T}/\mathscr{N} \to \mathscr{P}$ and $N: \mathscr{T}/\mathscr{P} \to \mathscr{N}$, respectively, that are inverse (up to isomorphism) to the functors in* 2.9.5.
2.9.7. *The functors $P: \mathscr{T}/\mathscr{N} \to \mathscr{T}$ and $N: \mathscr{T}/\mathscr{P} \to \mathscr{T}$ are left and right adjoint to the localisation functors $\mathscr{T} \to \mathscr{T}/\mathscr{N}$ and $\mathscr{T} \to \mathscr{T}/\mathscr{P}$, respectively; that is, we have natural isomorphisms*

$$\mathscr{T}(P(A), B) \cong \mathscr{T}/\mathscr{N}(A, B), \quad \mathscr{T}(A, N(B)) \cong \mathscr{T}/\mathscr{P}(A, B),$$

for all $A, B \in \mathscr{T}$.

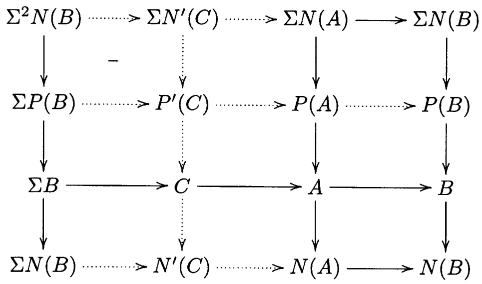

Fig. 1. Exactness of P and N.

Proof. We can exchange the roles of $\mathscr{P}$ and $\mathscr{N}$ by passing to opposite categories. Hence it suffices to prove the various assertions about one of them.

By hypothesis, $N \in \mathscr{N}$ implies $\mathscr{T}(P, N) = 0$ for all $P \in \mathscr{P}$. Conversely, suppose $\mathscr{T}(P, A) = 0$ for all $P \in \mathscr{P}$. Let $\Sigma N \to P \to A \to N$ be an exact triangle with $P \in \mathscr{P}$ and $N \in \mathscr{N}$. The map $P \to A$ vanishes by hypothesis. Lemma 2.2 implies $N \cong A \oplus \Sigma^{-1} P$. Since $\mathscr{N}$ is thick, $A \in \mathscr{N}$. This proves 2.9.1.

Let $\Sigma N \to P \to A \to N$ and $\Sigma N' \to P' \to A' \to N'$ be exact triangles with $P, P' \in \mathscr{P}$ and $N, N' \in \mathscr{N}$ and let $f \in \mathscr{T}(A, A')$. Since $\mathscr{T}(P, N') = 0$, the map $P' \to A'$ induces an isomorphism $\mathscr{T}(P, P') \cong \mathscr{T}(P, A')$. Hence there is a unique and hence natural way to lift the composite map $P \to A \to A'$ to a map $P \to P'$. By the axioms of a triangulated category, there exists a morphism of exact triangles from $\Sigma N \to P \to A \to N$ to $\Sigma N' \to P' \to A' \to N'$ that extends $f: A \to A'$ and its lifting $P(f): P \to P'$. An argument as above shows that there is a unique way to lift f to a map $N \to N'$. Thus the morphism of triangles that extends f is determined uniquely, so that the triangle $\Sigma N \to P \to A \to N$ depends functorially on A. This proves 2.9.2.

Next, we show that P is a triangulated functor on $\mathscr{T}$. Let $\Sigma B \to C \to A \to B$ be an exact triangle. Consider the solid arrows in the diagram in Fig. 1. We can find objects $N'(C)$ and $P'(C)$ of $\mathscr{T}$ and the dotted arrows in this diagram so that all rows and columns are exact and such that the diagram commutes except for the square marked with a $-$, which anti-commutes (see [8, Proposition 1.1.11]). Since $\mathscr{P}$ and $\mathscr{N}$ are triangulated subcategories and the rows in this diagram are exact triangles, we get $P'(C) \in \mathscr{P}$ and $N'(C) \in \mathscr{N}$. Hence the column over C is as in the definition of the functors P and N. Therefore, we can replace this column by the exact triangle $\Sigma N(C) \to P(C) \to C \to N(C)$. Our proof of 2.9.2 shows that the rows must be obtained by applying the functors P and N to the given exact triangle $\Sigma B \to C \to A \to B$. Since the rows are exact triangles by construction, the functors P and N preserve exact triangles. They evidently commute with suspensions. This proves 2.9.3

Next we construct a candidate $\mathscr{T}'$ for the localisation $\mathscr{T}/\mathscr{N}$. We let $\mathscr{T}'$ have the same objects as $\mathscr{T}$ and morphisms $\mathscr{T}'(A, B) := \mathscr{T}(P(A), P(B))$. The identity map on objects and the map P on morphisms define a canonical functor $\mathscr{T} \to \mathscr{T}'$. We define the suspension on $\mathscr{T}'$ to be the same as for $\mathscr{T}$. A triangle in $\mathscr{T}'$ is called exact if it is isomorphic to the image of an exact triangle in $\mathscr{T}$. We claim that $\mathscr{T}'$ with this additional structure is a triangulated category and that the functor $\mathscr{T} \to \mathscr{T}'$ is the localisation functor at $\mathscr{N}$.

The uniqueness of the exact triangle $\Sigma N(A) \to P(A) \to A \to N(A)$ yields that the natural map $P(A) \to A$ is an isomorphism for $A \in \mathscr{P}$. Therefore, the map $\mathscr{T}(A, B) \to \mathscr{T}'(A, B)$ is an isomorphism for $A, B \in \mathscr{P}$. That is, the restriction of the functor $\mathscr{T} \to \mathscr{T}'$ to $\mathscr{P}$ is fully faithful and identifies $\mathscr{P}$ with a full subcategory of $\mathscr{T}'$. Moreover, since $P(A) \in \mathscr{P}$, the map $P^2(A) \to P(A)$ is an isomorphism. This implies that the map $P(A) \to A$ is mapped to an isomorphism in $\mathscr{T}'$. Thus any object of $\mathscr{T}'$ is isomorphic to one in the full subcategory $\mathscr{P}$. Therefore, the category $\mathscr{T}'$ is equivalent to the subcategory $\mathscr{P}$. Using that P is a triangulated functor on $\mathscr{T}$, one shows easily that both functors $\mathscr{P} \to \mathscr{T}'$ and $\mathscr{T}' \to \mathscr{P}$ map exact triangles to exact triangles. They commute with suspensions anyway. Since they are equivalences of categories and since $\mathscr{P}$ is a triangulated category, the category $\mathscr{T}'$ is triangulated and the equivalence $\mathscr{P} \cong \mathscr{T}'$ is compatible with the triangulated category structure.

We define the functor $P: \mathscr{T}' \to \mathscr{P}$ to be P on objects and the identity on morphisms. This functor is clearly inverse to the above equivalence $\mathscr{P} \to \mathscr{T}'$ and has the property that the composition $\mathscr{T} \to \mathscr{T}' \xrightarrow{P} \mathscr{P} \subseteq \mathscr{T}$ agrees with $P: \mathscr{T} \to \mathscr{T}$. Moreover, we have observed already above that $\mathscr{T}(P(A), (B)) \cong \mathscr{T}(P(A), P(B))$ for all $A, B \in \mathscr{T}$. Hence all the remaining assertions follow once we show that $\mathscr{T}'$ has the universal property of $\mathscr{T}/\mathscr{N}$. It is easy to see that $\mathscr{N}$ is equal to the kernel of $\mathscr{T} \to \mathscr{T}'$. If $F: \mathscr{T} \to \mathscr{T}''$ is a triangulated functor with kernel $\mathscr{N}$, then the maps $P(A) \to A$ induce isomorphisms $F(P(A)) \to F(A)$ by Lemma 2.2. Therefore, $\mathscr{T}' \xrightarrow{P} \mathscr{P} \subseteq \mathscr{T} \xrightarrow{F} \mathscr{T}''$ is the required factorisation of F through $\mathscr{T}'$. □

We call the map $P(A) \to A$ an *$\mathscr{N}$-projective resolution* or a *$\mathscr{P}$-simplicial approximation* of A. The first term comes from homological algebra, the second one from homotopy theory. We prefer the terminology from homotopy theory because it gives a more accurate analogy for the Baum–Connes assembly map.

Finally, we consider the localisation of functors. Let $F: \mathscr{T} \to \mathscr{T}'$ be a covariant triangulated functor to another triangulated category $\mathscr{T}'$. Then its *localisation* or *left derived functor* $\mathbb{L}F: \mathscr{T}/\mathscr{N} \to \mathscr{T}'$ is, in general, defined by a certain universal property. In the case of a complementary pair of subcategories, it is given simply by $\mathbb{L}F \cong F \circ P$. This makes sense for any functor F, triangulated or not. If F is triangulated, then so is $\mathbb{L}F$. If F is (co)homological, then so is $\mathbb{L}F$. Both assertions follow from Proposition 2.9.3. In the following discussion, we assume F to be triangulated or homological.

The functor $\mathbb{L}F$ descends to the category $\mathscr{T}/\mathscr{N}$ and comes equipped with a natural transformation $\mathbb{L}F \to F$ which comes from the natural transformation $P(A) \to A$. The universal property that characterises $\mathbb{L}F$ is the following. If $F': \mathscr{T}/\mathscr{N} \to \mathscr{T}'$ is any functor together with a natural transformation $F' \to F$, then this natural transformation factorises uniquely through $\mathbb{L}F$. This factorisation is obtained as the composition

$$F'(A) \xleftarrow{\cong} F'(P(A)) \to F(P(A)) \cong \mathbb{L}F(A).$$

Thus we may view $\mathbb{L}F$ as the best approximation to F that factors through $\mathscr{T}/\mathscr{N}$. In particular, we have $\mathbb{L}F \cong F$ if and only if $\mathscr{N} \subseteq \ker F$ if and only if F maps $\mathscr{N}$-weak equivalences to isomorphisms in $\mathscr{T}'$.

Alternatively, we may view $\mathbb{L}F(A)$ as the best approximation to $F(A)$ that uses only the restriction of F to $\mathscr{P}$. The simplicial approximation $P(A) \to A$ has the universal property that any map $B \to A$ with $B \in \mathscr{P}$ factors uniquely through $P(A)$. In this sense, $P(A)$ is the best possible approximation to A inside $\mathscr{P}$ and $F(P(A))$ is the best guess we can make for $F(A)$ if we want the guess to be of the form $F(B)$ for some $B \in \mathscr{P}$.

We can also use the functor $N: \mathscr{T}/\mathscr{P} \to \mathscr{T}$ to define an *obstruction functor* $\operatorname{Obs} F := F \circ N$. It comes equipped with a natural transformation $F \to \operatorname{Obs} F$. Proposition 2.9.2 shows that if the functor F is triangulated then $\mathbb{L}F$, F and $\operatorname{Obs} F$ are related by a natural exact triangle

$$\Sigma \operatorname{Obs} F(A) \to \mathbb{L}F(A) \to F(A) \to \operatorname{Obs} F(A).$$

Thus Obs $F(A)$ measures the lack of invertibility of the map $\mathbb{L}F(A) \to F(A)$. In particular, Obs $F(A)=0$ if and only if $\mathbb{L}F(A) \cong F(A)$. Similar remarks apply if F is homological. In that case, the functors $\mathbb{L}F$, F and Obs F are related by a long exact sequence.

3. Preliminaries on compact subgroups and some functors

We first recall some structural results about compact subgroups in locally compact groups. Then we recall the well-known formal properties of tensor product, restriction and induction functors. We discuss them in some detail because they are frequently used. We apply the universal property of KK-theory to treat them. This has the advantage that proofs do not require the definition of KK.

3.1. Compact subgroups

Let G be a locally compact group. Let $G_0 \subseteq G$ be the connected component of the identity element. We call G *almost totally disconnected* if G_0 is compact, and *almost connected* if G/G_0 is compact. If G is almost totally disconnected, then G contains a compact open subgroup (and vice versa) by [21, Theorem 7.5]. Therefore, if G is arbitrary, then there exists an open almost connected subgroup $U \subseteq G$: take the preimage of a compact open subgroup in G/G_0. Almost connected groups are very closely related to Lie groups (with finitely many connected components) by [35]: if U is almost connected, then each neighbourhood of the identity element contains a compact normal subgroup $N \subseteq U$ such that U/N is a Lie group (the smooth structure on U/N is unique if it exists).

Let U be almost connected and let $K \subseteq U$ be maximal compact. We recall some structural results about U/K from [1]. Let $\mathfrak{k} \subseteq \mathfrak{u}$ be the Lie algebras of K and U, respectively, and let $\mathfrak{p} := \mathfrak{u}/\mathfrak{k}$. This quotient is a finite dimensional $\mathbb{R}$-vector space, on which K acts linearly by conjugation. There exists a K-equivariant homeomorphism $U/K \cong \mathfrak{p}$. Thus U/K as a K-space is homeomorphic to a linear action of K on a real vector space. This fact is crucial for our purposes. Moreover, Abels shows in [1] that U/K is a universal proper U-space. This contains the assertion that any compact subgroup of U is subconjugate to K (because it fixes a point in U/K). Especially, any two maximal compact subgroups are conjugate.

We define some classes of special compact subgroups that we shall use later. Let $H \subseteq G$ be a compact subgroup. We call H *strongly smooth* if its normaliser $N_G H \subseteq G$ is open in G and $N_G H/H$ is a Lie group. We call H *smooth* if it contains a strongly smooth subgroup of G. Finally, we call H *large* if it is a maximal compact subgroup of some open almost connected subgroup of G. We let LC = LC(G) be the set of large compact subgroups of G.

Of course, strongly smooth subgroups are smooth. Large subgroups are also smooth because if $L \subseteq U$ is maximal compact and $N \subseteq U$ is a smooth, compact normal subgroup, then NL is a compact subgroup as well by normality. Hence $N \subseteq L$ by maximality.

Lemma 3.1. *Any compact subgroup of G is contained in a large compact subgroup.*

If $H \subseteq G$ is a large compact subgroup, then the open almost connected subgroup $U \subseteq G$ in which H is maximal is unique and denoted by U_H.

Suppose $H, L \in$ LC satisfy $H \subseteq L$. Then $H = U_H \cap L$, so that H is open in L. The natural map $U_H/H \to U_L/L$ is a homeomorphism.

If $H \subseteq G$ is smooth, then the homogeneous space G/H is a smooth manifold in a canonical way.

Proof. We claim that any compact subgroup H of a totally disconnected group G is contained in a compact open one. Let $U \subseteq G$ be any compact open subgroup. Then $H \cap U$ has finite index in H. Therefore, $U' := \bigcap_{h \in H} hUh^{-1}$ is again a compact open subgroup of G. By construction, it is normalised by H, so that HU' is again a subgroup. It is compact and open and contains H. Since HU' is almost connected, H is contained in some maximal compact subgroup of HU'. This yields the first assertion.

Suppose H is maximal compact in the open almost connected subgroups U and V of G. We claim that $U = V$. Since H is still maximal compact in $U \cap V$, we may assume that $U \subseteq V$. Hence V/H is a disjoint union of $U : V$ copies of U/H. However, V/H is homeomorphic to a vector space and therefore connected, forcing $U = V$. Let H and L be large compact subgroups of G that satisfy $H \subseteq L$. Then $H \subseteq U_H \cap L \subseteq U_H$, so that $H = U_H \cap L$ by maximality. Hence the natural map $U_H/H \to U_L/L$ is injective. Its image is both open and closed and hence must be all of U_L/L by connectedness.

Let $H \subseteq G$ be smooth. Then we can find an almost connected open subgroup $U \subseteq G$ that contains H and a subgroup $N \subseteq H$ that is normal in U such that U/N is a Lie group. Write G/N as a disjoint union of copies of the Lie group U/N. This reveals that G/H is a disjoint union of copies of the homogeneous space $(U/N)/(H/N)$ and hence a smooth manifold in a canonical way. □

3.2. Functors on Kasparov categories

The (minimal) C^*-tensor product gives rise to bifunctors

$$\mathrm{KK}^{G\ltimes X} \times \mathrm{KK}^{G} \to \mathrm{KK}^{G\ltimes X}, \quad (A, B) \mapsto A \otimes B,$$

$$\mathrm{KK}^{G\ltimes X} \times \mathrm{KK}^{G\ltimes X} \to \mathrm{KK}^{G\ltimes X}, \quad (A, B) \mapsto A \otimes_X B,$$

see [28, Definition 2.12, Proposition 2.21]. We briefly recall how $A \otimes_X B$ looks like. If $A, B \in \mathrm{KK}^{G\ltimes X}$, then $A \otimes B$ is a $G\ltimes(X \times X)$-C^*-algebra, and $A \otimes_X B$ is defined as its "restriction" to the diagonal. That is, we divide out elements of the form $f \cdot a$ with $f \in C_0(X \times X)$, $f(x, x) = 0$ for all $x \in X$, $a \in A \otimes_X B$. See also [28, Definition 1.6].

The full and reduced descent functors $\mathrm{KK}^{G\ltimes X} \to \mathrm{KK}$ are defined in [28, pp. 170–173]. On objects, they act by $A \mapsto (G\ltimes X)\ltimes A$ and $A \mapsto (G\ltimes X)\ltimes_{\mathrm{r}} A$. We remark that the space X has no effect here, that is, $(G\ltimes X)\ltimes A = G\ltimes A$ and $(G\ltimes X)\ltimes_{\mathrm{r}} A = G\ltimes_{\mathrm{r}} A$ (see also [15]).

Let $H \subseteq G$ be a closed subgroup. Then we have functors

$$\mathrm{Res}_G^H : \mathrm{KK}^{G\ltimes X} \to \mathrm{KK}^{H\ltimes X},$$
$$\mathrm{Ind}_H^G : \mathrm{KK}^{H\ltimes X} \to \mathrm{KK}^{G\ltimes X},$$

called *restriction* and *induction*, respectively. The restriction functor is a special case of the functoriality of $\mathrm{KK}^{G\ltimes X}$ in G: any group homomorphism $H \to G$ induces a functor $\mathrm{KK}^{G\ltimes X} \to \mathrm{KK}^{H\ltimes X}$ by [28, Definition 3.1]. The induction functor is introduced in [28, Section 3.6], see also [11].

Finally, a G-equivariant continuous map $f: X \to Y$ induces functors

$$f_*: \mathrm{KK}^{G\ltimes X} \to \mathrm{KK}^{G\ltimes Y},$$
$$f^*: \mathrm{KK}^{G\ltimes Y} \to \mathrm{KK}^{G\ltimes X}.$$

The functor f_* is just a forgetful functor: to view a $G\ltimes X$-C^*-algebra A as a $G\ltimes Y$-C^*-algebra, compose $f^*: C_0(Y) \to C_b(X)$ and the canonical extension of the structural homomorphism $C_b(X) \to Z\mathscr{M}(A)$.

The functor f^* is defined on objects by $f^*(A) := C_0(X) \otimes_Y A$. Clearly, $\mathrm{id}_* \cong \mathrm{id}$, $\mathrm{id}^* \cong \mathrm{id}$ and $f_* g_* \cong (fg)_*$, $g^* f^* \cong (fg)^*$ if f and g are composable.

Nowadays, we can treat these functors much more easily than in [28] using the universal property of Kasparov theory. In the non-equivariant case, Nigel Higson has shown that the functor from C^*-algebras to KK is the universal split exact stable homotopy functor, that is, any functor from C^*-algebras to some category with these properties factors uniquely through KK. This result has been extended by Klaus Thomsen to the equivariant case and also works $G \ltimes X$-equivariantly by [34]. The above functors on KK-categories all come from functors F between categories of C^*-algebras, which are much easier to describe.

Let F be a functor from $G \ltimes X$-C^*-algebras to $H \ltimes Y$-C^*-algebras. The relevant functors F satisfy $F(A \otimes B) \approx F(A) \otimes B$ for any nuclear C^*-algebra B equipped with the trivial representation of G (recall that $\approx$ denotes isomorphism of $G \ltimes X$-C^*-algebras). This implies immediately that F is stable and homotopy invariant and commutes with suspensions. Suppose, in addition, that F maps extensions with a completely positive, contractive, G-equivariant linear section again to such extensions. This is the case in the above examples. By the universal property, F induces a functor $\mathrm{KK}^{G \ltimes X} \to \mathrm{KK}^{H \ltimes Y}$. Our mild exactness hypothesis guarantees that F maps mapping cone triangles again to mapping cone triangles. This suffices to conclude that we have got a triangulated functor. This argument provides a very quick existence proof for the functors above and also shows that they are triangulated. It is also easy to check that they commute with countable direct sums on $\mathrm{KK}^{G \ltimes X}$ (recall that direct sums in $\mathrm{KK}^{G \ltimes X}$ are just C^*-direct sums).

Green's Imprimitivity Theorem and its reduced version assert that

$$G \ltimes \mathrm{Ind}_H^G(A) \sim_M H \ltimes A, \quad G \ltimes_{\mathrm{r}} \mathrm{Ind}_H^G(A) \sim_M H \ltimes_{\mathrm{r}} A. \tag{9}$$

The functors f_* and f^* are compatible with $\otimes$ (without X) in the evident sense: $f_*(A \otimes B) \cong f_*(A) \otimes B$, $f^*(A \otimes B) \cong f^*(A) \otimes B$. We have a natural $G \ltimes Y$-equivariant isomorphism

$$f_*(A) \otimes_Y B \approx f_*(A \otimes_X f^*(B)) \tag{10}$$

for $f: X \to Y$ and $A \in \mathrm{KK}^{G \ltimes X}$, $B \in \mathrm{KK}^{G \ltimes Y}$ because $\otimes_X$ is associative and $A \otimes_X C_0(X) \approx A$. Eq. (10) asserts for the constant map $p_X: X \to \star$ that

$$A \otimes_X p_X^*(B) \approx A \otimes B. \tag{11}$$

The isomorphisms in (10) and (11) are natural, even in the formal sense. For (11), naturality means that the isomorphisms intertwine

$$x \otimes_X p_X^*(y) \in \mathrm{KK}^G(A \otimes_X p_X^* B, A' \otimes_X p_X^* B') \quad \text{and} \quad x \otimes y \in \mathrm{KK}^G(A \otimes B, A' \otimes B')$$

if $x \in \mathrm{KK}^G(A, A')$, $y \in \mathrm{KK}^G(B, B')$. By the universal property of KK, it suffices to verify this in the (easy) special case where x and y are ordinary $*$-homomorphisms; the general case then follows because two functors agree on KK once they agree for ordinary $*$-homomorphisms. All the isomorphisms that follow are also natural in this sense, for the same reason.

There are obvious compatibility conditions

$$\mathrm{Res}_G^H(A \otimes_{(X)} B) \approx \mathrm{Res}_G^H(A) \otimes_{(X)} \mathrm{Res}_G^H(B), \tag{12}$$

where we write $\otimes_{(X)}$ for either $\otimes_X$ or $\otimes$, and

$$\mathrm{Ind}_H^G \circ f_* \approx f_* \circ \mathrm{Ind}_H^G, \quad \mathrm{Res}_G^H \circ f_* \approx f_* \circ \mathrm{Res}_G^H, \quad \mathrm{Res}_G^H \circ f^* \approx f^* \circ \mathrm{Res}_G^H, \tag{13}$$

because in each case one of the functors is a forgetful functor. The relation $\mathrm{Ind}_H^G \circ f^* \approx f^* \circ \mathrm{Ind}_H^G$ also holds. The easiest way to prove this isomorphism and many others is to replace Ind_H^G by a forgetful functor as follows.

The groupoid $G \ltimes (G/H \times X)$ is Morita equivalent to $H \ltimes X$. Therefore, the categories of $H \ltimes X$-C^*-algebras and of $G \ltimes (G/H \times X)$-C^*-algebras are equivalent. We may view induction as a functor between these two categories. This is an equivalence of categories. Its inverse simply restricts a $G \ltimes (G/H \times X)$-C^*-algebra to $\{H\} \times X \subseteq G/H \times X$. By the universal property of KK, these functors induce an equivalence of categories $\mathrm{KK}^{G \ltimes (G/H \times X)} \cong \mathrm{KK}^{H \ltimes X}$ (see also [32]).

Let $\pi_X: G/H \times X \to X$ be the projection. When we reinterpret Ind_H^G and Res_G^H as functors between $\mathrm{KK}^{G \ltimes X}$ and $\mathrm{KK}^{G \ltimes (G/H \times X)}$, we get

$$\mathrm{Ind}_H^G \approx \pi_{X,*}, \quad \mathrm{Res}_G^H \approx \pi_X^*. \tag{14}$$

This is useful for understanding the formal properties of these functors. Using the properties of f_* and f^* shown above we get

$$\mathrm{Ind}_H^G \circ f^* \approx f^* \circ \mathrm{Ind}_H^G, \tag{15}$$

$$(\mathrm{Ind}_H^G A) \otimes_{(X)} B \approx \mathrm{Ind}_H^G (A \otimes_{(X)} \mathrm{Res}_G^H B) \tag{16}$$

$$\mathrm{Ind}_H^G \circ \mathrm{Res}_G^H (A) \approx C_0(G/H) \otimes A. \tag{17}$$

Our next goal is to prove the adjointness relation

$$\mathrm{KK}^{G \ltimes X}(f^*(A), B) \cong \mathrm{KK}^{G \ltimes Y}(A, f_*(B)) \tag{18}$$

for a *proper* continuous G-map $f: X \to Y$, $A \in \mathrm{KK}^{G \ltimes Y}$, $B \in \mathrm{KK}^{G \ltimes X}$. Experts on KK-theory can verify (18) easily by showing that both sides are defined by equivalent classes of cycles. Category theorists may prefer the following argument, which requires no knowledge of KK except the existence of f_* and f^* as functors on KK. Let $f: X \to Y$ be a continuous G-map. Let B be a $G \ltimes X$-C^*-algebra. There is a natural homomorphism

$$\pi_B: f^* f_*(B) \cong C_0(X) \otimes_Y B \to C_0(X) \otimes_X B \cong B.$$

Let A be a $G \ltimes Y$-C^*-algebra. We have a natural map ι_A from A to the multiplier algebra of $f^*(A) = f_* f^*(A)$, which sends $a \mapsto 1 \otimes a \in C_b(X) \otimes_Y A$ or, equivalently, $h \cdot a \mapsto h \otimes_Y a$ for $h \in C_0(Y)$, $a \in A$. The second description shows that we have a map $\iota_A: A \to f_* f^*(A)$ if f is proper. The composite maps

$$f^*(A) \xrightarrow{f^*(\iota_A)} f^*(f_* f^*(A)) = f^* f_*(f^* A) \xrightarrow{\pi_{f^*(A)}} f^* A,$$
$$f_*(B) \xrightarrow{\iota_{f_* B}} f_* f^*(f_* B) = f_*(f^* f_* B) \xrightarrow{f_*(\pi_B)} f_*(B)$$

are the identity. Thus the maps π_B and ι_A form (co)units of adjunction between the functors f^* and f_* (see [33] for this notion). This holds regardless of whether we use homomorphism or KK-morphisms.

Thus we get the desired adjointness relation (18) and a corresponding statement about equivariant $*$-homomorphisms.

Combining (18) with (14), we get a Frobenius reciprocity isomorphism

$$\mathrm{KK}^{G\ltimes X}(A, \mathrm{Ind}_H^G B) \cong \mathrm{KK}^{H\ltimes X}(\mathrm{Res}_G^H A, B) \tag{19}$$

if $H \subseteq G$ is a *cocompact* closed subgroup and $A \in \mathrm{KK}^{G\ltimes X}$, $B \in \mathrm{KK}^{H\ltimes X}$. Dually, there is a natural isomorphism

$$\mathrm{KK}^{G\ltimes X}(\mathrm{Ind}_U^G A, B) \cong \mathrm{KK}^{U\ltimes X}(A, \mathrm{Res}_G^U B) \tag{20}$$

for an *open* subgroup $U \subseteq G$. This can also be proven by writing down explicitly the units of adjunction. We can decompose $\mathrm{Res}_G^U \mathrm{Ind}_U^G(A)$ as a direct sum of $U \ltimes X$-C^*-algebras over the discrete space of double cosets $G//U$. The summand for the identity coset can be identified with A, so that we get a natural map $\iota_A: A \to \mathrm{Res}_G^U \mathrm{Ind}_U^G(A)$. We can represent $C_0(G/U)$ on the Hilbert space $\ell^2(G/U)$ by multiplication operators. Since U is open in G, this maps $C_0(G/U)$ into the C^*-algebra of compact operators $\mathbb{K}(\ell^2(G/U))$. Hence we get a natural morphism

$$\mathrm{Ind}_U^G \mathrm{Res}_G^U(B) \approx C_0(G/U) \otimes B \to \mathbb{K}(\ell^2(G/U)) \otimes B \sim_M B$$

for $B \in \mathrm{KK}^{G\ltimes X}$. This defines an element $\pi_B \in \mathrm{KK}^{G\ltimes X}(\mathrm{Ind}_U^G \mathrm{Res}_G^U(B), B)$ because KK is stable. One verifies easily that the morphisms π_B and ι_A are units of adjunction, so that we get (20).

After these purely formal manipulations of functors, we now come to a much deeper assertion, which is due to Gennadi Kasparov.

Proposition 3.2 (*Kasparov [28, Theorem 5.8]*). *Let G be almost connected and let $H \subseteq G$ be a maximal compact subgroup. If one of X, A and B is a proper G-space or a proper G-C^*-algebra, then*

$$\mathrm{Res}_G^H: \mathrm{KK}^{G\ltimes X}(A, B) \to \mathrm{KK}^{H\ltimes X}(\mathrm{Res}_G^H A, \mathrm{Res}_G^H B) \tag{21}$$

is an isomorphism.

Lemma 3.3. *Let $H \subseteq G$ be a large compact subgroup and let $U := U_H$. There is a natural isomorphism*

$$\mathrm{KK}^{G\ltimes X}(\mathrm{Ind}_H^G A, B) \cong \mathrm{KK}^{H\ltimes X}(\mathrm{Res}_U^H \mathrm{Ind}_H^U A, \mathrm{Res}_G^H B). \tag{22}$$

Define $J_H^G(A) := \mathrm{Ind}_H^G(C_0((U/H)^7) \otimes A)$. Then there is a natural isomorphism

$$\mathrm{KK}^{G\ltimes X}(J_H^G A, B) \cong \mathrm{KK}^{H\ltimes X}(A, \mathrm{Res}_G^H B)$$

if A belongs to the essential range of the functor Res_U^H. Furthermore, the functors

$$\mathrm{Res}_U^H: \mathrm{KK}^{U\ltimes X} \to \mathrm{KK}^{H\ltimes X}, \quad \mathrm{Res}_U^H \mathrm{Ind}_H^U: \mathrm{KK}^{H\ltimes X} \to \mathrm{KK}^{H\ltimes X},$$

have the same essential range.

The *essential range* of a functor $F: \mathfrak{C} \to \mathfrak{C}'$ is defined as the class of all objects of $\mathfrak{C}'$ that are isomorphic to an object of the form $F(X)$ with X an object of $\mathfrak{C}$.

Proof. Induction in stages and (20) yield

$$\mathrm{KK}^{G\ltimes X}(\mathrm{Ind}_H^G A, B) \cong \mathrm{KK}^{U\ltimes X}(\mathrm{Ind}_H^U A, \mathrm{Res}_G^U B).$$

Since $\mathrm{Ind}_H^U A$ is proper, Proposition 3.2 yields (22). Let us abbreviate $R := \mathrm{Res}_U^H$ and $I := \mathrm{Ind}_H^U$. If A belongs to the essential range of R, then $RI(A) \cong C_0(U/H) \otimes A$ by (17). Recall that U/H is H-equivariantly diffeomorphic to a real vector space with a linear action of H. Hence Bott periodicity provides a KK^H-equivalence between $(RI)^8(A) \cong C_0((U/H)^8) \otimes A$ and A. This yields the second isomorphism and shows that the essential range of R is contained in the essential range of RI. The converse inclusion is trivial. □

Hence Ind_H^G and Res_G^H for a large compact subgroup $H \subseteq G$ become adjoint functors if we replace $\mathrm{KK}^{H\ltimes X}$ by the essential range of Res_U^H. There is no analogue of this for arbitrary compact subgroups.

4. A decomposition of the Kasparov category

Definition 4.1. We call $A \in \mathrm{KK}^{G\ltimes X}$ *weakly contractible* if $\mathrm{Res}_G^H(A) \cong 0$ for all compact subgroups $H \subseteq G$. Let $\mathscr{CC} \subseteq \mathrm{KK}^{G\ltimes X}$ be the full subcategory of weakly contractible objects.

A morphism f in $\mathrm{KK}^{G\ltimes X}(A, B)$ is called a *weak equivalence* if $\mathrm{Res}_G^H(f)$ is invertible in $\mathrm{KK}^{H\ltimes X}$ for all compact subgroups $H \subseteq G$. We say that f *vanishes for compact subgroups* if $\mathrm{Res}_G^H(f) = 0$ for all compact subgroups $H \subseteq G$.

We call a $G\ltimes X$-C^*-algebra *compactly induced* if it is isomorphic in $\mathrm{KK}^{G\ltimes X}$ to $\mathrm{Ind}_H^G(A)$ for some compact subgroup $H \subseteq G$ and some $A \in \mathrm{KK}^{H\ltimes X}$. We let $\mathscr{CI} \subseteq \mathrm{KK}^{G\ltimes X}$ be the full subcategory of compactly induced objects.

In all these definitions, it suffices to consider large compact subgroups because any compact subgroup is contained in a large one by Lemma 3.1. Our next goal is to prove that $(\langle\mathscr{CI}\rangle, \mathscr{CC})$ is a complementary pair of localising subcategories of $\mathrm{KK}^{G\ltimes X}$, so that we can apply Proposition 2.9.

Lemma 4.2. *The subcategories $\mathscr{CC}$ and $\langle\mathscr{CI}\rangle$ of $\mathrm{KK}^{G\ltimes X}$ are localising.*

The subcategories $\mathscr{CC}$, $\mathscr{CI}$ and $\langle\mathscr{CI}\rangle$ are closed under tensor products with arbitrary objects of KK^G and $\mathrm{KK}^{G\ltimes X}$.

Proof. Since the functor Res_G^H is triangulated and commutes with direct sums, its kernel is localising. Hence $\mathscr{CC}$ is localising as an intersection of localising subcategories. The subcategory $\langle\mathscr{CI}\rangle$ is localising by construction. The subcategories $\mathscr{CC}$ and $\mathscr{CI}$ are closed under tensor products because of the compatibility of restriction and induction with tensor products discussed in Section 3. Since the functor $\sqcup\otimes_{(X)} B$ is triangulated and commutes with direct sums, it leaves $\langle\mathscr{CI}\rangle$ invariant as well. □

Lemma 4.3. *A morphism in $\mathrm{KK}^{G\ltimes X}$ is a weak equivalence if and only if its mapping cone is weakly contractible.*

Proof. Since the functor Res_G^H is triangulated, it maps an exact triangle $\Sigma B \to C \to A \xrightarrow{f} B$ again to an exact triangle. Lemma 2.2 implies that $\mathrm{Res}_G^H f$ is invertible if and only if $\mathrm{Res}_G^H C \cong 0$. □

Proposition 4.4. *An object N of $\mathrm{KK}^{G\ltimes X}$ is weakly contractible if and only if $\mathrm{KK}^{G\ltimes X}(P,N)\cong 0$ for all $P\in\mathscr{CI}$.*

A morphism $f\in\mathrm{KK}^{G\ltimes X}(A,B)$ is a weak equivalence if and only if it induces an isomorphism $f_*\colon \mathrm{KK}^{G\ltimes X}(P,A)\to\mathrm{KK}^{G\ltimes X}(P,B)$ for all $P\in\mathscr{CI}$.

A morphism $f\in\mathrm{KK}^{G\ltimes X}(A,B)$ vanishes for compact subgroups if and only if it induces the zero map $f_*\colon \mathrm{KK}^{G\ltimes X}(P,A)\to\mathrm{KK}^{G\ltimes X}(P,B)$ for all $P\in\mathscr{CI}$.

In the first two assertions, we can replace $\mathscr{CI}$ by $\langle\mathscr{CI}\rangle$.

Proof. Let $H\in\mathrm{LC}$ be maximal compact and let $U:=U_H$. Let $P=\mathrm{Ind}_H^G A$ for some $A\in\mathrm{KK}^{H\ltimes X}$. Any object of $\mathscr{CI}$ is of this form for some H, A by Lemma 3.1. We use (22) to rewrite $\mathrm{KK}^{G\ltimes X}(P,N)\cong\mathrm{KK}^{H\ltimes X}(A',\mathrm{Res}_G^H N)$ with $A':=\mathrm{Res}_U^H\mathrm{Ind}_H^U A$. If $\mathrm{Res}_G^H N\cong 0$, then the right hand side vanishes, so that $\mathrm{KK}^{G\ltimes X}(P,N)=0$. Conversely, if $\mathrm{KK}^{G\ltimes X}(P,N)=0$ for all $P\in\mathscr{CI}$, then $\mathrm{KK}^{H\ltimes X}(\mathrm{Res}_G^H N,\mathrm{Res}_G^H N)=0$ and hence $\mathrm{Res}_G^H N=0$. We have used that $\mathrm{Res}_G^H N$ belongs to the essential range of $\mathrm{Res}_U^H\mathrm{Ind}_H^U$ by Lemma 3.3. This proves the first assertion. We can enlarge $\mathscr{CI}$ to $\langle\mathscr{CI}\rangle$ because the class of objects P with $\mathrm{KK}^{G\ltimes X}(P,N)=0$ for all $N\in\mathscr{CC}$ is localising. The remaining assertions are proven similarly. □

Definition 4.5. A *$\mathscr{CI}$-simplicial approximation* of $A\in\mathrm{KK}^{G\ltimes X}$ is a weak equivalence $\tilde{A}\to A$ with $\tilde{A}\in\langle\mathscr{CI}\rangle$. A $\mathscr{CI}$-simplicial approximation of $C_0(X)$ is also called a *Dirac morphism* for $G\ltimes X$.

Proposition 4.6. *A Dirac morphism exists for any $G\ltimes X$.*

The existence of a Dirac morphism is the main technical result needed for our approach to the Baum–Connes conjecture. Logically, we should now prove the existence of the Dirac morphism (we postpone this until Section 6) and then compute the localisation $\mathrm{KK}^{G\ltimes X}/\mathscr{CC}$ (we do this in Section 7) before we dare to localise the functor $\mathrm{K}_*(G\ltimes_{\mathrm{r}}\sqcup)$. Instead, we head for the Baum–Connes assembly map as quickly as possible.

The following theorem uses the notation of Proposition 2.9.

Theorem 4.7. *The localising subcategories $\langle\mathscr{CI}\rangle$, $\mathscr{CC}$ of $\mathrm{KK}^{G\ltimes X}$ are complementary. Let $\mathsf{D}\in\mathrm{KK}^{G\ltimes X}(\mathsf{P},C_0(X))$ be a Dirac morphism for $G\ltimes X$ and form the exact triangle*

$$\Sigma\mathsf{N}\to\mathsf{P}\xrightarrow{\mathsf{D}}C_0(X)\to\mathsf{N}. \tag{23}$$

Then $P(A)\cong\mathsf{P}\otimes_X A$ and $N(A)\cong\mathsf{N}\otimes_X A$ naturally for all $A\in\mathrm{KK}^{G\ltimes X}$, and the natural transformations $\Sigma N(A)\to P(A)\to A\to N(A)$ are induced by the maps in (23). *We have $A\in\langle\mathscr{CI}\rangle$ if and only if $\mathrm{KK}^{G\ltimes X}(A,B)=0$ for all $B\in\mathscr{CC}$ if and only if $\mathsf{P}\otimes_X A\cong A$; and $B\in\mathscr{CC}$ if and only if $\mathrm{KK}^{G\ltimes X}(A,B)=0$ for all $A\in\langle\mathscr{CI}\rangle$* if *and only if $\mathsf{P}\otimes_X B\cong 0$. In particular, $\mathsf{P}\otimes_X\mathsf{P}\cong\mathsf{P}$.*

Proof. Since D is a weak equivalence, N is weakly contractible by Lemma 4.3. The tensor product of (23) with A is another exact triangle because $\sqcup\otimes_X A$ is a triangulated functor. Since $\mathscr{CC}$ and $\langle\mathscr{CI}\rangle$ are closed under tensor products by Lemma 4.2, we get an exact triangle as in the definition of a complementary pair of subcategories in Section 2.6. This yields the assertions in the first paragraph. Those in the second paragraph follow from Proposition 2.9. □

Definition 4.8. An exact triangle as in (23) is called a *Dirac triangle*.

Using Proposition 2.9.7, we can now compute localisations and obstruction functors from a Dirac triangle. The morphisms in $\mathrm{KK}^G/\mathscr{CC}$ are given by

$$\mathrm{KK}^{G\ltimes X}/\mathscr{CC}(A,B)\cong\mathrm{KK}^{G\ltimes X}(\mathsf{P}\otimes_X A,B).$$

The localisation and the obstruction functor of a functor $F\colon\mathrm{KK}^{G\ltimes X}\to\mathfrak{C}$ are

$$\mathbb{L}F(A)\cong F(\mathsf{P}\otimes_X A),\quad \mathrm{Obs}\,F(A)\cong F(\mathsf{N}\otimes_X A),$$

and the natural transformations $\mathbb{L}F(A)\to F(A)\to\mathrm{Obs}\,F(A)$ are induced by the maps $\mathsf{P}\to\star\to\mathsf{N}$ in the Dirac triangle.

We are particularly interested in the functor

$$\mathrm{KK}^{G\ltimes X}\to\mathrm{KK},\quad A\mapsto (G\ltimes X)\ltimes_{\mathrm{r}}A.$$

We denote its localisation and obstruction functor by $A\mapsto (G\ltimes X)\ltimes_{\mathrm{r}}^{\mathbb{L}}A$ and $A\mapsto (G\ltimes_{\mathrm{r}}X)\ltimes_{\mathrm{r}}^{\mathrm{Obs}}A$, respectively.

5. The Baum–Connes assembly map

We now relate our analysis of $\mathrm{KK}^{G\ltimes X}$ to the Baum–Connes assembly map. Since we consider transformation groups $G\ltimes X$, we first have to do some work related to the space X. Chabert, Echterhoff and Oyono-Oyono show in [15] that there is a commutative diagram

$$\begin{array}{ccc}
\mathrm{K}_*^{\mathrm{top}}(G\ltimes X,A) & \xrightarrow{\mu_A} & \mathrm{K}_*((G\ltimes X)\ltimes_{\mathrm{r}}A)\\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong}\\
\mathrm{K}_*^{\mathrm{top}}(G,A) & \xrightarrow{\mu_A} & \mathrm{K}_*(G\ltimes_{\mathrm{r}}A).
\end{array}$$

That is, the Baum–Connes assembly map just ignores the space X. We need a similar result in our setup.

Lemma 5.1. *The functor $p_X^*\colon\mathrm{KK}^G\to\mathrm{KK}^{G\ltimes X}$ maps $\mathscr{CC},\mathscr{CI},\langle\mathscr{CI}\rangle\subseteq\mathrm{KK}^G$ to the corresponding subcategories in $\mathrm{KK}^{G\ltimes X}$. If $f\in\mathrm{KK}^G(A,B)$ is a weak equivalence or vanishes for compact subgroups, so does $p_X^*(f)$. If $\mathsf{D}\in\mathrm{KK}^G(\mathsf{P},\star)$ is a Dirac morphism for G, then $p_X^*(\mathsf{D})\in\mathrm{KK}^{G\ltimes X}(C_0(X,\mathsf{P}),C_0(X))$ is a Dirac morphism for $G\ltimes X$. There are natural isomorphisms*

$$(G\ltimes X)\ltimes_{\mathrm{r}}^{\mathbb{L}}A\cong G\ltimes_{\mathrm{r}}^{\mathbb{L}}A,\quad (G\ltimes X)\ltimes_{\mathrm{r}}^{\mathrm{Obs}}A\cong G\ltimes_{\mathrm{r}}^{\mathrm{Obs}}A.$$

Proof. Recall from Section 3 that the functor p_X^* is compatible with restriction and induction. This implies $p_X^*(\mathscr{CC})\subseteq\mathscr{CC}$ and $p_X^*(\mathscr{CI})\subseteq\mathscr{CI}$. The same holds for $\langle\mathscr{CI}\rangle$ because p_X^* is triangulated and commutes with direct sums. This implies the assertions about weak equivalences and Dirac morphisms. Now (11) yields

$$(G\ltimes X)\ltimes_{\mathrm{r}}^{\mathbb{L}}A\cong(G\ltimes X)\ltimes_{\mathrm{r}}(p_X^*(\mathsf{P})\otimes_X A)\approx G\ltimes_{\mathrm{r}}(\mathsf{P}\otimes A)\cong G\ltimes_{\mathrm{r}}^{\mathbb{L}}A.$$

For the same reason, $(G\ltimes X)\ltimes_{\mathrm{r}}^{\mathrm{Obs}}A\cong G\ltimes_{\mathrm{r}}^{\mathrm{Obs}}A$. $\square$

For our purposes, we do not need the details of the standard definition of the Baum–Connes conjecture. We only need to know the following two facts: the Baum–Connes conjecture holds for compactly induced coefficient algebras (see [11]), and weak equivalences induce isomorphisms on $\mathrm{K}_*^{\mathrm{top}}(G, \sqcup)$ (see [16]). This second assertion also follows immediately from the definition of $\mathrm{K}_*^{\mathrm{top}}$ and Corollary 7.2 below. Thus the only substantial result about the standard definition of the Baum–Connes conjecture that we have to import is that it holds for compactly induced coefficient algebras.

Theorem 5.2. *Let $\tilde{A} \to A$ be a $\mathscr{CI}$-simplicial approximation of $A \in \mathrm{KK}^{G\ltimes X}$. Then the indicated maps in the commutative diagram*

$$\begin{array}{ccc} \mathrm{K}_*^{\mathrm{top}}(G \ltimes X, \tilde{A}) & \xrightarrow{\cong} & \mathrm{K}_*^{\mathrm{top}}(G \ltimes X, A) \\ \cong \downarrow \mu_{\tilde{A}} & & \downarrow \mu_A \\ \mathrm{K}_*((G \ltimes X) \ltimes_{\mathrm{r}} \tilde{A}) & \longrightarrow & \mathrm{K}_*((G \ltimes X) \ltimes_{\mathrm{r}} A) \end{array}$$

are isomorphisms. Hence the Baum–Connes assembly map is naturally isomorphic to the canonical map $\mathrm{K}_*((G\ltimes X)\ltimes_{\mathrm{r}}^{\mathbb{L}} A) \to \mathrm{K}_*((G\ltimes X)\ltimes_{\mathrm{r}} A)$. It is an isomorphism if and only if $\mathrm{K}_*((G\ltimes X)\ltimes_{\mathrm{r}}^{\mathrm{Obs}} A) \cong 0$.

Proof. Lemma 5.1 shows that we may assume without loss of generality that $X = \star$. The left vertical map is the assembly map for the coefficient algebra $\tilde{A}$. Since $\mathrm{K}_*^{\mathrm{top}}(G, \sqcup)$ is a homological functor that commutes with direct sums, the class of coefficient algebras B for which μ_B is an isomorphism is a localising subcategory of KK^G. Therefore, the Baum–Connes conjecture holds for all coefficient algebras in $\langle \mathscr{CI} \rangle$ because it holds for compactly induced coefficient algebras by [11]. As a result, the left vertical map in our diagram is an isomorphism. It is shown in [16] that weak equivalences induce isomorphisms on $\mathrm{K}_*^{\mathrm{top}}(G, \sqcup)$. Hence the top horizontal map is an isomorphism as well. □

Therefore, it is legitimate to view the map $\mathbb{L}F(A) \to F(A)$ for a covariant functor F defined on KK^G as an assembly map for $F(A)$.

As we explained in Section 2.6, there are two pictures of $\mathbb{L}F(A)$: either as the best possible approximation to $F(A)$ that vanishes on $\mathscr{CC}$ or as the best possible approximation to $F(A)$ that only uses the values $F(B)$ for $B \in \langle \mathscr{CI} \rangle$. In particular, the map $\mathbb{L}F(A) \to F(A)$ is an isomorphism for all $A \in \mathrm{KK}^{G\ltimes X}$ if and only if $F|_{\mathscr{CC}} = 0$. As we remarked in the introduction, this yields an equivalent, elementary formulation of the Baum–Connes conjecture when applied to $\mathrm{K}_*(G\ltimes_{\mathrm{r}} A)$.

Alain Connes has asked recently whether it is possible to improve upon the Baum–Connes conjecture, finding better approximations to $\mathrm{K}_*(G\ltimes_{\mathrm{r}} A)$. Like the approaches in [3–5,18], our construction of the assembly map is sufficiently flexible to say something about this, though our answer may not be very satisfactory. The Baum–Connes conjecture asserts that the objects of $\langle \mathscr{CI} \rangle$ are general enough to predict everything that happens in the K-theory of reduced crossed products. If it fails, this means that there are some phenomena in $\mathrm{K}_*(G\ltimes_{\mathrm{r}} A)$ that do not yet occur for $A \in \langle \mathscr{CI} \rangle$. To get a better conjecture, we have to add some of the coefficient algebras for which Baum–Connes fails to $\mathscr{CI}$. Then the general machinery of localisation yields again a best possible approximation to $\mathrm{K}_*(G\ltimes_{\mathrm{r}} A)$ based on what happens for coefficients in $\langle \mathscr{CI}' \rangle$. The new conjecture expresses $\mathrm{K}_*(G\ltimes_{\mathrm{r}} A)$ for arbitrary A in terms of $\mathrm{K}_*(G\ltimes_{\mathrm{r}} A)$ for $A \in \mathscr{CI}'$. However, such a reduction of the problem is only as good as our understanding of what happens for $A \in \mathscr{CI}'$. At the moment, it does not seem that we have a sufficient understanding of the failure of the Baum–Connes conjecture to make any progress in this direction.

6. The Brown Representability Theorem and the Dirac morphism

Recall that a morphism $\mathsf{D} \in \mathrm{KK}^G(\mathsf{P}, \star)$ is a weak equivalence if and only if the induced map $\mathrm{KK}^G(A, \mathsf{P}) \to \mathrm{KK}^G(A, \star)$ is an isomorphism for all $A \in \langle \mathscr{CI} \rangle$. Since P is supposed to lie in the same subcategory $\langle \mathscr{CI} \rangle$, the Dirac morphism exists if and only if the functor$A \mapsto \mathrm{KK}^G(A, \star)$ on the category $\langle \mathscr{CI} \rangle$ is representable. In the classical case of simplicial approximation of arbitrary topological spaces by simplicial complexes, one can either write down explicitly such a representing object or appeal to the Brown Representability Theorem. We shall prove the existence of the Dirac morphism using the second method.

There are several representability theorems for triangulated categories that use different hypotheses. It seems that none of them applies directly to the category $\langle \mathscr{CI} \rangle$ that we need. To circumvent this, we choose a smaller set of generators $\mathscr{CI}_0 \subseteq \mathscr{CI}$ which is small enough so that a general representability theorem is available in $\langle \mathscr{CI}_0 \rangle$ and large enough so that the representing object in $\langle \mathscr{CI}_0 \rangle$ actually represents the functor on the whole of $\langle \mathscr{CI} \rangle$. A byproduct of this proof technique is that we get $\mathsf{P} \in \langle \mathscr{CI}_0 \rangle$. This is used in Section 9.

Since KK^G only has countable direct sums, we have to do some cardinality bookkeeping. Let $\mathscr{T}$ be a triangulated category and let $\aleph$ be an infinite regular cardinal number. We will only need the countable cardinal number $\aleph_0$. We suppose that $\mathscr{T}$ has direct sums of cardinality $\aleph$.

Recall that Ab denotes the category of Abelian groups. A contravariant functor $F: \mathscr{T} \to \mathsf{Ab}$ is called *representable* if it is isomorphic to $X \mapsto \mathscr{T}(X, Y)$ for some $Y \in \mathscr{T}$. Representable functors are cohomological and compatible with direct sums of cardinality $\aleph$. We now formulate conditions on $\mathscr{T}$ that ensure that these necessary conditions plus an extra cardinality hypothesis are also sufficient.

An object $X \in \mathscr{T}$ is called $\aleph$-*compact* if $\mathscr{T}(X, Y)$ has cardinality at most $\aleph$ for all $Y \in \mathscr{T}$ and the covariant functor $\mathscr{T}(X, \sqcup)$ is compatible with direct sums of cardinality $\aleph$. The reader who consults [38] on direct sums and representability should beware that our notation differs slightly. The axiom (TR5$\aleph$) and the notion of $\aleph$-compactness in [38] deal with direct sums of cardinality *strictly less* than $\aleph$.

Theorem 6.1. *Let $\aleph$ be an infinite regular cardinal number and let $\mathscr{T}$ be a triangulated category with direct sums of cardinality $\aleph$. Let $\mathscr{G}$ be a set of $\aleph$-compact objects of $\mathscr{T}$ with $|\mathscr{G}| \leqslant \aleph$. Suppose that $\mathscr{T}(X, Y) = 0$ for all $X \in \mathscr{G}$ already implies $Y = 0$. Let $F: \mathscr{T} \to \mathsf{Ab}$ be an additive, contravariant functor.*

Then F is representable if and only if it satisfies the following conditions:

(i) *F is cohomological*;
(ii) *F is compatible with $\aleph$-direct sums*;
(iii) *$F(C)$ has cardinality at most $\aleph$ for all $C \in \mathscr{G}$.*

Moreover, the hypothesis that $\mathscr{T}(X, Y) = 0$ for all $X \in \mathscr{G}$ implies $Y = 0$ can be replaced by the hypothesis that $\mathscr{T} = \langle \mathscr{G} \rangle^{\aleph}$.

Proof. Conditions (i)–(iii) are clearly necessary. The interesting assertion is that they are also sufficient. If we leave out the cardinality restriction $\aleph$, this is proven by Neeman in [37], and it also follows from

[38, Theorem 8.3.3]. Since we do not have direct sums of arbitrary cardinality in $\mathscr{T}$, we have to check that the proof does not require direct sums of cardinality strictly greater than $\aleph$. This is indeed the case, as the determined reader may check for himself. It turns out that the largest sums we need have cardinality at most $\mathbb{N} \times \aleph \times \aleph$. This is dominated by $\aleph$ because $\aleph$ is a regular cardinal. □

6.1. Construction of Dirac morphisms

Now we introduce a subcategory of $\langle \mathscr{CI} \rangle$ with a hand-selected set of generators $\mathscr{CI}_0$. Let $H \in \mathrm{LC}$ and let $U := U_H$ as in Lemma 3.1. We define J_H^G as in Lemma 3.3 and let

$$\mathsf{R}_H := J_H^G(\star) = \mathrm{Ind}_H^G C_0((U/H)^7) \cong C_0(G \times_H (U/H)^7). \tag{24}$$

This compactly induced G-C^*-algebra satisfies

$$\mathrm{KK}^G(\mathsf{R}_H, B) \cong \mathrm{KK}^H(\star, B) \cong \mathrm{K}(H \ltimes B). \tag{25}$$

by Lemma 3.3. Eq. (25) says that R_H (co)represents the covariant functor $\mathrm{K}(H \ltimes \sqcup)$ and thus determines R_H uniquely up to KK^G-equivalence. Eq. (24) merely is a convenient choice of representing object. For an arbitrary compact subgroup, the functor $\mathrm{K}(H \ltimes \sqcup)$ may fail to be representable. This is why we work with large compact subgroups.

If $H, H' \subseteq U$ are two maximal compact subgroups, they are conjugate in U by Lemma 3.1, so that the G-C^*-algebras R_H and $\mathsf{R}_{H'}$ are isomorphic. Hence it suffices to choose one maximal compact subgroup in each almost connected open subgroup. We let $\mathscr{CI}_0$ be the set of R_H for the chosen subgroups H.

Lemma 6.2. *The set $\mathscr{CI}_0$ is (at most) countable and consists of $\aleph_0$-compact objects of KK^G, where $\aleph_0$ denotes the countable cardinal.*

Proof. It is well-known that $\mathrm{K}_*(A)$ is countable if A is a separable C^*-algebra and that K-theory for C^*-algebras commutes with direct sums [9]. Hence (25) implies that $\mathrm{KK}^G(\mathsf{R}_H, B)$ is countable for each $B \in \mathrm{KK}^G$ and that $\mathrm{KK}^G(\mathsf{R}_H, \sqcup)$ commutes with countable direct sums. That is, $\mathscr{CI}_0$ consists of $\aleph_0$-compact objects. Since the objects of $\mathscr{CI}_0$ are in bijection with the almost connected open subgroups of G, it remains to prove that there are at most countably many such subgroups. Let $G_0 \subseteq G$ be the connected component of the identity element. The open almost connected subgroups of G are in bijection with the compact open subgroups of G/G_0. Since the latter group is second countable as a topological space, even the set of all compact open sub*sets* of G/G_0 is countable. □

Corollary 6.3. *For any $B \in \mathrm{KK}^G$, there is $\tilde{B} \in \langle \mathscr{CI}_0 \rangle$ and $f \in \mathrm{KK}^G(\tilde{B}, B)$ such that $f_*: \mathrm{KK}^G(A, \tilde{B}) \to \mathrm{KK}^G(A, B)$ is an isomorphism for all $A \in \langle \mathscr{CI}_0 \rangle$.*

Proof. Lemma 6.2 implies that the category $\langle \mathscr{CI}_0 \rangle$ with generating set $\mathscr{CI}_0$ satisfies the conditions of Theorem 6.1. The functor $F(A) := \mathrm{KK}^G(A, B)$ fulfils the necessary and sufficient conditions for representability because it is already represented on the larger category KK^G. Hence it is representable on $\langle \mathscr{CI}_0 \rangle$. □

We can now prove the existence of the Dirac morphism.

Proof of Proposition 4.6. We may assume $X = \star$ by Lemma 5.1. Corollary 6.3 for $B = \star$ yields $\mathsf{D} \in \mathrm{KK}^G(\mathsf{P}, \star)$ with $\mathsf{P} \in \langle \mathscr{CI}_0 \rangle \subseteq \langle \mathscr{CI} \rangle$. We claim that D is a weak equivalence. Any compact subgroup $H \subseteq G$ is subconjugate to a subgroup $L \subseteq G$ with $\mathsf{R}_L \in \mathscr{CI}_0$ by Lemma 3.1. Hence it suffices to prove that $\mathrm{Res}_G^L(\mathsf{D})$ is invertible for those L. By construction of D, it induces an isomorphism on $\mathrm{KK}_*^G(\mathsf{R}_L, \sqcup)$. Eq. (25) yields that D induces an isomorphism $\mathrm{KK}^L(\star, \mathsf{P}) \to \mathrm{KK}^L(\star, \star)$. To check that D is an isomorphism in KK^L, it suffices to check that D induces an isomorphism $\mathrm{KK}^L(\mathsf{P}, \mathsf{P}) \to \mathrm{KK}^L(\mathsf{P}, \star)$ as well. Since $\mathsf{P} \in \langle \mathscr{CI}_0 \rangle$, this follows if we have isomorphisms $\mathrm{KK}_*^L(A, \mathsf{P}) \cong \mathrm{KK}_*^L(A, \star)$ for all $A \in \mathscr{CI}_0$. Thus we have to fix another large compact subgroup H and show that D induces an isomorphism $\mathrm{KK}_*^L(\mathsf{R}_H, \mathsf{P}) \to \mathrm{KK}_*^L(\mathsf{R}_H, \star)$.

Let $V := U/H$, then we have $\mathsf{R}_H = C_0(G \times_H V^7) \cong C_0(G \times_U V^8)$. We only need the action of L on this space. The space $G \times_U V^8$ decomposes into a disjoint union of the spaces $LgU \times_U V^8$ over the double cosets $g \in L\backslash G/U$. We have a natural isomorphism $LgU \times_U V^8 \cong L \times_{L \cap gUg^{-1}} V^8$, where we use the conjugation automorphism $gUg^{-1} \cong U$ to let gUg^{-1} act on V. Since the action of $L \cap gUg^{-1}$ on V is diffeomorphic to a linear action, equivariant Bott periodicity yields that $C_0(LgU \times_U V^8)$ is KK^L-equivalent to $\mathrm{Ind}_{L \cap gUg^{-1}}^L(\star)$. Thus

$$\mathrm{Res}_G^L \mathsf{R}_H \cong \bigoplus_{g \in L\backslash G/U} \mathrm{Ind}_{L \cap gUg^{-1}}^L(\star)$$

The subgroups $L \cap gUg^{-1}$ are open in L and again large by Lemma 3.1. It follows from (20) that

$$\begin{aligned} \mathrm{KK}_*^L(\mathsf{R}_H, B) &\cong \bigoplus_{g \in L\backslash G/U} \mathrm{KK}_*^L(\mathrm{Ind}_{L \cap gUg^{-1}}^L(\star), B) \\ &\cong \bigoplus_{g \in L\backslash G/U} \mathrm{KK}_*^{L \cap gUg^{-1}}(\star, B) \cong \bigoplus_{g \in L\backslash G/U} \mathrm{KK}_*^G(\mathsf{R}_{L \cap gUg^{-1}}, B). \end{aligned}$$

By construction, D induces an isomorphism on the right hand side and hence on $\mathrm{KK}_*^L(\mathsf{R}_H, \sqcup)$. □

Remark 6.4. Incidentally, the proof above shows that

$$J_L^G \mathrm{Res}_G^L J_H^G(\star) = J_L^G \mathrm{Res}_G^L(\mathsf{R}_H) \cong \bigoplus_{g \in L\backslash G/U} \mathsf{R}_{L \cap gUg^{-1}}.$$

6.2. A localisation related to the Universal Coefficient Theorem

Definition 6.5. A C^*-algebra A is called K-*contractible* if $\mathrm{K}_*(A) = 0$. A morphism $f \in \mathrm{KK}(A, B)$ is called a K-*equivalence* if $f_*: \mathrm{K}_*(A) \to \mathrm{K}_*(B)$ vanishes.

We write $\mathscr{N} \subseteq \mathrm{KK}$ for the full subcategory of K-contractible objects. This subcategory is localising because K-theory is a homological functor compatible with direct sums. A morphism is a K-equivalence if and only if its mapping cone is K-contractible.

Theorem 6.6. *The localising subcategories* $\langle \star \rangle$ *and* $\mathscr{N}$ *in* KK *are complementary.*

Proof. We have $B \in \mathcal{N}$ if and only if $\mathrm{KK}_*(\star, B) \cong \mathrm{K}_*(B) = 0$ for all $* \in \mathbb{Z}$, if and only if $\mathrm{KK}(A, B) = 0$ for all $A \in \langle\star\rangle$. Similarly, $f \in \mathrm{KK}(\tilde{B}, B)$ is a K-equivalence if and only if $f_*: \mathrm{KK}(A, \tilde{B}) \to \mathrm{KK}(A, B)$ is an isomorphism for all $A \in \langle\star\rangle$. We have to construct a K-equivalence $\tilde{B} \to B$ with $\tilde{B} \in \langle\star\rangle$ for any $B \in \mathrm{KK}$. This is equivalent to the representability of the functor $A \mapsto \mathrm{KK}(A, B)$ on $\langle\star\rangle$. The object $\star$ of KK is clearly compact and generates $\langle\star\rangle$ by definition. Hence we can apply the Brown Representability Theorem 6.1 to get the assertion. □

As we observed in Section 2.5, $\langle\star\rangle \subseteq \mathrm{KK}$ is just the bootstrap category. The simplicial approximations in this context are usually called *geometric resolutions* (see [9]). Let $\mathrm{UCT} := \mathrm{KK}/\mathcal{N}$ be the localisation of KK at the K-contractible objects. This is a triangulated category with countable direct sums and equipped with a triangulated functor $\mathrm{KK} \to \mathrm{UCT}$ commuting with direct sums. It has the same objects as KK. Morphisms are computed using geometric resolutions:

$$\mathrm{UCT}(A, B) \cong \mathrm{KK}(\tilde{A}, B) \cong \mathrm{KK}(\tilde{A}, \tilde{B}).$$

The group $\mathrm{UCT}(A, B)$ can always be computed using the Universal Coefficient Theorem (see [9]) because the latter applies to $\tilde{A} \in \langle\star\rangle$. Moreover, A satisfies the Universal Coefficient Theorem if and only if $\mathrm{KK}(A, B) \cong \mathrm{UCT}(A, B)$ for all B if and only if $A \in \langle\star\rangle$. Thus we have translated the Universal Coefficient Theorem as an isomorphism statement. This is often convenient.

The functor $A \otimes \sqcup$ preserves K-equivalences for $A \in \langle\star\rangle$ because this holds for the generator $\star$. Hence the natural maps from $\tilde{A} \otimes \tilde{B}$ to $A \otimes \tilde{B}$ and $\tilde{A} \otimes B$ are both K-equivalences, so that the various ways of localising $A \otimes B$ give the same result $A \otimes^{\mathbb{L}} B$ in the category UCT. Since $A \otimes^{\mathbb{L}} B$ only involves C^*-algebras from the bootstrap category, $\mathrm{K}_*(A \otimes^{\mathbb{L}} B)$ can always be computed by the Künneth Formula (see [9]). We remark also that $\langle\star\rangle \otimes \langle\star\rangle \subseteq \langle\star\rangle$ because $\star \otimes \star = \star$.

Thus localisation of KK at $\mathcal{N}$ yields a natural map $\mathrm{KK}_*(A, B) \to \mathrm{UCT}_*(A, B)$, which is an isomorphisms for all B if and only if A satisfies the Universal Coefficient Theorem, and a natural map $\mathrm{K}_*(A \otimes^{\mathbb{L}} B) \to \mathrm{K}_*(A \otimes B)$, which is an isomorphism for all B if and only if A satisfies the Künneth Formula. We want to emphasise that, on a formal level, these maps are analogous to the Baum–Connes assembly map $\mathrm{K}_*(G \ltimes^{\mathbb{L}}_{\mathrm{r}} A) \to \mathrm{K}_*(G \ltimes_{\mathrm{r}} A)$.

7. The derived category and proper actions

We want to describe the localisation of $\mathrm{KK}^{G \ltimes X}$ at $\mathcal{CC}$ in more classical terms. Let A and B be $G \ltimes X$-C^*-algebras. Let Y be a locally compact G-space. Generalising (2) slightly, we let

$$\mathrm{RKK}^{G \ltimes X}(Y; A, B) := \mathrm{KK}^{G \ltimes (X \times Y)}(C_0(Y, A), C_0(Y, B)).$$

We let $\mathrm{RKK}^{G \ltimes X}(Y)$ be the category with the same objects as $\mathrm{KK}^{G \ltimes X}$ and with morphisms as above. That is, $\mathrm{RKK}^{G \ltimes X}(Y)$ identifies with the range of the functor

$$p_Y^*: \mathrm{KK}^{G \ltimes X} \to \mathrm{KK}^{G \ltimes (X \times Y)} \tag{26}$$

induced by the projection map $X \times Y \to X$ and thus with a subcategory of $\mathrm{KK}^{G \ltimes (X \times Y)}$. (There is no reason to expect this subcategory to be triangulated.)

Let $f: Y \to Y'$ be a continuous G-equivariant map. Since $p_{Y'} \circ f = p_Y$, the functor $f^*: \mathrm{KK}^{G \ltimes (X \times Y')} \to \mathrm{KK}^{G \ltimes (X \times Y)}$ yields natural maps

$$f^*: \mathrm{RKK}^{G \ltimes X}(Y'; A, B) \to \mathrm{RKK}^{G \ltimes X}(Y; A, B). \tag{27}$$

For the constant map $p_Y: Y \to \star$ this reproduces the functor p_Y^* in (26). The maps in (27) turn $Y \mapsto \mathrm{RKK}^{G \ltimes X}(Y; A, B)$ into a contravariant functor.

If S is a compact G-space, then (18) yields a natural isomorphism

$$\mathrm{RKK}^{G \ltimes X}(Y \times S; A, B) \cong \mathrm{KK}^{G \ltimes X}(Y; A, C(S, B)).$$

For $S = [0, 1]$, we see that homotopy invariance of $\mathrm{KK}^G(A, B)$ in the variable B implies homotopy invariance in the variable Y; that is, $f_1^* = f_2^*$ if $f_1, f_2: Y \to Y'$ are G-equivariantly homotopic. Let $\mathscr{E}G$ be a second countable, locally compact universal proper G-space. Then $X \times \mathscr{E}G$ is a universal proper $G \ltimes X$-space. The category $\mathrm{RKK}^{G \ltimes X}(\mathscr{E}G)$ and the functor $p^*_{\mathscr{E}G}: \mathrm{KK}^{G \ltimes X} \to \mathrm{RKK}^{G \ltimes X}(\mathscr{E}G)$ do not depend on the choice of $\mathscr{E}G$ because $\mathscr{E}G$ is unique up to homotopy.

Theorem 7.1. *The functor $p^*_{\mathscr{E}G}: \mathrm{KK}^{G \ltimes X} \to \mathrm{RKK}^{G \ltimes X}(\mathscr{E}G)$ descends to an isomorphism of categories $\mathrm{KK}^{G \ltimes X}/\mathscr{CC} \cong \mathrm{RKK}^{G \ltimes X}(\mathscr{E}G)$.*

More explicitly, let $\pi: \tilde{A} \to A$ be a $\mathscr{CI}$-simplicial approximation of $A \in \mathrm{KK}^{G \ltimes X}$. Then the indicated maps in the following commutative diagram are isomorphisms:

$$\begin{array}{ccc}
\mathrm{KK}^{G \ltimes X}(\tilde{A}, B) & \xleftarrow{\ \pi^*\ } & \mathrm{KK}^{G \ltimes X}(A, B) \\
{\scriptstyle p^*_{\mathcal{E}G}}\downarrow\cong & & \downarrow{\scriptstyle p^*_{\mathcal{E}G}} \\
\mathrm{RKK}^{G \ltimes X}(\mathcal{E}G; \tilde{A}, B) & \underset{\pi^*}{\xleftarrow{\ \cong\ }} & \mathrm{RKK}^{G \ltimes X}(\mathcal{E}G; A, B).
\end{array}$$

Proof. The first assertion follows from the second one and Proposition 2.9.7. Hence we only have to prove that the two indicated maps are isomorphisms. Consider $p^*_{\mathscr{E}G}$ first. Fix B. Both $\mathrm{KK}^{G \ltimes X}(\llcorner\!\lrcorner, B)$ and $\mathrm{RKK}^{G \ltimes X}(\mathscr{E}G; \llcorner\!\lrcorner, B)$ are cohomological functors compatible with direct sums. Thus the class of objects $\tilde{A}$ for which the natural transformation $p^*_{\mathscr{E}G}$ between them is an isomorphism is localising. Hence we have an isomorphism for all $\tilde{A} \in \langle \mathscr{CI} \rangle$ once we have an isomorphism for $\tilde{A} \in \mathscr{CI}$. This is what we shall prove. Thus we let $\tilde{A} := \mathrm{Ind}_H^G A'$ for some large compact subgroup H and some $A' \in \mathrm{KK}^{H \ltimes X}$. Let $U := U_H$ and let Y be a G-space as above. We use Lemma 3.3 and the compatibility of Ind_H^G with p_Y^* to rewrite

$$\begin{aligned}
&\mathrm{RKK}^{G \ltimes X}(Y; \mathrm{Ind}_H^G A', B) \\
&\quad = \mathrm{KK}^{G \ltimes (X \times Y)}(\mathrm{Ind}_H^G p_Y^* A', p_Y^* B) \\
&\quad \cong \mathrm{KK}^{H \ltimes (X \times Y)}(p_Y^* \mathrm{Ind}_H^U A', p_Y^* B) = \mathrm{RKK}^{H \ltimes X}(Y; \mathrm{Res}_U^H \mathrm{Ind}_H^U A', \mathrm{Res}_G^H B);
\end{aligned}$$

we have dropped restriction functors from the notation except in the final result. These isomorphisms are natural and especially compatible with the functoriality in Y. Since H is compact and $\mathscr{E}G$ is H-equivariantly contractible, homotopy invariance implies that $p^*_{\mathscr{E}G}$ is an isomorphism

$$\mathrm{RKK}^{H\ltimes X}(\star; \mathrm{Res}^H_U \mathrm{Ind}^U_H A', \mathrm{Res}^H_G B) \xrightarrow{\cong} \mathrm{RKK}^{H\ltimes X}(\mathscr{E}G; \mathrm{Res}^H_U \mathrm{Ind}^U_H A', \mathrm{Res}^H_G B).$$

Hence $p^*_{\mathscr{E}G}: \mathrm{RKK}^{G\ltimes X}(\star; \mathrm{Ind}^G_H A', B) \to \mathrm{RKK}^{G\ltimes X}(\mathscr{E}G; \mathrm{Ind}^G_H A', B)$ is an isomorphism as well.

We claim that any weak equivalence $\pi: \tilde{A} \to A$ induces an isomorphism

$$\mathrm{RKK}^{G\ltimes X}(\mathscr{E}G; \tilde{A}, B) \cong \mathrm{RKK}^{G\ltimes X}(\mathscr{E}G; A, B).$$

The proof of this claim will finish the proof of the theorem. We remark that the usual definition of $\mathrm{K}^{\mathrm{top}}_*(G, A)$ is functorial for elements of $\mathrm{RKK}^G(\mathscr{E}G; A, A')$ in a rather obvious way. Therefore, $f \in \mathrm{KK}^G(A, A')$ induces an isomorphism on $\mathrm{K}^{\mathrm{top}}_*(G, A)$ once $p^*_{\mathscr{E}G}(f)$ is an isomorphism in $\mathrm{RKK}^G(\mathscr{E}G; A, A')$. Thus the claim above implies that weak equivalences induce isomorphisms on $\mathrm{K}^{\mathrm{top}}_*(G, A)$. We have already used this result of [16] in the proof of Theorem 5.2 above.

The claim is equivalent to $\mathrm{RKK}^{G\ltimes X}(\mathscr{E}G; A, B) = 0$ for $A \in \mathscr{CC}$ by Lemma 4.3 because $\mathrm{RKK}^{G\ltimes X}(\mathscr{E}G; \sqcup, B)$ is a cohomological functor. This condition for all B is equivalent to $p^*_{\mathscr{E}G}(A) = 0$ for $A \in \mathscr{CC}$, that is, $\mathscr{CC} \subseteq \ker p^*_{\mathscr{E}G}$. Let $\mathscr{S}$ be the class of proper G-spaces Y for which $\mathscr{CC} \subseteq \ker p^*_Y$. We shall use the following trivial observation. If $Y \to Y'$ is a G-equivariant map and $Y' \in \mathscr{S}$, then $Y \in \mathscr{S}$ as well because p^*_Y factors through $p^*_{Y'}$. Therefore, $\mathscr{E}G \in \mathscr{S}$ if and only if $Y \in \mathscr{S}$ for all proper G-spaces Y. This is what we are going to prove.

Let $H \subseteq G$ be a compact subgroup and let Y' be a locally compact H-space. Then we can form a G-space $Y = G\times_H Y'$. We call such G-spaces *compactly induced*. The groupoid $G\ltimes(X \times Y)$ is Morita equivalent to $H\ltimes(X \times Y')$. This yields an isomorphism

$$\mathrm{RKK}^{G\ltimes X}(G\times_H Y'; A, B) \cong \mathrm{RKK}^{H\ltimes X}(Y'; \mathrm{Res}^H_G A, \mathrm{Res}^H_G B),$$

(see [28, Theorem 3.6]) and hence factors p^*_Y through Res^H_G. Thus $Y \in \mathscr{S}$, that is, $\mathscr{S}$ contains all compactly induced G-spaces.

Any locally compact proper G-space is locally compactly induced, that is, can be covered by open G-invariant subsets that are isomorphic to compactly induced spaces. This result of Abels [2] is rediscovered in [13]. It ought to imply our claim by a Mayer–Vietoris argument. However, the proof is somewhat delicate because it is unclear whether $\mathrm{RKK}^{G\ltimes X}(Y; A, B)$ as a functor of Y is excisive.

We let $\mathfrak{C}_n$ be the class of proper G-spaces that can be covered by at most n compactly induced, G-invariant open subsets. Thus $\mathfrak{C}_1$ consists of the compactly induced G-spaces. We prove $\mathfrak{C}_n \subseteq \mathscr{S}$ by induction on n. We already know $\mathfrak{C}_1 \subseteq \mathscr{S}$. If $Y \in \mathfrak{C}_n$, then $Y = Y_0 \cup Y_1$ with open subsets Y_0, Y_1 such that $Y_0 \in \mathfrak{C}_1$ and $Y_1 \in \mathfrak{C}_{n-1}$. Hence $Y_0, Y_1 \in \mathscr{S}$ by induction hypothesis. Let $Y_\cap := Y_0 \cap Y_1$. Then $Y_\cap \in \mathscr{S}$ as well because $Y_\cap$ maps to Y_0. The idea of the following proof is to replace Y by a homotopy push-out Z of the diagram $Y_0 \leftarrow Y_\cap \rightarrow Y_1$. It is easy to see that $Z \in \mathscr{S}$. Since there is a G-equivariant map $Y \to Z$ this implies $Y \in \mathscr{S}$.

In detail, let ϕ_0, ϕ_1 be a G-invariant partition of unity subordinate to the covering Y_0, Y_1. This can be constructed by working in $G\backslash Y$. Let

$$Z := (Y_0 \sqcup Y_1 \sqcup ([0, 1] \times Y_\cap)) / \sim$$

where we identify $Y_\cap \subseteq Y_j$ with $\{j\} \times Y_\cap \subseteq [0, 1] \times Y_\cap$ for $j = 0, 1$. We define a map $\phi_*: Y \to Z$ by $\phi_*(y) := (\phi_1(y), y) \in [0, 1] \times Y_\cap$ for $y \in Y_\cap$, $\phi_*(y) := y \in Y_0$ for $y \in Y_0 \backslash Y_\cap$ and $\phi_*(y) := y \in Y_1$ for $y \in Y_1 \backslash Y_\cap$. Notice that this is a continuous, G-equivariant map. Thus $Z \in \mathscr{S}$ implies $Y \in \mathscr{S}$.

A cycle for $\mathrm{RKK}^{G\ltimes X}(Z; A, A)$ is a triple $(f_0, f_1, f_\cap)$, where f_j is a cycle for $\mathrm{RKK}^{G\ltimes X}(Y_j; A, A)$ for $j=0, 1$ and $f_\cap$ is a homotopy in $\mathrm{RKK}^{G\ltimes X}(Y_\cap; A, A)$ between $f_0|_{Y_\cap}$ and $f_1|_{Y_\cap}$. Fix any such cycle. The cycles f_0 and f_1 are homotopic to 0 because $Y_0, Y_1 \in \mathscr{S}$ and $A \in \mathscr{CC}$. This yields a homotopy between $(f_0, f_1, f_\cap)$ and $(0, 0, f'_\cap)$, where $f'_\cap$ is some cycle for $\mathrm{RKK}^{G\ltimes X}(Y_\cap \times [0, 1]; A, A)$ whose restrictions to 0 and 1 vanish. Thus $f'_\cap$ is equivalent to a cycle for

$$\mathrm{RKK}^{G\ltimes X\times[0,1]}(Y_\cap; C([0, 1], A), \Sigma A) \cong \mathrm{RKK}^{G\ltimes X}(Y_\cap; A, \Sigma A).$$

Apply (18) to the coordinate projection $X \times Y_\cap \times [0, 1] \to X \times Y_\cap$ to get this isomorphism. We have $\mathrm{RKK}^{G\ltimes X}(Y_\cap; A, \Sigma A) = 0$ because $Y_\cap \in \mathscr{S}$. Thus $f'_\cap$ is homotopic to 0 and $\mathrm{RKK}^{G\ltimes X}(Z; A, A) = 0$ for all $A \in \mathscr{CC}$, that is, $Z \in \mathscr{S}$.

So far we have proven that $\mathfrak{C}_n \subseteq \mathscr{S}$ for all $n \in \mathbb{N}$. Now let Y be an arbitrary proper G-space. Since Y is locally compactly induced, there is a locally finite covering by compactly induced G-invariant open subsets $(U_j)_{j\in\mathbb{N}}$. We let $Y_j = \bigcup_{k=0}^{j} U_j$ and write $Y = \cup Y_n$. Thus $Y_n \in \mathfrak{C}_n \subseteq \mathscr{S}$ for all $n \in \mathbb{N}$. We use the following variant of the mapping telescope (compare Section 2.4):

$$Z := \{(y, t) \in Y \times \mathbb{R}_+ \mid y \in Y_m \text{ whenever } t < m + 1\} = \bigcup_{m\in\mathbb{N}} Y_m \times [m, m + 1].$$

This is a closed G-invariant subset of $Y \times \mathbb{R}_+$. There exists a partition of unity by G-invariant functions subordinate to (U_j) because $G\backslash Y$ is paracompact. We use this to construct a G-invariant function $\phi: Y \to \mathbb{R}_+$ with $\phi(y) \geqslant m$ for all $y \in Y_m \backslash Y_{m-1}$. We get an embedding $Y \to Z$, $y \mapsto (y, \phi(y))$. Thus $Y \in \mathscr{S}$ follows if $Z \in \mathscr{S}$. The proof of $Z \in \mathscr{S}$ is analogous to the argument in the preceding paragraph. Therefore, we are rather brief.

A cycle for $\mathrm{RKK}^{G\ltimes X}(Z; A, A)$ is equivalent to sequences of cycles $(f_m)_{m\in\mathbb{N}}$ for $\mathrm{RKK}^{G\ltimes X}(Y_m; A, A)$ and homotopies $(H_m)_{m\in\mathbb{N}}$ between f_m and $f_{m+1}|_{Y_m}$. The assumption that $Y_m \in \mathscr{S}$ for all m allows us to find a homotopy between f_m and 0 for all m. Thus the cycle described by the data $(f_m, H_m)_{m\in\mathbb{N}}$ is homotopic to a cycle $(0, H'_m)_{m\in\mathbb{N}}$. Each H'_m is equivalent to a cycle for $\mathrm{RKK}^{G\ltimes X}(Y_m; A, \Sigma A) \cong 0$ and thus homotopic to 0. Hence $\mathrm{RKK}^{G\ltimes X}(Z; A, A) = 0$, that is, $Z \in \mathscr{S}$. This finishes the proof. □

Corollary 7.2. *An object of* $\mathrm{KK}^{G\ltimes X}$ *is weakly contractible if and only if its image in* $\mathrm{KK}^{G\ltimes(X\times\mathscr{E}G)}$ *is isomorphic to* 0, *if and only if its image in* $\mathrm{KK}^{G\ltimes(X\times Y)}$ *is isomorphic to* 0 *for all proper* G*-spaces* Y. *A morphism in* $\mathrm{KK}^{G\ltimes X}$ *is a weak equivalence if and only if its image in* $\mathrm{KK}^{G\ltimes(X\times\mathscr{E}G)}$ *is invertible*, *if and only if its image in* $\mathrm{KK}^{G\ltimes(X\times Y)}$ *is invertible for all proper* G*-spaces* Y.

Proof. By the universal property of $\mathscr{E}G$ the map $Y \to \star$ for any proper G-space factors through $\mathscr{E}G$. Hence assertions about $\mathrm{RKK}^G(\mathscr{E}G)$ as in the statement of the corollary imply the corresponding assertions about $\mathrm{RKK}^G(Y)$ for all proper G-spaces Y. An object is weakly contractible if and only if its image in the localisation vanishes and a morphism is a weak equivalence if and only if its image in the localisation is invertible. Thus everything follows from Theorem 7.1. □

Recall that a $G\ltimes X$-C^*-algebra is called *proper* if it is a $G\ltimes(X \times Y)$-algebra for some proper G-space Y.

Corollary 7.3. *All proper* $G\ltimes X$-C^**-algebras belong to* $\langle\mathscr{CI}\rangle$.

Proof. Let A be a proper $G\ltimes X$-C^*-algebra. Then A is a $G\ltimes(X\times\mathscr{E}G)$-$C^*$-algebra. Let $\mathsf{D}\colon\mathsf{P}\to C_0(X)$ be a Dirac morphism for $G\ltimes X$. Since D is a weak equivalence, $p^*_{\mathscr{E}G}(\mathsf{D})$ is an invertible morphism in $\mathrm{KK}^{G\ltimes(X\times\mathscr{E}G)}$ by Corollary 7.2. Hence

$$p^*_{\mathscr{E}G}(\mathsf{D})\otimes_{X\times\mathscr{E}G}\mathrm{id}_A\in\mathrm{KK}^{G\ltimes(X\times\mathscr{E}G)}(p^*_{\mathscr{E}G}(\mathsf{P})\otimes_{X\times\mathscr{E}G}A,\,C_0(X\times\mathscr{E}G)\otimes_{X\times\mathscr{E}G}A)$$

is invertible. If we forget the $\mathscr{E}G$-structure, we still have an invertible element in $\mathrm{KK}^{G\ltimes X}$. Eq. (11) implies $C_0(X\times\mathscr{E}G)\otimes_{X\times\mathscr{E}G}A\cong A$ and $p^*_{\mathscr{E}G}(\mathsf{P})\otimes_{X\times\mathscr{E}G}A\cong\mathsf{P}\otimes_X A\in\langle\mathscr{C}\mathscr{I}\rangle$. These isomorphisms identify $p^*_{\mathscr{E}G}(\mathsf{D})\otimes_{X\times\mathscr{E}G}\mathrm{id}_A$ with $\mathsf{D}_*\in\mathrm{KK}^G(\mathsf{P}\otimes_X A,A)$. Thus D is invertible and $A\in\langle\mathscr{C}\mathscr{I}\rangle$. □

We do not know whether, conversely, any object in $\langle\mathscr{C}\mathscr{I}\rangle$ is isomorphic in $\mathrm{KK}^{G\ltimes X}$ to a proper G-C^*-algebra. Since $A\in\langle\mathscr{C}\mathscr{I}\rangle$ implies $A\cong\mathsf{P}\otimes_X A$, this holds if and only if the source P of the Dirac morphism for $G\ltimes X$ has this property. Thus the question is whether we can find a Dirac morphism whose source is proper. This can be done for many groups. For instance, if G is almost connected with maximal compact subgroup K, then the cotangent bundle $T^*(G/K)$ always has a K-equivariant spin structure, so that its Dirac operator is defined. It is indeed a Dirac morphism for G by results of Kasparov [28]. This is where our terminology comes from. Generalising this construction to non-Hausdorff manifolds, one can construct concrete Dirac morphisms of a similar sort for totally disconnected groups with finite dimensional $\mathscr{E}G$ (see [19,29]).

8. Dual Dirac morphisms

Let $\Sigma\mathsf{N}\to\mathsf{P}\xrightarrow{\mathsf{D}}C_0(X)\to\mathsf{N}$ be a Dirac triangle.

Definition 8.1. We call $\eta\in\mathrm{KK}^{G\ltimes X}(C_0(X),\mathsf{P})$ a *dual Dirac morphism* for $G\ltimes X$ if $\eta\circ\mathsf{D}=\mathrm{id}_{\mathsf{P}}$. The composition $\gamma:=\mathsf{D}\eta\in\mathrm{KK}^{G\ltimes X}(C_0(X),C_0(X))$ is called a *γ-element* for $G\ltimes X$.

Kasparov's Dirac dual Dirac method is the main tool for proving injectivity and bijectivity of the Baum–Connes assembly map. The following theorem shows that a dual Dirac morphism in the above sense exists whenever the Dirac dual Dirac method in the usual sense applies. Our reformulation has the advantage that the Dirac morphism is fixed, so that we only have to find one piece of structure. This is quite useful for analysing the existence of a dual Dirac morphism (see [19]).

Theorem 8.2. *Let A be a $\mathbb{Z}/2$-graded $G\ltimes X$-C^*-algebra, $\alpha\in\mathrm{KK}^{G\ltimes X}(A,C_0(X))$ and $\beta\in\mathrm{KK}^{G\ltimes X}(C_0(X),A)$. If $\gamma:=\alpha\beta\in\mathrm{KK}^{G\ltimes X}(C_0(X),C_0(X))$ satisfies $p^*_{\mathscr{E}G}(\gamma)=1$ and A is proper, then there is a dual Dirac morphism for $G\ltimes X$. Moreover, γ is equal to the γ-element.*

If, in addition, A is trivially graded and $\beta\alpha=1$, then α and β themselves are Dirac and dual Dirac morphisms for G.

Proof. Let $\mathsf{D}\in\mathrm{KK}^{G\ltimes X}(\mathsf{P},C_0(X))$ be a Dirac morphism. Even if A is graded, the same argument as in the proof of Corollary 7.3 shows that $\mathsf{D}_*\in\mathrm{KK}^G(\mathsf{P}\otimes_X A,A)$ is invertible—provided A is proper. We claim that the composite morphism

$$\eta\colon C_0(X)\xrightarrow{\beta}A\xrightarrow{\mathsf{D}_*^{-1}}\mathsf{P}\otimes_X A\xrightarrow{\alpha_*}\mathsf{P}$$

is a dual Dirac morphism, that is, $\eta \circ \mathsf{D} = 1_{\mathsf{P}}$. We have $\mathsf{D} \circ \eta = \beta \circ \alpha = \gamma$ because exterior products are graded commutative. Since D is a weak equivalence, $p^*_{\mathscr{E}G}(\mathsf{D})$ is invertible. Since $1 = p^*_{\mathscr{E}G}(\gamma) = p^*_{\mathscr{E}G}(\eta\mathsf{D})$, we get $p^*_{\mathscr{E}G}(\eta) = p^*_{\mathscr{E}G}(\mathsf{D})^{-1}$. Therefore, $p^*_{\mathscr{E}G}(\eta\mathsf{D}) = 1 = p^*_{\mathscr{E}G}(1)$. The map

$$p^*_{\mathscr{E}G}\colon \mathrm{KK}^{G\ltimes X}(\mathsf{P}, \mathsf{P}) \to \mathrm{RKK}^{G\ltimes X}(\mathscr{E}G; \mathsf{P}, \mathsf{P})$$

is bijective by Theorem 7.1 because $\mathsf{P} \in \langle \mathscr{C}\mathscr{I} \rangle$. Hence $\eta\mathsf{D} = 1$.

If A is trivially graded, then $A \in \langle \mathscr{C}\mathscr{I} \rangle$ by Corollary 7.3. The morphisms α and β are weak equivalences because $\beta\alpha = 1$ and $\alpha\beta = \gamma$ are. This implies that α is a Dirac morphism and that β is a dual Dirac morphism. □

Theorem 8.3. *The following assertions are equivalent*:

8.3.1. *there is a dual Dirac morphism* ($\eta \in \mathrm{KK}^{G\ltimes X}(C_0(X), \mathsf{P})$ *with* $\eta\mathsf{D} = \mathrm{id}_{\mathsf{P}}$);
8.3.2. $\mathrm{KK}^{G\ltimes X}_*(\mathsf{N}, \mathsf{P}) = 0$ (*for all* $* \in \mathbb{Z}$);
8.3.3. *the natural map* $p^*_{\mathscr{E}G}\colon \mathrm{KK}^{G\ltimes X}_*(C_0(X), \mathsf{P}) \to \mathrm{RKK}^{G\ltimes X}_*(\mathscr{E}G; C_0(X), \mathsf{P})$ *is an isomorphism* (*for all* $* \in \mathbb{Z}$);
8.3.4. $\mathrm{KK}^{G\ltimes X}_*(A, B) = 0$ *for all* $A \in \mathscr{C}\mathscr{C}$, $B \in \langle \mathscr{C}\mathscr{I} \rangle$;
8.3.5. *the natural map* $\mathrm{KK}^{G\ltimes X}_*(A, B) \to \mathrm{RKK}^{G\ltimes X}_*(\mathscr{E}G; A, B)$ *is an isomorphism for all* $A \in \mathrm{KK}^{G\ltimes X}$, $B \in \langle \mathscr{C}\mathscr{I} \rangle$;
8.3.6. *there is an equivalence of triangulated categories* $\mathrm{KK}^{G\ltimes X} \cong \langle \mathscr{C}\mathscr{I} \rangle \times \mathscr{C}\mathscr{C}$.

Suppose these equivalent conditions to be satisfied and let

$$\gamma := \mathsf{D} \circ \eta \in \mathrm{KK}^{G\ltimes X}(C_0(X), C_0(X)).$$

Then $\gamma_A := \gamma \otimes_X A \in \mathrm{KK}^{G\ltimes X}(A, A)$ *is an idempotent for all* $A \in \mathrm{KK}^{G\ltimes X}$. *We have* $\gamma_A = 0$ *if and only if* $A \in \mathscr{C}\mathscr{C}$ *and* $\gamma_A = \mathrm{id}$ *if and only if* $A \in \langle \mathscr{C}\mathscr{I} \rangle$.

Proof. We often use the isomorphism $\mathrm{RKK}^{G\ltimes X}_*(\mathscr{E}G; A, B) \cong \mathrm{KK}^{G\ltimes X}_*(\mathsf{P} \otimes_X A, B)$ proven in Theorem 7.1. A long exact sequence argument shows that 8.3.2 and 8.3.3 are equivalent. Conditions 8.3.4 and 8.3.5 are two ways of expressing that objects of $\langle \mathscr{C}\mathscr{I} \rangle$ are $\mathscr{C}\mathscr{C}$-injective and hence equivalent. The implications 8.3.6⟹ 8.3.4⟹8.3.2 and 8.3.3⟹8.3.1 are trivial. It remains to prove that 8.3.1 implies 8.3.6. Along the way we show the additional assertions about γ (and part of the following corollary).

Since $\eta\mathsf{D} = \mathrm{id}_{\mathsf{P}}$, the map $\Sigma\mathsf{N} \to \mathsf{P}$ in the Dirac triangle vanishes. Hence Lemma 2.2 yields an isomorphism $C_0(X) \cong \mathsf{P} \oplus \mathsf{N}$ such that the maps $\mathsf{P} \to C_0(X) \to \mathsf{N}$ become the obvious ones. Any two choices for η differ by a morphism $\mathsf{N} \to \mathsf{P}$. Therefore, 8.3.2 implies that η is unique. We cannot use this so far because we still have to prove that 8.3.2 follows from 8.3.1. We may, however, choose η such that $\gamma = \mathsf{D}\eta$ is the orthogonal projection onto P that vanishes on N. We get a direct sum decomposition (in KK)

$$A \cong C_0(X) \otimes_X A \cong \mathsf{P} \otimes_X A \oplus \mathsf{N} \otimes_X A$$

such that $\mathsf{D} \otimes_X \mathrm{id}_A$ is the inclusion of the first summand and γ_A is the orthogonal projection onto $\mathsf{P} \otimes_X A$. Theorem 4.7 yields $\gamma_A = 1$ if and only if $A \in \langle \mathscr{C}\mathscr{I} \rangle$, and $\gamma_A = 0$ if and only if $A \in \mathscr{C}\mathscr{C}$. Since

$$\gamma_B \circ f = \gamma \otimes_X f = f \circ \gamma_A$$

for all $f \in \mathrm{KK}^{G\ltimes X}(A, B)$, there are no non-zero morphisms between $\mathscr{CC}$ and $\langle \mathscr{CI} \rangle$. The above decomposition of A respects suspensions and exact triangles because the tensor product functors $\mathsf{P}\otimes_X \sqcup$ and $\mathsf{N}\otimes_X \sqcup$ are triangulated. Hence we get an equivalence of triangulated categories $\langle \mathscr{CI} \rangle \times \mathscr{CC} \cong \mathrm{KK}^G$. □

Corollary 8.4. *Fix a Dirac morphism* $\mathsf{D} \in \mathrm{KK}^{G\ltimes X}(\mathsf{P}, C_0(X))$. *Then the dual Dirac morphism and the* γ*-element are unique if they exist.*

A morphism $\eta \in \mathrm{KK}^{G\ltimes X}(C_0(X), \mathsf{P})$ *is a dual Dirac morphism if and only if* $p^*_{\mathscr{E}G}(\eta)$ is inverse to $p^*_{\mathscr{E}G}(\mathsf{D})$ *if and only if* $p^*_{\mathscr{E}G}(\mathsf{D}\eta) = 1$.

Proof. We have already shown the uniqueness of η and hence of γ during the proof of Theorem 8.3. The map $p^*_{\mathscr{E}G}\colon \mathrm{KK}^{G\ltimes X}_*(\mathsf{P}, \mathsf{P}) \to \mathrm{RKK}^{G\ltimes X}_*(\mathscr{E}G; \mathsf{P}, \mathsf{P})$ is an isomorphism by Theorem 7.1. Hence $\eta\mathsf{D} = \mathrm{id}$ if and only if $p^*_{\mathscr{E}G}(\eta\mathsf{D}) = \mathrm{id}$. Since D is a weak equivalence, $p^*_{\mathscr{E}G}(\mathsf{D})$ is an isomorphism. Hence there is no difference between left, right and two-sided inverses for $p^*_{\mathscr{E}G}(\mathsf{D})$. □

Suppose now that a dual Dirac morphism exists. It induces a canonical section for the map $\mathsf{P}\otimes_X A \to A$. Hence the natural transformation $\mathbb{L}F(A) \to F(A)$ for a covariant functor F is naturally split injective. Similarly, the natural transformation $F(A) \to F(\mathsf{P}\otimes_X A)$ is naturally split surjective for a contravariant functor F.

It is clear from Theorem 8.3 that $\gamma = 1$ if and only if $\mathscr{CC} = 0$. In this case, $\mathbb{L}F(A) \cong F(A)$ for any functor on $\mathrm{KK}^{G\ltimes X}$, that is, any functor F satisfies the analogue of the Baum–Connes conjecture. Higson and Kasparov show in [23] that all groups with the Haagerup property and in particular all amenable groups have a dual Dirac element and satisfy $\gamma = 1$. Tu generalises their argument to groupoids that satisfy an analogue of the Haagerup property in [45]. In particular, this applies to the special groupoids $G\ltimes X$. We get:

Theorem 8.5. *Suppose that the groupoid* $G\ltimes X$ *is amenable or, more generally, acts continuously and isometrically on a continuous field of affine Euclidean spaces over X. Then weakly contractible objects of* $\mathrm{KK}^{G\ltimes X}$ *are already isomorphic to* 0 *and weak equivalences are isomorphisms. The assembly map is an isomorphism for any functor defined on* $\mathrm{KK}^{G\ltimes X}$.

8.1. Approximate dual Dirac morphisms

In some cases of interest, for instance, for groups acting on bolic spaces, one cannot construct an actual dual Dirac morphism but only approximations to one.

Definition 8.6. Suppose that for each G-compact proper G-space Y there is $\eta_Y \in \mathrm{KK}^G(\star, \mathsf{P})$ such that $p^*_Y(\mathsf{D}\circ\eta_Y) = 1 \in \mathrm{RKK}^G(Y; \star, \star)$. Then we call the family (η_Y) an *approximate dual Dirac morphism* for G. We also let $\gamma_Y := \mathsf{D}\circ\eta_Y$.

Lemma 8.7. *Suppose that for each G-compact proper G-space Y there are a possibly* $\mathbb{Z}/2$*-graded, proper G-*C^**-algebra* A_Y *and* $\alpha_Y \in \mathrm{KK}^G(A_Y, \star)$, $\beta_Y \in \mathrm{KK}^G(\star, A_Y)$ *such that* $p^*_Y(\alpha_Y\circ\beta_Y) = 1 \in \mathrm{RKK}^G(Y; \star, \star)$. *Then G has an approximate dual Dirac morphism with* $\gamma_Y = \alpha_Y\beta_Y$.

Proof. Proceed as in the proof of Theorem 8.2. □

The situation of Lemma 8.7 occurs in [30]. It follows that a discrete group G has an approximate dual Dirac morphism if it acts properly and by isometries on a weakly bolic, weakly geodesic metric space. Clearly, G has an approximate dual Dirac morphism once it has a dual Dirac morphism. The converse holds if G does not have too many compact subgroups:

Proposition 8.8. *Suppose that there exist finitely many compact subgroups of G such that any compact subgroup is subconjugate to one of them. If G has an approximate dual Dirac morphism, then it already has a dual Dirac morphism.*

Proof. Let $\mathsf{D} \in \mathrm{KK}^G(\mathsf{P}, \star)$ be a Dirac morphism for G. Let S be a finite set of compact subgroups such that any other compact subgroup is subconjugate to one of them. Let Y be the disjoint union of the spaces G/H for $H \in S$. By hypothesis, there is $\eta_Y \in \mathrm{KK}^G(\star, \mathsf{P})$ such that $\gamma_Y := \mathsf{D} \circ \eta_Y$ satisfies $p_Y^*(\gamma_Y) = 1$. This means that $p_{G/H}^*(\gamma_Y) = 1$ for all $H \in S$. By (14), this is equivalent to $\mathrm{Res}_G^H(\gamma_Y) = 1$ for all $H \in S$. By hypothesis, any compact subgroup of G is subconjugate to one in S. Thus γ_Y is a weak equivalence. Since D is a weak equivalence as well, it follows that η_Y is a weak equivalence. Hence the composition $\eta_Y \circ \mathsf{D} \in \mathrm{KK}^G(\mathsf{P}, \mathsf{P})$ is a weak equivalence. Since $\mathsf{P} \in \langle \mathscr{C}\mathscr{I} \rangle$, it is projective with respect to weak equivalences by Proposition 4.4. Hence $\eta_Y \circ \mathsf{D}$ is invertible; $\eta := (\eta_Y \circ \mathsf{D})^{-1}\eta_Y \in \mathrm{KK}^G(\star, \mathsf{P})$ is the desired dual Dirac morphism for G. □

It is unclear whether the condition on compact subgroups in Proposition 8.8 can be removed. Our next goal is a weakening of Theorem 8.3.5, which still holds if G has an approximate dual Dirac morphism and which is used in [19].

Lemma 8.9. *Let* $\mathsf{D} \in \mathrm{KK}^G(\mathsf{P}, \star)$ *be a Dirac morphism and let* $\alpha \in \mathrm{KK}^G(\star, \mathsf{P})$. *Define* $\beta := \mathsf{D} \circ \alpha \in \mathrm{KK}^G(\star, \star)$ *and* $\beta_A := \beta \otimes \mathrm{id}_A \in \mathrm{KK}^G(A, A)$ *for all* $A \in \mathrm{KK}^G$. *Then* $\alpha \circ \mathsf{D} = \beta_{\mathsf{P}}$. *For* $A, B \in \mathrm{KK}^G$, *the composites*

$$\alpha^*\mathsf{D}^*\colon \mathrm{KK}^G(A, B) \to \mathrm{KK}^G(\mathsf{P} \otimes A, B) \to \mathrm{KK}^G(A, B),$$
$$\mathsf{D}^*\alpha^*\colon \mathrm{KK}^G(\mathsf{P} \otimes A, B) \to \mathrm{KK}^G(A, B) \to \mathrm{KK}^G(\mathsf{P} \otimes A, B),$$

are both given by $f \mapsto \beta_B \circ f$.

Proof. Since D is a weak equivalence, the map

$$\mathsf{D}_*\colon \mathrm{KK}^G(\mathsf{P}, \mathsf{P}) \to \mathrm{KK}^G(\mathsf{P}, \star)$$

is an isomorphism. It maps both $\alpha \circ \mathsf{D}$ and β_{P} to $\beta \circ \mathsf{D} = \mathsf{D} \otimes \beta$. Hence $\alpha \circ \mathsf{D} = \beta_{\mathsf{P}}$. The second assertion now follows from $\beta_B \circ f = \beta \otimes f = f \circ \beta_{A'}$ for all $A', B \in \mathrm{KK}^G$, $f \in \mathrm{KK}^G(A', B)$ (applied to $A' = A$ and $A' = \mathsf{P} \otimes A$). □

Lemma 8.10. *Let* $\beta \in \mathrm{KK}^G(\star, \star)$, *let Y be a locally compact G-space and let A be a* $G \ltimes Y$-C^*-*algebra. If* $p_Y^*(\beta) = 1$, *then* $\beta_A = 1$.

Proof. We have a natural isomorphism $B \otimes A \cong p_Y^*(B) \otimes_Y A$ for all A and B. Hence $\beta_A := \beta \otimes \mathrm{id}_A = p_Y^*(\beta) \otimes_Y \mathrm{id}_A = 1 \otimes_Y \mathrm{id}_A = 1$. □

For a finite set of compact subgroups S, let $\mathscr{CI}(S) \subseteq \mathscr{CI}$ be the class of G-C^*-algebras that are KK^G-equivalent to $\mathrm{Ind}_H^G(A)$ for some $H \in S$ and some $A \in \mathrm{KK}^H$. Let $\langle \mathscr{CI}(S) \rangle$ be the localising subcategory generated by $\mathscr{CI}(S)$. These subcategories form a directed set of localising subcategories. Let $\mathscr{CIF}$ be their union, that is, $A \in \mathscr{CIF}$ if and only if $A \in \mathscr{CI}(S)$ for some *finite* set of compact subgroups S. This is a thick, triangulated subcategory of KK^G, but it need not be localising: it is only closed under countable direct sums if all summands lie in the same category $\mathscr{CI}(S)$ for some S. The hypothesis of Proposition 8.8 ensures that $\mathscr{CIF} = \mathscr{CI}(S)$ for some S, so that $\mathscr{CIF}$ is localising as well. Thus $\mathscr{CIF} = \langle \mathscr{CI} \rangle$ in this case. In general, $\mathscr{CI} \subseteq \mathscr{CIF} \subseteq \langle \mathscr{CI} \rangle$ and both containments may be strict.

Proposition 8.11. *If G has an approximate dual Dirac morphism, then the map*

$$p_{\mathscr{E}G}^*: \mathrm{KK}_*^G(A, B) \to \mathrm{RKK}_*^G(\mathscr{E}G; A, B) \tag{28}$$

is an isomorphism for all $B \in \mathscr{CIF}$, $A \in \mathrm{KK}^G$.

Proof. Fix $B \in \mathscr{CIF}$ and let S be a finite set of compact subgroups of G such that $B \in \langle \mathscr{CI}(S) \rangle$. Let Y be the disjoint union of the spaces G/H for $H \in S$. This is a G-compact proper G-space. Since G has an approximate dual Dirac morphism, there is $\eta_Y \in \mathrm{KK}^G(\star, \mathsf{P})$ such that $\gamma := \mathsf{D}\eta_Y$ satisfies $p_Y^*(\gamma) = 1$. This yields $\gamma_{B'} = 1$ in $\mathrm{KK}^G(B', B')$ for $B' \in \mathscr{CI}(S)$ by Lemma 8.10. In particular, $\gamma_{B'}$ is invertible if $B' \in \mathscr{CI}(S)$. The class of $B' \in \mathrm{KK}^G$ for which $\gamma_{B'}$ is invertible is localising by the five lemma and the functoriality of direct sums. Hence $\gamma_{B'}$ is invertible for all $B' \in \langle \mathscr{CI}(S) \rangle$ and, especially, for our chosen B. By Lemma 8.9, $\mathsf{D}^*: \mathrm{KK}^G(A, B) \to \mathrm{KK}^G(\mathsf{P} \otimes A, B)$ is invertible because both $\mathsf{D}^*\eta_Y^*$ and $\eta_Y^*\mathsf{D}^*$ are equal to invertible maps of the form $(\gamma_B)_*$.Theorem 7.1 allows us to replace D^* by the map $p_{\mathscr{E}G}^*$ in (28). □

9. The strong Baum–Connes conjecture

Definition 9.1. We say that G satisfies the *strong Baum–Connes conjecture with coefficients* $A \in \mathrm{KK}^G$ if the assembly map $G \ltimes_{\mathrm{r}}^{\mathbb{L}} A \to G \ltimes_{\mathrm{r}} A$ is a KK-equivalence.

The strong Baum–Connes conjecture implies that the assembly map is an isomorphism for any functor defined on KK. In particular, this covers K-theory, K-homology and local cyclic homology and cohomology of the reduced crossed product.

Suppose that G has a dual Dirac morphism and resulting γ-element γ. Applying descent, we get $G \ltimes_{\mathrm{r}} \gamma_A \in \mathrm{KK}(G \ltimes_{\mathrm{r}} A, G \ltimes_{\mathrm{r}} A)$. The strong Baum–Connes conjecture amounts to the assertion that $G \ltimes_{\mathrm{r}} \gamma_A = 1$. This is known to be false for quite some time if $A = \star$ and G is a discrete subgroup of finite covolume in $\mathrm{Sp}(n, 1)$ [41], even though the Baum–Connes conjecture itself holds in this case by [26].

For general G, the Baum–Connes conjecture with coefficients A holds if and only if $G \ltimes_{\mathrm{r}}^{\mathrm{Obs}} A$ is K-contractible as in Definition 6.5, whereas the *strong* Baum–Connes conjecture with coefficients A holds if and only if $G \ltimes_{\mathrm{r}}^{\mathrm{Obs}} A \cong 0$ in KK. These two assertions are equivalent if $G \ltimes_{\mathrm{r}}^{\mathrm{Obs}} A$ belongs to the bootstrap category $\langle \star \rangle$. A sufficient condition for $G \ltimes_{\mathrm{r}}^{\mathrm{Obs}} A \in \langle \star \rangle$ is that both $G \ltimes_{\mathrm{r}}^{\mathbb{L}} A$ and $G \ltimes_{\mathrm{r}} A$ belong to the bootstrap category.

Now we use the notion of smooth compact subgroup introduced in Section 3.1. If G is discrete, any finite subgroup of G is smooth. Let $\mathscr{CI}_1 \subseteq \mathscr{CI}$ be the set of all G-C^*-algebras of the form $C_0(G/H)$

for smooth, compact subgroups $H \subseteq G$. This is a variant of the subcategory $\mathscr{CI}_0 \subseteq \mathscr{CI}$ introduced in Section 6.1.

The following lemma is motivated by work of Chabert and Echterhoff (see, for instance, [16, Lemma 4.20]).

Proposition 9.2. *The localising category $\langle \mathscr{CI}_1 \rangle$ generated by $\mathscr{CI}_1$ contains $\langle \mathscr{CI}_0 \rangle$ and hence also contains the source of the Dirac morphism.*

Proof. Our existence proof for Dirac morphisms shows that $\mathsf{P} \in \langle \mathscr{CI}_0 \rangle$. If the generators R_H defined in (24) belong to $\langle \mathscr{CI}_1 \rangle$, then $\langle \mathscr{CI}_0 \rangle \subseteq \langle \mathscr{CI}_1 \rangle$, and we are done. Let $H \subseteq G$ be a large compact subgroup, $U := U_H$ and $V := U/H$. Recall that $\mathsf{R}_H = \mathrm{Ind}_H^G C_0(V^7)$. Since U is almost connected, there is a compact normal subgroup $N \subseteq U$ such that U/N is a Lie group. By maximality, $H = NH \supseteq N$. The quotient H/N is a compact Lie group. It acts linearly on the $\mathbb{R}$-vector space V^7. One can show that V^7 is an H/N-CW-complex; this is a special case of [24]. Hence $C_0(V^7)$ belongs to the localising subcategory of $\mathrm{KK}^{H/N}$ generated by $C_0(H/K)$ with $N \subseteq K \subseteq H$. Hence R_H belongs to the localising subcategory of KK^G generated by $\mathrm{Ind}_H^G C_0(H/K) \cong C_0(G/K)$ for such K. Since N is a strongly smooth compact subgroup of G contained in each K, the assertion follows. □

Theorem 9.3. *For any $A \in \mathrm{KK}^G$, the C^*-algebra $G \ltimes_{\mathrm{r}}^{\mathbb{L}} A$ belongs to the localising subcategory of* KK *generated by $H \ltimes A$ for smooth compact subgroups $H \subseteq G$. In particular, $G \ltimes_{\mathrm{r}}^{\mathbb{L}} A \in \langle \star \rangle$ once $H \ltimes A \in \langle \star \rangle$ for all smooth compact subgroups H.*

If $H \ltimes A \cong 0$ in KK *for all smooth compact subgroups H, then $G \ltimes_{\mathrm{r}}^{\mathbb{L}} A \cong 0$ as well. If $f \in \mathrm{KK}^G(A, B)$ induces* KK*-equivalences $H \ltimes A \cong H \ltimes B$ for all smooth compact subgroups H, then it induces a* KK*-equivalence $G \ltimes_{\mathrm{r}}^{\mathbb{L}} A \cong G \ltimes_{\mathrm{r}}^{\mathbb{L}} B$.*

If $H \ltimes A$ is K*-contractible for all smooth compact subgroups H, so is $G \ltimes_{\mathrm{r}}^{\mathbb{L}} A$. If $f \in \mathrm{KK}^G(A, B)$ induces a* K*-equivalence $H \ltimes A \to H \ltimes B$ for all smooth compact subgroups H, then it induces a* K*-equivalence $G \ltimes_{\mathrm{r}}^{\mathbb{L}} A \to G \ltimes_{\mathrm{r}}^{\mathbb{L}} B$.*

Proof. Proposition 9.2 implies that $G \ltimes_{\mathrm{r}}^{\mathbb{L}} A \cong G \ltimes_{\mathrm{r}} (\mathsf{P} \otimes A)$ belongs to the localising subcategory of KK generated by $G \ltimes_{\mathrm{r}} C_0(G/H, A)$ for smooth compact subgroups $H \subseteq G$. Eq. (9) yields $G \ltimes_{\mathrm{r}} C_0(G/H, A) \sim_M H \ltimes A$. This implies the criteria for $G \ltimes_{\mathrm{r}}^{\mathbb{L}} A$ to be in $\langle \star \rangle$, to be KK-contractible and to be K-contractible because all these conditions define localising subcategories of KK. The assertions about morphisms follow if we replace f by its mapping cone. □

The following corollary is originally due to Tu [45]. It applies to amenable groups by Theorem 8.5.

Corollary 9.4. *Let G be a locally compact group, let X be a G-space, and let $A \in \mathrm{KK}^{G \ltimes X}$. Suppose that $G \ltimes X$ has a dual Dirac morphism with $\gamma = 1$ or, more generally, $G \ltimes_{\mathrm{r}} \gamma_A = 1 \in \mathrm{KK}(G \ltimes_{\mathrm{r}} A, G \ltimes_{\mathrm{r}} A)$. If $H \ltimes A \in \langle \star \rangle$ for all smooth compact subgroups H, then $G \ltimes_{\mathrm{r}} A \in \langle \star \rangle$.*

Proof. If $\gamma = 1 \in \mathrm{KK}^{G \ltimes X}(C_0(X), C_0(X))$, then $(G \ltimes X) \ltimes_{\mathrm{r}} \gamma_A = 1$. This implies $(G \ltimes X) \ltimes_{\mathrm{r}}^{\mathbb{L}} A \cong (G \ltimes X) \ltimes_{\mathrm{r}} A$ in KK. Now use Lemma 5.1 to get rid of the space X and apply Theorem 9.3. □

Theorem 9.3 describes other interesting localising subcategories of KK^G on which $\mathrm{K}_*^{\mathrm{top}}(G, \sqcup)$ vanishes. Hence it gives a variant of the rigidity formulation of the Baum–Connes conjecture. Namely, G

satisfies the Baum–Connes conjecture with arbitrary coefficients if and only if $\mathrm{K}_*(G \ltimes_{\mathrm{r}} A) \cong 0$ whenever $A \in \mathrm{KK}^G$ satisfies $\mathrm{K}_*(H \ltimes_{\mathrm{r}} A) \cong 0$ for all smooth compact subgroups $H \subseteq G$.

Proposition 9.5. *If the G-C^*-algebra A is commutative (or just type I), it then $G \ltimes_{\mathrm{r}}^{\mathbb{L}} A \in \langle \star \rangle$. In particular, $G \ltimes_{\mathrm{r}}^{\mathbb{L}} \star \in \langle \star \rangle$. Suppose $G \ltimes_{\mathrm{r}}^{\mathbb{L}} A \in \langle \star \rangle$ (for instance, $A = \star$). Then the strong Baum–Connes conjecture with coefficients A holds if and only if $G \ltimes_{\mathrm{r}} A \in \langle \star \rangle$ and the usual Baum–Connes conjecture with coefficients A holds.*

Proof. If A is a type I C^*-algebra and H is compact, then $H \ltimes A$ is a type I C^*-algebra as well (this follows easily from [42, Theorem 6.1]). Therefore, it belongs to $\langle \star \rangle$ (see [9, 22.3.5]). Thus $G \ltimes_{\mathrm{r}}^{\mathbb{L}} A \in \langle \star \rangle$ by Theorem 9.3. The strong Baum–Connes conjecture is stronger than the Baum–Connes conjecture and implies that $G \ltimes_{\mathrm{r}} A \in \langle \star \rangle$ once $G \ltimes_{\mathrm{r}}^{\mathbb{L}} A \in \langle \star \rangle$. The converse also holds because a K-equivalence between objects of $\langle \star \rangle$ is already a KK-equivalence. □

Therefore, if we already know that $C_{\mathrm{r}}^*(G) \in \langle \star \rangle$, then the strong and the usual Baum–Connes conjecture with trivial coefficients are equivalent. Chabert et al. show in [16] that $C_{\mathrm{r}}^*(G) \in \langle \star \rangle$ if G is almost connected or a linear algebraic group over the p-adic numbers or over the adele ring of a number field. The Baum–Connes conjecture with trivial coefficients for these groups is also known, see [14,16]. Hence these groups satisfy the strong Baum–Connes conjecture with trivial coefficients.

10. Permanence properties of the (strong) Baum–Connes conjecture

Let $\mathscr{T}$ and $\mathscr{T}'$ be triangulated categories, let $F: \mathscr{T} \to \mathscr{T}'$ be a triangulated functor, and let $(\mathscr{N}, \mathscr{P})$ and $(\mathscr{N}', \mathscr{P}')$ be complementary pairs of localising subcategories in $\mathscr{T}$ and $\mathscr{T}'$, respectively. Suppose $F(\mathscr{P}) \subseteq \mathscr{P}'$. Then $\mathbb{L}(F' \circ F) \cong \mathbb{L}F' \circ \mathbb{L}F$ up to isomorphism for any covariant functor F' defined on $\mathscr{T}'$. This trivial observation has lots of applications. When applied to restriction and induction functors, partial crossed product functors and the complexification functor, we get permanence properties of the (strong) Baum–Connes conjecture. We remark that Lemma 5.1 is another such result that logically belongs into this section.

10.1. Restriction and induction

Proposition 10.1. *Let $H \subseteq G$ be a closed subgroup. The functors*

$$\mathrm{Res}_G^H: \mathrm{KK}^{G \ltimes X} \to \mathrm{KK}^{H \ltimes X} \quad \textit{and} \quad \mathrm{Ind}_H^G: \mathrm{KK}^{H \ltimes X} \to \mathrm{KK}^{G \ltimes X}$$

preserve weak contractibility and weak equivalences and map $\langle \mathscr{CI} \rangle$ to $\langle \mathscr{CI} \rangle$. Therefore, Res_G^H maps a Dirac triangle for $G \ltimes X$ to a Dirac triangle for $H \ltimes X$ and Ind_H^G maps a Dirac triangle for $H \ltimes X$ to a Dirac triangle for $G \ltimes X$.

Proof. Restriction and induction in stages yield $\mathrm{Res}_G^H(\mathscr{CC}) \subseteq \mathscr{CC}$ and $\mathrm{Ind}_H^G(\mathscr{CI}) \subseteq \mathscr{CI}$ and hence $\mathrm{Ind}_H^G(\langle \mathscr{CI} \rangle) \subseteq \langle \mathscr{CI} \rangle$. To prove $\mathrm{Res}_G^H(\langle \mathscr{CI} \rangle) \subseteq \langle \mathscr{CI} \rangle$, it suffices to show $\mathrm{Res}_G^H(\mathscr{CI}) \subseteq \langle \mathscr{CI} \rangle$ because Res_G^H is triangulated and commutes with direct sums. It is clear that Res_G^H maps compactly induced G-C^*-algebras to proper H-C^*-algebras. This implies the assertion by Corollary 7.3. As a consequence, Res_G^H maps a Dirac triangle for $G \ltimes X$ to one for $H \ltimes X$.

Next, we prove that $\mathrm{Ind}_H^G(\mathscr{CC}) \subseteq \mathscr{CC}$. Let $\mathsf{D} \in \mathrm{KK}^{G\ltimes X}(\mathsf{P}, C_0(X))$ be a Dirac morphism for $G\ltimes X$. We have just seen that $\mathrm{Res}_G^H \mathsf{D}$ is a Dirac morphism for $H\ltimes X$. Let $A \in \mathrm{KK}^{H\ltimes X}$. Eq. (16) yields

$$\mathsf{P}\otimes_X \mathrm{Ind}_H^G A \approx \mathrm{Ind}_H^G(\mathrm{Res}_G^H \mathsf{P}\otimes_X A).$$

By Theorem 4.7, $\mathrm{Ind}_H^G A \in \mathscr{CC}$ is equivalent to $\mathsf{P}\otimes_X \mathrm{Ind}_H^G A \cong 0$ and $A \in \mathscr{CC}$ is equivalent to $\mathrm{Res}_G^H \mathsf{P}\otimes_X A \cong 0$. Thus $\mathrm{Ind}_H^G(\mathscr{CC}) \subseteq \mathscr{CC}$. As a consequence, Ind_H^G maps a Dirac triangle for $H\ltimes X$ to one for $G\ltimes X$. □

It follows immediately from Proposition 10.1 that

$$\mathbb{L}(F\circ \mathrm{Ind}_H^G) \cong (\mathbb{L}F)\circ \mathrm{Ind}_H^G, \quad \mathrm{Obs}(F\circ \mathrm{Ind}_H^G) \cong (\mathrm{Obs}\, F)\circ \mathrm{Ind}_H^G,$$
$$\mathbb{L}(F\circ \mathrm{Res}_G^H) \cong (\mathbb{L}F)\circ \mathrm{Res}_G^H, \quad \mathrm{Obs}(F\circ \mathrm{Res}_G^H) \cong (\mathrm{Obs}\, F)\circ \mathrm{Res}_G^H.$$

Since $G\ltimes_{\mathrm{r}} \mathrm{Ind}_H^G A \sim_M H\ltimes_{\mathrm{r}} A$ by (9), this yields natural KK-equivalences

$$G\ltimes_{\mathrm{r}}^{\mathbb{L}} \mathrm{Ind}_H^G A \cong H\ltimes_{\mathrm{r}}^{\mathbb{L}} A, \quad G\ltimes_{\mathrm{r}}^{\mathrm{Obs}} \mathrm{Ind}_H^G A \cong H\ltimes_{\mathrm{r}}^{\mathrm{Obs}} A. \tag{29}$$

Hence the (strong) Baum–Connes conjectures for $G\ltimes_{\mathrm{r}} \mathrm{Ind}_H^G A$ and $H\ltimes_{\mathrm{r}} A$ are equivalent. As a result, the (strong) Baum–Connes conjecture with coefficients and the (strong) Baum–Connes conjecture with commutative coefficients are both hereditary for subgroups. For the usual Baum–Connes conjecture, this is due to Chabert and Echterhoff [11].

10.2. Full and reduced crossed products and functoriality

Let $\phi: G_1 \to G_2$ be a continuous group homomorphism. It induces a functor $\phi^*: \mathrm{KK}^{G_2} \to \mathrm{KK}^{G_1}$. Of course, $\phi^*(\star) = \star$. If ϕ is open, then the universal property of full crossed products yields a natural transformation

$$\phi_*: G_1\ltimes \phi^*(A) \to G_2\ltimes A \tag{30}$$

for $A \in \mathrm{KK}^{G_2}$ (if ϕ is not open, we only get a map to the multiplier algebra of $G_2\ltimes A$). There is no analogue of (30) for reduced crossed products. For instance, the homomorphism from G to the trivial group induces a homomorphism $C_{\mathrm{r}}^*(G) \to C_{\mathrm{r}}^*(\{1\})$ if and only if G is amenable. Nevertheless, $\mathrm{K}^{\mathrm{top}}(G)$ has the same functoriality as full crossed products. We can reprove this easily in our setup.

Theorem 10.2. *The natural map $G\ltimes A \to G\ltimes_{\mathrm{r}} A$ is a* KK*-equivalence for $A \in \langle \mathscr{CI}\rangle$. Hence $G\ltimes^{\mathbb{L}} A \cong G\ltimes_{\mathrm{r}}^{\mathbb{L}} A$ (in* KK*) for any $A \in \mathrm{KK}^G$.*

Proof. Since full and reduced crossed products agree for compact groups, (9) yields that the map $G\ltimes A \to G\ltimes_{\mathrm{r}} A$ is an isomorphism in KK for $A \in \mathscr{CI}$. Since both crossed products are triangulated functors that commute with direct sums, this extends from $\mathscr{CI}$ to $\langle \mathscr{CI}\rangle$. This implies the second statement because the localisations only see $\langle \mathscr{CI}\rangle$. □

Corollary 10.3. *There exists a natural map $\phi_*: G_1\ltimes_{\mathrm{r}}^{\mathbb{L}} \phi^*(A) \to G_2\ltimes_{\mathrm{r}}^{\mathbb{L}} A$ for any open, continuous group homomorphism $\phi: G_1 \to G_2$ and any $A \in \mathrm{KK}^{G_2}$.*

Proof. Let $\tilde{A} \to A$ be a $\mathscr{CI}$-simplicial approximation in KK^{G_2}, so that $G_2 \ltimes \tilde{A} \cong G_2 \ltimes^{\mathbb{L}} A$. Since ϕ maps compact subgroups in G_1 to compact subgroups in G_2, the functor $\phi^*\colon \mathrm{KK}^{G_2} \to \mathrm{KK}^{G_1}$ preserves weak equivalences. Hence $\phi^*(\tilde{A}) \to \phi^*(A)$ is a weak equivalence in KK^{G_1}. As such it induces an isomorphism on $\mathbb{L}F$ for any functor F. Theorem 10.2 and (30) yield canonical maps

$$\begin{aligned} G_1 \ltimes_{\mathrm{r}}^{\mathbb{L}} \phi^*(A) &\cong G_1 \ltimes^{\mathbb{L}} \phi^*(A) \cong G_1 \ltimes^{\mathbb{L}} \phi^*(\tilde{A}) \\ &\to G_1 \ltimes \phi^*(\tilde{A}) \to G_2 \ltimes \tilde{A} \cong G_2 \ltimes^{\mathbb{L}} A \cong G_2 \ltimes_{\mathrm{r}}^{\mathbb{L}} A. \qquad \square \end{aligned}$$

10.3. Unions of open subgroups

Let $G = \cup G_n$ be a union of a sequence of open subgroups. For instance, adelic groups are of this form. Then $G \ltimes_{\mathrm{r}} A \cong \varinjlim G_n \ltimes_{\mathrm{r}} A$ for any $A \in \mathrm{KK}^G$. Since restriction to $G_n \subseteq G$ is a completely positive map $G \ltimes_{\mathrm{r}} A \to G_n \ltimes_{\mathrm{r}} A$, the inductive system $(G_n \ltimes_{\mathrm{r}} A)_{n \in \mathbb{N}}$ is admissible. Hence we can replace the direct limit by the homotopy direct limit (see Section 2.4).

Let $\Sigma \mathsf{N} \to \mathsf{P} \xrightarrow{\mathsf{D}} \star \to \mathsf{N}$ be a Dirac triangle for G. By Proposition 10.1, the functor $\mathrm{Res}_G^{G_n}$ maps this to a Dirac triangle in KK^{G_n}. Hence

$$G_n \ltimes_{\mathrm{r}} (\mathsf{P} \otimes A) \cong G_n \ltimes_{\mathrm{r}}^{\mathbb{L}} A, \quad G_n \ltimes_{\mathrm{r}} (\mathsf{N} \otimes A) \cong G_n \ltimes_{\mathrm{r}}^{\mathrm{Obs}} A.$$

Taking limits, we obtain

$$G \ltimes_{\mathrm{r}}^{\mathbb{L}} A \cong \text{ho-}\varinjlim G_n \ltimes_{\mathrm{r}}^{\mathbb{L}} A, \quad G \ltimes_{\mathrm{r}}^{\mathrm{Obs}} A \cong \text{ho-}\varinjlim G_n \ltimes_{\mathrm{r}}^{\mathrm{Obs}} A. \tag{31}$$

We have omitted restriction functors from our notation to avoid clutter. The following result is due to Baum et al. [7] for the usual Baum–Connes conjecture.

Theorem 10.4. *If the groups G_n satisfy the (strong) Baum–Connes conjecture with coefficients A for all $n \in \mathbb{N}$, then so does G.*

Proof. Recall that G satisfies the Baum-Conjecture (or the strong Baum–Connes conjecture) with coefficients A if and only if $G \ltimes_{\mathrm{r}}^{\mathrm{Obs}} A$ is K-contractible (or KK-contractible). Since the category of K-contractible C^*-algebras is localising, it is closed under homotopy direct limits. Hence the assertions follow from (31). $\square$

10.4. Direct products of groups

Let G_1 and G_2 be locally compact groups and let $G := G_1 \times G_2$. Let $\mathsf{D}_j \in \mathrm{KK}^{G_j}(\mathsf{P}_j, \star)$ be Dirac morphisms for the factors. Then $\mathsf{D}_1 \otimes \mathsf{D}_2 \in \mathrm{KK}^{G_1 \times G_2}(\mathsf{P}_1 \otimes \mathsf{P}_2, \star)$ is a Dirac morphism for $G_1 \times G_2$ because

$$\mathscr{CI}(G_1) \otimes \mathscr{CI}(G_2) \subseteq \mathscr{CI}(G_1 \times G_2) \quad \text{and} \quad \mathscr{CC}(G_1) \otimes \mathscr{CC}(G_2) \subseteq \mathscr{CC}(G_1 \times G_2).$$

Let $A_j \in \mathrm{KK}^{G_j}$ for $j = 1, 2$ and put $A := A_1 \otimes A_2 \in \mathrm{KK}^G$. We have a natural isomorphism

$$G \ltimes_{\mathrm{r}} A \approx (G_1 \ltimes_{\mathrm{r}} A_1) \otimes (G_2 \ltimes_{\mathrm{r}} A_2)$$

(because we use minimal C^*-tensor products) and hence

$$G \ltimes_{\mathrm{r}}^{\mathbb{L}} A \cong G \ltimes_{\mathrm{r}} (A_1 \otimes \mathsf{P}_1) \otimes (A_2 \otimes \mathsf{P}_2)$$
$$\approx (G_1 \ltimes_{\mathrm{r}} A_1 \otimes \mathsf{P}_1) \otimes (G_2 \ltimes_{\mathrm{r}} A_2 \otimes \mathsf{P}_2) \cong (G_1 \ltimes_{\mathrm{r}}^{\mathbb{L}} A_1) \otimes (G_2 \ltimes_{\mathrm{r}}^{\mathbb{L}} A_2).$$

Furthermore, the assembly map $G \ltimes_{\mathrm{r}}^{\mathbb{L}} A \to G \ltimes_{\mathrm{r}} A$ is the exterior tensor product of the assembly maps $G_j \ltimes_{\mathrm{r}}^{\mathbb{L}} A_j \to G_j \ltimes_{\mathrm{r}} A_j$ for the factors.

There are, of course, similar isomorphisms for $G \ltimes_{\mathrm{r}}^{\mathrm{Obs}} A$. Therefore, if the strong Baum–Connes conjecture holds for $G_1 \ltimes_{\mathrm{r}} A_1$ and $G_2 \ltimes_{\mathrm{r}} A_2$, then also for $G \ltimes_{\mathrm{r}} A$. The corresponding assertion about the usual Baum–Connes conjecture needs further hypotheses (see [16]) because we cannot always compute the K-theory of a tensor product by the Künneth Formula. We can formulate this as

$$(G_1 \ltimes_{\mathrm{r}}^{\mathrm{Obs}} A_1) \otimes (G_2 \ltimes_{\mathrm{r}}^{\mathrm{Obs}} A_2) \cong (G_1 \ltimes_{\mathrm{r}}^{\mathrm{Obs}} A_1) \otimes^{\mathbb{L}} (G_2 \ltimes_{\mathrm{r}}^{\mathrm{Obs}} A_2),$$

using the localised tensor product $\otimes^{\mathbb{L}}$ introduced in Section 6.2.

Combining the results on finite direct products and unions of groups, we get assertions about restricted direct products as in [16].

10.5. Group extensions

Next we consider a group extension $N \rightarrowtail G \twoheadrightarrow G/N$. If A is a G-C^*-algebra, then $N \ltimes_{\mathrm{r}} A$ carries a canonical twisted action of G/N. In [11], Chabert and Echterhoff use this to construct a *partial crossed product functor*

$$N \ltimes_{\mathrm{r}} \sqcup : \mathrm{KK}^G \to \mathrm{KK}^{G/N}.$$

This functor is triangulated and commutes with direct sums. We have a natural isomorphism $G/N \ltimes_{\mathrm{r}} (N \ltimes_{\mathrm{r}} A) \cong G \ltimes_{\mathrm{r}} A$ in KK. The following result is due to Chabert et al. [16] for the usual Baum–Connes conjecture.

Theorem 10.5. *The functor $N \ltimes_{\mathrm{r}} \sqcup : \mathrm{KK}^G \to \mathrm{KK}^{G/N}$ maps $\mathscr{CI}$ to $\mathscr{CI}$ and hence $\langle \mathscr{CI} \rangle$ to $\langle \mathscr{CI} \rangle$. Therefore, there is a natural isomorphism*

$$G/N \ltimes_{\mathrm{r}}^{\mathbb{L}} (N \ltimes_{\mathrm{r}}^{\mathbb{L}} A) \cong G \ltimes_{\mathrm{r}}^{\mathbb{L}} A,$$

which is compatible with the isomorphism $G/N \ltimes_{\mathrm{r}} (N \ltimes_{\mathrm{r}} A) \cong G \ltimes_{\mathrm{r}} A$.

Suppose that the (*strong*) *Baum–Connes conjecture holds for $HN \subseteq G$ with coefficients A for any smooth compact subgroup $H \subseteq G/N$. Then the* (*strong*) *Baum–Connes conjecture holds for G with coefficients A if and only if it holds for G/N with coefficients $N \ltimes_{\mathrm{r}} A$.*

Suppose that G/N and HN for compact subgroups $H \subseteq G/N$ have a dual Dirac morphism and satisfy $\gamma = 1$. Then the same holds for G.

Proof. Let A be compactly induced from, say, the compact subgroup $H \subseteq G$. By (14), this means that A is a $G \ltimes G/H$-C^*-algebra. We still have a canonical homomorphism from $C_0(G/HN)$ to the central multiplier algebra of $N \ltimes_{\mathrm{r}} A$. This means that $N \ltimes_{\mathrm{r}} A$ is compactly induced as a G/N-algebra. Therefore, $N \ltimes_{\mathrm{r}} \sqcup$ preserves $\mathscr{CI}$ and hence $\langle \mathscr{CI} \rangle$. This implies $G/N \ltimes_{\mathrm{r}}^{\mathbb{L}} (N \ltimes_{\mathrm{r}}^{\mathbb{L}} A) \cong G \ltimes_{\mathrm{r}}^{\mathbb{L}} A$.

Proposition 10.1implies that a Dirac morphism for HN is one for N as well. Hence the hypothesis of the second paragraph is equivalent to the condition that the assembly map $N \ltimes_{\mathrm{r}}^{\mathbb{L}} A \to N \ltimes_{\mathrm{r}} A$ in $\mathrm{KK}^{G/N}$

induces a K-equivalence (or a KK-equivalence) $H\ltimes(N\ltimes_{\mathrm{r}}^{\mathbb{L}}A) \to H\ltimes(N\ltimes_{\mathrm{r}}A)$ for all smooth compact subgroups $H \subseteq G/N$. By Theorem 9.3, the map $G/N\ltimes_{\mathrm{r}}^{\mathbb{L}}(N\ltimes_{\mathrm{r}}^{\mathbb{L}}A) \to G/N\ltimes_{\mathrm{r}}^{\mathbb{L}}(N\ltimes_{\mathrm{r}}A)$ is a K-equivalence (or a KK-equivalence) as well. Together with $G\ltimes_{\mathrm{r}}^{\mathbb{L}}A \cong G/N\ltimes_{\mathrm{r}}^{\mathbb{L}}(N\ltimes_{\mathrm{r}}^{\mathbb{L}}A)$ this yields the assertions in the second paragraph.

Now assume that G/N and the subgroups $HN \subseteq G$ for $H \subseteq G/N$ compact have dual Dirac morphisms and satisfy $\gamma = 1$. We show that G has the same properties. Recall that this is equivalent to $\langle\mathscr{CI}(G)\rangle = \mathrm{KK}^G$. The group homomorphism $\pi: G \to G/N$ induces a triangulated functor commuting with direct sums $\pi^*: \mathrm{KK}^{G/N} \to \mathrm{KK}^G$. Of course, $\pi^*(\star) = \star$. Since $\langle\mathscr{CI}(G/N)\rangle = \mathrm{KK}^{G/N}$, the essential range of π^* is generated by objects of the form $\pi^*(\mathrm{Ind}_H^{G/N} A)$, where $H \subseteq G/N$ is compact. We have $\pi^*(\mathrm{Ind}_H^{G/N} A) \cong \mathrm{Ind}_{HN}^G \pi_H^*(A)$, where $\pi_H: HN \to H$ is the restriction of π. Hence $\star \in \mathrm{KK}^G$ belongs to the localising subcategory generated by the ranges of the functors Ind_{HN}^G for compact subgroups $H \subseteq G/N$.

By hypothesis, $\mathrm{KK}^{HN} = \langle\mathscr{CI}(HN)\rangle$. Since induction is a triangulated functor that commutes with direct sums, the range of Ind_{HN}^G is contained in the localising subcategory of KK^G generated by objects of the form $\mathrm{Ind}_{HN}^G \mathrm{Ind}_L^{HN}(D) \cong \mathrm{Ind}_L^G(D)$ for compact subgroups $L \subseteq HN$ and $D \in \mathrm{KK}^L$. As a result, $\star \in \mathrm{KK}^G$ belongs to $\langle\mathscr{CI}(G)\rangle$. This implies that the Dirac morphism is invertible, that is, G has a dual Dirac morphism and $\gamma = 1$. □

10.6. *Real versus complex assembly maps*

Now we reprove a result of Baum and Karoubi [6] and Schick [39]. In order to compare the real and complex assembly maps, we have to distinguish between the real and complex Kasparov theories in our notation. We denote them by $\mathrm{KK}_{\mathbb{R}}^{G\ltimes X}$ and $\mathrm{KK}_{\mathbb{C}}^{G\ltimes X}$, respectively. We write $A \mapsto A_{\mathbb{C}}$ for the complexification functor $\mathrm{KK}_{\mathbb{R}}^{G\ltimes X} \to \mathrm{KK}_{\mathbb{C}}^{G\ltimes X}$. This functor is obviously triangulated and commutes with direct sums and tensor products, that is, $(A\otimes_{(X)}B)_{\mathbb{C}} \cong A_{\mathbb{C}}\otimes_{(X)}B_{\mathbb{C}}$.

Proposition 10.6. *The complexification functor* $\mathrm{KK}_{\mathbb{R}}^{G\ltimes X} \to \mathrm{KK}_{\mathbb{C}}^{G\ltimes X}$ *preserves weak contractibility and weak equivalences, and it maps* $\langle\mathscr{CI}\rangle$ *to* $\langle\mathscr{CI}\rangle$. *Hence it maps a Dirac triangle in* $\mathrm{KK}_{\mathbb{R}}^{G\ltimes X}$ *to one in* $\mathrm{KK}_{\mathbb{C}}^{G\ltimes X}$.

Proof. Since complexification commutes with restriction and induction, it maps $\mathscr{CC}_{\mathbb{R}}$ to $\mathscr{CC}_{\mathbb{C}}$ and $\mathscr{CI}_{\mathbb{R}}$ to $\mathscr{CI}_{\mathbb{C}}$. Being triangulated and compatible with direct sums, it also maps $\langle\mathscr{CI}_{\mathbb{R}}\rangle$ to $\langle\mathscr{CI}_{\mathbb{C}}\rangle$. This implies the assertion about Dirac triangles. □

There is a long exact sequence that relates real and complex K-theory. This exact sequence is generalised in [39] to a similar long exact sequences

$$\begin{aligned}\cdots \xrightarrow{\delta} \mathrm{KK}_{\mathbb{R},q-1}^{G\ltimes X}(A, B) \xrightarrow{\chi} \mathrm{KK}_{\mathbb{R},q}^{G\ltimes X}(A, B) \xrightarrow{c} \mathrm{KK}_{\mathbb{C},q}^{G\ltimes X}(A_{\mathbb{C}}, B_{\mathbb{C}})\\ \xrightarrow{\delta} \mathrm{KK}_{\mathbb{R},q-2}^{G\ltimes X}(A, B) \xrightarrow{\chi} \mathrm{KK}_{\mathbb{R},q-1}^{G\ltimes X}(A, B) \xrightarrow{c} \mathrm{KK}_{\mathbb{C},q-1}^{G\ltimes X}(A_{\mathbb{C}}, B_{\mathbb{C}}) \xrightarrow{\delta} \cdots,\end{aligned} \tag{32}$$

for any $A, B \in \mathrm{KK}_{\mathbb{R}}^{G\ltimes X}$. The map c is the complexification functor, χ is the product with the generator of $\mathrm{KK}_1(\mathbb{R}, \mathbb{R}) \cong \mathbb{Z}/2$ and δ is the composition of the inverse of the Bott periodicity isomorphism with "forgetting the complex structure". In [39], (32) is only written down for KK^G. The same proof works for $\mathrm{KK}^{G\ltimes X}$, even for equivariant Kasparov theory for groupoids. It is easy to see that (32) is natural with respect to morphisms in $\mathrm{KK}^{G\ltimes X}_{\mathbb{R}}$ (see [39]). Hence the maps

$$\mathrm{KK}^{G\ltimes X}_{\mathbb{C},q}(A_{\mathbb{C}}, B_{\mathbb{C}}) \to \mathrm{KK}^{G\ltimes X}_{\mathbb{C},q}(A'_{\mathbb{C}}, B'_{\mathbb{C}})$$

induced by elements of $\mathrm{KK}^{G}_{\mathbb{R}}(A', A)$ and $\mathrm{KK}^{G}_{\mathbb{R}}(B, B')$ are isomorphisms for all $q \in \mathbb{Z}$ once the corresponding maps

$$\mathrm{KK}^{G\ltimes X}_{\mathbb{R},q}(A, B) \to \mathrm{KK}^{G\ltimes X}_{\mathbb{R},q}(A', B')$$

are isomorphisms for all $q \in \mathbb{Z}$. Remarkably, the converse also holds by [39, Lemma 3.1]. A special case is Karoubi's result that $\mathrm{K}_*(A)\cong 0$ if and only if $\mathrm{K}_*(A_{\mathbb{C}})\cong 0$ [27]. Moreover, $A\cong 0$ in $\mathrm{KK}^{G\ltimes X}_{\mathbb{R}}$ if and only if $A_{\mathbb{C}}\cong 0$ in $\mathrm{KK}^{G\ltimes X}_{\mathbb{C}}$ (because $A\cong 0$ if and only if 0 induces the identity map on $\mathrm{KK}^{G\ltimes X}_{*}(A, A)$).

Theorem 10.7. *Let $A \in \mathrm{KK}^{G}_{\mathbb{R}}$. The (strong) Baum–Connes conjecture for G holds with coefficients A if and only if it holds with coefficients $A_{\mathbb{C}}$.*

Proof. The (strong) Baum–Connes conjecture with coefficients A is equivalent to the statement that $\mathrm{K}_*(G\ltimes_{\mathrm{r}}^{\mathrm{Obs}} A)\cong 0$ (or $G\ltimes_{\mathrm{r}}^{\mathrm{Obs}} A\cong 0$ in KK). Proposition 10.6 implies $G\ltimes_{\mathrm{r}}^{\mathrm{Obs}} A_{\mathbb{C}}\cong (G\ltimes_{\mathrm{r}}^{\mathrm{Obs}} A)_{\mathbb{C}}$. Hence the assertion follows from the results of [39] discussed above. □

Theorem 10.8. *Let G be a locally compact group and X a locally compact G-space. If there is a dual Dirac morphism in $\mathrm{KK}^{G\ltimes X}_{\mathbb{C}}$, then there is one in $\mathrm{KK}^{G\ltimes X}_{\mathbb{R}}$, and vice versa. In this case, we have $\gamma_{\mathbb{C}} = 1$ if and only if $\gamma_{\mathbb{R}} = 1$.*

Proof. By Theorem 8.3, a dual Dirac morphism exists if and only if D induces an isomorphism $\mathrm{KK}^{G\ltimes X}_{*}(C_0(X), \mathsf{P})\cong \mathrm{KK}^{G\ltimes X}_{*}(\mathsf{P}, \mathsf{P})$. This holds both in the real and complex case. By Proposition 10.6, the Dirac morphism in $\mathrm{KK}^{G\ltimes X}_{\mathbb{C}}$ is the complexification of the Dirac morphism in $\mathrm{KK}^{G\ltimes X}_{\mathbb{R}}$. Hence the existence of a dual Dirac morphism in $\mathrm{KK}^{G\ltimes X}_{\mathbb{C}}$ and $\mathrm{KK}^{G\ltimes X}_{\mathbb{R}}$ are equivalent by the results of [39] discussed above. Since the complexification of a dual Dirac morphism in $\mathrm{KK}^{G\ltimes X}_{\mathbb{R}}$ is one in $\mathrm{KK}^{G\ltimes X}_{\mathbb{C}}$, $\gamma_{\mathbb{C}}$ is the complexification of $\gamma_{\mathbb{R}}$. We have $\gamma = 1$ if and only if γ is invertible if and only if multiplication by γ on $\mathrm{KK}^{G\ltimes X}_{*}(C_0(X), C_0(X))$ is an isomorphism. Again it follows from [39] that $\gamma_{\mathbb{R}} = 1$ if and only if $\gamma_{\mathbb{C}} = 1$. □

Appendix A. The equivariant Kasparov category is triangulated

We have defined a translation automorphism and a class of exact triangles on $\widetilde{\mathrm{KK}}^{G\ltimes X}$ in Section 3. Here we prove that these data verify the axioms of a triangulated category (see [38]). More precisely, we prove the equivalent assertion that the opposite category of $\widetilde{\mathrm{KK}}^{G\ltimes X}$ is triangulated.

By definition, the class of exact triangles is closed under isomorphism and the translation functor is an automorphism. The zeroth axiom (TR 0) requires triangles of the form $\Sigma X \to 0 \to X \xrightarrow{\mathrm{id}_X} X$ to be exact. This follows from the contractibility of $\mathrm{cone}(\mathrm{id}_X)\cong C_0(]0, 1]) \otimes X$.

Axiom (TR 1) asks that for any morphism $f\colon A \to B$ there should be an exact triangle $\Sigma B \to C \to A \xrightarrow{f} B$. If f is an equivariant $*$-homomorphism, we may take the mapping cone triangle of f. In general, we claim that any morphism in $\widetilde{\mathrm{KK}}^G$ is isomorphic to an equivariant $*$-homomorphism. We can first replace f by a morphism in KK^G because KK^G and $\widetilde{\mathrm{KK}}^G$ are equivalent categories. By [34] we can represent f by an equivariant $*$-homomorphism $f_*: q_s A \to q_s B$, where

$$q_s A := \mathbb{K}(L^2(G \times \mathbb{N})) \otimes q(\mathbb{K}(L^2 G) \otimes A).$$

If $X = \star$, then the C^*-algebra qA is the usual one from the Cuntz picture for Kasparov theory. Otherwise, we have to modify its definition so as to get a $G \ltimes X$-C^*-algebra. Namely, let $A *_X A$ be the free product of A with itself in the category of $G \ltimes X$-C^*-algebras. That is, it comes equipped with two natural maps $\iota_1, \iota_2: A \to A *_X A$ with the universal property that pairs of $G \ltimes X$-equivariant $*$-homomorphisms $A \to B$ correspond bijectively to $G \ltimes X$-equivariant $*$-homomorphisms $A *_X A \to B$. We can construct $A *_X A$ as the quotient of $A * A$ by the ideal generated by the relations $\iota_1(fa_1)\iota_2(a_2) \sim \iota_1(a_1)\iota_2(fa_2)$ for all $a_1, a_2 \in A$, $f \in C_0(X)$. The pair $(\mathrm{id}_A, \mathrm{id}_A)$ induces a natural homomorphism $A *_X A \to A$. Let $q_X A$ be its kernel. With this modified definition of qA, the assertions of [34] remain true for $\mathrm{KK}^{G \ltimes X}$. In particular, there is a natural $\mathrm{KK}^{G \ltimes X}$-equivalence $q_s A \cong A$. Therefore, any morphism in KK is isomorphic to an equivariant $*$-homomorphism. Thus axiom (TR 1) holds.

Axiom (TR 2) asks that a triangle $\Sigma B \to C \to A \to B$ be exact if and only if $\Sigma A \to \Sigma B \to C \to A$ (with certain signs) is exact. It suffices to prove one direction because suspensions and desuspensions evidently preserve exact triangles. Thus axiom (TR 2) is equivalent to the statement that the rotated mapping cone triangle

$$\Sigma A \xrightarrow{-\Sigma f} \Sigma B \xrightarrow{\iota} \mathrm{cone}(f) \xrightarrow{\varepsilon} A$$

is exact for any equivariant $*$-homomorphism $f: A \to B$. We claim that this triangle is the extension triangle for the natural extension

$$0 \longrightarrow \Sigma B \xrightarrow{\iota} \mathrm{cone}(f) \xrightarrow{\varepsilon} A \longrightarrow 0$$

and hence exact. Build the diagram (5) for this extension. The resulting map $\Sigma B \to \mathrm{cone}(\varepsilon)$ is a homotopy equivalence in a natural and hence equivariant fashion. Thus the above extension is admissible and gives rise to an exact triangle. One easily identifies the map $\Sigma A \to \Sigma B$ in the extension triangle with $-\Sigma f$. This finishes the proof of axiom (TR 2).

Suppose that we are given the solid arrows in the diagram

$$\begin{array}{ccccccc}
\Sigma B & \longrightarrow & C & \longrightarrow & A & \longrightarrow & B \\
\downarrow{\scriptstyle \Sigma\beta} & & \vdots{\scriptstyle \gamma} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\
\Sigma B' & \longrightarrow & C' & \longrightarrow & A' & \longrightarrow & B'
\end{array} \tag{33}$$

and that the rows in this diagram are exact triangles. Axiom (TR 3) asks that we can find γ making the diagram commute. We may first assume that the rows are mapping cone triangles for certain maps $f: A \to B$ and $f': A' \to B'$ because any exact triangle is isomorphic to one of this form.

We represent α and β by Kasparov cycles, which we again denote by α and β. Since (33) commutes, the Kasparov cycles $f'_*(\alpha)$ and $f^*(\beta)$ are homotopic. Choose a homotopy H between them. Now we glue together β, H and α to obtain a cycle for $\mathrm{KK}^{G \ltimes X}(\mathrm{cone}(f), \mathrm{cone}(f'))$ with the required properties. Since $(\mathrm{ev}_1)_*(H) = f'_*(\alpha)$, the pair (H, α) defines a Kasparov cycle for A and $\mathrm{cyl}(f')$. The constant family

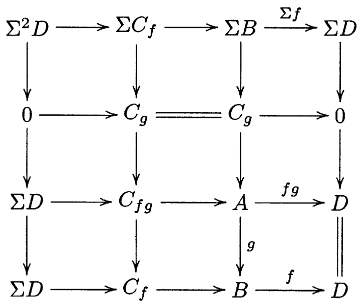

Fig. 2. The octahedral axiom.

β defines a cycle $C\beta$ for $\mathrm{KK}^{G\ltimes X}(C_0(]0,1],B), C_0(]0,1],B'))$. Reparametrisation gives a canonical isomorphism

$$\mathrm{cone}(f') \cong \{(x,y) \in C_0(]0,1],B') \oplus \mathrm{cyl}(f') \mid x(1) = \tilde{f}'(y)\}.$$

Since $\tilde{f}'_*(H,\alpha) = (\mathrm{ev}_0)_*(H) = f^*(\beta)$, we can glue together (H,α) and $C\beta$ to get a cycle for $\mathrm{KK}^{G\ltimes X}(\mathrm{cone}(f), \mathrm{cone}(f'))$. It is straightforward to see that it has the required properties. This finishes the verification of axiom (TR 3).

It remains to verify Verdier's octahedral axiom, which is crucial to localise triangulated categories. Neeman formulates it rather differently in [38]. We shall use Verdier's original octahedral axiom (see [46] or [38, Proposition 1.4.6]) because it can be applied more directly and because its meaning is more transparent in the applications we have met so far.

Proposition A.1. *For any pair of morphisms $f \in \mathrm{KK}^G(B,D)$, $g \in \mathrm{KK}^G(A,B)$ there is a commuting diagram as in Fig. 2 whose rows and columns are exact triangles. Moreover, the two maps $\Sigma B \to \Sigma D \to C_{fg}$ and $\Sigma B \to C_g \to C_{fg}$ in this diagram coincide.*

Proof. Replacing all C^*-algebras by appropriate universal algebras, we can achieve that f and g are equivariant $*$-homomorphisms. We assume this in the following. We shall use the mapping cones and mapping cylinders defined in Section 2. We define a natural G-C^*-algebra

$$\mathrm{cyl}(f,g) := \{(a,b,d) \in A \oplus C([0,1],B) \oplus C([0,1],D) \mid g(a) = b(1),\ \ f(b(0)) = d(1)\}$$

and natural equivariant $*$-homomorphisms

$$p_A\colon \mathrm{cyl}(f,g) \to A, \quad (a,b,d) \mapsto a,$$
$$j_A\colon A \to \mathrm{cyl}(f,g), \quad a \mapsto (a, \mathrm{const}\, g(a), \mathrm{const}\, fg(a)),$$
$$\tilde{g}\colon \mathrm{cyl}(f,g) \to \mathrm{cyl}(f), \quad (a,b,d) \mapsto (b(0), d).$$

We have $p_A j_A = \mathrm{id}_A$, and $j_A p_A$ is homotopic to the identity map in a natural and hence equivariant way. Thus $\mathrm{cyl}(f,g)$ is homotopy equivalent to A. Moreover, $\tilde{g} j_A = j_B g$, where $j_B\colon B \to \mathrm{cyl}(f)$ is the natural map, which is a homotopy equivalence. That is, the map $\tilde{g}$ is isomorphic to $g\colon A \to B$. Recall also that the map $\tilde{f}\colon \mathrm{cyl}(f) \to D$ is isomorphic to $f\colon B \to D$.

The maps $\tilde{g}\colon \mathrm{cyl}(f, g) \to \mathrm{cyl}(f)$, $\tilde{f}\colon \mathrm{cyl}(f) \to D$ and $\tilde{f} \circ \tilde{g}\colon \mathrm{cyl}(f, g) \to D$ are all surjective. The kernel of $\tilde{f}$ is $\mathrm{cone}(f)$, the kernel of $\tilde{g}$ is naturally isomorphic to $\mathrm{cone}(g)$. We let $\mathrm{cone}(f, g)$ be the kernel of $\tilde{f}\tilde{g}$. Thus we obtain a commutative diagram of G-C^*-algebras whose rows and columns are extensions:

$$\begin{array}{ccccc}
\mathrm{cone}(g) & = & \mathrm{cone}(g) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{cone}(f,g) & \rightarrowtail & \mathrm{cyl}(f,g) & \overset{\tilde{f}\tilde{g}}{\twoheadrightarrow} & D \\
\downarrow & & \downarrow{\scriptstyle \tilde{g}} & & \| \\
\mathrm{cone}(f) & \rightarrowtail & \mathrm{cyl}(f) & \overset{\tilde{f}}{\twoheadrightarrow} & D.
\end{array} \tag{34}$$

We claim that all rows and columns in this diagram are admissible extensions. (Even more, the maps $K \to \mathrm{cone}(p)$ in (5) for these extensions are all homotopy equivalences.) We have already observed this for the third row in Section 2.3, and the argument for the second row is similar. The assertion is trivial for the first row and the third column. The remaining two columns can be treated in a similar fashion. A conceptual reason for this is that they are pull backs of the standard extension $\mathrm{cone}(g)\rightarrowtail\mathrm{cyl}(g)\twoheadrightarrow B$ along the natural projections $\mathrm{cone}(f) \to B$ and $\mathrm{cyl}(f) \to B$, respectively. The projection $\mathrm{cyl}(g) \to B$ is a cofibration in the notation of [40]; this property implies that $K \to \mathrm{cone}(p)$ is a homotopy equivalence and is hereditary for pull backs (see [40]).

We can now write down extension triangles for the rows and columns in (34) and replace A and B by the homotopy equivalent algebras $\mathrm{cyl}(f, g)$ and $\mathrm{cyl}(f)$, respectively. This yields a diagram as in Fig. 2.

The composite map $\Sigma B \to \mathrm{cone}(g) \to \mathrm{cone}(f, g)$ is just the restriction of the canonical map $\mathrm{cone}(g) \to \mathrm{cone}(f, g)$ to ΣB. There is a natural homotopy from this map to the composition $\Sigma B \to \Sigma D \to \mathrm{cone}(f, g)$ via translations involving f. This finishes the proof of Proposition A.1. □

We have now verified that $\widetilde{\mathrm{KK}}^{G\ltimes X}$ is a triangulated category.

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