Cantor sets determined by partial quotients of continued fractions of Laurent series

Xue-Hai Hu a,*, Bao-Wei Wang a, Jun Wu b, Yue-Li Yu a

a Department of Mathematics, Wuhan University, Wuhan, Hubei 430072, PR China
b Department of Mathematics, Huazhong University of Science and Technology, Wuhan, Hubei 430074, PR China

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Abstract

In this paper, two types of general sets determined by partial quotients of continued fractions over the field of formal Laurent series with coefficients from a given finite field are studied. The Hausdorff dimensions of \{x: \deg A_n(x) \geq \phi(n), \text{ for infinitely many } n\} and \{x: \deg A_n(x) \geq \phi(n), \forall n \geq 1\} are determined completely, where \(A_n(x)\) denotes the partial quotients in the continued fraction expansion (in case of Laurent series) of \(x\) and \(\phi(n)\) is a positive valued function defined on natural numbers \(\mathbb{N}\).

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1. Introduction

It is known that the Gauss transformation \(T: [0, 1] \rightarrow [0, 1]\) given by

\[
T(0) := 0, \quad T(x) := \frac{1}{x} \pmod{1}, \quad \text{for } x \in (0, 1)
\]

(1.1)

leads to the continued fraction expansions over the real field, i.e., for any irrational number \(x \in [0, 1]\), it can be uniquely expanded into the infinite form

* Corresponding author.

E-mail addresses: jackyhxuehai@hotmail.com (X.-H. Hu), bwei_wang@yahoo.com.cn (B.-W. Wang), wujunyu@public.wh.hb.cn (J. Wu), yuyuel08@hotmail.com (Y.-L. Yu).

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\[ x = \frac{1}{a_1(x) + \frac{1}{a_2(x) + \frac{1}{a_3(x) + \cdots}}} \]  

(1.2)

where \( a_1(x) = \lfloor 1/x \rfloor \), and \( a_n(x) = a_1(T^{n-1}(x)) \) (for all \( n \geq 2 \)) are called the partial quotients of \( x \).

The metrical theory of continued fractions is one of the major subjects in the study of continued fractions. It has close connections with dynamical system, ergodic theory, probability theory, Diophantine approximation and many others. Among them, a well-known theorem is the Borel–Bernstein theorem, see Borel [4, 5], Bernstein [2], (also be called Khintchine ‘0–1’ law, see Khintchine [12, Theorem 30]), which says that for almost all \( x \in [0, 1] \), \( a_n(x) \geq \phi(n) \) holds for infinitely many \( n \)’s or only for finite many \( n \)’s according as the series \( \sum_{n=1}^{\infty} 1/\phi(n) \) diverges or not. As a consequence of this theorem, many sets of points obeying some simple restriction to their partial quotients are of zero Lebesque measure. The fractal structure of such sets have been extensively investigated. It seems that the first published work is due to Jarnik [11], who studied the set \( E \) of points whose partial quotients are bounded. Later, I.J. Good [9] gave an exhaustive investigation of these sets, including the set \( \{ x \in [0, 1) : a_n(x) \to \infty \} \). Also he attempted to consider the general set \( \{ x \in [0, 1] : a_n(x) \geq \phi(n) \}, \) for infinitely many \( n \), but unfortunately he did not give the exact value of its Hausdorff dimension (the Hausdorff dimension of such general sets has been well solved in [22]). After this, many authors have tried to expand Good’s work, such as Hirst [10], Moorthy [15] and Lúczak [14].

In this paper, we study the analogous sets after Good’s work over the field of formal Laurent series. We know (Artin [1]) that every formal Laurent series can be developed into the so-called continued fraction expansion provided by the Gauss transformation over the field of formal Laurent series. The metrical and ergodic theory of such an expansion was studied by H. Niederreiter [17], Berthé and Nakada [3]. Also, H. Niederreiter [17] got an analogous result as Borel–Bernstein theorem over this field. Moreover, H. Niederreiter and M. Vielhaber [18] determined, for any positive integer \( d \), the Hausdorff dimension of the set \( \{ x : \deg A_n(x) \leq d, \forall n \geq 1 \} \). The aim of this paper is to derive the Hausdorff dimension of the set \( \{ x : \deg A_n(x) \geq \phi(n), \) for infinitely many \( n \} \) and \( \{ x : \deg A_n(x) \geq \phi(n), \forall n \geq 1 \} \).

The paper is organized as follows. In Section 2, we fix up some notations and state our main results. Section 3 is devoted to collecting some basic properties of continued fraction over the field of formal Laurent series. We respectively prove Theorem 2.2 in Section 4, Theorem 2.1 in Section 5 and Theorem 2.3 in Section 6. Before proving Theorem 2.4 in the last section, which is the most important result of this paper, we will get the Hausdorff dimension of the set \( \{ x : \deg A_n(x) \geq \alpha n, \) for infinitely many \( n \} \) in Section 7.

2. Statements of main results

At first we fix up some notations and describe the continued fraction expansion of Laurent series introduced in [21].

Let \( \mathbb{F}_q \) be a finite field of \( q \) elements, where \( q \) is a power of some prime number \( p \). Let \( \mathbb{F}_q((z^{-1})) \) denote the field of all formal Laurent series \( B = \sum_{n=v}^{\infty} c_n z^{-n} \) in an indeterminate \( z \), with coefficients \( c_n \) all lying in the field \( \mathbb{F}_q \). Recall that \( \mathbb{F}_q[z] \) denotes the ring of polynomials in \( z \) with coefficients in \( \mathbb{F}_q \).
For the above formal Laurent series $B$, we may assume that $c_v \neq 0$. Then the integer $v = v(B)$ is called the order of $B$. The norm (or valuation) of $B$ is defined to be

$$|B|_{\infty} = q^{-v(B)}.$$ 

It is known that $|\cdot|_{\infty}$ is a non-Archimedean norm on the field $F_q((z^{-1}))$ and $F_q((z^{-1}))$ is a complete metric space under the metric $\rho$ defined by $\rho(B_1, B_2) = |B_1 - B_2|_{\infty}$. We denote the ball $B(x, r) := \{y \in I : |y - x|_{\infty} \leq r\}$. For $B = \sum_{n=v}^{\infty} c_n z^{-n} \in F_q((z^{-1}))$, let $[B] = \sum_{v \leq n \leq 0} c_n z^{-n} \in F_q[z]$, which is called the integral part of $B$. It is evident that the integer $-v(B) := -v$ is equal to the degree $\deg [B]$ of the polynomial $[B]$ provided $v \leq 0$, i.e., $[B] \neq 0$.

Let $I$ denote the valuation ideal of $F_q((z^{-1}))$. It consists of all formal series $\sum_{n=1}^{\infty} c_n z^{-n}$. The ideal $I$ is compact because it is isomorphic to $\prod_{n=1}^{\infty} F_q$. A natural measure on $I$ is the normalized Haar measure on $\prod_{n=1}^{\infty} F_q$, which we denote by $P$.

Consider the Gauss transformation from $I$ to $I$ defined by

$$Tx := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \quad T0 := 0.$$ 

Then the Gauss transformation leads to the continued fraction expansion over the field of formal Laurent series, which was first well introduced by Artin [1]. As in the real case, every $x \in I$ has the following continued fraction expansion:

$$x = \frac{1}{A_1(x) + \frac{1}{A_2(x) + \frac{1}{A_3(x) + \cdots}} := [0; A_1(x), A_2(x), A_3(x), \ldots],$$

where the digits $A_i(x)$ are polynomials of strictly positive degree defined by

$$A_n(x) = \left\lfloor \frac{1}{T^{n-1}(x)} \right\rfloor, \quad \forall n \geq 1.$$ 

Throughout this paper, we are willing to use $|E|$ to denote the diameter of the set $E$, $\lfloor \cdot \rfloor$ to denote the integer part of a real number, $H^t$ to denote the $t$-dimensional Hausdorff measure, “i.o.” to denote “for infinitely many,” $\dim H$ to denote the Hausdorff dimension, respectively.

Now we state our main results.

**Theorem 2.1.** $\dim_H \{x \in I : \deg A_n(x) \to \infty\} = \frac{1}{2}$.

**Theorem 2.2.** For any $a > 1$, we have

$$\dim_H \{x \in I : \deg A_n(x) \geq a^n, \ \forall n \geq 1\} = \dim_H \{x \in I : \deg A_n(x) \geq a^n, \ \text{i.o.} \ n\} = \frac{1}{a + 1}.$$
In the sequel, we always assume that $\phi(n)$ is a positive valued function defined on the natural numbers $\mathbb{N}$. Then we have

**Theorem 2.3.** Let $A = \{x \in I: \deg A_n(x) \geq \phi(n), \forall n \geq 1\}$. Assume that $\phi(n) \to \infty$ as $n \to \infty$ and

$$\limsup_{n \to \infty} \frac{\log \phi(n)}{n} = \log a.$$ 

Then we have

$$\dim_H A = \frac{1}{a + 1}.$$ 

**Theorem 2.4.** The set $E = \{x \in I: \deg A_n(x) \geq \phi(n), \ i.o. \ n\}$ has the Hausdorff dimension given as follows. Suppose $\liminf_{n \to \infty} \frac{\phi(n)}{n} = \alpha$. 

1. $\alpha = 0$, $\dim_H E = 1$.
2. $0 < \alpha < \infty$, $\dim_H E = s_\alpha$, where $s_\alpha$ is the unique solution to the equation

$$\sum_{k=1}^{\infty} (q - 1)q^k \left( \frac{1}{q^{2k+\alpha}} \right)^s = 1.$$ 

3. $\alpha = \infty$. Suppose that $\liminf_{n \to \infty} \log \phi(n)/n = \log b$.
   (i) $b = 1$, $\dim_H E = \frac{1}{2}$.
   (ii) $1 < b < \infty$, $\dim_H E = \frac{1}{1+b}$.
   (iii) $b = \infty$, $\dim_H E = 0$.

3. **Preliminaries**

In this section, we collect some basic properties of continued fractions over the field of formal Laurent series. For more details, we refer to the results in [3,7,13,19].

As in the real case, let

$$x = [0; A_1(x), A_2(x), \ldots] = \frac{1}{A_1(x) + \frac{1}{A_2(x) + \ddots}}.$$ 

Similarly, $P_n(x)$ and $Q_n(x)$ are obtained by the following recurrence formula:

$$P_{-1} = 1, \quad P_0 = 0, \quad P_n = A_n(x)P_{n-1}(x) + P_{n-2}(x) \quad (n \geq 2),$$

$$Q_{-1} = 0, \quad Q_0 = 1, \quad Q_n = A_n(x)Q_{n-1}(x) + Q_{n-2}(x) \quad (n \geq 2).$$

We call $P_n(x)/Q_n(x)$ the $n$th convergents of $x$, since
\[
\frac{P_n(x)}{Q_n(x)} = \frac{1}{A_1(x)} + \frac{1}{A_2(x)} + \cdots + \frac{1}{A_n(x)}.
\]

The following three propositions are due to H. Niederreiter [17,18], also we can find them in [7].

**Proposition 3.1.** (See [7,17,18].) Let \( x \in \mathbb{F}((X)^{-1}) \). Then one has:

1. \( |Q_k|_\infty = \prod_{i=1}^{k} |A_i|_\infty \).
2. \( \left| x - \frac{P_n(x)}{Q_n(x)} \right|_\infty = \frac{1}{|Q_n(x)Q_{n+1}(x)|_\infty} = \frac{1}{|A_{n+1}(x)Q_n(x)^2|_\infty} < \frac{1}{|Q_n^2|_\infty} \).

**Definition 3.2.** Let \( A_1, A_2, \ldots, A_n \in F_q[z] \) be of strictly positive degree. Call the set
\[
I(A_1, A_2, \ldots, A_n) = \{ x \in I : A_1(x) = A_1, A_2(x) = A_2, \ldots, A_n(x) = A_n \}
\]
an \( n \)th order cylinder (in the case of formal power series).

**Proposition 3.3.** (See [7,17,18].) Every \( n \)th order cylinder \( I(A_1, A_2, \ldots, A_n) \) is a closed disc with diameter equal to
\[
|I(A_1, A_2, \ldots, A_n)| = q^{-2\sum_{k=1}^{n} \deg A_k - 1}
\]
and
\[
P(I(A_1, A_2, \ldots, A_n)) = q^{-2\sum_{k=1}^{n} \deg A_k}.
\]

For more details about the Haar measure \( P \), one can refer to the work of Sprindžuk [20], where he gave a quite accurate characterization.

**Proposition 3.4** (Borel–Cantelli lemma for formal Laurent series). (See [7,17,18].)
\[
P(|A_n(x)|_\infty \geq \phi(n) \text{ for infinitely many } n) = 0 \text{ or } 1
\]
according as \( \sum_{n=1}^{\infty} 1/\phi(n) \) converges or not.

**Remark 3.5.** Since the valuation \( |\cdot|_\infty \) is non-Archimedean, it follows that if two cylinders intersect, then one contains the other.

**Remark 3.6.** From Proposition 3.4, we know that the sets in Theorems 2.1–2.4 are of zero Haar measure.

Next, we prove our results. It is easy to see that Theorem 2.2 will supply a lower bound for the Hausdorff dimension of the set in Theorem 2.1, so we prove Theorem 2.1 after the proof of Theorem 2.2.
4. Proof of Theorem 2.2

In this section, we establish Theorem 2.2.

4.1. Lower bound

To get the lower bound of Hausdorff dimension of the set in Theorem 2.2, we need to construct a kind of Cantor set, called a homogeneous Moran set. Here we recall the definition and a basic dimension result of the homogeneous Moran set.

Let \( \{n_k\}_{k \geq 1} \) be a sequence of positive integers and \( \{c_k\}_{k \geq 1} \) be a sequence of positive numbers satisfying \( n_k \geq 2, 0 < c_k < 1, n_1 c_1 \leq \delta \) and \( n_k c_k \leq 1 \) \((k \geq 2)\), where \( \delta \) is some positive number. Let

\[
D = \bigcup_{k \geq 0} D_k \quad \text{with} \ D_0 = \{\emptyset\}, \ D_k = \{(i_1, \ldots, i_k); \ 1 \leq i_j \leq n_j, \ 1 \leq j \leq k\}.
\]

If \( \sigma = (\sigma_1, \ldots, \sigma_k) \in D_k, \ \tau = (\tau_1, \ldots, \tau_m) \in D_m, \) we define

\[
\sigma \ast \tau = (\sigma_1, \ldots, \sigma_k, \tau_1, \ldots, \tau_m).
\]

Suppose that \( J \) is an interval of length \( \delta \). A collection \( \mathcal{F} = \{J_\sigma: \ \sigma \in D\} \) of subintervals of \( J \) is said to have a homogeneous Moran structure if it satisfies:

1. \( J_\emptyset = J \).
2. For any \( k \geq 0 \) and \( \sigma \in D_k, \ J_{\sigma \ast 1}, J_{\sigma \ast 2}, \ldots, J_{\sigma \ast n_k+1} \) are subintervals of \( J_\sigma \) and \( J_{\sigma \ast i} \cap J_{\sigma \ast j} = \emptyset \) \((i \neq j)\).
3. For any \( k \geq 1 \) and any \( \sigma \in D_{k-1}, 1 \leq j \leq n_k \), we have

\[
\frac{|J_{\sigma \ast j}|}{|J_\sigma|} = c_k.
\]

If \( \mathcal{F} \) is such a collection, \( E := \bigcap_{k \geq 1} \bigcup_{\sigma \in D_k} J_\sigma \) is called a homogeneous Moran set determined by \( \mathcal{F} \).

Proposition 4.1. (See [6].) For the homogeneous Moran set defined above, we have

\[
\dim_H E \geq \liminf_{n \to \infty} \frac{\log n_1 n_2 \cdots n_k}{- \log c_1 c_2 \cdots c_k + 1 n_k+1}.
\]

Lemma 4.2. For any \( a > 1 \), let \( E = \{x \in I: \deg A_n(x) = \lfloor a^n \rfloor + 1\} \). Then we have \( \dim_H E \geq \frac{1}{a+1} \).

Proof. Let \( D_n = \{\sigma: \ \sigma = (A_1, \ldots, A_n) \in \mathbb{F}[z]^n, \ \deg A_k = \lfloor a^k \rfloor + 1, \ 1 \leq k \leq n\} \). Put

\[
E_0 = I, \quad E_n = \bigcup_{(\sigma_1, \ldots, \sigma_n) \in D_n} I(\sigma_1, \ldots, \sigma_n), \quad \forall n \geq 1.
\]
Then

\[ E = \bigcap_{n=1}^{+\infty} E_n. \]

We denote \( m_k = (q - 1)q^{\lceil a^k \rceil + 1}, \epsilon_k = q^{-2(\lceil a^k \rceil + 1)}. \) It is easy to know that every element \( I(\sigma_1, \ldots, \sigma_{k-1}) \) in \( E_{k-1} \) contains \( m_k \) many elements \( I(\sigma_1, \ldots, \sigma_k) \) in \( E_k \) with the same ratio \( \epsilon_k. \) Thus \( E \) is a standard homogeneous Moran set. By Proposition 4.1, we have

\[ \dim_H E \geq \lim inf_{n \to +\infty} \frac{\log (m_1 \cdots m_{k-1})}{-\log(\epsilon_1 \cdots \epsilon_k m_k)} = \frac{1}{a+1}. \]

For more results on Moran sets, one can refer to the work of Moran [16], Feng et al. [6].

4.2. Upper bound

Let \( T = \{ x \in I : \deg A_n(x) \geq a^n, \text{ i.o. } n \}. \) To get the upper bound of \( \dim_H T \), we construct a special covering system. It should be mentioned that the ideas are due to T. Lúczak [14].

Fix \( 1 < b < a. \) Define

\[ I_{n,x} = B \left( \frac{P_n(x)}{Q_n(x)}, q^{-(1+b)\deg Q_n(x)} \right), \quad J_{n,x} = B \left( \frac{P_n(x)}{Q_n(x)}, q^{-2(1+b)\deg Q_n(x)} \right), \]

and for any \( k \geq 1, \) let

\[ \mathcal{I}_k = \left\{ I_{n,x}: \deg Q_n(x) = k \text{ and } n \leq \frac{\log 3k}{b} \right\}, \quad \mathcal{J}_k = \left\{ J_{n,x}: \deg Q_n(x) = k \right\}. \]

Next we state a few propositions which will be used to estimate the upper bound of \( \dim_H T. \)

**Proposition 4.3.** For any \( x \in I, \) \( \deg Q_{n+1}(x) > \max\{ b \deg Q_n(x), b^{(n+1)} \} \) holds for infinitely many \( n \)’s.

**Proof.** Fix \( x \in T. \) Since \( 1 < b < a, \) for any \( m \geq 1, \) there exists \( k > m \) such that \( \deg Q_m(x) < a^k b^{m-k} \) and \( \deg A_k(x) \geq a^k. \) Let \( f(t) = a^k b^{t-k}. \) Then

\[ \deg Q_m(x) < f(m) \quad \text{and} \quad \deg Q_k(x) \geq \deg A_k(x) \geq a^k = f(k). \]

Denote by \( n \) the minimal integer such that \( \deg Q_n(x) < f(n) \) and \( \deg Q_{n+1}(x) \geq f(n+1). \) Then we have

\[ \deg Q_{n+1}(x) \geq f(n+1) = b(\deg Q_n(x)) > \max\{ b \deg Q_n(x), b^{n+1} \}. \]

**Proposition 4.4.** For each \( m \geq 1, \) the family \( \bigcup_{k=m}^{+\infty} \mathcal{I}_k \cup \bigcup_{k=m}^{+\infty} \mathcal{J}_k \) covers \( T. \)
Proof. Fix $m \geq 1$. For any $x \in T$, by Proposition 4.3, choose $n$ such that

$$\deg Q_n(x) > m \quad \text{and} \quad \deg Q_{n+1}(x) > \max\{b \deg Q_n(x), b^{n+1}\}.$$ 

Recall that

$$\left| x - \frac{P_n(x)}{Q_n(x)} \right| = \frac{1}{|Q_n(x)Q_{n+1}(x)|}.$$ 

Let $k = \deg Q_n(x)$. If $k \geq \frac{1}{3}b^n$, $x \in I_{n,x} \in \mathcal{I}_k$. On the other hand, if $k < \frac{1}{3}b^n$, it follows that

$$(1 + 2b) \deg Q_n(x) < (1 + 2b)\frac{1}{3}b^n < b^{n+1} < \deg Q_{n+1}(x).$$

Thus $x \in J_{n,x} \in \mathcal{J}_k$. Therefore $\bigcup_{k=m}^{+\infty} \mathcal{I}_k \cup \bigcup_{k=m}^{+\infty} \mathcal{J}_k$ covers $T$. \hfill \Box

**Proposition 4.5.** Let $\psi(k) := \#\{I_{n,x}: I_{n,x} \in \mathcal{I}_k\}$ and $\tilde{\psi}(k) := \#\{J_{n,x}: J_{n,x} \in \mathcal{J}_k\}$. Then we have:

(i) $\psi(k) = O(q^{k(1+\epsilon)})$, $\forall \epsilon > 0$.

(ii) $\tilde{\psi}(k) = O(q^{k(2+\epsilon)})$, $\forall \epsilon > 0$.

**Proof.** We show assertion (i) only, since the second one follows similarly. Notice that $I_{n,x}$ is determined by $P_n(x)$ and $Q_n(x)$, while both of them are wholly determined by $A_1(x), \ldots, A_n(x)$. Therefore

$$\psi(k) = \#\left\{ (A_1, \ldots, A_n): \deg A_1 + \cdots + A_n = k, \quad k \geq \frac{1}{3}b^n \right\}$$

$$= \sum_{\deg A_1 + \cdots + \deg A_n = k, \ n \leq t} 1 \quad (\text{where } t = \lfloor \log_b 3k \rfloor)$$

$$= \sum_{n=1}^{t} \sum_{k_1 + \cdots + k_n = k} \left( \sum_{\deg A_i = k_i, \ 1 \leq i \leq n} 1 \right)$$

$$= \sum_{n=1}^{t} \sum_{k_1 + \cdots + k_n = k} q^k(q - 1)^n = O(q^{k(1+\epsilon)}). \hfill \Box$$

**Lemma 4.6.** $\dim_H T \leq \frac{1}{a+1}$. 

**Proof.** By Proposition 4.3 for any $m \geq 1$, the family $\bigcup_{k=m}^{+\infty} \mathcal{I}_k \cup \bigcup_{k=m}^{+\infty} \mathcal{J}_k$ is a covering system of $T$. Then for any $\forall \epsilon > 0$ and $s > \frac{1+\epsilon}{1+b}$, we have

$$\mathcal{H}^s(T) \leq \liminf_{m \to \infty} \left( \sum_{k=m}^{+\infty} \psi(k)q^{-(1+b)ks} + \sum_{k=m}^{+\infty} \tilde{\psi}(k)q^{-2(1+b)ks} \right)$$

$$\leq c \liminf_{m \to \infty} \left( \sum_{k=m}^{+\infty} q^{k(1+\epsilon)}q^{-(1+b)ks} + \sum_{k=m}^{+\infty} q^{2(k+\epsilon)}q^{-2(1+b)ks} \right) < \infty.$$
Thus we obtain \( \dim_H T \leq \frac{1+\epsilon}{b+1} \). By the arbitrariness of \( \epsilon > 0 \) and \( b < a \), we have
\[
\dim_H T \leq \frac{1}{a+1}. \quad \square
\]

**Proof of Theorem 2.2.** Combining Lemmas 4.6 and 4.2, we get the result of Theorem 2.2. \( \square \)

5. Proof of Theorem 2.1

We show a lemma first, which will imply that changing only finitely many terms on the restriction of the partial quotients will not influence the original Hausdorff dimension.

**Lemma 5.1.** Let \( B = \{ x \in I : A_n(x) \in B_n, \ n \geq 1 \} \), where \( B_n \) is a subset of all polynomials with strictly positive degree for each \( n \). Let \( B_N = \{ x \in I : A_n(x) \in B_n, \ \forall n \geq N \} \). Then
\[
\dim_H B = \dim_H B_N.
\]

**Proof.** Fix a sequence of polynomials \( A_1, \ldots, A_n \), define \( f_{A_1, \ldots, A_n} : x \rightarrow [0; A_1, A_2, \ldots, A_n + x] \). Recall that \( f \) is called bi-Lipschitz if there exist two constants \( c_1, c_2 \) such that
\[
c_1|x - y| \leq |f(x) - f(y)| \leq c_2|x - y|.
\]
Then it is easy to see that \( f_{A_1, \ldots, A_n} \) is bi-Lipschitz, since
\[
|f_{A_1, \ldots, A_n}(x) - f_{A_1, \ldots, A_n}(y)|_\infty = \frac{|x - y|_\infty}{|Q_n|_\infty^2}.
\]
By the invariant properties under bi-Lipschitz transformation and \( \sigma \)-stability of Hausdorff dimension [8], we get the desired result. \( \square \)

Here we cite a result due to J. Wu [21], which will be used later.

**Proposition 5.2.** Let \( S \) be a non-empty finite set of polynomials with strictly positive degree and coefficients lying in \( \mathbb{F}_q \), say \( S = \{ a_1, a_2, \ldots, a_m \} \). Write
\[
E_S = \{ x \in I : A_i(x) \in S \text{ for } i \geq 1 \}.
\]
Then \( \dim_H E_S = t \), where \( t \) is given by
\[
\sum_{k=1}^{m} q^{-2r \deg a_k} = 1. \quad (5.1)
\]
Let \( E = \{ x \in I : \deg A_n(x) \rightarrow \infty \} \). It is quite evident that
\[
E = \bigcap_{m=1}^{\infty} \bigcup_{N=1}^{\infty} E_N(m),
\]
where \( E_N(m) = \{ x \in I: \deg A_n(x) \geq m, \forall n \geq N \} \). In the light of Lemma 5.1, we know \( \dim_H E \leq \inf_{m \geq 1} \dim_H E_1(m) \). Thus, in the sequel, we concentrate on estimating \( \dim_H E_1(m) \).

**Lemma 5.3.** \( \dim_H E_1(m) = s_m \), where \( s_m \) is the unique solution to the equation

\[
\sum_{k=m}^{\infty} (q - 1)q^k q^{-2ks} = 1. \tag{5.2}
\]

**Proof.** It is clear that

\[
E_1(m) = \bigcap_{n=1}^{+\infty} \{ x \in I: \deg A_k(x) \geq m, \forall k \leq n \}
\]

\[
= \bigcap_{n=1}^{+\infty} \bigcup_{A_1, \ldots, A_n} \{ x \in I: A_1(x) = A_1, \ldots, A_n(x) = A_n \},
\]

where the union takes over all \( (A_1, \ldots, A_n) \in F[z]^n \) with \( \deg A_j \geq m, 1 \leq j \leq n \). As a consequence, we have

\[
\mathcal{H}^{d}(E_1(m)) \leq \liminf_{n \to \infty} \sum_{A_1, \ldots, A_n} \left( \frac{1}{q^2 \sum_{i=1}^{n} \deg A_i} \right)^{s_m} = \liminf_{n \to \infty} \prod_{i=1}^{n} \left( \frac{1}{q^2 \deg A_i} \right)^{s_m}
\]

\[
\leq \liminf_{n \to \infty} \left( \sum_{k=m}^{n} (q - 1)q^k q^{-2ks_m} \right)^{n} = 1.
\]

So \( \dim_H E_1(m) \leq s_m \).

On the other hand, define

\[
E(m, \alpha) = \{ x \in I: m \leq \deg A_n(x) \leq \alpha, \forall n \geq 1 \}.
\]

By Proposition 5.2, we have \( \dim_H E(m, \alpha) = s_m(\alpha) \), where \( s_m(\alpha) \) is the unique solution of \( \sum_{k=m}^{\alpha} (q - 1)q^k q^{-2ks} = 1 \). As a result, \( \dim_H E_1(m) \geq s_m(\alpha), \forall \alpha \geq m \). Notice that \( s_m(\alpha) \to s_m \) as \( \alpha \to \infty \), so we have \( \dim_H E_1(m) \geq s_m \). This completes the proof. \( \square \)

**Proposition 5.4.**

\[
\lim_{m \to \infty} s_m = \frac{1}{2}.
\]

**Proof.** We can find that \( s_m \) is monotonously decreasing as \( m \to \infty \) and \( s_m \geq \frac{1}{2} \) for all \( m \geq 1 \) from (5.2). For any \( \epsilon > 0 \), the series \( \sum_{k=m}^{\infty} (q - 1)q^k q^{-2k(1+\epsilon)} \) converges, so we can find \( m \) large enough such that \( \sum_{k=m}^{\infty} (q - 1)q^k q^{-2k(1+\epsilon)} \leq 1 \). This means that \( s_m \leq \frac{1}{2} + \epsilon \) for \( m \) large enough. By the arbitrariness of \( \epsilon \) we get the result of proposition. \( \square \)
Proof of Theorem 2.1. Lemmas 5.1, 5.3 and Proposition 5.4 give \( \dim_H E \leq 1/2 \), while the other direction of the inequality is just a consequence of Theorem 2.2. \( \square \)

6. Proof of Theorem 2.3

In this section we show Theorem 2.3 in detail. We divide it into three parts according as \( a = 1 \), \( 1 < a < \infty \) and \( a = \infty \).

**Case I.** \( a = 1 \). In this case, we have for any \( \epsilon > 0 \), \( \phi(n) < (1 + \epsilon)^n \) for all \( n \) sufficiently large. Then

\[
F \supset \{ x \in I : \deg A_n(x) \geq \phi(n), \ 1 \leq n < n_0, \ \deg A_n(x) \geq (1 + \epsilon)^n, \ n \geq n_0 \},
\]

for some \( n_0 = n_0(\epsilon) \). In light of Lemma 5.1 and Theorem 2.2, we have

\[
\dim_H F \geq \dim_H \{ x \in I : \deg A_n(x) \geq (1 + \epsilon)^n, \ n \geq 1 \} = \frac{1}{1 + 1 + \epsilon}.
\]

By the arbitrariness of \( \epsilon > 0 \), we have \( \dim_H F \geq 1/2 \). The upper bound of \( \dim_H F \) follows from Theorem 2.1, since \( F \subset \{ x \in I : \deg A_n(x) \to \infty \} \). As a result, we have

\[
\dim_H F = \frac{1}{2} = \frac{1}{a + 1}.
\] (6.1)

**Case II.** \( 1 < a < \infty \). The lower bound of \( \dim_H F \) can be done with the same argument as in Case I. For the upper bound of \( \dim_H F \), it should be noticed that for any \( \epsilon > 0 \), \( \phi(n) > (a - \epsilon)^n \) holds for infinitely many \( n \)'s. This gives that

\[
F \subset \{ x \in I : \deg A_n(x) \geq (a - \epsilon)^n, \ \text{i.o.} \ n \}
\]
as a consequence of Theorem 2.2, we have

\[
\dim_H F \leq \frac{1}{1 + a - \epsilon}.
\]

**Case III.** \( a = \infty \). In this case, we have for any \( a' > 1 \),

\[
F \subset \{ x \in I : \deg A_n(x) \geq (a' - \epsilon)^n, \ \text{i.o.} \ n \}.
\]

Thus it follows

\[
\dim_H F \leq \frac{1}{a' + 1}, \ \forall a' \geq 1.
\]

Cases I, II together with III give the desired claims.
7. Hausdorff dimension of \( \{ x \in I : \deg A_n(x) \geq \alpha n, \ i.o. \ n \} \)

Finally, we will get the Hausdorff dimension of \( \{ x \in I : \deg A_n(x) \geq \alpha n, \ i.o. \ n \} \), which will lead to the result of Theorem 2.4.

**Lemma 7.1.** Let \( s(\alpha) \) be the unique solution of

\[
\sum_{k=1}^{\infty} (q-1)q^k \left( \frac{1}{q^{2k+\alpha}} \right)^s = 1.
\]

Then \( s(\alpha) \) is continuous with respect to \( \alpha \). Furthermore,

\[
\lim_{\alpha \to 0} s(\alpha) = 1, \quad \lim_{\alpha \to \infty} s(\alpha) = \frac{1}{2}.
\]

**Proof.** (i) Fix \( \alpha > 0 \). For any \( \epsilon > 0 \), when \( \alpha < \alpha' < \alpha + \alpha \epsilon \), we will show that \( s(\alpha') < s(\alpha) < s(\alpha') + \epsilon \). The first inequality is obvious since \( s(\cdot) \) is monotonously decreasing. For the other one, it follows from the fact that

\[
\sum_{k=1}^{\infty} (q-1)q^k \left( \frac{1}{q^{2k+\alpha}} \right)^{s(\alpha') + \epsilon} \leq \frac{1}{q^\alpha \epsilon} \sum_{k=1}^{\infty} (q-1)q^k \left( \frac{1}{q^{2k+\alpha}} \right)^s(\alpha') = \frac{1}{q^\alpha \epsilon} q^{(\alpha'-\alpha)s(\alpha')} \leq q^{\alpha'-\alpha(1+\epsilon)} < 1.
\]

(ii) We show \( \lim_{\alpha \to 0} s(\alpha) = 1 \) only, since the other assertion can be established similarly. It should be noticed first that \( s(\alpha) < 1 \), for all \( \alpha > 0 \), since \( f_\alpha(1) = q^{-\alpha} < 1 \). On the other hand, for any \( \epsilon > 0 \), when \( \alpha < 2 \epsilon \), we have \( s(\alpha) > 1 - \epsilon \). This is the fact because

\[
\sum_{k=1}^{\infty} (q-1)q^k \left( \frac{1}{q^{2k+\alpha}} \right)^{1-\epsilon} \geq \sum_{k=1}^{\infty} (q-1)q^k \frac{1}{q^{2k}} = 1,
\]

where the second inequality follows from \( (2k+\alpha)(1-\epsilon) \leq 2k \), for all \( k \geq 1 \), when \( \alpha < 2 \epsilon \).

**Theorem 7.2.** For any \( \alpha > 0 \), let \( F = \{ x \in I : \deg A_n(x) \geq \alpha n, \ i.o. \ n \} \). Then \( \dim_H F = s(\alpha) \), where \( s(\alpha) \) is given in Lemma 7.1.

7.1. Upper bound

**Lemma 7.3.** \( \dim_H F \leq s(\alpha) \).

**Proof.** By the definition of \( F \), we have

\[
F = \limsup_{n \to \infty} \{ x \in I : \deg A_n(x) \geq \alpha n \}.
\]
\[= \bigcap_{N=1}^{+\infty} \bigcup_{n=N}^{+\infty} \{ x \in I : \deg A_{n+1}(x) \geq \alpha(n+1) \} \]
\[= \bigcap_{N=1}^{+\infty} \bigcup_{n=N}^{+\infty} \bigcup_{A_1, \ldots, A_n} \{ x \in I : A_k(x) = A_k, \ 1 \leq k \leq n, \ \deg A_{n+1}(x) \geq \alpha(n+1) \}, \]

where the third union takes over all \((A_1, \ldots, A_n) \in F[z]^n\) with strictly positive degree. Let

\[J(A_1, \ldots, A_n) = \bigcup_{\deg A_{n+1}(x) \geq \alpha(n+1)} I(A_1, \ldots, A_n, A_{n+1}).\]

Then we have

\[|J(A_1, \ldots, A_n)| = q^{-\alpha(n+1) - 2 \sum_{i=1}^n \deg A_i}. \quad (7.2)\]

For any \(x, y \in J(A_1, \ldots, A_n)\), let \(x \in I(A_1, \ldots, A_n, A_{n+1})\) and \(y \in I(A_1, \ldots, A_n, \tilde{A}_{n+1})\) with \(\deg A_{n+1} \leq \deg \tilde{A}_{n+1}\). Then

\[|x - y| = \left| \frac{A_{n+1} - \tilde{A}_{n+1}}{A_{n+1} \tilde{A}_{n+1} Q_n^2} \right| \leq \left| \frac{1}{A_{n+1} \tilde{A}_{n+1} Q_n^2} \right|, \quad (7.3)\]

and (7.3) will be met with equality if \(\deg A_{n+1} \neq \deg \tilde{A}_{n+1}\).

Since for any \(N \geq 1, \bigcup_{n \geq N} \bigcup_{A_1, \ldots, A_n} J(A_1, \ldots, A_n)\) is a covering system of \(F\), for any \(t > s(\alpha)\) we have

\[\mathcal{H}^t(F) \leq \liminf_{N \to \infty} \sum_{n=N}^{\infty} \sum_{A_1, \ldots, A_n} \left( \frac{1}{q^{\alpha(n+1) + 2 \sum_{i=1}^n \deg A_i}} \right)^t \]
\[= \liminf_{N \to \infty} \sum_{n=N}^{\infty} \frac{1}{q^{\alpha(n+1)t}} \prod_{j=1}^n \frac{1}{\sum_{j=1}^n \deg A_j} \left( \frac{1}{q^{2\deg A_j}} \right)^t \]
\[= \liminf_{N \to \infty} \sum_{n=N}^{\infty} \frac{1}{q^{\alpha(n+1)t}} \left( \sum_{k=1}^{\infty} (q - 1)q^k \frac{1}{q^{2kt}} \right)^n \]
\[\leq \liminf_{N \to \infty} \sum_{n=N}^{\infty} \left( \frac{q^{\alpha s(\alpha)}}{q^{\alpha t}} \right)^n < \infty.\]

This implies \(\dim_H F \leq t\). Since \(t\) is arbitrary, we have \(\dim_H F \leq s(\alpha)\). \(\square\)

### 7.2. Lower bound

In order to obtain lower bound of Hausdorff dimensions, we need the mass distribution principle, see [8, Proposition 4.2].
Lemma 7.4 (Mass distribution principle). Suppose $E \subset I$, and $\mu$ is a measure with $\mu(E) > 0$. If there exist constants $c > 0$ and $\delta > 0$ such that

$$\mu(D) \leq c|D|^s$$

(7.4)

for all discs $D$ with diameter $|D| \leq \delta$, then

$$\dim E \geq s.$$ 

Lemma 7.5. $\dim_H F \geq s(\alpha)$.

Proof. To get the lower bound, we will construct a subset of $F$ and give a probability measure supported on it. In the light of mass distribution principle, we need to show the given measure satisfies (7.4). To make the proof more clear, we divide it into several parts, and every part has its own obvious aim.

7.2.1. A subset $F_\beta$ of $F$

In this part, we construct $F_\beta$—a more regular subset of $F$. This means the lower bound of Hausdorff dimension of $F_\beta$ is surely a lower bound of $\dim_H F$.

Fix an integer sequence $n_0, n_1, \ldots, n_k, \ldots$ satisfying

$$n_0 = 1, \quad n_{k+1} \geq (k+1)n_k, \quad \forall k \geq 0.$$ 

Let $F_\beta = \{x \in I: 1 \leq \deg A_n(x) \leq \beta, \forall 1 \leq n \neq n_k, \deg A_nk = [\alpha n_k] + 1, \forall k \geq 1\}$. It is obvious that $F_\beta \subset F$. Let $s_\beta(\alpha)$ be the unique solution of

$$\sum_{k=1}^{\beta} (q - 1)q^k q^{-2ks} = q^{ks}.$$ 

We will show that $\dim_H F_\beta \geq s_\beta$.

7.2.2. Fractal structure of $F_\beta$

In the sequel we often make use of the code space to express the fractal structure of $F_\beta$ clearly.

For any $n \geq 1$, denote

$$D_n = \{(\sigma_1, \ldots, \sigma_n) \in \mathbb{F}_q[z]^n: 1 \leq \deg \sigma_j(x) \leq \beta, \deg \sigma_nk = [\alpha n_k] + 1, 1 \leq j \neq n_k \leq n\},$$

$$D = \bigcup_{n=0}^{\infty} D_n.$$ 

For any $n \geq 1$, $(\sigma_1, \ldots, \sigma_n) \in D_n$, define

$$J(\sigma_1, \ldots, \sigma_n) = \bigcup_{\sigma_{n+1}} I(\sigma_1, \ldots, \sigma_{n+1}).$$
where the union takes over all $\sigma_{n+1}$ such that $(\sigma_1, \ldots, \sigma_{n+1}) \in D_{n+1}$ and call it an $n$th order admissible cylinder. Then

$$F_{\beta} = \bigcap_{n \geq 1} \bigcup_{\sigma_1, \ldots, \sigma_n \in D_n} J(\sigma_1, \ldots, \sigma_n).$$

### 7.2.3. The measure $\mu$ on $F_{\beta}$

In this part, we define a probability measure $\mu$ supported on $F_{\beta}$. We will eventually show that the measure satisfies (7.4).

Let $m_1 = n_1 - 1$, $m_k = n_k - n_{k-1} - 1$, $\ldots$. Now we define a set function $\mu : \{J(\sigma), \sigma \in D\} \rightarrow \mathbb{R}^+$ given as follows. For $(\sigma_1, \ldots, \sigma_{n_1}, \sigma_{n_1+1}, \ldots, \sigma_{n_k})$, we will denote

$$q_{m_1}(\sigma_{n_1+1}, \ldots, \sigma_{n_k}) = q^{2 \sum_{k=n_{j-1}+1}^{n_j} \deg \sigma_k}.$$

Without causing any confusion, we will write $q_{m_j} = q_{m_1}(\sigma_{n_{j-1}+1}, \ldots, \sigma_{n_j})$ for simplicity.

For $n = n_0 = 1$,

$$\mu(J(\sigma_1)) = \frac{1}{(q-1)q^{\lceil \alpha \rceil}+1}. \quad (7.5)$$

For $n = n_k$,

$$\mu(J(\sigma_1, \ldots, \sigma_{n_k})) = \frac{1}{(q-1)q^{\lceil \alpha n_k \rceil}+1} \mu(J(\sigma_1, \ldots, \sigma_{n_k-1})). \quad (7.6)$$

For $n_{k-1} < n < n_k$, for some $k \geq 1$, set

$$\mu(J(\sigma_1, \ldots, \sigma_n)) = \frac{1}{(q^\alpha q^{2 \deg \sigma_n}s_{\beta})} \mu(J(\sigma_1, \ldots, \sigma_{n-1})). \quad (7.7)$$

By the definition of $\mu$, immediately we have

$$\mu(J(\sigma_1, \ldots, \sigma_{n_{k-1}})) = \left(\frac{1}{q^\alpha m_k} q_{m_k}^{2} \right)^{s_{\beta}} \mu(J(\sigma_1, \ldots, \sigma_{n_{k-1}})).$$

Thus,

$$\mu(J(\sigma_1, \ldots, \sigma_{n_k})) = \frac{1}{(q-1)q^{\lceil \alpha n_k \rceil}+1} \left(\frac{1}{q^\alpha m_k} q_{m_k}^{2} \right)^{s_{\beta}} \mu(\sigma_1, \ldots, \sigma_{n_{k-1}})$$

$$= \prod_{j=1}^{k} \frac{1}{(q-1)q^{\lceil \alpha n_j \rceil}+1} \left(\frac{1}{q^\alpha m_j} q_{m_j}^{2} \right)^{s_{\beta}},$$

$$\mu(J(\sigma_1, \ldots, \sigma_{n_k-1})) = \prod_{j=1}^{k-1} \frac{1}{(q-1)q^{\lceil \alpha n_j \rceil}+1} \left(\frac{1}{q^\alpha m_j} q_{m_j}^{2} \right)^{s_{\beta}} \cdot \left(\frac{1}{q^\alpha m_k} q_{m_k}^{2} \right)^{s_{\beta}} \cdot \mu(J(\sigma_1, \ldots, \sigma_{n_{k-1}})).$$

It is easy to check that the measure $\mu$ is well defined, because by (7.1)
Thus by Kolmogorov extension theorem, the set function $\mu$ can be extended into a probability measure supported on $F_\beta$.

7.2.4. The measure on cylinders

In this part, we estimate the measure on cylinders and show (7.4) is true on cylinders.

For any $\epsilon > 0$, there exists a $k_0$ such that for all $k > k_0$,

$$2^{2(|\alpha n_k| + 1)}s_\beta \leq \alpha n_k + \alpha m_k s_\beta, \quad \alpha m_k s_\beta \geq (s_\beta - \epsilon)(|\alpha n_k| + 1). \quad (7.8)$$

Let $c_0 = \prod_{j=1}^{k_0} q^{2(|\alpha n_j| + 1)} q^{2n_k \beta}$. By all these assumptions, we have

$$\mu(J(\sigma_1, \ldots, \sigma_{n_k})) = c_0 \prod_{j=1}^k \left( \frac{1}{q - 1} q^{2(|\alpha n_j| + 1)} \left( \frac{1}{q^{\alpha m_j} q^{2m_j}} \right)^s \right) \leq \prod_{j=k_0}^k \left( \frac{1}{q^{2(|\alpha n_j| + 1)} q^{2m_j}} \right)^s \leq c_0 \prod_{j=1}^k \left( q^{2(|\alpha n_j| + 1)} q^{2m_j} \right)^s = c_0 (q^{-2 \sum_{j=1}^{n_k} \deg \sigma_j}s_\beta).$$

Notice that following the method in showing (7.2), we have

$$|J(\sigma_1, \ldots, \sigma_{n_k})| = q^{-2 \sum_{j=1}^{n_k} \deg \sigma_j - 1}.$$ 

As a result, we have

$$\mu(J(\sigma_1, \ldots, \sigma_{n_k})) \leq c_0 q |J(\sigma_1, \ldots, \sigma_{n_k})|^{s_\beta}. \quad (7.9)$$

Similarly, we have

$$\mu(J(\sigma_1, \ldots, \sigma_{n_k-1})) \leq c_0 \prod_{j=1}^{k-1} \left( q^{2(|\alpha n_j| + 1)} q^{2m_j} \right)^s \leq c_0 (q^{-2 \sum_{j=1}^{n_k-1} \deg \sigma_j - (|\alpha n_k| + 1)}s_\beta - \epsilon). \quad (7.10)$$

Since the length of $J(\sigma_1, \ldots, \sigma_{n_k-1})$ is just $q^{-2 \sum_{j=1}^{n_k-1} \deg \sigma_j - (|\alpha n_k| + 1)}$, we have

$$\mu(J(\sigma_1, \ldots, \sigma_{n_k-1})) \leq c_0 |J(\sigma_1, \ldots, \sigma_{n_k-1})|^{s_\beta - \epsilon}. \quad (7.11)$$

For $n_k - 1 < n < n_k - 1$, we have
\[
\mu(J(\sigma_1, \ldots, \sigma_n)) = \prod_{j=1}^{k-1} \frac{1}{(q-1)q^{[\alpha]_j} + 1} \left( q^{\alpha_m} q_m^2 \right)^{s_j} \prod_{j=n_k-1}^{n} \left( q^{\alpha q^{2\deg j}} \right)^{s_j} \leq c_0 q^{-2 \sum_{j=1}^{n_k-1} \deg \sigma_j} \left( q^{2 \sum_{j=n_k-1+1}^{n} \deg \sigma_j} \right)^{s_j}.
\]

Since the length of \( J(\sigma_1, \ldots, \sigma_n) \) (\( n \neq n_k - 1 \)) is \( q^{-2 \sum_{j=1}^{n_k-1} \deg \sigma_j - 1} \), we have
\[
\mu(J(\sigma_1, \ldots, \sigma_n)) \leq c_0 |J(\sigma_1, \ldots, \sigma_n)|^{s_j}.
\] (7.12)

Combining (7.9), (7.11) and (7.12), we have, for any \( n \geq 1 \), \( (\sigma_1, \ldots, \sigma_n) \in D_n \),
\[
\mu(J(\sigma_1, \ldots, \sigma_n)) \leq c |J(\sigma_1, \ldots, \sigma_n)|^{s_j - \epsilon}.
\] (7.13)

### 7.2.5. The measure on an arbitrary ball

In this part, we estimate the measure \( \mu \) on an arbitrary ball and show (7.4) is true. This will then complete the proof of Theorem 7.2.

For any \( x \in F_{\beta}, r \leq \min_{\sigma \in D_{n_k}} \| J(\sigma) \| \) (\( J(\sigma) \), there exists \( (\sigma_1, \sigma_2, \ldots) \in D \) such that \( x \in J(\sigma_1, \ldots, \sigma_k) \) for any \( k \geq 1 \), and for some \( n \geq n_k \), we have \( |J(\sigma_1, \ldots, \sigma_{n+1})| \leq r < |J(\sigma_1, \ldots, \sigma_n)| \).

**Case I.** \( n = n_k - 1 \), \( |J(\sigma_1, \ldots, \sigma_{n_k})| \leq r < |J(\sigma_1, \ldots, \sigma_{n_k-1})| \).

On one hand, since \( r < |J(\sigma_1, \ldots, \sigma_{n_k})| \leq |I(\sigma_1, \ldots, \sigma_{n_k-1})| \), then we have \( B(x, r) \subseteq I(\sigma_1, \ldots, \sigma_{n_k-1}) \). This implies
\[
\mu(B(x, r)) = \mu(I(\sigma_1, \ldots, \sigma_{n_k-1})) = \mu(J(\sigma_1, \ldots, \sigma_{n_k-1})).
\] (7.14)

On the other hand, since \( B(x, r) \subseteq I(\sigma_1, \ldots, \sigma_{n_k-1}) \), we consider how many \( n_k \) admissible cylinders contained in \( I(\sigma_1, \ldots, \sigma_{n_k}) \) can intersect \( B(x, r) \), and denote the number by \( N_r \). Since \( |I(\sigma_1, \ldots, \sigma_{n_k})| = |J(\sigma_1, \ldots, \sigma_{n_k})| = |J(\sigma_1, \ldots, \sigma_{n_k})|, \) for all \( (\sigma_1, \ldots, \sigma_{n_k}) \in D_{n_k} \) with \( \sigma_{n_k} \neq \sigma_{n_k} \), as a result, if \( J(\sigma_1, \ldots, \sigma_{n_k}) \cap B(x, r) \neq \emptyset \), then \( I(\sigma_1, \ldots, \sigma_{n_k}) \subseteq B(x, r) \). Thus
\[
P(B(x, r)) \geq \sum_{I(\sigma_1, \ldots, \sigma_{n_k}) \subseteq B(x, r) \neq \emptyset} P(I(\sigma_1, \ldots, \sigma_{n_k})) = N_r q^{-2 \sum_{j=1}^{n_k} \deg \sigma_j}.
\]

From this it follows that
\[
N_r \leq q r q^{2 \sum_{j=1}^{n_k} \deg \sigma_j}.
\]

As a consequence,
\[
\mu(B(x, r)) \leq \sum_{I(\sigma_1, \ldots, \sigma_{n_k}) \subseteq B(x, r) \neq \emptyset} \mu(J(\sigma_1, \ldots, \sigma_{n_k})) = N_r \mu(J(\sigma_1, \ldots, \sigma_{n_k}))
\]
\[
\leq q r q^{2 \sum_{j=1}^{n_k} \deg \sigma_j} \prod_{j=1}^{k} \frac{1}{(q-1)q^{[\alpha]_j} + 1} \left( q^{\alpha_m} q_m^2 \right)^{s_j} \leq q r q^{2 \sum_{j=1}^{n_k-1} \deg \sigma_j + \deg \sigma_{n_k}} \frac{1}{q-1} \mu(J(\sigma_1, \ldots, \sigma_{n_k-1})).
\]
Combining this and (7.14), we have

$$\mu(B(x, r)) \leq \mu(J(\sigma_1, \ldots, \sigma_{n_k-1})) \min \left\{ 1, \frac{q}{q-1}rq^{-2 \sum_{j=1}^{n_k-1} \deg \sigma_j + \deg \sigma_{n_k}} \right\}.$$ 

By (7.10) and the inequality that $\min \{a, b\} \leq asb$ for any $0 \leq s \leq 1$, we have

$$\mu(B(x, r)) \leq c_0 \left( \frac{qr}{q-1} \right)^{s\beta - \epsilon} \leq c_0 \frac{q}{q-1}r^{s\beta - \epsilon}. \quad (7.15)$$

**Case II.** $n_{k-1} \leq n < n_k - 1.$

In this case, by the definition of $\mu$ (see (7.6)), thus we have

$$\mu(J(\sigma_1, \ldots, \sigma_{n+1})) \leq \frac{1}{q^{\alpha + 2\beta}} \mu(J(\sigma_1, \ldots, \sigma_n)).$$

So we have

$$\mu(B(x, r)) \leq \mu(J(\sigma_1, \ldots, \sigma_n)) \leq \frac{1}{q^{\alpha + 2\beta}} \mu(J(\sigma_1, \ldots, \sigma_{n+1}))$$

$$\leq \frac{1}{q^{\alpha + 2\beta}}c_1 |J(\sigma_1, \ldots, \sigma_{n+1})|^{s\beta - \epsilon} \leq \frac{1}{q^{\alpha + 2\beta}}c_1 r^{s\beta - \epsilon}.$$ 

Combining Cases I and II, we have

$$\mu(B(x, r)) \leq cr^{s\beta - \epsilon},$$

where $c$ is some constant which only depends on $q$, $\beta$, $\alpha$.

By Lemma 7.4, $\dim_H F_\beta \geq s_\beta - \epsilon$. By the arbitrariness of $\epsilon$ and the fact that $F \supset F_\beta$, we have $\dim_H F \geq s_\beta$ for any $\beta \geq 1$. Letting $\beta \rightarrow \infty$, we get $\dim_H F \geq s(\alpha)$. This completes the proof of Lemma 7.5. \qed

7.3. Proof of Theorem 7.2

**Proof of Theorem 7.2.** Lemma 7.3 together with Lemma 7.5 complete the proof. \qed

**Corollary 7.6.** Let $\mathcal{L} \subset \mathbb{N}$ be an infinite integer set. Then

$$\dim_H E_\mathcal{L} = \left\{ x \in I : \deg A_n(x) \geq \alpha n, n \in \mathcal{L} \right\} = s(\alpha).$$

**Proof.** This follows just by choosing the integer sequence $\{n_k\}$ from $\mathcal{L}$. \qed
8. Proof of Theorem 2.4

In this section, we prove Theorem 2.4 by using the result of Theorem 7.2. At first, we show a lemma which will be used in the proof.

**Lemma 8.1.** Let \( G = \{ x : x \in I, \ deg A_n(x) \geq \phi(n) \ for \ infinitely \ many \ n \} \), and if

\[
\lim_{n \to +\infty} \frac{\phi(n+1)}{n} = +\infty.
\]

Then \( \dim_H G \leq \frac{1}{2} \).

**Proof.** Without loss of generality, we can suppose \( \phi \) is an integer valued function. We define

\[
G_n = \{ x \in I: deg A_{n+1} \geq \phi(n+1) \}
\]

\[
= \bigcup_{A_1, \ldots, A_n} \{ x \in I(A_1, \ldots, A_n): deg A_{n+1}(x) \geq \phi(n+1) \},
\]

and let

\[
G = \bigcap_{m=1}^{+\infty} \bigcup_{n=m}^{+\infty} G_n.
\]

Similar to (7.2), we have

\[
\left| \left\{ x \in I(A_1, \ldots, A_n): deg A_{n+1}(x) \geq \phi(n+1) \right\} \right| = q^{-2(deg A_1 + \cdots + deg A_n) - \phi(n+1)}.
\]

For any \( 1/2 < s < 1 \), set

\[
\zeta = \sum_A q^{-2 \deg A_s} = \sum_{k=1}^{\infty} (q-1)q^k \frac{1}{q^{2ks}} < \infty.
\]

As a consequence, we have

\[
\mathcal{H}^s(G) \leq \liminf_{m \to +\infty} \sum_{n=m}^{+\infty} \sum_{A_1, \ldots, A_n} q^{-2(deg A_1 + \cdots + deg A_n)s - \phi(n+1)s}
\]

\[
= \liminf_{m \to +\infty} \sum_{n=m}^{+\infty} q^{-\phi(n+1)s} \zeta^n < \infty,
\]

where the last assertion follows from \( \frac{\phi(n+1)}{n} \to +\infty \). Thus \( \dim_H G \leq \frac{1}{2} \). \( \square \)
Proof of Theorem 2.4.

(1) $\alpha = 0$, i.e. $\liminf_{n \to \infty} \frac{\phi(n)}{n} = 0$.

Since $\alpha = 0$, then for all $\epsilon > 0$, $\phi(n) < \epsilon n$ holds for infinitely many $n$’s. Let $\mathcal{L} = \{n : \phi(n) < \epsilon n\}$. Then $\mathcal{L}$ is an infinite integer sequence. Furthermore, $E \supset \{x \in I : \deg A_n(x) \geq n\epsilon, \ n \in \mathcal{L}\}$.

By Corollary 7.6, we have

$$\dim_H E \geq \dim_H \{x \in I : \deg A_n(x) \geq n\epsilon, \ n \in \mathcal{L}\} = s(\epsilon).$$

By the continuity of $s(\alpha)$ (we saw this in Lemma 6.3), we get $\dim_H E = 1$.

(2) $0 < \alpha < \infty$, it is same as (1), we have $\phi(n) < (\alpha + \epsilon)n$ holds for infinitely many $n$’s.

$$\dim_H E \geq \dim_H \{x \in I : \deg A_n(x) \geq n(\alpha + \epsilon), \ n \in \mathcal{L}\} = s(\alpha).$$

We also have $\phi(n) > (\alpha - \epsilon)n$ for all $n$ large enough. Then

$$\dim_H E \leq \dim_H \{x \in I : \deg A_n(x) \geq n(\alpha - \epsilon), \ \text{i.o.} \ n\} = s(\alpha - \epsilon).$$

As a result, $\dim_H E \leq s(\alpha)$.

(3) $\alpha = \infty$, we consider $\liminf_{n \to \infty} \log \phi(n)/n = \log b$.

(i) $b = 1$. We have $\forall \epsilon > 0, \phi(n) < (1 + \epsilon)^n$ holds for infinitely many $n$’s. By Theorem 2.2, $\dim_H E \geq \frac{1}{1 + 1 + \epsilon}$. Hence $\dim_H E \geq \frac{1}{2}$. On the other hand, since $\liminf_{n \to \infty} \phi(n)/n = \infty$, by Lemma 8.1, we have $\dim_H E \leq \frac{1}{2}$.

(ii) $1 < b < \infty$. At this case $\forall \epsilon > 0, \phi(n) < (b + \epsilon)^n$ holds for infinitely many $n$’s. Then we have

$$\dim_H E \geq \dim_H \{x \in I : \deg A_n(x) \geq (b + \epsilon)^n, \ \text{i.o.} \ n\} = \frac{1}{1 + b + \epsilon}.$$  

We also have $\forall \epsilon > 0, \phi(n) > (b - \epsilon)^n$ holds for all $n$ large enough. This gives that

$$\dim_H E \leq \dim_H \{x \in I : \deg A_n(x) \geq (b - \epsilon)^n, \ \text{i.o.} \ n\} = \frac{1}{1 + b - \epsilon}.$$  

Hence we get $\dim_H E = \frac{1}{1 + b}$.

(iii) $b = \infty$. We have $\forall \epsilon > 0, \forall M > 0, \phi(n) > (M - \epsilon)^n$ holds for all $n$ large enough. Then

$$\dim_H E \leq \frac{1}{1 + M - \epsilon}.$$  

Since $M$ is arbitrary, $\dim_H E = 0$.

This completes the proof. $\square$
References