Discrete Applied Mathematics 160 (2012) 1822–1836

Contents lists available at SciVerse ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

The hamiltonicity and path t-coloring of Sierpiński-like graphs[☆]

Bing Xue[a], Liancui Zuo[b,*], Guojun Li[a]

[a] School of Mathematics, Shandong University, Jinan, 250100, China

[b] College of Mathematical Science, Tianjin Normal University, Tianjin, 300387, China

ARTICLE INFO

Article history:
Received 15 June 2011
Received in revised form 18 March 2012
Accepted 19 March 2012
Available online 18 April 2012

Keywords:
Hamiltonicity
Path t-coloring
Vertex linear arboricity
Sierpiński graph
Matching

ABSTRACT

A mapping ϕ from $V(G)$ to $\{1, 2, \ldots, t\}$ is called a *path t-coloring* of a graph G if each $G[\phi^{-1}(i)]$, for $1 \le i \le t$, is a linear forest. The vertex linear arboricity of a graph G, denoted by $\mathrm{vla}(G)$, is the minimum t for which G has a path t-coloring. Graphs $S[n, k]$ are obtained from the Sierpiński graphs $S(n, k)$ by contracting all edges that lie in no induced K_k. In this paper, the hamiltonicity and path t-coloring of Sierpiński-like graphs $S(n, k)$, $S^+(n, k)$, $S^{++}(n, k)$ and graphs $S[n, k]$ are studied. In particular, it is obtained that $\mathrm{vla}(S(n, k)) = \mathrm{vla}(S[n, k]) = \lceil k/2 \rceil$ for $k \ge 2$. Moreover, the numbers of edge disjoint Hamiltonian paths and Hamiltonian cycles in $S(n, k)$, $S^+(n, k)$ and $S^{++}(n, k)$ are completely determined, respectively.

1. Introduction

In this paper, all graphs considered are simple. Given a graph G, we use $V(G)$, $E(G)$, $\Delta(G)$ to denote the vertex set, the edge set and the maximum degree of G, respectively. For any $v \in V(G)$, let $N_G(v)$ be the set of all vertices adjacent to v in G. For a real number x, $\lceil x \rceil$ is denoted by the least integer not less than x. For any integers a and b, we use the symbol $[a, b]$ to denote the set $\{a, a+1, \ldots, b\}$, where $a \le b$. A complete graph with n vertices is denoted by K_n. A matching of a graph G is a set of edges in which any two edges are not adjacent to each other. A Hamiltonian path (cycle) of a graph G is a path (cycle) which contains every vertex of G. A *partition* of a graph G is a set of non-empty vertex subsets $\{V_1, V_2, \ldots, V_s\}$ such that $V_1 \cup V_2 \cup \cdots \cup V_s = V(G)$ and $V_i \cap V_j = \emptyset$ for all $i \ne j$. A *decomposition* of a graph G is a set of graphs $\{G_1, G_2, \ldots, G_s\}$ such that $E(G_1) \cup E(G_2) \cup \cdots \cup E(G_s) = E(G)$ and $E(G_i) \cap E(G_j) = \emptyset$ for all $i \ne j$. If a graph G has a decomposition $\{G_1, G_2, \ldots, G_s\}$, then we say that $\{G_1, G_2, \ldots, G_s\}$ decomposes G, or G can be decomposed into $\{G_1, G_2, \ldots, G_s\}$. A linear forest is such a graph that every connected component is a path.

A *t-coloring* of a graph G is a mapping ϕ from $V(G)$ to $[1, t]$. With respect to a given t-coloring ϕ, $\phi^{-1}(i)$ is denoted by the set of all vertices of G colored i, and $G[\phi^{-1}(i)]$ is denoted by the subgraph induced by $\phi^{-1}(i)$ in G. If $\phi^{-1}(i)$ is an independent set for any $1 \le i \le t$, then ϕ is called a proper t-coloring. The chromatic number $\chi(G)$ of a graph G is the minimum t for which G has a proper t-coloring. If every $G[\phi^{-1}(i)]$, for any $1 \le i \le t$, is a linear forest, then ϕ is called a path t-coloring. The vertex linear arboricity of a graph G, denoted by $\mathrm{vla}(G)$, is the minimum t for which G has a path t-coloring. In other words, the vertex linear arboricity $\mathrm{vla}(G)$ of a graph G is the minimum number of subsets into which the vertex set $V(G)$ can be partitioned so that each subset induces a subgraph which is a linear forest.

Graphs of "Sierpiński type" play an important part in many different areas of mathematics as well as in several other scientific fields. The graphs $S(n, 3)$ were generalized to the Sierpiński graphs $S(n, k)$ in [12]. The motivation for generalization

☆ Supported by NSFC with codes of 61070095, 60873207 and 61103073.

* Corresponding author. Tel.: +86 0 22 23684013.
E-mail addresses: zuolc@vip.sina.com, zuolc@yahoo.com.cn, lczuo@mail.tjnu.edu.cn (L. Zuo).

0166-218X/$ – see front matter Crown Copyright © 2012 Published by Elsevier B.V. All rights reserved.
doi:10.1016/j.dam.2012.03.022

came from topological studies of the Lipscomb's space [16,18]. In fact, the Sierpiński graphs were also independently studied in [20]. The graphs $S(n,k)$ have many interesting properties and were studied from different points of view. In [13], unique 1-perfect codes in Sierpiński graphs are studied. Alternative arguments for uniqueness of 1-perfect codes in $S(n,k)$ were presented in [5] in order to determine their optimal $L(2,1)$-labeling. Recently, covering codes and equitable $L(2,1)$-labelings of Sierpiński graphs were studied in [1,4]. An appealing relation is that $S(n,3)$ is isomorphic to the graphs of the Tower of Hanoi puzzle with n disks [6,12] and had been extensively studied in [7,21]. In [14], $S^{+}(n,k)$ and $S^{++}(n,k)$ are introduced and the crossing numbers of Sierpiński-like graphs are completely determined. In [8,10], vertex, edge and total colorings of Sierpiński-like graphs are studied. Moreover, the hub number of Sierpiński graphs is obtained in [15]. Besides the mentioned properties, several metric properties are also studied in [9,19].

In this paper, the hamiltonicity and path t-coloring of Sierpiński-like graphs $S(n,k), S^{+}(n,k), S^{++}(n,k)$ and graphs $S[n,k]$ are studied. It is obtained that $S(n,k)$ can be decomposed into edge disjoint union of $k/2$ Hamiltonian paths, $S^{+}(n,k)$ and $S^{++}(n,k)$ can be decomposed into edge disjoint union of $k/2$ Hamiltonian cycles for even $k \geq 2$, respectively, and $S(n,k), S^{+}(n,k)$ and $S^{++}(n,k)$ possess $(k-1)/2$ edge disjoint Hamiltonian cycles for odd $k \geq 3$, respectively. Moreover, we have shown that $\mathrm{vla}(S(n,k)) = \mathrm{vla}(S[n,k]) = \lceil k/2 \rceil$ for $k \geq 2$. Furthermore, for $S(n,k)$, we proved that (1) if $k \geq 2$ is even, then there must exist a path $k/2$-coloring ϕ from $V(G)$ to $[1,k/2]$ satisfying: $(a)|\phi^{-1}(i)| = 2k^{n-1}$ and $(b) G[\phi^{-1}(i)]$ is a path for each $i \in [1,k/2]$; (2) if $k \geq 3$ is odd, then there must exist a path $(k+1)/2$-coloring φ from $V(G)$ to $[1,(k+1)/2]$ satisfying: (a) $|\varphi^{-1}(i)| = 2k^{n-1}$ for every $i \in [1,(k-1)/2]$, and $|\varphi^{-1}((k+1)/2)| = k^{n-1}$, and (b) each $G[\varphi^{-1}(i)]$, for every $i \in [1,(k-1)/2]$, is a path, and $G[\varphi^{-1}((k+1)/2)]$ is a subgraph consisting of $(k^{n-1}-1)/2$ isolated edges and one extreme vertex.

2. Preliminaries

As the preparation, the following lemmas are needed.

Lemma 2.1 (*[2]*). *For $n \geq 3$, the complete graph K_n is decomposable into edge disjoint Hamiltonian cycles if and only if n is odd. For $n \geq 2$, the complete graph K_n is decomposable into edge disjoint Hamiltonian paths if and only if n is even.*

Lemma 2.2 (*[3]*). *Let $G = K_{2n}$ and $V(G) = \{v_0, v_1, \ldots, v_{2n-1}\}$. For $i \in [0, n-1]$, put*

$$P_i = v_{0+i}v_{1+i}v_{2n-1+i}v_{2+i}v_{2n-2+i}\cdots v_{n+1+i}v_{n+i},$$

where the indices of v_j are taken modulo $2n$. Then P_i, $i \in [0, n-1]$, are disjoint Hamiltonian paths of G.

Corollary 2.3. *In Lemma 2.2, we have $N_{P_i}(v_j) = \{v_{j+1}\}$ if $j \in \{i, i+n\}$, and $N_{P_i}(v_j) = \{v_{j+2(i-j)}, v_{j+2(i-j)+1}\}$ if $j \notin \{i, i+n\}$, where $v_t \in \{v_0, v_1, \ldots, v_{2n-1}\}$ and the indices of v_t are taken modulo $2n$.*

Lemma 2.4 (*[22]*). *Let $G = K_{2n+1}$ and $V(G) = \{v_0, v_1, \ldots, v_{2n}\}$. For $i \in [0, n-1]$, let*

$$P_i = v_{0+i}v_{1+i}v_{2n+i}v_{2+i}v_{2n-1+i}\cdots v_{n+i}v_{n+1+i},$$

and $M = \{v_{n-i}v_{n+i} : i \in [1, n]\}$, where the indices of v_j are taken modulo $2n+1$. Then P_i, $i \in [0, n-1]$, are edge disjoint Hamiltonian paths, and M is a maximum matching of G.

In Lemma 2.4, note that there exists only one vertex v_n which is not incident with any edge of M.

Corollary 2.5. *In Lemma 2.4, we have $N_{P_i}(v_j) = \{v_{j+1}\}$ if $j = i$, $N_{P_i}(v_j) = \{v_{j-1}\}$ if $j = i+n+1$, and $N_{P_i}(v_j) = \{v_{j+2(i-j)}, v_{j+2(i-j)+1}\}$ if $j \notin \{i, i+n+1\}$, where $v_t \in \{v_0, v_1, \ldots, v_{2n}\}$ and the indices of v_t are taken modulo $2n+1$.*

Lemma 2.6 (*[22]*). *Let $G = K_{2n}$ and $V(G) = \{v_0, v_1, \ldots, v_{2n-1}\}$. For $i \in [0, 2n-2]$, let*

$$M_i = \{v_i v_{2n-1}\} \cup \{v_{i+j}v_{i-j} \mid j \in [1, n-1]\},$$

where the indices of v_t are taken modulo $2n-1$ except for the index $2n-1$ of v_{2n-1}. Then $M_p \cup M_q$ forms a Hamiltonian cycle of G for any $p \neq q$. Thus, K_{2n} contains $n-1$ edge disjoint Hamiltonian cycles.

Lemma 2.7 (*[17]*). *For every finite graph G, $\mathrm{vla}(G) \leq \lceil (\Delta(G)+1)/2 \rceil$, moreover, if $\Delta(G)$ is even, then $\mathrm{vla}(G) = \lceil (\Delta(G)+1)/2 \rceil$ if and only if G is the complete graph of order $\Delta(G)+1$ or a cycle.*

Definition 2.8 (*[12]*). The Sierpiński graph $S(n,k)$ is defined as follows. For $n \geq 1$ and $k \geq 1$, the vertex set of $S(n,k)$ consists of all n-tuples of integers $1, 2, \ldots, k$, that is, $V(S(n,k)) = [1,k]^n$. Two different vertices $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ are adjacent if and only if there exists an $h \in [1, n]$ such that

(a) $u_t = v_t$ for $t \in [1, h-1]$;
(b) $u_h \neq v_h$;
(c) $u_t = v_h$ and $v_t = u_h$ for $t \in [h+1, n]$.

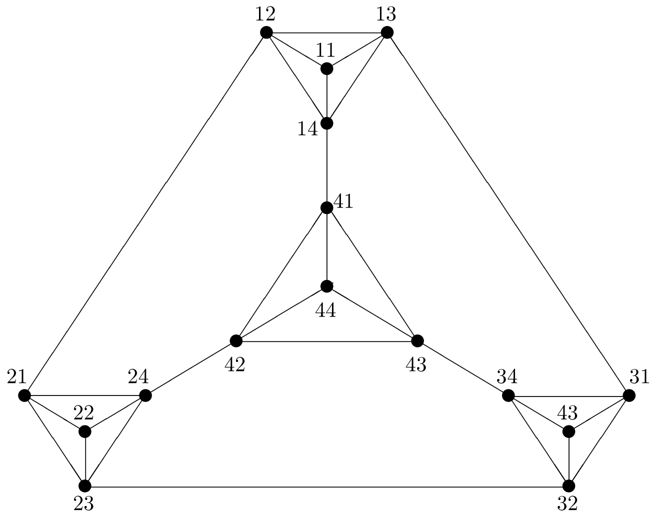

Fig. 1. The Sierpiński graph $S(2, 4)$.

We denote $(u_1, u_2, \ldots, u_n)$ by $\langle u_1u_2 \cdots u_n \rangle$ or $u_1u_2 \cdots u_n$ briefly. The vertices $\langle ii \cdots i \rangle$, $i \in [1, k]$, are called the *extreme vertices* of $S(n, k)$. For $i \in [1, k]$ and $n \geq 2$, let $S_i(n, k)$ denote the subgraph of $S(n, k)$ induced by the vertices of the form $\langle i \cdots \rangle$. Clearly, $S_i(n, k)$ is isomorphic to $S(n-1, k)$. The edges of $S(n, k)$ that lie in no induced K_k are called *bridge edges*. Note that $S_i(n, k)$ and $S_j(n, k)$, $i \neq j$, are connected by a single bridge edge between vertices $ij \cdots j$ and $ji \cdots i$. The Sierpiński graph $S(2, 4)$ is shown in Fig. 1.

Definition 2.9 (*[14]*)**.** The graph $S^+(n, k)$ is defined as follows. For $n \geq 1$ and $k \geq 1$, the graph $S^+(n, k)$ is obtained from Sierpiński graph $S(n, k)$ by adding a new vertex w and edges joining w with k extreme vertices of $S(n, k)$.

Definition 2.10 (*[14]*)**.** The graph $S^{++}(n, k)$ is defined as follows. For $n = 1$, let $S^{++}(1, k) = K_{k+1}$. For $n \geq 2$ and $k \geq 1$, the graph $S^{++}(n, k)$ is obtained from the disjoint union of $k + 1$ copies of $S(n-1, k)$ in which the extreme vertices in distinct copies of $S(n-1, k)$ are connected as the complete graph K_{k+1}.

Equivalently, the graph $S^{++}(n, k)$ can be obtained from the vertex disjoint union of a copy of $S(n, k)$ and a copy $S(n-1, k)$ such that the extreme vertices of $S(n, k)$ and the extreme vertices of $S(n-1, k)$ are connected by a matching.

In [11], graphs $S[n, k]$ are introduced, moreover, the hamiltonicity and chromatic number of $S[n, k]$ are studied. Clearly, the graph $S[n, 3]$ is the Sierpiński gasket graph S_n.

Graphs $S[n, k]$ are obtained from the Sierpiński graphs $S(n, k)$ by contracting all bridge edges. Let $u_1u_2 \cdots u_rjl \cdots l$ and $u_1u_2 \cdots u_rlj \cdots j$, $1 \leq r \leq n-2$, be two adjacent vertices in $S(n, k)$. In $S[n, k]$, $u_1u_2 \cdots u_rjl \cdots l$ and $u_1u_2 \cdots u_rlj \cdots j$ are identified in one vertex which is denoted by $u_1u_2 \cdots u_r\{j, l\}$, where $j \neq l$ and $j, l \in [1, k]$. In particular, vertices $lj \cdots j$ and $jl \cdots l$ of $S(n, k)$ are identified in one vertex denoted by $\{l, j\}$ in $S[n, k]$. The graph $S[2, 4]$ is shown in Fig. 2.

Since graph $S(n, k) (n \geq 2)$ is obtained by k copies of $S(n-1, k)$ in which any two different $S(n-1, k)$ are connected by a single bridge edge, the graph $S[n, k]$ is also constructed of k copies of $S[n-1, k]$. Each copy is denoted by $S_i[n, k]$ which is isomorphic to $S[n-1, k]$ for $n \geq 2$, i.e., $S_i[n, k]$ corresponds to $S_i(n, k)$. Clearly, $S_i[n, k]$ and $S_j[n, k]$, $i \neq j$, share one common vertex $\{i, j\}$.

Since $S(1, k)$ is isomorphic to K_k, $S[1, k]$ is also isomorphic to K_k.

In [11], it is obtained that graph $S[n, k]$ has $(k^n + k)/2$ vertices and $k^{n-1}\binom{k}{2}$ edges. It is not difficult to see that $S[n, k]$ has k extreme vertices of degree $k-1$ and $(k^n - k)/2$ vertices of degree $2k-2$.

3. The hamiltonicity in $S(n, k)$, $S^+(n, k)$ and $S^{++}(n, k)$

In [12], Klavžar and Milutinović proved that Sierpiński graphs $S(n, k)$ are Hamiltonian for $n \geq 1$ and $k \geq 3$. In this section, we will continue to discuss the hamiltonicity of Sierpiński-like graphs. Let G be a graph. For any two different vertices $v_i, v_j \in V(G)$, we use the notation $H_{(v_i,v_j)}$ to denote a Hamiltonian path of G whose end vertices are v_i and v_j.

Theorem 3.1. *For even $k \geq 2$, $S(n, k)$ can be decomposed into edge disjoint union of $k/2$ Hamiltonian paths of which the end vertices are extreme vertices.*

Proof. For $k = 2$, $S(n, k)$ is isomorphic to the path P_{2^n}.

In the sequel, let $k \geq 4$ be even and we will prove the theorem by induction on n.

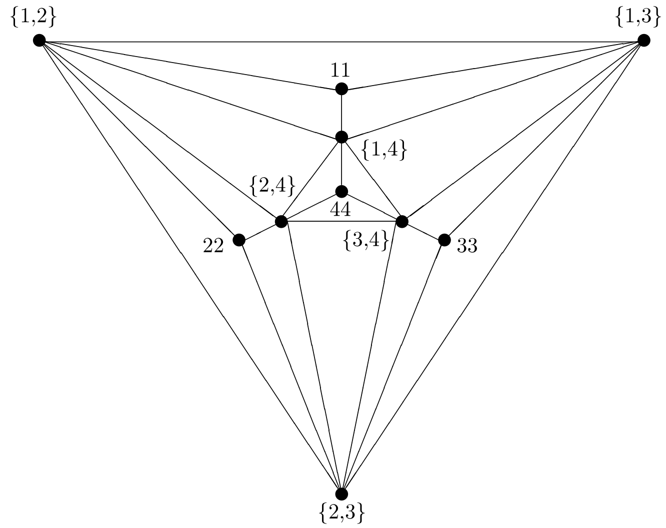

Fig. 2. The graph $S[2, 4]$.

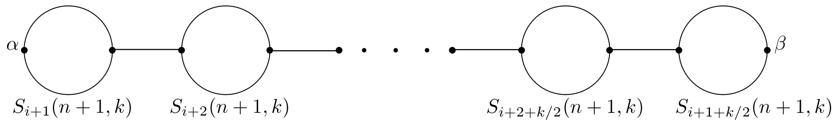

Fig. 3. The subgraph $\mathcal{H}_i$ of $S(n+1, k)$ for even k.

For $n = 1$, $S(1, k)$ is isomorphic to the complete graph K_k. By Lemmas 2.1 and 2.2, the theorem holds.

Assume that $S(n, k)$ can be decomposed into edge disjoint union of $k/2$ Hamiltonian paths of which the end vertices are extreme vertices. In the following, we will prove that $S(n + 1, k)$ can be also decomposed into edge disjoint union of $k/2$ Hamiltonian paths of which the end vertices are extreme vertices.

Obviously,

$$S(n + 1, k) = S_1(n + 1, k) \cup S_2(n + 1, k) \cup \cdots \cup S_k(n + 1, k) \cup \mathcal{B},$$

where $\mathcal{B}$ is the set of all bridge edges connecting $S_i(n + 1, k)$ and $S_j(n + 1, k)$ for any $i \neq j$. Moreover, $S_i(n + 1, k)$ is isomorphic to the graph $S(n, k)$ for $i \in [1, k]$. For each $S_i(n + 1, k)$, we identify all vertices $\langle i \cdots \rangle$ and denote them by s_{i-1}. Then we obtain a complete graph $H = K_k$ with vertex set $V(H) = \{s_0, s_1, \ldots, s_{k-1}\}$ and edge set $E(H) = \mathcal{B}$. Thus every vertex s_i of H corresponds to the subgraph $S_{i+1}(n + 1, k)$ of $S(n + 1, k)$. Since $k(\geq 4)$ is even, by Lemma 2.2, H has a decomposition $\{H_0, H_1, \ldots, H_{k/2-1}\}$, where

$$H_i = s_{0+i}s_{1+i}s_{k-1+i}s_{2+i}s_{k-2+i} \cdots s_{k/2+1+i}s_{k/2+i}, \quad 0 \leq i \leq k/2 - 1,$$

are edge disjoint Hamiltonian paths of H and the indices of s_j are taken modulo k. All edges of each $H_i (i \in [0, k/2 - 1])$ form a matching M_i with $|M_i| = k - 1$ in $S(n + 1, k)$, and all $S_i(n + 1, k)$, $i \in [1, k]$, are connected by all edges of M_i. Therefore, each $H_i (i \in [0, k/2 - 1])$ corresponds to a subgraph

$$\mathcal{H}_i = S_1(n + 1, k) \cup S_2(n + 1, k) \cup \cdots \cup S_k(n + 1, k) \cup M_i$$

of $S(n + 1, k)$. The subgraph $\mathcal{H}_i$ is shown in Fig. 3.

In the following, all indices j of each $S_j(n + 1, k)$ and each $H_{i,j}$ are taken modulo k and the modulo values are all in $[1, k]$ (i.e., we define $j(\text{mod } k) := k$ if $j \equiv 0(\text{mod } k)$ and $j(\text{mod } k)$ is the ordinary modulo value otherwise), as well as each index j_l in every vertex $\langle j_1 j_2 \cdots j_{n+1} \rangle$, where $l \in [1, n + 1]$. Now let $t = 2(i - j + 1)$.

By Corollary 2.3, in $\mathcal{H}_i$, for every $j \in [1, k]$, we have

(1) if $j \in \{i + 1, i + k/2 + 1\}$, then $S_j(n + 1, k)$ and $S_{j+1}(n + 1, k)$ are connected by a single bridge edge whose two end vertices are $\langle j(j + 1) \cdots (j + 1) \rangle$ and $\langle (j + 1)j \cdots j \rangle$, and

(2) if $j \notin \{i + 1, i + k/2 + 1\}$, then $S_j(n + 1, k)$ and $S_{j+t}(n + 1, k)$ are connected by a single bridge edge whose two end vertices are $\langle j(j + t) \cdots (j + t) \rangle$ and $\langle (j + t)j \cdots j \rangle$, and $S_j(n + 1, k)$ and $S_{j+t+1}(n + 1, k)$ are connected by a single bridge edge whose two end vertices are $\langle j(j + t + 1) \cdots (j + t + 1) \rangle$ and $\langle (j + t + 1)j \cdots j \rangle$.

For a fixed $\mathcal{H}_i$, we will look for a Hamiltonian path H'_i of $S(n + 1, k)$ in the following way. In each $S_j(n + 1, k)$, for $j \notin \{i + 1, i + k/2 + 1\}$, we find a Hamiltonian path $H_{i,j}$ of $S_j(n + 1, k)$ whose two end vertices are $\langle j(j + t) \cdots (j + t) \rangle$ and $\langle j(j + t + 1) \cdots (j + t + 1) \rangle$ that are incident with two edges of M_i, respectively.

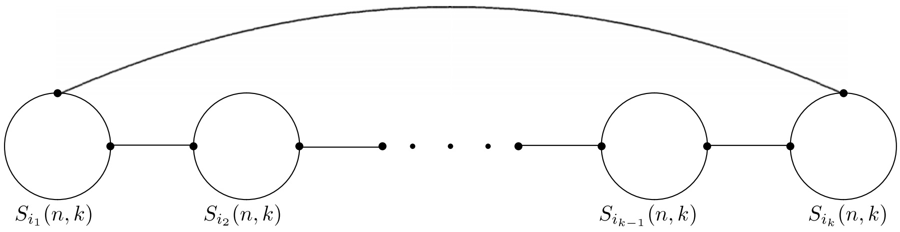

Fig. 4. The subgraph $\mathcal{C}_i$.

In $S_{i+1}(n+1, k)$, we choose a Hamiltonian path $H_{i,i+1}$ of $S_{i+1}(n+1, k)$ such that one end vertex is $\langle (i+1)(i+2)\cdots(i+2)\rangle$ incident with an edge of M_i, and the other is the extreme vertex

$$\alpha = \langle (i+1)(i+1)\cdots(i+1)\rangle$$

of $S(n+1, k)$. In $S_{i+1+k/2}(n+1, k)$, we choose a Hamiltonian path $H_{i,i+1+k/2}$ of $S_{i+1+k/2}(n+1, k)$ such that one end vertex is $\langle (i+1+k/2)(i+2+k/2)\cdots(i+2+k/2)\rangle$ incident with an edge of M_i and the other is the extreme vertex

$$\beta = \langle (i+1+k/2)(i+1+k/2)\cdots(i+1+k/2)\rangle$$

of $S(n+1, k)$.

Thus, from each $\mathcal{H}_i (i \in [0, k/2-1])$, we obtain a Hamiltonian path

$$H'_i = H_{i,1} \cup H_{i,2} \cup \cdots \cup H_{i,k} \cup M_i$$

of $S(n+1, k)$.

Let $\mathcal{V}$ be the set of all extreme vertices of $S(n, k)$. By the induction assumption and the symmetry of $S(n, k)$, it is not difficult to see that there must exist a decomposition

$$\{H_{(v_1,v_2)}, H_{(v_3,v_4)}, \ldots, H_{(v_{k-1},v_k)}\}$$

of $S(n, k)$ for any set $\{(v_1, v_2), (v_3, v_4), \ldots, (v_{k-1}, v_k)\}$, where $v_i \in \mathcal{V}$ for $i \in [1, k]$ and $v_p \neq v_q$ for $p \neq q$. Therefore, we can properly choose the $k/2$ Hamiltonian paths $H_{i,j}$ from each $S_j(n+1, k)$ such that $\{H_{0,j}, H_{1,j}, \ldots, H_{k/2-1,j}\}$ is a decomposition of $S_j(n+1, k)$. Furthermore,

$$\{H'_0, H'_1, \ldots, H'_{k/2-1}\}$$

is a decomposition of $S(n+1, k)$ and two end vertices of each H'_i are extreme vertices of $S(n+1, k)$.

This completes the proof. □

Theorem 3.2. *For even $k \geq 2$, $S(n, k)$ contains $k/2-1$ edge disjoint Hamiltonian cycles.*

Proof. For $n = 1$, by Lemma 2.6, the result holds since $S(1, k)$ is isomorphic to K_k; for $k = 2$, the result holds since $S(n, 2)$ is isomorphic to the path P_{2^n}. In the following, suppose that $n \geq 2$ and $k \geq 4$.

By the definition of $S(n, k)$,

$$S(n, k) = S_1(n, k) \cup S_2(n, k) \cup \cdots \cup S_k(n, k) \cup \mathcal{B},$$

where $\mathcal{B}$ is the set of all bridge edges connecting $S_i(n, k)$ and $S_j(n, k)$ for any $i \neq j$. Moreover, $S_i(n, k)$ is isomorphic to the graph $S(n-1, k)$ for $i \in [1, k]$. For each $S_i(n, k)$, we identify all vertices $\langle i\cdots\rangle$ and denote them by s_i. Then we obtain a complete graph $H = K_k$ with vertex set $V(H) = \{s_1, s_2, \ldots, s_k\}$ and edge set $E(H) = \mathcal{B}$. Since $k(\geq 4)$ is even, by Lemma 2.6, H contains $k/2-1$ edge disjoint Hamiltonian cycles $C_1, C_2, \ldots, C_{k/2-1}$. All edges of each $C_i (i \in [1, k/2-1])$ form a matching M_i with $|M_i| = k$ in $S(n, k)$. Furthermore, all $S_i(n+1, k)$, $i \in [1, k]$, are connected by all edges of M_i. Therefore, each $C_i (i \in [1, k/2-1])$ corresponds to a subgraph

$$\mathcal{C}_i = S_{i_1}(n, k) \cup S_{i_2}(n, k) \cup \cdots \cup S_{i_k}(n, k) \cup M_i$$

of $S(n, k)$, where $i_1 i_2 \cdots i_k$ is some permutation of $1, 2, \ldots, k$. The subgraph $\mathcal{C}_i$ is shown in Fig. 4.

For a fixed $\mathcal{C}_i$, we will look for a Hamiltonian cycle of $S(n, k)$ in the following way. In each $S_j(n, k)$, for any $j \in [1, k]$, we find a Hamiltonian path $H_{i,j}$ of $S_j(n, k)$ whose two end vertices are incident with two edges of M_i. Let $C'_i = H_{i,1} \cup H_{i,2} \cup \cdots \cup H_{i,k} \cup M_i$. By Theorem 3.1 and the symmetry of $S(n, k)$, we can properly choose $H_{i,j}$ $(j \in [1, k/2-1])$ such that $H_{1,j}, H_{2,j}, \ldots, H_{k/2-1,j}$ are edge disjoint Hamiltonian paths of $S_j(n, k)$. Thus, all C'_i, $i \in [1, k/2-1]$, are edge disjoint Hamiltonian cycles of $S(n, k)$. □

Theorem 3.3. *$S^+(n, k)$ can be decomposed into edge disjoint union of $k/2$ Hamiltonian cycles for even $k \geq 2$, i.e., $S^+(n, k)$ has a decomposition $\{C_0, C_1, \ldots, C_{k/2-1}\}$, where each C_i is a Hamiltonian cycle for $i \in [0, k/2-1]$ and $k \geq 2$ is even.*

Proof. According to Theorem 3.1, we know that $S(n, k)$ can be decomposed into edge disjoint union of $k/2$ Hamiltonian paths of which the end vertices are extreme vertices. That is, $S(n, k)$ has a decomposition $\{H_0, H_1, \ldots, H_{k/2-1}\}$, where each H_i is a Hamiltonian path of $S(n, k)$ and two end vertices of each H_i are extreme vertices. Since $S^+(n, k)$ is obtained from Sierpiński graph $S(n, k)$ by adding a new vertex w and edges joining w with k extreme vertices of $S(n, k)$, $C_i = wH_iw$ must be a Hamiltonian cycle of $S^+(n, k)$, where $i \in [0, k/2 - 1]$. □

Theorem 3.4. *$S^{++}(n, k)$ can be decomposed edge disjoint union of $k/2$ Hamiltonian cycles for even $k \geq 2$, i.e., $S^{++}(n, k)$ has a decomposition $\{C'_0, C'_1, \ldots, C'_{k/2-1}\}$, where each C'_i is a Hamiltonian cycle for $i \in [0, k/2 - 1]$ and $k \geq 2$ is even.*

Proof. For $n = 1$, the statement is obvious since $S^{++}(1, k)$ is isomorphic to K_{k+1}. In the following, suppose that $n \geq 2$.

Since $S^{++}(n, k)$ is obtained from the vertex disjoint union of a copy of $S(n, k)$ and a copy $S(n-1, k)$ such that the extreme vertices of $S(n, k)$ and the extreme vertices of $S(n - 1, k)$ are connected by a matching M, in what follows, we will properly look for Hamiltonian paths from $S(n, k)$ and $S(n - 1, k)$.

According to Theorem 3.1, we know that $S(n, k)$ can be decomposed into edge disjoint union of $k/2$ Hamiltonian paths of which the end vertices are extreme vertices. That is, $S(n, k)$ has a decomposition $\{H_0, H_1, \ldots, H_{k/2-1}\}$, where each H_i is a Hamiltonian path of $S(n, k)$ and two end vertices of each H_i are extreme vertices. Without loss of generality, suppose that $H_i = H_{(u_i,v_i)}$. Next, by Theorem 3.1 and the symmetry of $S(n - 1, k)$, we can properly look for a decomposition $\{H'_0, H'_1, \ldots, H'_{k/2-1}\}$ of $S(n - 1, k)$ such that $H'_i = H_{(u'_i,v'_i)}$ and $u_iu'_i, v_iv'_i \in E(S^{++}(n, k))$, where $i \in [0, k/2 - 1]$. Thus $C'_i = H_i \cup H'_i \cup \{u_iu'_i, v_iv'_i\}$ must be a Hamiltonian cycle of $S^{++}(n, k)$, where $i \in [0, k/2 - 1]$. □

Theorem 3.5. *For $k \geq 3$ is odd, there exists $(k - 1)/2$ edge disjoint Hamiltonian paths whose two end vertices are extreme vertices, in $S(n, k)$.*

Proof. Using the same method than in the proof of Theorem 3.1 and by Lemma 2.4 and Corollary 2.5, the result holds immediately. □

Theorem 3.6. *For odd $k \geq 3$, $S(n, k)$, $S^+(n, k)$ and $S^{++}(n, k)$ contain $(k - 1)/2$ edge disjoint Hamiltonian cycles, respectively.*

Proof. Using the same method than in the proof of Theorems 3.2–3.4 and by Theorem 3.5, the result holds immediately. □

4. The path t-colorings of $S(n, k)$, $S^+(n, k)$ and $S^{++}(n, k)$

The following result is obvious.

Theorem 4.1. *For every $k \geq 3$, $\text{vla}(S(n, k)) = \text{vla}(S^+(n, k)) = \text{vla}(S^{++}(n, k)) = \lceil k/2 \rceil$. For $k = 2$, $\text{vla}(S(n, 2)) = 1$ and $\text{vla}(S^+(n, 2)) = \text{vla}(S^{++}(n, 2)) = 2$.*

Proof. The theorem follows from Lemma 2.7 and the fact that $S(n, k)$, $S^+(n, k)$ and $S^{++}(n, k)$ contain complete graphs K_k as their subgraphs, respectively. □

In the sequel, we will obtain several stronger results.

Theorem 4.2. *Let $k \geq 2$ be even and $G = S(n, k)$. Then there must exist a path $k/2$-coloring ϕ from $V(G)$ to $[1, k/2]$ satisfying:*

(a) *$|\phi^{-1}(i)| = 2k^{n-1}$ for every $i \in [1, k/2]$, and*
(b) *$G[\phi^{-1}(i)]$ is a path whose end vertices are two extreme vertices, where $i \in [1, k/2]$.*

Proof. In order to prove the theorem, we need to show that $S(n, k)$, for even $k \geq 2$, has a partition $\{V_i : i \in [0, k/2 - 1]\}$ satisfying:

(a′) $|V_i| = 2k^{n-1}$ for each $i \in [0, k/2 - 1]$, and
(b′) $G[V_i]$ is a path whose end vertices are two extreme vertices, where $i \in [0, k/2 - 1]$.

The statement is clear for $k = 2$ since $S(n, 2)$ is the path on 2^n vertices.

In the following, let $k \geq 4$ and we will prove the theorem by induction on n.

For $n = 1$, the result is obvious since $S(1, k)$ is isomorphic to the complete graph K_k.

Assume that $S(n, k)$ has a partition $\{V_i : i \in [0, k/2 - 1]\}$ that satisfies (a′) and (b′). In the following, we will prove that $G' = S(n + 1, k)$ has also a partition $\{W_i : i \in [0, k/2 - 1]\}$ satisfying (a′) and (b′).

Obviously,

$$S(n + 1, k) = S_1(n + 1, k) \cup S_2(n + 1, k) \cup \cdots \cup S_k(n + 1, k) \cup \mathcal{B},$$

where $\mathcal{B}$ is the set of all bridge edges connecting $S_i(n + 1, k)$ and $S_j(n + 1, k)$ for any $i \neq j$. Moreover, $S_i(n + 1, k)$ is isomorphic to the graph $S(n, k)$ for $i \in [1, k]$. For each $S_i(n + 1, k)$, we identify all vertices $\langle i \cdots \rangle$ and denote them by s_{i-1}. Then we obtain a complete graph $H = K_k$ with vertex set $V(H) = \{s_0, s_1, \ldots, s_{k-1}\}$ and edge set $E(H) = \mathcal{B}$. Thus every vertex s_i of H corresponds to the subgraph $S_{i+1}(n+1, k)$ of $S(n+1, k)$. Since $k(\geq 4)$ is even, by Lemma 2.2, H has a decomposition $\{H_0, H_1, \ldots, H_{k/2-1}\}$, where

$$H_i = s_{0+i}s_{1+i}s_{k-1+i}s_{2+i}s_{k-2+i}\cdots s_{k/2+1+i}s_{k/2+i}, \quad 0 \leq i \leq k/2-1,$$

are edge disjoint Hamiltonian paths of H and the indices of s_j are taken modulo k. All edges of each $H_i (i \in [0, k/2-1])$ form a matching M_i with $|M_i| = k-1$ in $S(n+1, k)$ and all $S_i(n+1, k)$, $i \in [1, k]$, are connected by all edges of M_i. Therefore, each $H_i (i \in [0, k/2-1])$ corresponds to a subgraph

$$\mathcal{H}_i = S_1(n+1, k) \cup S_2(n+1, k) \cup \cdots \cup S_k(n+1, k) \cup M_i$$

of $S(n+1, k)$. The subgraph $\mathcal{H}_i$ is similar as in Fig. 3.

In the following, all indices j of each $S_j(n+1, k)$ and each $W_{i,j}$ are taken modulo k and the modulo values are all in $[1, k]$, as well as each index j_l in every vertex $\langle j_1 j_2 \cdots j_{n+1}\rangle$, where $l \in [1, n+1]$. Now let $t = 2(i-j+1)$.

Just as in Theorem 3.1, in $\mathcal{H}_i$, for every $j \in [1, k]$, we have

(1) if $j \in \{i+1, i+k/2+1\}$, then $S_j(n+1, k)$ and $S_{j+1}(n+1, k)$ are connected by a single bridge edge whose two end vertices are $\langle j(j+1)\cdots(j+1)\rangle$ and $\langle (j+1)j\cdots j\rangle$, and
(2) if $j \notin \{i+1, i+k/2+1\}$, then $S_j(n+1, k)$ and $S_{j+t}(n+1, k)$ are connected by a single bridge edge whose two end vertices are $\langle j(j+t)\cdots(j+t)\rangle$ and $\langle (j+t)j\cdots j\rangle$, and $S_j(n+1, k)$ and $S_{j+t+1}(n+1, k)$ are connected by a single bridge edge whose two end vertices are $\langle j(j+t+1)\cdots(j+t+1)\rangle$ and $\langle (j+t+1)j\cdots j\rangle$.

For a fixed $\mathcal{H}_i$, we will look for a vertex subset W_i of $S(n+1, k)$ in the following way. In each $S_j(n+1, k)$, for $j \notin \{i+1, i+k/2+1\}$, we choose a vertex subset $W_{i,j}$ with $|W_{i,j}| = 2k^{n-1}$, which contains the two vertices $\langle j(j+t)\cdots(j+t)\rangle$ and $\langle j(j+t+1)\cdots(j+t+1)\rangle$ that are incident with two edges of M_i, respectively. In $S_{i+1}(n+1, k)$, we choose a vertex subset $W_{i,i+1}$ with $|W_{i,i+1}| = 2k^{n-1}$, which contains two vertices such that one is $\langle (i+1)(i+2)\cdots(i+2)\rangle$ incident with an edge of M_i and the other is the extreme vertex

$$\alpha = \langle (i+1)(i+1)\cdots(i+1)\rangle$$

of $S(n+1, k)$. In $S_{i+1+k/2}(n+1, k)$, we also choose a vertex subset $W_{i,i+1+k/2}$ with $|W_{i,i+1+k/2}| = 2k^{n-1}$, which contains two vertices such that one is $\langle (i+1+k/2)(i+2+k/2)\cdots(i+2+k/2)\rangle$ incident with an edge of M_i and the other is the extreme vertex

$$\beta = \langle (i+1+k/2)(i+1+k/2)\cdots(i+1+k/2)\rangle$$

of $S(n+1, k)$.

Since each $S_i(n+1, k) (i \in [1, k])$ is isomorphic to $S(n, k)$, by induction assumption and the symmetry of $S(n, k)$, we can properly choose the vertex set $W_{j,i}$ $(i \in [1, k])$ such that $\{W_{0,i}, W_{1,i}, \ldots, W_{k-1,i}\}$ is a partition of $S_i(n+1, k)$ and satisfies (a′) and (b′).

Thus, from each $\mathcal{H}_i (i \in [0, k/2-1])$, we obtain a vertex subset

$$W_i = W_{i,1} \cup W_{i,2} \cup \cdots \cup W_{i,k},$$

where $i \in [0, k/2-1]$. It is not difficult to check that $\{W_i : i \in [0, k/2-1]\}$ satisfy (a′) and (b′).

In fact, by induction assumption,

$$|W_i| = |W_{i,1}| + |W_{i,2}| + \cdots + |W_{i,k}| = k \cdot 2k^{n-1} = 2k^n.$$

Clearly,

$$G'[W_i] = G'[W_{i,1}] \cup G'[W_{i,2}] \cup \cdots \cup G'[W_{i,k}] \cup G'[M_i]$$

is a path whose end vertices are extreme vertices since all paths $G'[W_{i,j}]$, $j \in [1, k]$, are properly connected by all edges of M_i. □

Theorem 4.3. *Let $k \geq 3$ be odd and $G = S(n, k)$. Then there must exist a path $(k+1)/2$-coloring φ from $V(G)$ to $[1, (k+1)/2]$ satisfying:*

(a) $|\varphi^{-1}(i)| = 2k^{n-1}$ *for each* $i \in [1, (k-1)/2]$, *and* $|\varphi^{-1}((k+1)/2)| = k^{n-1}$, *and*
(b) *each* $G[\varphi^{-1}(i)]$ $(i \in [1, (k-1)/2])$ *is a path of which end vertices are two extreme vertices, and* $G[\varphi^{-1}((k+1)/2)]$ *is a subgraph consisting of* $(k^{n-1}-1)/2$ *isolated edges and one extreme vertex.*

Proof. Using the same method than in the proof of Theorem 4.2 and by Lemma 2.4 and Corollary 2.5, the result holds immediately. □

Theorem 4.4. *Let $k \geq 3$ be odd and $H = S^+(n, k)$. Then there must exist a path $(k+1)/2$-coloring φ from $V(H)$ to $[1, (k+1)/2]$ satisfying:*

(a) $|\varphi^{-1}(i)| = 2k^{n-1}$ *for every* $i \in [1, (k-1)/2]$, *and* $|\varphi^{-1}((k+1)/2)| = k^{n-1}+1$, *and*

(b) *each* $H[\varphi^{-1}(i)]$ ($i \in [1, (k-1)/2]$) *is a path, and* $H[\varphi^{-1}((k+1)/2)]$ *is a matching consisting of* $(k^{n-1}+1)/2$ *isolated edges.*

Proof. By Theorem 4.3, $G = S(n,k)$, for odd $k \geq 3$, has a partition $\{V_1, V_2, \ldots, V_{(k-1)/2}, \mathcal{V}\}$ satisfying:

(a′) $|\mathcal{V}| = k^{n-1}$ and $|V_i| = 2k^{n-1}$ for every $i \in [1, (k-1)/2]$, and
(b′) each $G[V_i]$ ($i \in [1, (k-1)/2]$) is a path whose end vertices are two extreme vertices, and $G[\mathcal{V}]$ is a subgraph consisting of one extreme vertex and $(k^{n-1}-1)/2$ isolated edges.

Since $S^+(n,k)$ is obtained from Sierpiński graph $S(n,k)$ by adding a new vertex w and edges joining w with k extreme vertices of $S(n,k)$, $\{V_1, V_2, \ldots, V_{(k-1)/2}, \mathcal{V} \cup \{w\}\}$ is a partition of $S^+(n,k)$. Let $\varphi^{-1}(i) = V_i$ for each $i \in [1, (k-1)/2]$, and $\varphi^{-1}((k+1)/2) = \mathcal{V} \cup \{w\}$. Then φ satisfies (a) and (b). □

Theorem 4.5. *Let* $k \geq 4$ *be even and* $H = S^{++}(n,k)(n \geq 2)$. *Then there must exist a path* $k/2$*-coloring* ϕ *from* $V(H)$ *to* $[1, k/2]$ *satisfying:*

(a) $|\phi^{-1}(i)| = 2(k^{n-1} + k^{n-2})$ *for each* $i \in [1, k/2]$, *and*
(b) $H[\phi^{-1}(i)]$ *is a path for every* $i \in [1, k/2]$.

Proof. By Theorem 4.2, $G = S(n,k)$, for even $k \geq 2$, has a partition $\{V_i : i \in [1, k/2]\}$ satisfying:

(a′) $|V_i| = 2k^{n-1}$ for each $i \in [1, k/2]$, and
(b′) $G[V_i]$, for every $i \in [1, k/2]$, is a path whose end vertices are two extreme vertices.

Let $P_{(u_i,u_i')} = G[V_i]$ be the path whose end vertices are u_i and u_i', where $i \in [1, k/2]$. It is well known that the graph $S^{++}(n,k)$ is obtained from the vertex disjoint union of a copy of $G = S(n,k)$ and a copy $G' = S(n-1,k)$ such that the extreme vertices of $S(n,k)$ and the extreme vertices of $S(n-1,k)$ are connected by a matching. We will properly choose a partition $\{V_i' : i \in [1, k/2]\}$ of $S(n-1,k)$ such that ϕ satisfies (a) and (b), where $\phi^{-1}(i) = V_i \cup V_i'$ for every $i \in [1, k/2]$.

Let $Q_{(v_i,v_i')} = G'[V_i']$ be the path whose end vertices are v_i and v_i' for every $i \in [1, k/2]$. Then, by the symmetry of $S(n-1,k)$ and Theorem 4.2, we can properly choose a partition $\{V_i' : i \in [1, k/2]\}$ of $S(n-1,k)$ such that $u_iv_i \in E(S^{++}(n,k))$, $u_i'v_i' \notin E(S^{++}(n,k))$ and $\{V_i' : i \in [1, k/2]\}$ satisfies (a′) and (b′). Thus, let $\phi^{-1}(i) = V_i \cup V_i'$ for any $i \in [1, k/2]$. It is easy to check that ϕ satisfies (a) and (b). In fact, $H[V_i \cup V_i'] = P_{(u_i,u_i')} \cup Q_{(v_i,v_i')} \cup \{u_iv_i\}$ is a path with end vertices u_i' and v_i'. □

Theorem 4.6. *Let* $k \geq 3$ *be odd and* $H = S^{++}(n,k)(n \geq 2)$. *Then there must exist a path* $(k+1)/2$*-coloring* φ *from* $V(H)$ *to* $[1, (k+1)/2]$ *satisfying:*

(a) $|\varphi^{-1}(i)| = 2(k^{n-1} + k^{n-2})$ *for every* $i \in [1, (k-1)/2]$, *and* $|\varphi^{-1}((k+1)/2)| = k^{n-1} + k^{n-2}$, *and*
(b) *each* $H[\varphi^{-1}(i)](i \in [1, (k-1)/2])$ *is a path and* $H[\varphi^{-1}((k+1)/2)]$ *is a matching consisting of* $(k^{n-1} + k^{n-2})/2$ *isolated edges.*

Proof. Using the same method than in the proof of Theorem 4.5 and by Theorem 4.3, the result holds immediately. □

5. The vertex linear arboricity of $S[n,k]$

In this section, all indices j of each $S_j[n+1,k]$ and each $W_{i,j}$ are taken modulo k and the modulo values are all in $[1,k]$, as well as each index j_l in every vertex $\langle j_1 j_2 \cdots j_{n+1} \rangle$, where $l \in [1, n+1]$. Now let $t = 2(i-j+1)$.

Theorem 5.1. *Let* $k \geq 2$ *and* $G = S[n,k]$. *Then* $\text{vla}(G) = \lceil k/2 \rceil$.

Proof. Since $S[n,k]$ contains complete graphs K_k as its subgraphs, $\text{vla}(G) \geq \lceil k/2 \rceil$. In the sequel, we will show that $\text{vla}(G) \leq \lceil k/2 \rceil$ which will follow from the following two stronger claims.

Claim 1. $S[n,k]$, *for even* $k \geq 2$, *has a partition* $\{V_i : i \in [0, k/2-1]\}$ *satisfying:*

(a) $|V_i| = k^{n-1} + 1$ *for each* $i \in [0, k/2-1]$, *and*
(b) $G[V_i]$ *is a path whose end vertices are two extreme vertices, where* $i \in [0, k/2-1]$.

The statement is clear for $k = 2$ since $S[n,2]$ is the path on $2^{n-1}+1$ vertices.

Let $k \geq 4$. We will prove the claim by induction on n.

For $n = 1$, it is obvious since $S[1,k]$ is isomorphic to the complete graph K_k.

Assume that $S[n,k]$ has a partition $\{V_i : i \in [0, k/2-1]\}$ that satisfies (a) and (b). In the sequel, we will prove that $G' = S[n+1,k]$ has also a partition $\{V_i' : i \in [0, k/2-1]\}$ that satisfies (a) and (b).

First, we construct a new graph $\overline{G}$ which is obtained from $S[n+1,k]$ by adding all bridge edges between $S_i[n+1,k]$ and $S_j[n+1,k]$ for $i \neq j$. Let $\mathcal{B}$ be the edge set of all bridge edges connecting $S_i[n+1,k]$ and $S_j[n+1,k]$ for $i \neq j$.

Clearly,

$$\overline{G} = S_1[n+1,k] \cup S_2[n+1,k] \cup \cdots \cup S_k[n+1,k] \cup \mathcal{B}.$$

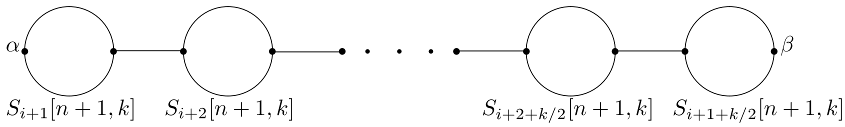

Fig. 5. The subgraph $\mathcal{H}_i$ of $S[n+1,k]$ for even k.

Moreover, $S_i[n+1,k]$ is isomorphic to the graph $S[n,k]$ for $i \in [1,k]$. For each $S_i[n+1,k]$, we identify all vertices of $S_i[n+1,k]$ and denote them by s_{i-1}. Then we obtain a complete graph $H = K_k$ with vertex set $V(H) = \{s_0, s_1, \ldots, s_{k-1}\}$ and edge set $E(H) = \mathcal{B}$.

Thus every vertex s_i of H corresponds to the subgraph $S_{i+1}[n+1,k]$ of $S[n+1,k]$. Since $k(\geq 4)$ is even, by Lemma 2.2, H has a decomposition $\{H_0, H_1, \ldots, H_{k/2-1}\}$, where

$$H_i = s_{0+i}s_{1+i}s_{k-1+i}s_{2+i}s_{k-2+i}\cdots s_{k/2+1+i}s_{k/2+i}, \quad 0 \leq i \leq k/2-1,$$

are edge disjoint Hamiltonian paths of H and the indices of s_j are taken modulo k. All edges of each $H_i(i \in [0,k/2-1])$ constitute a matching M_i with $|M_i| = k-1$ in $\overline{G}$ and all $S_i[n+1,k]$, $i \in [1,k]$, are connected by all edges of M_i. Therefore, each $H_i(i \in [0,k/2-1])$ corresponds to a subgraph

$$\mathcal{H}_i = S_1[n+1,k] \cup S_2[n+1,k] \cup \cdots \cup S_k[n+1,k] \cup M_i$$

of $\overline{G}$. The subgraph $\mathcal{H}_i$ is shown in Fig. 5.

Just as in Theorem 3.1, in $\mathcal{H}_i$, for every $j \in [1,k]$, we have

(1) if $j \in \{i+1, i+k/2+1\}$, then $S_j[n+1,k]$ and $S_{j+1}[n+1,k]$ are connected by a single bridge edge whose two end vertices are $\langle j(j+1)\cdots(j+1)\rangle$ and $\langle (j+1)j\cdots j\rangle$, and

(2) if $j \notin \{i+1, i+k/2+1\}$, then $S_j[n+1,k]$ and $S_{j+t}[n+1,k]$ are connected by a single bridge edge whose two end vertices are $\langle j(j+t)\cdots(j+t)\rangle$ and $\langle (j+t)j\cdots j\rangle$, and $S_j[n+1,k]$ and $S_{j+t+1}[n+1,k]$ are connected by a single bridge edge whose two end vertices are $\langle j(j+t+1)\cdots(j+t+1)\rangle$ and $\langle (j+t+1)j\cdots j\rangle$.

For a fixed $\mathcal{H}_i$, we will search a vertex subset $\mathcal{W}_i$ of $\overline{G}$ in the following way. In each $S_j[n+1,k]$, for $j \notin \{i+1, i+k/2+1\}$, we choose a vertex subset $\mathcal{W}_{i,j}$ with $|\mathcal{W}_{i,j}| = k^{n-1}+1$, which contains two vertices $\langle j(j+t)\cdots(j+t)\rangle$ and $\langle j(j+t+1)\cdots(j+t+1)\rangle$ that are incident with two edges of M_i, respectively. In $S_{i+1}[n+1,k]$, we choose a vertex subset $\mathcal{W}_{i,i+1}$ with $|\mathcal{W}_{i,i+1}| = k^{n-1}+1$, which contains two vertices such that one is $\langle (i+1)(i+2)\cdots(i+2)\rangle$ incident with an edge of M_i and the other is the extreme vertex

$$\alpha = \langle (i+1)(i+1)\cdots(i+1)\rangle$$

of $S[n+1,k]$. In $S_{i+1+k/2}[n+1,k]$, we also choose a vertex subset $\mathcal{W}_{i,i+1+k/2}$ with $|\mathcal{W}_{i,i+1+k/2}| = k^{n-1}+1$, which contains two vertices such that one is $\langle (i+1+k/2)(i+2+k/2)\cdots(i+2+k/2)\rangle$ incident with an edge of M_i and the other is the extreme vertex

$$\beta = \langle (i+1+k/2)(i+1+k/2)\cdots(i+1+k/2)\rangle$$

of $S[n+1,k]$.

Since each $S_i[n+1,k](i \in [1,k])$ is isomorphic to $S[n,k]$, by induction assumption and the symmetry of $S[n,k]$, we can properly choose the vertex set $\mathcal{W}_{j,i}$ $(i \in [1,k])$ such that $\{\mathcal{W}_{0,i}, \mathcal{W}_{1,i}, \ldots, \mathcal{W}_{k-1,i}\}$ is a partition of $S_i[n+1,k]$ and satisfies (a) and (b).

Thus, from each $\mathcal{H}_i(i \in [0,k/2-1])$, we obtain a vertex subset

$$\mathcal{W}_i = \mathcal{W}_{i,1} \cup \mathcal{W}_{i,2} \cup \cdots \cup \mathcal{W}_{i,k},$$

where $i \in [0, k/2-1]$. By induction assumption, it is not difficult to check that $\{\mathcal{W}_i : i \in [0, k/2-1]\}$ is a partition of $\overline{G}$ satisfying

(1) $|\mathcal{W}_i| = k \cdot (k^{n-1}+1) = k^n + k$ for each $i \in [0, k/2-1]$, and
(2) $\overline{G}[\mathcal{W}_i]$ is a path whose end vertices are two extreme vertices, where $i \in [0, k/2-1]$.

Let X_i be the vertex set consisting of all end vertices of edges in M_i. Since all edges of M_i are contracted in $S[n+1, k]$ and the two end vertices of every edge in M_i are identified in a single vertex. Let Y_i be the vertex set consisting of new vertices after contracting all edges of M_i.

Let $V_i' = (\mathcal{W}_i \setminus X_i) \cup Y_i$. Then $\{V_i' : i \in [0, k/2] - 1\}$ is a partition of $S[n+1, k]$ satisfying (a) and (b). In fact, Since $|X_i| = 2(k-1)$ and $|Y_i| = k-1$,

$$|V_i'| = |\mathcal{W}_i| - |X_i'| + |Y_i'| = k^n + k - 2(k-1) + k - 1 = k^n + 1,$$

where $i \in [0, k/2-1]$. $G[V_i']$ is a path obtained from $\overline{G}[\mathcal{W}_i]$ by contracting all edges of M_i.

Claim 2. *$S[n, k]$, for each odd $k \geq 3$, has a partition $\{V_0, V_1, \ldots, V_{(k-1)/2-1}, \mathcal{V}\}$ satisfying:*

(a) *$|\mathcal{V}| = (k^{n-1}+1)/2$ and $|V_i| = k^{n-1}+1$ for every $i \in [0, (k-1)/2-1]$, and*
(b) *each $G[V_i]$ ($i \in [0, (k-1)/2-1]$) is a path whose end vertices are two extreme vertices, and $\mathcal{V}$ is an independent set.*

Using the same method than in the proof of Claim 1 and by Lemma 2.4 and Corollary 2.5, Claim 2 holds.
This completes the proof. □

The vertex arboricity $\text{va}(G)$ of a graph G is the minimum number of subsets into which the vertex set $V(G)$ can be partitioned so that every subset induces a forest.

Theorem 5.2. *Let $k \geq 2$. Then $\text{va}(S(n, k)) = \text{va}(S[n, k]) = \lceil k/2 \rceil$.*

Proof. The theorem follows from Theorems 4.1 and 5.1, the fact that $S(n, k)$ and $S[n, k]$ contain complete graphs as their subgraphs, and $\text{va}(G) \leq \text{vla}(G)$ for every graph G. □

Acknowledgments

We thank the referees for providing very helpful suggestions for this paper.

Appendix

The proof of Theorem 3.5. We will prove the theorem by induction on n.

For $n = 1$, $S(1, k)$ is isomorphic to the complete graph K_k. By Lemma 2.4, the theorem holds.

Assume that $S(n, k)$ possesses $(k-1)/2$ edge disjoint Hamiltonian paths of which the end vertices are extreme vertices. In the sequel, we will prove that $S(n+1, k)$ also possesses $(k-1)/2$ edge disjoint Hamiltonian paths of which the end vertices are extreme vertices.

Since

$$S(n+1, k) = S_1(n+1, k) \cup S_2(n+1, k) \cup \cdots \cup S_k(n+1, k) \cup \mathcal{B}',$$

where $\mathcal{B}'$ is the set of all bridge edges between $S_i(n+1, k)$ and $S_j(n+1, k)$ for any $i \neq j$. Moreover, $S_i(n+1, k)$ is isomorphic to graph $S(n, k)$ for $i \in [1, k]$. For each $S_i(n+1, k)$, we identify all vertices $\langle i \cdots \rangle$ and denote them by s_{i-1}. Then we obtain a complete graph $H = K_k$ with vertex set $V(H) = \{s_0, s_1, \ldots, s_{k-1}\}$ and edge set $E(H) = \mathcal{B}'$.

Since $k \geq 3$ is odd, by Lemma 2.4, H has a decomposition $\{H_0, H_1, \ldots, H_{(k-1)/2-1}, M\}$, where

$$H_i = s_{0+i}s_{1+i}s_{k-1+i}s_{2+i}s_{k-2+i} \cdots s_{(k-1)/2+i}s_{(k+1)/2+i}, \quad 0 \leq i \leq (k-1)/2-1,$$

are edge disjoint Hamiltonian paths of H, $M = \{s_{(k-1)/2-i}s_{(k-1)/2+i} : i \in [1, (k-1)/2]\}$ is a maximum matching of H and the indices of s_j are taken modulo k. All edges of each $H_i (i \in [0, (k-1)/2-1])$ form a matching M_i with $|M_i| = k-1$ in $S(n+1, k)$ and all $S_i(n+1, k)$, $i \in [1, k]$, are connected by all edges of M_i. Therefore, each $H_i (i \in [0, (k-1)/2-1])$ corresponds to a subgraph

$$\mathcal{H}_i = S_1(n+1, k) \cup S_2(n+1, k) \cup \cdots \cup S_k(n+1, k) \cup M_i$$

of $S(n+1, k)$. The subgraph $\mathcal{H}_i$ is shown in Fig. 6.

In the following, all indices j of each $S_j(n+1, k)$ and each $H_{i,j}$ are taken modulo k and the modulo values are all in $[1, k]$, as well as each index j_l in every vertex $\langle j_1 j_2 \cdots j_{n+1} \rangle$, where $l \in [1, n+1]$. Now let $t = 2(i-j+1)$.

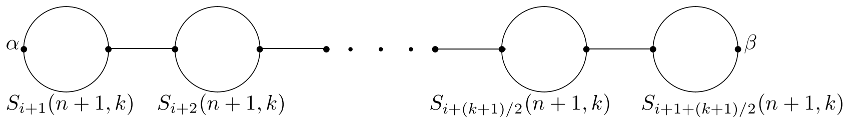

Fig. 6. The subgraph $\mathcal{H}_i$ of $S(n+1,k)$ for odd k.

By Corollary 2.5, in $\mathcal{H}_i$, for each $j \in [1,k]$, we have

(1) if $j = i+1$, then $S_j(n+1,k)$ and $S_{j+1}(n+1,k)$ are connected by a single bridge edge whose two end vertices are $\langle j(j+1)\cdots(j+1)\rangle$ and $\langle (j+1)j\cdots j\rangle$; if $j = i+(k+1)/2+1$, then $S_j(n+1,k)$ and $S_{j-1}(n+1,k)$ are connected by a single bridge edge whose two end vertices are $\langle j(j-1)\cdots(j-1)\rangle$ and $\langle (j-1)j\cdots j\rangle$, and
(2) if $j \notin \{i+1, i+(k+1)/2+1\}$, then $S_j(n+1,k)$ and $S_{j+t}(n+1,k)$ are connected by a single bridge edge whose two end vertices are $\langle j(j+t)\cdots(j+t)\rangle$ and $\langle (j+t)j\cdots j\rangle$, and $S_j(n+1,k)$ and $S_{j+t+1}(n+1,k)$ are connected by a single bridge edge whose two end vertices are $\langle j(j+t+1)\cdots(j+t+1)\rangle$ and $\langle (j+t+1)j\cdots j\rangle$.

For a fixed $\mathcal{H}_i$, we will look for a Hamiltonian path H'_i of $S(n+1,k)$ in the following way. In each $S_j(n+1,k)$, for $j \notin \{i+1, i+(k+1)/2+1\}$, we find a Hamiltonian path $H_{i,j}$ of $S_j(n+1,k)$ whose two end vertices are $\langle j(j+t)\cdots(j+t)\rangle$ and $\langle j(j+t+1)\cdots(j+t+1)\rangle$ that are incident with two edges of M_i, respectively. In $S_{i+1}(n+1,k)$, we choose a Hamiltonian path $H_{i,i+1}$ of $S_{i+1}(n+1,k)$ such that one end vertex is $\langle (i+1)(i+2)\cdots(i+2)\rangle$ incident with one edge of M_i and the other is the extreme vertex

$$\alpha = \langle (i+1)(i+1)\cdots(i+1)\rangle$$

of $S(n+1,k)$. In $S_{i+1+(k+1)/2}(n+1,k)$, we choose a Hamiltonian path $H_{i,i+1+(k+1)/2}$ such that one end vertex is $\langle (i+1+(k+1)/2)(i+(k+1)/2)\cdots(i+(k+1)/2)\rangle$ incident with one edge of M_i and the other is the extreme vertex

$$\beta = \langle (i+1+(k+1)/2)(i+1+(k+1)/2)\cdots(i+1+(k+1)/2)\rangle$$

of $S(n+1,k)$.

Thus, from each $\mathcal{H}_i (i \in [0,(k-1)/2-1])$, we obtain a Hamiltonian path

$$H'_i = H_{i,1} \cup H_{i,2} \cup \cdots \cup H_{i,k} \cup M_i$$

of $S(n+1,k)$.

Let $\mathcal{V}$ be the set of all extreme vertices of $S(n,k)$. By the induction assumption, $S(n,k)$ has $(k-1)/2$ edge disjoint Hamiltonian paths of which the end vertices are extreme vertices. According to the symmetry of $S(n,k)$, there must exist $(k-1)/2$ edge disjoint Hamiltonian paths

$$H_{(t_1,t_2)}, H_{(t_3,t_4)}, \ldots, H_{(t_{k-2},t_{k-1})}$$

of $S(n,k)$ for any set $\{(t_1,t_2),(t_3,t_4),\ldots,(t_{k-2},t_{k-1})\}$, where $t_i \in \mathcal{V}$ for any $i \in [1,k-1]$ and $t_p \neq t_q$ for $p \neq q$. Therefore, we can properly choose the $(k-1)/2$ Hamiltonian paths $H_{j,i}$ of $S_i(n+1,k)$ such that $H_{0,i}, H_{1,i}, \ldots, H_{(k-1)/2-1,i}$ are edge disjoint each other. Furthermore, $H'_0, H'_1, \ldots, H'_{(k-1)/2-1}$ are edge disjoint Hamiltonian paths of $S(n+1,k)$ and two end vertices of each H'_i are extreme vertices of $S(n+1,k)$.

This completes the proof. □

The proof of Theorem 3.6. First, we prove that $S(n,k)$ contains $(k-1)/2$ edge disjoint Hamiltonian cycles.

For $n=1$, the result holds since $S(1,k)$ is isomorphic to K_k. In the following, suppose that $n \geq 2$.

By the definition of $S(n,k)$,

$$S(n,k) = S_1(n,k) \cup S_2(n,k) \cup \cdots \cup S_k(n,k) \cup \mathcal{B}',$$

where $\mathcal{B}'$ is the set of all bridge edges connecting $S_i(n,k)$ and $S_j(n,k)$ for any $i \neq j$. Moreover, $S_i(n,k)$ is isomorphic to the graph $S(n-1,k)$ for $i \in [1,k]$. For each $S_i(n,k)$, we identify all vertices $\langle i\cdots\rangle$ and denote them by s_i. Then we obtain a complete graph $H = K_k$ with vertex set $V(H) = \{s_1,s_2,\ldots,s_k\}$ and edge set $E(H) = \mathcal{B}'$. Since $k \geq 3$ is odd, by Lemma 2.1, H can be decomposed into edge disjoint union of $(k-1)/2$ Hamiltonian cycles $C_1, C_2, \ldots, C_{(k-1)/2}$. All edges of each $C_i (i \in [1,(k-1)/2])$ form a matching M_i with $|M_i| = k$ in $S(n,k)$. Furthermore, all $S_i(n,k)$, $i \in [1,k]$, are connected by these edges of M_i. Therefore, each $C_i (i \in [1,(k-1)/2])$ corresponds to a subgraph

$$\mathcal{C}_i = S_1(n,k) \cup S_2(n,k) \cup \cdots \cup S_k(n,k) \cup M_i$$

of $S(n+1,k)$. For a fixed $\mathcal{C}_i$, we will look for a Hamiltonian cycle of $S(n,k)$ in the following way. In each $S_j(n,k)$ for any $j \in [1,k]$, we find a Hamiltonian path $H_{i,j}$ of $S_j(n,k)$ whose two end vertices are incident with two edges of M_i. Let $C'_i = H_{i,1} \cup H_{i,2} \cup \cdots \cup H_{i,k} \cup M_i$, where $i \in [1,(k-1)/2]$. By Theorem 3.5 and the symmetry of $S(n,k)$, we can properly choose $H_{i,j}$ $(j \in [1,(k-1)/2])$ such that $H_{1,j}, H_{2,j}, \ldots, H_{(k-1)/2,j}$ are edge disjoint Hamiltonian paths of $S_j(n,k)$. Thus, C'_i, $i \in [1,(k-1)/2]$, are edge disjoint Hamiltonian cycles of $S(n,k)$.

Next, we show that $S^+(n, k)$ contains $(k-1)/2$ edge disjoint Hamiltonian cycles.

According to Theorem 3.5, we know that $S(n, k)$ contains $(k-1)/2$ edge disjoint Hamiltonian paths $H_i (i \in [0, (k-1)/2-1])$ of which the end vertices are extreme vertices. Since $S^+(n, k)$ is obtained from Sierpiński graph $S(n, k)$ by adding a new vertex w and all edges joining w with k extreme vertices of $S(n, k)$, $C_i = wH_iw$ must be a Hamiltonian cycle of $S^+(n, k)$, where $i \in [0, (k-1)/2-1]$.

Finally, we prove that $S^{++}(n, k)$ contains $(k-1)/2$ edge disjoint Hamiltonian cycles.

For $n = 1$, the statement is clear since $S^{++}(1, k)$ is isomorphic to K_{k+1}. In the following, let $n \geq 2$.

Since $S^{++}(n, k)$ is obtained from the vertex disjoint union of a copy of $S(n, k)$ and a copy $S(n-1, k)$ such that the extreme vertices of $S(n, k)$ and the extreme vertices of $S(n-1, k)$ are connected by a matching M, in what follows, we will properly look for Hamiltonian paths from $S(n, k)$ and $S(n-1, k)$.

According to Theorem 3.5, we know that $S(n, k)$ contains $(k-1)/2$ edge disjoint Hamiltonian paths $H_1, H_2, \ldots, H_{(k-1)/2}$ such that each end vertex of H_i $(i \in [1, (k-1)/2])$ is an extreme vertex. Without loss of generality, suppose that $H_i = H_{(u_i, v_i)}$. Next, by Theorem 3.5 and the symmetry of $S(n-1, k)$, we can look for $(k-1)/2$ edge disjoint Hamiltonian paths $\{H'_1, H'_2, \ldots, H'_{(k-1)/2}\}$ of $S(n-1, k)$ such that $H'_i = H_{(u'_i, v'_i)}$ and $u_iu'_i, v_iv'_i \in E(S^{++}(n, k))$ for $i \in [1, (k-1)/2]$. Thus $C'_i = H_i \cup H'_i \cup \{u_iu'_i, v_iv'_i\}$ must be a Hamiltonian cycle of $S^{++}(n, k)$, where $i \in [1, (k-1)/2]$. □

The proof of Theorem 4.3. In order to prove the theorem, we need to show that $S(n, k)$, for each odd $k \geq 3$, has a partition $\{V_0, V_1, \ldots, V_{(k-1)/2-1}, \mathcal{V}\}$ satisfying:

(a′) $|\mathcal{V}| = k^{n-1}$ and $|V_i| = 2k^{n-1}$ for every $i \in [0, (k-1)/2-1]$,
(b′) each $G[V_i](i \in [0, (k-1)/2-1])$ is a path whose end vertices are two extreme vertices, and $G[\mathcal{V}]$ is a subgraph consisting of one extreme vertex and $(k^{n-1}-1)/2$ isolated edges.

By induction on n.

For $n = 1$, the statement is obvious since $S(1, k)$ is isomorphic to the complete graph K_k.

Assume that $S(n, k)$ has a partition $\{V_0, V_1, \ldots, V_{(k-1)/2-1}, \mathcal{V}\}$ that satisfies (a′) and (b′). In the following, we will prove that $G' = S(n+1, k)$ has also a partition $\{W_0, W_1, \ldots, W_{(k-1)/2-1}, \mathcal{W}\}$ which satisfies (a′) and (b′).

Obviously,

$$S(n+1, k) = S_1(n+1, k) \cup S_2(n+1, k) \cup \cdots \cup S_k(n+1, k) \cup \mathcal{B},$$

where $\mathcal{B}$ is the set of all bridge edges connecting $S_i(n+1, k)$ and $S_j(n+1, k)$ for any $i \neq j$. Moreover, $S_i(n+1, k)$ is isomorphic to the graph $S(n, k)$ for $i \in [1, k]$. For each $S_i(n+1, k)$, we identify all vertices $\langle i \cdots \rangle$ and denote them by s_{i-1}. Then we obtain a complete graph $H = K_k$ with vertex set $V(H) = \{s_0, s_1, \ldots, s_{k-1}\}$ and edge set $E(H) = \mathcal{B}$. Since $k \geq 3$ is odd, by Lemma 2.4, H has a decomposition $\{H_0, H_1, \ldots, H_{(k-1)/2-1}, M\}$, where

$$H_i = s_{0+i}s_{1+i}s_{k-1+i}s_{2+i}s_{k-2+i} \cdots s_{(k-1)/2+i}s_{(k+1)/2+i}, \quad 0 \leq i \leq (k-1)/2-1,$$

are edge disjoint Hamiltonian paths of H, $M = \{s_{(k-1)/2-i}s_{(k-1)/2+i} : i \in [1, (k-1)/2]\}$ is a maximum matching of H, and the indices of s_j are taken modulo k. Clearly, there exists only one vertex $s_{(k-1)/2}$ which is not incident with any edge of M. The matching M corresponds to a matching $\mathcal{M}$ with $|\mathcal{M}| = (k-1)/2$ in $S(n+1, k)$. All edges of each $H_i(i \in [0, (k-1)/2-1])$ form a matching M_i with $|M_i| = k-1$ in $S(n+1, k)$ and all $S_i(n+1, k)$, $i \in [1, k]$, are connected by all edges of M_i. Therefore, each $H_i(i \in [0, (k-1)/2-1])$ corresponds to a subgraph

$$\mathcal{H}_i = S_1(n+1, k) \cup S_2(n+1, k) \cup \cdots \cup S_k(n+1, k) \cup M_i$$

of $S(n+1, k)$. The subgraph $\mathcal{H}_i$ is similar as in Fig. 6.

In the following, all indices j of each $S_j(n+1, k)$ and each $W_{i,j}$ are taken modulo k and the modulo values are all in $[1, k]$, as well as each index j_l in every vertex $\langle j_1j_2 \cdots j_{n+1} \rangle$, where $l \in [1, n+1]$. Finally, let $t = 2(i-j+1)$.

Just as in Theorem 3.5, in $\mathcal{H}_i$, for any $j \in [1, k]$, we have

(1) if $j = i+1$, then $S_j(n+1, k)$ and $S_{j+1}(n+1, k)$ are connected by a single bridge edge whose two end vertices are $\langle j(j+1) \cdots (j+1) \rangle$ and $\langle (j+1)j \cdots j \rangle$; if $j = i+(k+1)/2+1$, then $S_j(n+1, k)$ and $S_{j-1}(n+1, k)$ are connected by a single bridge edge whose two end vertices are $\langle j(j-1) \cdots (j-1) \rangle$ and $\langle (j-1)j \cdots j \rangle$, and
(2) if $j \notin \{i+1, i+(k+1)/2+1\}$, then $S_j(n+1, k)$ and $S_{j+t}(n+1, k)$ are connected by a single bridge edge whose two end vertices are $\langle j(j+t) \cdots (j+t) \rangle$ and $\langle (j+t)j \cdots j \rangle$, and $S_j(n+1, k)$ and $S_{j+t+1}(n+1, k)$ are connected by a single bridge edge whose two end vertices are $\langle j(j+t+1) \cdots (j+t+1) \rangle$ and $\langle (j+t+1)j \cdots j \rangle$.

For a fixed $\mathcal{H}_i$, we will look for a vertex subset W_i of $S(n+1, k)$ in the following way. In each $S_j(n+1, k)$, for $j \notin \{i+1, i+(k+1)/2+1\}$, we choose a vertex subset $W_{i,j}$ with $|W_{i,j}| = 2k^{n-1}$, which contains the two vertices $\langle j(j+t) \cdots (j+t) \rangle$ and $\langle j(j+t+1) \cdots (j+t+1) \rangle$ that are incident with two edges in M_i. In $S_{i+1}(n+1, k)$, we choose a vertex subset $W_{i,i+1}$ with $|W_{i,i+1}| = 2k^{n-1}$, which contains two vertices such that one is $\langle (i+1)(i+2) \cdots (i+2) \rangle$ incident with an edge in M_i and the other is the extreme vertex

$$\alpha = \langle (i+1)(i+1) \cdots (i+1) \rangle.$$

In $S_{i+1+(k+1)/2}(n+1,k)$, we also choose a vertex subset $W_{i,i+1+(k+1)/2}$ with $|W_{i,i+1+(k+1)/2}| = 2k^{n-1}$, which contains two vertices such that one is $\langle (i+1+(k+1)/2)(i+(k+1)/2)\cdots(i+(k+1)/2)\rangle$ incident with an edge in M_i and the other is the extreme vertex

$$\beta = \langle (i+1+(k+1)/2)(i+1+(k+1)/2)\cdots(i+1+(k+1)/2)\rangle.$$

Let $W^j = V(S_j(n+1,k)) \setminus \cup_{i=0}^{(k-1)/2-1} W_{i,j}$ and $\mathcal{W} = W^1 \cup W^2 \cup \cdots \cup W^k$.

Since each $S_i(n+1,k)(i \in [1,k])$ is isomorphic to $S(n,k)$, by induction assumption and the symmetry of $S(n,k)$, we can properly choose $W_{i,j}$ such that $\{W_{0,i}, W_{1,i}, \ldots, W_{(k-1)/2-1,i}, W^i\}$ is a partition of $S_i(n+1,k)$ satisfying (a′) and (b′).

Thus, from each $\mathcal{H}_i(i \in [0,(k-1)/2-1])$, we obtain a vertex subset

$$W_i = W_{i,1} \cup W_{i,2} \cup \cdots \cup W_{i,k},$$

where $i \in [0,(k-1)/2-1]$. It is not difficult to verify that $\{W_0, W_1, \ldots, W_{(k-1)/2-1}, \mathcal{W}\}$ satisfies (a′) and (b′).

In fact, by induction assumption,

$$|W_i| = |W_{i,1}| + |W_{i,2}| + \cdots + |W_{i,k}| = k \cdot 2k^{n-1} = 2k^n,$$

where $i \in [0,(k-1)/2-1]$, and

$$G'[W_i] = G'[W_{i,1}] \cup G'[W_{i,2}] \cup \cdots \cup G'[W_{i,k}] \cup G'[M_i]$$

is a path whose end vertices are extreme vertices since these paths $G'[W_{i,j}], j \in [1,k]$, are properly connected by all edges of M_i for $i \in [0,(k-1)/2-1]$. Moreover,

$$\begin{aligned}|\mathcal{W}| &= |W^1| + |W^2| + \cdots + |W^k| = k \cdot k^{n-1} = k^n,\\ |E(G'[\mathcal{W}])| &= |E(G'[W^1])| + |E(G'[W^2])| + \cdots + |E(G'[W^k])| + |E(G'[\mathcal{M}])|\\ &= k \cdot (k^{n-1}-1)/2 + (k-1)/2 = (k^n-1)/2,\end{aligned}$$

and $G'[\mathcal{W}]$ consists of an isolated vertex $\langle ((k+1)/2)((k+1)/2)\cdots((k+1)/2)\rangle$ and $(k^n-1)/2$ isolated edges. □

The proof of Theorem 4.6. By Theorem 4.3, $G = S(n,k)$, for odd $k \geq 3$, has a partition $\{V_1, V_2, \ldots, V_{(k-1)/2}, \mathcal{V}\}$ satisfying:

(a′) $|\mathcal{V}| = k^{n-1}$ and $|V_i| = 2k^{n-1}$ for every $i \in [1,(k-1)/2]$, and
(b′) each $G[V_i]$ $(i \in [1,(k-1)/2])$ is a path whose end vertices are two extreme vertices, and $G[\mathcal{V}]$ is a subgraph consisting of one extreme vertex and $(k^{n-1}-1)/2$ isolated edges.

Let $P_{(u_i,u_i')} = G[V_i]$ be the path whose end vertices are u_i and u_i' for each $i \in [1,(k-1)/2]$. Since the graph $S^{++}(n,k)$ can be obtained from the vertex disjoint union of a copy of $G = S(n,k)$ and a copy $G' = S(n-1,k)$ such that the extreme vertices of $S(n,k)$ and the extreme vertices of $S(n-1,k)$ are connected by a matching, in the following, we will properly choose a partition $\{V_1', V_2', \ldots, V_{(k-1)/2}', \mathcal{V}'\}$ of $S(n-1,k)$ such that φ satisfies (a) and (b), where $\varphi^{-1}(i) = V_i \cup V_i'$ for any $i \in [1,(k-1)/2]$, and $\varphi^{-1}((k+1)/2) = \mathcal{V} \cup \mathcal{V}'$.

Let $Q_{(v_i,v_i')} = G'[V_i']$ be the path whose end vertices are v_i and v_i' for any $i \in [1,(k-1)/2]$. Then, by the symmetry of $S(n-1,k)$ and Theorem 4.3, we can properly choose a partition $\{V_1', V_2', \ldots, V_{(k-1)/2}', \mathcal{V}'\}$ of $S(n-1,k)$ such that $u_iv_i \in E(S^{++}(n,k)), u_i'v_i' \notin E(S^{++}(n,k))$ and the extreme vertex u in $\mathcal{V}$ is adjacent to the extreme vertex v in $\mathcal{V}'$, moreover, $\{V_1', V_2', \ldots, V_{(k-1)/2}', \mathcal{V}'\}$ satisfies (a′) and (b′). Thus, let $\varphi^{-1}(i) = V_i \cup V_i'$ for any $i \in [1,(k-1)/2]$ and $\varphi^{-1}((k+1)/2) = \mathcal{V} \cup \mathcal{V}'$. It is easy to check that φ satisfies (a) and (b). In fact, $H[V_i \cup V_i'] = P_{(u_i,u_i')} \cup Q_{(v_i,v_i')} \cup \{u_iv_i\}$ is a path with end vertices u_i' and v_i'. $H[\mathcal{V} \cup \mathcal{V}'] = G[\mathcal{V}] \cup G'[\mathcal{V}'] \cup \{uv\}$ is a matching consisting of $(k^{n-1}+k^{n-2})/2$ edges. □

The proof of Claim 2 in Theorem 5.1. In the following, let $k \geq 3$ and we prove the claim by induction on n.

For $n = 1$, the statement is obvious since $S[1,k]$ is isomorphic to the complete graph K_k.

Assume that $S[n,k]$ has a partition $\{V_0, V_1, \ldots, V_{(k-1)/2-1}, \mathcal{V}\}$ that satisfies (a) and (b). In the sequel, we will prove that $G' = S[n+1,k]$ has also a partition $\{V_0', V_1', \ldots, V_{(k-1)/2-1}', \mathcal{V}'\}$ satisfying (a) and (b).

First, we construct a new graph $\overline{G}$, which is obtained from $S[n+1,k]$ by adding all bridge edges between $S_i[n+1,k]$ and $S_j[n+1,k]$ for $i \neq j$. Let $\mathcal{B}$ be the edge set of all bridge edges connecting $S_i[n+1,k]$ and $S_j[n+1,k]$ for $i \neq j$.

Obviously,

$$\overline{G} = S_1[n+1,k] \cup S_2[n+1,k] \cup \cdots \cup S_k[n+1,k] \cup \mathcal{B}.$$

Moreover, $S_i[n+1,k]$ is isomorphic to the graph $S[n,k]$ for $i \in [1,k]$. For each $S_i[n+1,k]$, we identify all vertices of $S_i[n+1,k]$ and denote them by s_{i-1}. Then we obtain a complete graph $H = K_k$ with vertex set $V(H) = \{s_0, s_1, \ldots, s_{k-1}\}$ and edge set $E(H) = \mathcal{B}$.

Thus every vertex s_i of H corresponds to the subgraph $S_{i+1}[n+1,k]$ of $S[n+1,k]$. Since $k(\geq 4)$ is odd, by Lemma 2.4, H has a decomposition $\{H_0, H_1, \ldots, H_{(k-1)/2-1}, M\}$, where

$$H_i = s_{0+i}s_{1+i}s_{k-1+i}s_{2+i}s_{k-2+i}\cdots s_{(k-1)/2+i}s_{(k+1)/2+i}, \quad 0 \leq i \leq (k-1)/2-1,$$

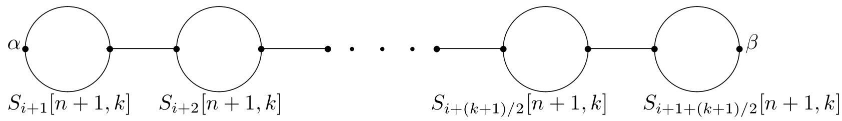

Fig. 7. The subgraph $\mathcal{H}_i$ of $S[n+1,k]$ for odd k.

are edge disjoint Hamiltonian paths of H, $M = \{s_{(k-1)/2-i}s_{(k-1)/2+i} : i \in [1, (k-1)/2]\}$ is a maximum matching of H and the indices of s_j are taken modulo k. All edges of each $H_i (i \in [0, (k-1)/2-1])$ constitute a matching M_i with $|M_i| = k-1$ in $\overline{G}$, and all $S_i[n+1,k]$, $i \in [1,k]$, are connected by all edges of M_i. Therefore, each $H_i (i \in [0, (k-1)/2-1])$ corresponds to a subgraph

$$\mathcal{H}_i = S_1[n+1,k] \cup S_2[n+1,k] \cup \cdots \cup S_k[n+1,k] \cup M_i$$

of $\overline{G}$. The subgraph $\mathcal{H}_i$ is shown in Fig. 7.

In the following, all indices j of each $S_j[n+1,k]$ and each $W_{i,j}$ are taken modulo k and the modulo values are all in $[1,k]$, as well as each index j_l in every vertex $\langle j_1 j_2 \cdots j_{n+1}\rangle$, where $l \in [1, n+1]$. Finally, let $t = 2(i-j+1)$.

Just as in Theorem 3.5, in $\mathcal{H}_i$, for any $j \in [1,k]$, we have

(1) if $j = i+1$, then $S_j(n+1,k)$ and $S_{j+1}(n+1,k)$ are connected by a single bridge edge whose two end vertices are $\langle j(j+1)\cdots(j+1)\rangle$ and $\langle (j+1)j\cdots j\rangle$; if $j = i+(k+1)/2+1$, then $S_j(n+1,k)$ and $S_{j-1}(n+1,k)$ are connected by a single bridge edge whose two end vertices are $\langle j(j-1)\cdots(j-1)\rangle$ and $\langle (j-1)j\cdots j\rangle$, and
(2) if $j \notin \{i+1, i+(k+1)/2+1\}$, then $S_j(n+1,k)$ and $S_{j+t}(n+1,k)$ are connected by a single bridge edge whose two end vertices are $\langle j(j+t)\cdots(j+t)\rangle$ and $\langle (j+t)j\cdots j\rangle$, and $S_j(n+1,k)$ and $S_{j+t+1}(n+1,k)$ are connected by a single bridge edge whose two end vertices are $\langle j(j+t+1)\cdots(j+t+1)\rangle$ and $\langle (j+t+1)j\cdots j\rangle$.

For a fixed $\mathcal{H}_i$, we will look for a vertex subset $\mathcal{W}_i$ of $\overline{G}$ in the following way. In each $S_j[n+1,k]$, for $j \notin \{i+1, i+(k+1)/2+1\}$, we choose a vertex subset $\mathcal{W}_{i,j}$ with $|\mathcal{W}_{i,j}| = k^{n-1}+1$, which contains two vertices $\langle j(j+t)\cdots(j+t)\rangle$ and $\langle j(j+t+1)\cdots(j+t+1)\rangle$ that are incident with two edges of M_i. In $S_{i+1}[n+1,k]$, we choose a vertex subset $\mathcal{W}_{i,i+1}$ with $|\mathcal{W}_{i,i+1}| = k^{n-1}+1$, which contains two vertices such that one is $\langle (i+1)(i+2)\cdots(i+2)\rangle$ incident with an edge of M_i and the other is the extreme vertex

$$\alpha = \langle (i+1)(i+1)\cdots(i+1)\rangle.$$

In $S_{i+1+(k+1)/2}[n+1,k]$, we also choose a vertex subset $\mathcal{W}_{i,i+1+(k+1)/2}$ with $|\mathcal{W}_{i,i+1+(k+1)/2}| = k^{n-1}+1$, which contains two vertices such that one is $\langle (i+1+(k+1)/2)(i+(k+1)/2)\cdots(i+(k+1)/2)\rangle$ incident with an edge of M_i and the other is the extreme vertex

$$\beta = \langle (i+1+(k+1)/2)(i+1+(k+1)/2)\cdots(i+1+(k+1)/2)\rangle.$$

Let $\mathcal{W}^i = V(S_i[n+1,k]) \setminus \cup_{j=0}^{(k-1)/2-1} \mathcal{W}_{j,i}$ and $\mathcal{W} = \cup_{i=1}^{k} \mathcal{W}^i$.

Since each $S_i[n+1,k] (i \in [1,k])$ is isomorphic to $S[n,k]$, by induction assumption and the symmetry of $S[n,k]$, we can properly choose the vertex set $\mathcal{W}_{j,i}$ $(i \in [1,k])$ such that $\{\mathcal{W}_{0,i}, \mathcal{W}_{1,i}, \ldots, \mathcal{W}_{(k-1)/2-1,i}, \mathcal{W}^i\}$ is a partition of $S_i[n+1,k]$ and satisfies (a) and (b).

Thus, from each $\mathcal{H}_i$, we obtain a vertex subset

$$\mathcal{W}_i = \mathcal{W}_{i,1} \cup \mathcal{W}_{i,2} \cup \cdots \cup \mathcal{W}_{i,k},$$

where $i \in [0, (k-1)/2-1]$. By induction assumption, it is not difficult to check that $\{\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_{(k-1)/2}, \mathcal{W}\}$ is a partition of $\overline{G}$ satisfying:

(1) $|\mathcal{W}| = (k^n+k)/2$ and $|\mathcal{W}_i| = k \cdot (k^{n-1}+1) = k^n + k$ for each $i \in [0, (k-1)/2-1]$, and
(2) $\overline{G}[\mathcal{W}_i]$ is a path whose end vertices are two extreme vertices, where $i \in [0, (k-1)/2-1]$, and $\overline{G}[\mathcal{W}]$ consists of $(k-1)/2$ isolated edges of M and $(k^n-k+2)/2$ isolated vertex.

Let X_i and X be the vertex set consisting of all end vertices of edges of M_i and M, respectively. Since all edges of M_i and M are contracted in $S[n+1,k]$ and the two end vertices of every edge of M_i and M are identified in a single vertex, we let Y_i and Y be the vertex sets consisting of new vertices after contracting all edges of M_i and M, respectively.

Let $V_i' = (\mathcal{W}_i \setminus X_i) \cup Y_i$ and $\mathcal{V}' = (\mathcal{W} \setminus X) \cup Y$. Then $\{V_0', V_1', \ldots, V_{(k-1)/2-1}', \mathcal{V}'\}$ is a partition of $S[n+1,k]$ satisfying (a) and (b). In fact, Since $|X_i| = 2(k-1)$ and $|Y_i| = k-1$,

$$|V_i'| = |\mathcal{W}_i| - |X_i| + |Y_i| = k^n + k - 2(k-1) + k - 1 = k^n + 1$$

for each $i \in [0, (k-1)/2-1]$. $G[V_i']$ is a path obtained from $\overline{G}[\mathcal{W}_i]$ by contracting all edges in M_i. Moreover,

$$|\mathcal{V}'| = |\mathcal{W}| - |X'| + |Y'| = (k^n+k)/2 - (k-1) + (k-1)/2 = (k^n+1)/2$$

since $|X| = k-1$ and $|Y| = (k-1)/2$. Clearly, $\mathcal{V}'$ is an independent set of $S[n+1,k]$. □

References

[1] L. Beaudou, S. Gravier, S. Klavžar, M. Kovše, M. Mollard, Covering codes in Sierpiński graphs, Discrete Math. Theor. Comput. Sci. 12 (2010) 63–74.
[2] Béla Bollobás, Modern Graph Theory, Springer-Verlag, New York, 1998.
[3] B.L. Chen, K.C. Huang, On the linear k-arboricity of K_n and $K_{n,n}$, Discrete Math. 254 (2002) 51–61.
[4] H.Y. Fu, D.Z. Xie, Equitable $L(2, 1)$-labelings of Sierpiński graphs, Australas. J. Combin. 46 (2010) 147–156.
[5] S. Gravier, S. Klavžar, M. Mollard, Codes and $L(2, 1)$-labelings in Sierpiński-like graphs, Taiwanese J. Math. 9 (2005) 671–681.
[6] A.M. Hinz, Pascal's triangle and the Tower of Hanoi, Amer. Math. Monthly 99 (1992) 538–544.
[7] A.M. Hinz, S. Klavžar, U. Milutinović, D. Parisse, C. Petr, Metric properties of the Tower of Hanoi graphs and Stern's diatomic sequence, Electron. J. Combin. 26 (2005) 693–708.
[8] A.M. Hinz, D. Parisse, Coloring Hanoi and Sierpiński graphs, Discrete Math. 312 (2012) 1521–1535.
[9] A.M. Hinz, D. Parisse, The average eccentricity of Sierpiński graphs, Graphs Combin. http://dx.doi.org/10.1007/s00373-011-1076-4 (in press).
[10] M. Jakavac, S. Klavžar, Vertex-, edge-, and total-coloring of Sierpiński-like graphs, Discrete Math. 309 (2009) 1548–1556.
[11] M. Jakovac, A 2-parametric generalization of Sierpiński gasket graphs, Ars Combin. (in press).
[12] S. Klavžar, U. Milutinović, Graphs $S(n, k)$ and a variant of the Tower of Hanoi problem, Czechoslovak Math. J. 47 (122) (1997) 95–104.
[13] S. Klavžar, U. Milutinović, C. Petr, 1-perfect codes in Sierpiński-like graphs, Bull. Aust. Math. Soc. 66 (2002) 369–384.
[14] S. Klavžar, B. Mohar, Crossing numbers of Sierpiński-like graphs, J. Graph Theory 50 (2005) 186–198.
[15] C.H. Lin, J.J. Liu, Y.L. Wang, W.C. Yen, The hub number of Sierpiński-like graphs, Theory Comput. Syst. 49 (2011) 588–600.
[16] S.L. Lipscomb, J.C. Perry, Lipscomb's $L(A)$ space fractalized in Hilbert's $l^2(A)$ space, Proc. Amer. Math. Soc. 115 (1992) 1157–1165.
[17] M. Matsumoto, Bounds for the vertex linear arboricity, J. Graph Theory 14 (1990) 73–75.
[18] U. Milutinović, Completeness of the Lipscomb space, Glas. Mat. Ser. III 27 (47) (1992) 343–364.
[19] D. Parisse, On some metric properties of the Sierpiński-like graphs $S(n, k)$, Ars Combin. 90 (2009) 145–160.
[20] T. Pisanski, T.W. Tucker, Growth in repeated truncations of maps, Atti Semin. Mat. Fis. Univ. Modena Reggio Emilia 49 (2001) 167–176.
[21] D. Romik, Shortest paths in the Tower of Hanoi graph and finite automata, SIAM J. Discrete Math. 20 (2006) 610–622.
[22] B. Xue, L. Zuo, On the linear $(n-1)$-arboricity of $K_{n(m)}$, Discrete Appl. Math. 158 (2010) 1546–1550.