Local and Global Hopf Bifurcation in a Scalar Delay Differential Equation

Fotios Giannakopoulos* and Andreas Zapp

Mathematical Institute, University of Cologne, Weyertal 86-90, D-50931 Cologne, Germany
E-mail: giannakopoulos@gmd.de, azapp@mi.uni-koeln.de

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1. INTRODUCTION

In this paper we study the bifurcation properties of Hopf branches in the scalar retarded functional differential equation

\[ \dot{x}(t) = -x(t) + F(x(t - \tau)) \]

with the time delay \( \tau > 0 \) as bifurcation parameter, where \( F \in C^k(\mathbb{R}, \mathbb{R}) \), \( k \geq 3 \), and \( F(0) = 0 \). Equation (1) is a prototype for a retarded functional differential equation which has many applications in sciences. Special cases of (1) have been proposed as models of both physical and physiological phenomena (see, for instance [1, 15, 21, 26]; see also [16, pp. 68–81; 24; 28]).

For a wide class of functions \( F \) (e.g., monotone \( F \)) the dynamics of (1) are determined by periodic solutions (see [25, 31]). In the applications periodic dynamics are also of particular interest (see [16, pp. 57–81]).

There exist many papers which are concerned with the existence of periodic solutions of (1) and explore their dependence on the time delay \( \tau > 0 \) (see [5, 17, 23, 27, 31, 32] and the references in [23, 32]). The existence of a sequence of time delays \( 0 < \tau_0 < \tau_1 < \cdots \), with \( \tau_k \to \infty \) as \( k \to \infty \), such that a local Hopf bifurcation occurs at \( \tau = \tau_k \), provided \( |F'(0)| > 1 \), is proved in [32] (see also [27]). In the case \( F'(0) < -1 \) a condition for supercritical Hopf bifurcation at \( \tau = \tau_0 \) is given in [3]. In [33]

*Present address: GMD-German Research Center for Information Technology, Schloss Birlinghoven, D-53754 Sankt Augustin, Germany.
conditions for both supercritical and subcritical Hopf bifurcations at \( \tau = \tau_0 \) are provided for \( F''(0) < -1, F''(0) = 0, \) and \( F''(0) \neq 0. \)

The global continuation for all \( \tau > \tau_k \) of the locally existing Hopf branches at \( \tau = \tau_k \) is demonstrated in [23], provided that there exists a compact interval \( I \subset \mathbb{R}, \) \( 0 \in \text{int}(I), \) such that \( F(I) \subset I \) (invariance condition), \( xF(x) < 0 \) for all \( x \in I \setminus \{0\}, \) and \( F'(0) < -1 \) (negative feedback condition).

However, many examples of functions \( F \) which do not satisfy the conditions mentioned above arise in applications (see [2; 15; 16, pp. 68–81; 21; 24]).

Furthermore, from the point of view of qualitative analysis, it is also of interest to know how the properties of Eq. (1) change if one of the conditions above fails.

The main objective of this work is to extend the results mentioned above to a more general class of functions \( F. \) More precisely, we give both sufficient and necessary conditions for the local existence and the bifurcation direction of Hopf branches and for the stability of the periodic solutions bifurcating from the trivial solution \( x = 0 \) at \( \tau = \tau_k, k \in \mathbb{N}_0, \) provided \( F \in C^2. \) The Hopf bifurcation formula derived here has a readily applicable form and provides all possible bifurcation scenarios. For the derivation of this formula we consider a system of two ordinary differential equations describing the flow of (1) on an appropriate center manifold (see Sect. 2). Furthermore, we present a result on the global continuation of the local Hopf branches for functions \( F \in C^2, \) where \( F'(x) \neq 1 \) for all fixed points \( x \) of \( F \) and \( |F'(0)| > 1 \) (see Sect. 3).

Finally, we discuss several examples which illustrate our results and indicate what can happen if the invariance condition or the negative feedback condition is violated (see Sect. 4).

## 2. LOCAL BEHAVIOR

Throughout this section we assume

\[ \text{(LH)} \quad F \in C^3, F(0) = 0, \text{ and } |F'(0)| > 1. \]

### 2.1. Main Results

In this subsection we state the main results on the local Hopf bifurcation. First we need some definitions.

**Definition 2.1.** For \( k \in \mathbb{N}_0 \) let

\[ b_k, \tau_k: \mathbb{R} \setminus [-1, 1] \rightarrow \mathbb{R}_+ \]
be positive functions of $\lambda$ defined by

$$b_k(\lambda) := \begin{cases} \arccos\left(\frac{1}{\lambda}\right) + 2k\pi & \text{for } \lambda < -1 \\ 2(k + 1)\pi - \arccos\left(\frac{1}{\lambda}\right) & \text{for } \lambda > 1 \end{cases}$$

(2)

$$\tau_k(\lambda) := \frac{b_k(\lambda)}{\sqrt{\lambda^2 - 1}}$$

(3)

with $\arccos(x)$ in $(0, \pi)$.

Furthermore we define

$$C_k(\lambda) := \begin{cases} \frac{11\lambda^2 + 6\lambda - 2}{(5\lambda + 4)(\lambda - 1)} + \frac{(\lambda + 1)^2}{(5\lambda + 4)(1 + \lambda^2\tau)} & \text{for } \lambda < -1 \\ \frac{11\lambda^3 + 35\lambda^2 + 24\lambda - 6}{(\lambda - 1)(5\lambda^2 + 15\lambda + 12)} - \frac{2(\lambda + 1)(\lambda^2 - 3)}{(5\lambda^2 + 15\lambda + 12)(1 + \lambda^2\tau)} & \text{for } \lambda > 1 \end{cases}$$

with $\tau = \tau_k(\lambda)$, $k \in \mathbb{N}_0$.

**Theorem 2.1 (Local Hopf Bifurcation).** Let $|\lambda| > 1$ with $\lambda = F'(0)$.

1. At $\bar{\tau} = \tau_k(\lambda)$, $k \in \mathbb{N}_0$, Eq. (1) undergoes a Hopf bifurcation; that is, in every small neighborhood of $(\bar{x} = 0, \bar{\tau} = \tau_k(\lambda))$ there is a unique branch of periodic solutions $x_k(t; \tau)$ with $x_k(t; \tau) \to 0$ as $\tau \to \tau_k(\lambda)$. For the period $p_k(\lambda, \tau)$ of $x_k(t; \tau)$, $p_k(\lambda, \tau) \to 2\pi \tau_k(\lambda)/b_k(\lambda) = 2\pi/\sqrt{\lambda^2 - 1} =: p(\lambda)$ as $\tau \to \tau_k(\lambda)$.

2. Assume

$$F''(0)F'(0) < F''(0)^2 C_k(\lambda).$$

Then the bifurcating branch of periodic solutions exists for $\tau > \tau_k(\lambda)$ (supercritical bifurcation). Moreover the arising periodic solutions are stable if

$$\lambda < -1 \quad \text{and} \quad k = 0$$

and they are unstable if

$$\lambda < -1 \quad \text{and} \quad k \geq 1 \quad \text{or} \quad \lambda > 1.$$

3. Assume

$$F''(0)F'(0) > F''(0)^2 C_k(\lambda).$$
Then the bifurcating branch of periodic solutions exists for $\tau < \tau_k(\lambda)$ (subcritical bifurcation). All periodic solutions on this branch are unstable.

**Proof.** The proof is given in Section 2.5.

**Remark 2.1.**
1. For $|\lambda| \leq 1$ there are no Hopf bifurcation points.
2. $\tau = \tau$ is a Hopf bifurcation point if and only if $\tau = \tau_k(\lambda)$ with $k \in \mathbb{N}_0$.

Using the properties of the function $C_k(\lambda)$ we get the following criteria for the direction of the bifurcating periodic solutions (see Fig. 1).

**Corollary 2.1.** Let $F''(0)$ and $F'''(0)$ be not simultaneously equal to zero and $|\lambda| > 1$ with $\lambda = F'(0)$.

1. If $F''(0)F'(0) \leq 0$ then all bifurcations are supercritical.
2. If $F''(0) = 0$ and $F'''(0)F'(0) > 0$ then all bifurcations are subcritical.
3. If $F''(0) \neq 0$ and $F'''(0)F'(0) > 0$ then
   (a) For $\lambda \in (-\infty, -1) \cup (\sqrt{3}, \infty)$ the Hopf bifurcations occurring at $\tau = \tau_k(\lambda)$ are either supercritical for all $k \in \mathbb{N}_0$ or subcritical for all $k \in \mathbb{N}_0$ or there is a unique $k_c \in \mathbb{N}$ such that the Hopf bifurcations occurring at $\tau = \tau_k(\lambda)$ are subcritical for $k < k_c$ and supercritical for $k > k_c$.
   
   (b) For $\lambda \in (1, \sqrt{3})$ the Hopf bifurcations occurring at $\tau = \tau_k(\lambda)$ are either supercritical for all $k \in \mathbb{N}_0$ or subcritical for all $k \in \mathbb{N}_0$, or there is a unique $k_c \in \mathbb{N}$ such that the Hopf bifurcations occurring at $\tau = \tau_k(\lambda)$ are supercritical for $k < k_c$ and subcritical for $k > k_c$.

**Fig. 1.** All possible bifurcation scenarios (see Corollary 2.1).
(c) For \( \lambda = \sqrt{3} \) and \( F''(0)F'(0) \neq F''(0)^2 C_k(\sqrt{3}) \) the Hopf bifurcations occurring at \( \tau = \tau_k(\lambda) \) are either supercritical or subcritical for all \( k \in \mathbb{N}_0 \).

**Proof.** The proof can be found in Section 2.4.

### 2.2. The Characteristic Equation

The objective of this section is to study the properties of the characteristic equation associated with the linearization of (1). For the sake of simplicity we rescale the time \( t \) by \( t \to \frac{t}{\tau} \). This provides a scalar delay differential equation with constant time delay 1,

\[
\dot{x}(t) = -\tau x(t) + \tau F(x(t - 1)).
\]

After linearization about the equilibrium \( x = 0 \) we get

\[
\dot{x}(t) = -\tau x(t) + \tau \lambda x(t - 1), \quad \lambda := F'(0).
\]

The characteristic equation of (5) is given by

\[
0 = \Delta(z; \tau, \lambda) := z + \tau - \tau \lambda \exp(-z).
\]

The analysis of (6) yields conditions for the stability of the equilibrium \( x = 0 \) and critical parameter values at which a Hopf bifurcation can occur. Much is known about the characteristic equation \( \Delta = 0 \); for instance, see [4, 9, 20, 23, 27]. Here we present only some relevant results for completeness.

**Lemma 2.1.** The characteristic equation (6) has a pair of purely imaginary, conjugate complex solutions \( z = \pm ib, \, b > 0 \), if and only if \( |\lambda| > 1 \) and \( b = b_k(\lambda), \, \tau = \tau_k(\lambda), \, k \in \mathbb{N}_0 \). \( b_k, \tau_k \) are defined by (2) and (3). The functions \( b_k, \tau_k \) are \( C^1 \), and \( b_k \) satisfy

\[
b_k(\lambda) \in \left(2k + \frac{1}{2}\right)\pi, (2k + 2)\pi \quad \text{for} \quad k \in \mathbb{N}_0, \, \lambda > 1,
\]

\[
b_k(\lambda) \in \left(2k + \frac{1}{2}\right)\pi, (2k + 1)\pi \quad \text{for} \quad k \in \mathbb{N}_0, \, \lambda < -1.
\]

**Proof.** See [9, pp. 305–309], Proposition A.2 in [23], or Lemma 3.1 in [32]. The estimates for \( b_k \) follow from Definition 2.1.

**Lemma 2.2 (Properties of \( \tau_k \)).** \( \tau_k \) is a decreasing function of \( \lambda > 1 \) and an increasing function of \( \lambda < -1 \), \( k \in \mathbb{N}_0 \). Furthermore, the following hold for \( k \in \mathbb{N}_0 \)

\[
\tau_{k+1}(\lambda) - \tau_k(\lambda) = \frac{2\pi}{\sqrt{\lambda^2 - 1}} \quad \forall \lambda \in \mathbb{R} \setminus [-1, 1] \quad \text{(monotonicity in} \, k)\]
and

\[ \lim_{\lambda \to \pm \infty} \tau_k(\lambda) = 0 \quad \lim_{\lambda \to \pm 1} \tau_k(\lambda) = \infty. \]

**Proof.** The assertions follow from Definition 2.1 and Lemma 2.1.

**Lemma 2.3.** Let \(|\lambda| > 1\). Then there are an \(\epsilon > 0\) and a simple characteristic root \(z(\tau) = a(\tau) + ib(\tau)\) of (6) for \(|\tau - \tau_k(\lambda)| < \epsilon\), \(k \in \mathbb{N}_0\). \(z\) is \(C^1\) in \(\tau\) and \(a(\tau_k(\lambda)) = 0\), \(b(\tau_k(\lambda)) = b_k(\lambda)\). Moreover \(\frac{\tau}{\pi}a(\tau_k(\lambda)) > 0\).

**Proof.** See Lemma 3.1 in [32].

**Lemma 2.4 (D-partition of the \((\lambda, \tau)\)-Plane).** The characteristic equation (6) has

1. only solutions with negative real part, provided \(\tau \geq 0\) and \(\lambda \in [-1, 1)\) or \(0 \leq \tau < \tau_0(\lambda)\) and \(\lambda < -1\);
2. exactly one solution with positive real part, provided \(0 \leq \tau \leq \tau_0(\lambda)\) with \(\lambda > 1\);
3. exactly \(2k\) solutions with positive real part, provided \(\tau_k - 1(\lambda) < \tau < \tau_k(\lambda)\) with \(\lambda < -1\), \(k \in \mathbb{N}\);
4. exactly \(2k + 1\) solutions with positive real part, provided \(\tau_k - 1(\lambda) < \tau \leq \tau_k(\lambda)\) with \(\lambda > 1\), \(k \in \mathbb{N}\).

(See Fig. 2.)

**Proof.** The proof is an application of XI.2 in [9]. (See also Proposition A.2 in [23].)

**Remark 2.2.** If \((\lambda, \tau) \in [-1, 1) \times [0, \tau_0(\lambda)]\) the equilibrium \(\bar{x} = 0\) is asymptotically stable. If \((\lambda, \tau) \in (1, \infty) \times [0, \tau_0(\lambda)]\) of \((\lambda, \tau) \in (1, \infty) \times [0, \tau_0(\lambda)]\) the equilibrium \(\bar{x} = 0\) is unstable.

### 2.3. Hopf Bifurcation

In the following theorem we show that a Hopf bifurcation occurs at \(\tau = \tau_k(\lambda)\). The proof is an application of the Hopf theorem for retarded functional differential equations (see [19, p. 332]). In [27, 32] the existence of Hopf bifurcations is proved in a similar way.

**Theorem 2.2.** Let \(|\lambda| > 1\) with \(\lambda = F'(0)\) and \(k \in \mathbb{N}_0\). Then a Hopf bifurcation for Eq. (1) occurs at \(\bar{\tau} = \tau_k(\lambda)\); i.e., in every small neighborhood of \((\bar{x} = 0, \bar{\tau} = \tau_k(\lambda))\) there is a unique branch of periodic solutions \(x_k(t; \tau)\) with \(x_k(t; \tau) \to 0\) as \(\tau \to \tau_k(\lambda)\). For the period \(p_k(\lambda, \tau)\) of \(x_k(t; \tau)\), \(p_k(\lambda, \tau) \to 2p_k(\tau_k(\lambda))/b_k(\lambda) = 2\pi/\sqrt{\lambda^2 - 1} = p(\lambda)\) as \(\tau \to \tau_k(\lambda)\).
Proof. Lemma 2.1 and Lemma 2.3 provide the assumptions of the Hopf–Theorem (see [19, p. 332]) and thus the above statements are proved.

Remark 2.3. By Lemma 2.1 we get that only at $\tau = \tau_0(\lambda)$ with $\lambda < -1$ can the Hopf bifurcation provide slowly oscillating periodic solutions (i.e., $p_0(\lambda, \tau) > 2\tau$). For $\lambda < -1$ and $k \geq 1$, $p_k(\lambda, \tau) \in \left[\mid \tau/(k + \frac{1}{2}) \mid, \tau/(k + \frac{1}{2}) \right]$, and for $\lambda > 1$ and $k \in \mathbb{N}_0$, $p_k(\lambda, \tau) \in \left[\mid \tau/(k + 1) \mid, \tau/(k + \frac{1}{2}) \right]$. For the definition of slowly oscillating periodic solutions see [23].

2.4. Direction of Hopf Bifurcation

In order to be able to analyse the Hopf bifurcation in more detail we compute the reduced system on the center manifold associated with the pair of conjugate complex, purely imaginary solutions $\Lambda = \{ib_\lambda(\lambda), -ib_\lambda(\lambda)\}$ of the characteristic equation (6). By this reduction we are able to determine the Hopf bifurcation direction, i.e., to answer the question of whether the bifurcating branch of periodic solution exists locally for
\[ \tau > \tau_k(\lambda) \) (supercritical bifurcation) or \( \tau < \tau_k(\lambda) \) (subcritical bifurcation). In the sequel we write \( b_k, \tau_k \) instead of \( b_k(\lambda) \) and \( \tau_k(\lambda) \). In general it is difficult to compute the center manifold itself, but in [12] a method is presented providing the reduced system on the center manifold in normal form without computing the manifold itself. A similar method to estimate the direction of the Hopf bifurcation is presented in [30] by using the method of Lyapunov and Schmidt.

Since we have two simple, purely imaginary characteristic solutions the center manifold considered is two-dimensional and thus, following [12, 13], we can compute the reduced system on the center manifold in polar coordinates \((\rho, \xi)\):

\[
\dot{\rho} = (\tau - \tau_k) \frac{d}{d\tau} a(\tau_k) \rho + K \rho^3 + O((\tau, \rho) |^4) \quad (7)
\]

\[
\dot{\xi} = -b_k + O((\tau, \rho)). \quad (8)
\]

Since \( \frac{d}{d\tau} a(\tau_k) > 0 \) (see Lemma 2.3) the sign of the factor \( K \) determines the direction of the Hopf bifurcation. A positive \( K \) implies a subcritical bifurcation; a negative \( K \) implies a supercritical bifurcation. Our aim is now to calculate the factor \( K \) in terms of the derivatives of \( F \) at zero. To do this let us first give the Taylor expansion of the right hand side of (4). Setting \( \alpha := \tau - \tau_k \), Eq. (4) becomes

\[
\dot{x}(t) = L_\alpha x + G(x(t)) \quad (4)
\]

with

\[
L_\alpha \varphi = - (\tau_k + \alpha) \varphi(0) + (\tau_k + \alpha) \lambda \varphi(-1)
\]

\[
= (\tau_k + \alpha) \int_{0}^{1} \eta \varphi(\eta) d\eta \quad (4)
\]

and

\[
G(u; \alpha) = (\tau_k + \alpha) F(u) - (\tau_k + \alpha) \lambda u.
\]

The function \( G \) satisfies

\[
G(0; \alpha) = 0, \quad DG(0; \alpha) = 0,
\]

\[
G(u; \alpha) = \tau_k \frac{F''(0)}{2} u^2 + \frac{F''(0)}{2} \alpha u^2 + \tau_k \frac{F''(0)}{3!} u^3 + O((u, \alpha) |^4).
\]

This implies

\[
G(u; 0) = \tau_k \frac{F''(0)}{2} u^2 + \tau_k \frac{F''(0)}{3!} u^3 + O(u^4).
\]
Proceeding as in [12, 13], in the next step we consider the decomposition of the phase space $\mathcal{V} = C([−1, 0], \mathbb{R}) = P \oplus Q$, where $P$ is the two-dimensional eigenspace associated with the set of simple eigenvalues $\Lambda = \{ib_k, -ib_k\}$. Let $\Phi = (\varphi_1, \varphi_2) = (e^{ib_k}, e^{-ib_k})$ be a basis of $P$ and $\Psi = \text{col}(\psi_1, \psi_2) = (\psi_1(0)e^{-ib_k}, \psi_2(0)e^{ib_k})$ a basis of the dual space $P^*$ in $\mathcal{V}^*$ with the property

$$\langle \Psi, \Phi \rangle := \langle (\psi_i, \varphi_j), i, j = 1, 2 \rangle = I_2.$$ 

The above bilinear form is given by (see [19, p. 211])

$$\langle \psi, \varphi \rangle = \psi(0)\varphi(0) - \int_{-1}^{0} \int_{0}^{\theta} \psi(\xi - \theta) d\eta(\theta) \varphi(\xi) d\xi.$$ 

$(\Psi, \Phi) = I_2$ yields (see [13])

$$\psi_i(0) = \left[1 - L_0(\theta e^{ib_k})\right]^{-1}, \quad \psi_2(0) = \overline{\psi_1(0)}. \quad (9)$$

Thus the factor $K$ in (7) is given by (see [13])

$$K = \frac{F''(0)\tau_k}{2} \left\{ \frac{-\text{Re}\left(e^{-ib_k}\psi_1(0)\right)}{L_0(1)} + \frac{1}{2} \frac{\text{Re}\left(e^{-3ib_k}\psi_1(0)\right)}{L_0(1) + \text{Re}\left(2ib_k - L_0(e^{2ib_k})\right)} \right\}. \quad (10)$$

Here we have

$$L_0(1) = \tau_k(\lambda - 1), \quad L_0(\theta e^{ib_k}) = -\tau_k \lambda e^{-ib_k},$$

$$L_0(e^{2ib_k}) = \tau_k(-1 + \lambda e^{-2ib_k}), \quad \psi_1(0) = \frac{1}{1 + \tau_k \lambda e^{-ib_k}}.$$ 

Substituting the above terms into Eq. (10) we obtain $K$ as function of $\lambda$. Now we are in a position to prove the following useful lemma.

**Lemma 2.5.** Let $\tau = \tau_k(\lambda), \lambda \in \mathbb{N}_0$. Then

$$K = \frac{F''(0)\tau_k}{2} \frac{1 + \lambda^2\tau}{\lambda(1 + 2\tau + \lambda^2\tau^2)} - \frac{F''(0)^2\tau_k^2}{2} \frac{\lambda^2(11\lambda^2 + 6\lambda - 2) + (2\lambda^3 + 13\lambda^2 + 4\lambda - 4)}{\lambda^2(5\lambda + 4)(\lambda - 1)(1 + 2\tau + \lambda^2\tau^2)}.$$
for $\lambda < -1$ and
\[
K = \frac{F''(0)\tau}{2} \frac{1 + \lambda^2}{\lambda(1 + 2\tau + \lambda^2\tau^2)} \frac{F'(0)^2}{2} \frac{\lambda^2(11\lambda^3 + 21\lambda^2 + 24\lambda - 6) + (-2\lambda^4 + 11\lambda^3 + 43\lambda^2 + 24\lambda - 12)}{\lambda^2(\lambda - 1)(5\lambda^2 + 15\lambda + 12)(1 + 2\tau + \lambda^2\tau^2)}
\]
for $\lambda > 1$.

**Proof.** The proof will be given in the Appendix.

To determine the direction of the Hopf bifurcation one can evaluate the term $K$ or one can use the following conditions which guarantee the sign of $K$:

**Lemma 2.6.** Let $|\lambda| > 1$ and $\tau = \tau_k(\lambda)$, $k \in \mathbb{N}_0$. Then
\[
\begin{align*}
K < 0 & \iff F''(0)F'(0) < F''(0)^2C_k(\lambda) \\
K > 0 & \iff F''(0)F'(0) > F''(0)^2C_k(\lambda) \\
K = 0 & \iff F''(0)F'(0) = F''(0)^2C_k(\lambda),
\end{align*}
\]
where
\[
C_k(\lambda) := \begin{cases}
\frac{11\lambda^2 + 6\lambda - 2}{(5\lambda + 4)(\lambda - 1)} + 2\left(\frac{\lambda + 1}{5\lambda + 4}(1 + \lambda^2\tau^2)\right) & \text{for } \lambda < -1 \\
\frac{11\lambda^2 + 35\lambda^2 + 24\lambda - 6}{(\lambda - 1)(5\lambda^2 + 15\lambda + 12)} - 2\left(\frac{\lambda + 1}{5\lambda^2 + 15\lambda + 12}(1 + \lambda^2\tau^2)\right) & \text{for } \lambda > 1.
\end{cases}
\]

**Proof.** Multiplying the term $K$ given in Lemma 2.5 by the positive factor $\lambda^2(2/\tau)(1 + 2\tau + \lambda^2\tau^2)/(1 + \lambda^2\tau^2)$ provides the results stated in the lemma.

**Corollary 2.2.** Let $|\lambda| > 1$ with $\lambda = \lambda(0)$ and $\tau = \tau_k(\lambda)$, $k \in \mathbb{N}_0$. Then the periodic solutions bifurcating at $\tau = \tau_k(\lambda)$ exist

1. for $\tau > \tau_k(\lambda)$ provided $F''(0)F'(0) < F''(0)^2C_k(\lambda)$ (supercritical bifurcation);
2. for $\tau < \tau_k(\lambda)$ provided $F''(0)F'(0) > F''(0)^2C_k(\lambda)$ (subcritical bifurcation).

**Proof.** Equations (7) and (8) and Lemma 2.6 provide the assertions.
The next lemma deals with the properties of the sequence \((C_k(\lambda))_{k \in \mathbb{N}_0}\).

**Lemma 2.7.** The following hold:

1. For \(\lambda < -1\) or \(\lambda > \sqrt{3}\) the sequence \((C_k(\lambda))_{k \in \mathbb{N}_0}\) is positive, bounded, and strictly increasing.
2. For \(1 < \lambda < \sqrt{3}\) the sequence \((C_k(\lambda))_{k \in \mathbb{N}_0}\) is positive, bounded, and strictly decreasing.
3. For \(\lambda = \sqrt{3}\) \((C_k(\lambda))_{k \in \mathbb{N}_0}\) is a constant sequence.

**Proof.** The statements are a consequence of the definition of \(C_k(\lambda)\).

**Remark 2.4.** One can prove that \((C_k(\lambda))_{k \in \mathbb{N}_0}\) is bounded by

\[
1.4887 \equiv 2 \frac{126 + 125\sqrt{7}}{(2 + 5\sqrt{7})(35 + 2\sqrt{7})} < C_k(\lambda) < \frac{11}{5} \text{ for } \lambda < -1
\]

and

\[
C_k(\lambda) > \frac{21}{10} \text{ for } \lambda > 1.
\]

**Remark 2.5.** The monotonicity and boundness of the sequence \((C_k(\lambda))_{k \in \mathbb{N}_0}\) provide readily applicable criteria for the direction of the Hopf bifurcation at \(\tau = \tau_k(\lambda)\). For instance:

The monotonicity implies the assertions of Corollary 2.1.3. From the boundness we get

1. If \(\lambda < -1\) and \(F''(0)F'(0) < 2(126 + 125\sqrt{7})/(2 + 5\sqrt{7})(35 + 2\sqrt{7})F''(0)^2\) then all bifurcations are supercritical \((K < 0)\).
2. If \(\lambda < -1\) and \(F''(0)F'(0) > \frac{11}{5}F''(0)^2\) then all bifurcations are subcritical \((K > 0)\).
3. If \(\lambda > 1\) and \(F''(0)F'(0) < \frac{21}{10}F''(0)^2\) then all bifurcations are supercritical \((K < 0)\).

**Remark 2.6.** If \(K = 0\), i.e., \(C = C_0(\lambda)\) (see proof of Corollary 2.1) or \(F''(0) = F''(0) = 0\), one has to calculate higher derivatives of \(F\) and to compute higher order terms of the normal form of the reduced system on the center manifold.

### 2.5. Proofs of the Main Results

**Proof of Theorem 2.1.** Theorem 2.2 provides the existence of Hopf bifurcation at \(\tau = \tau_k(\lambda)\). By applying Corollary 2.2 one gets the conditions for the direction of the Hopf bifurcation. The stability properties of the bifurcating periodic solutions follow from Lemma 2.4 and [7].
Proof of Corollary 2.1. If \( F''(0)F'(0) \leq 0 \) then \( K < 0 \) and therefore, in this case, all bifurcations are supercritical. If \( F''(0) = 0 \) and \( F'''(0) \neq 0 \) then \( \text{sign} \ K = \text{sign} \ F'''(0)F'(0) \), and thus all bifurcations are either supercritical or subcritical. Otherwise there is a unique \( C > 0 \) with \( F'''(0)F'(0) = F''(0)^2C \). If \( C < C_0(\lambda) \), then the bifurcation is supercritical; if \( C > C_0(\lambda) \) the bifurcation is subcritical. By applying Lemma 2.7 it is clear that there is at most one \( k \in \mathbb{N}_0 \) where the direction of the Hopf bifurcations can change, i.e., \( (C - C_{k-1}(\lambda))(C - C_{k+1}(\lambda)) < 0 \), provided \( \lambda \neq \sqrt{3} \).

3. GLOBAL CONTINUATION

In this section we study the global continuation of the local Hopf branches bifurcating from the points \((\bar{x}, \tau) = (0, \tau_0(\lambda)); k \in \mathbb{N}_0\). We prove that each of these branches lies on an unbounded continuum \( S^\lambda_k \) of periodic solutions with pairwise disjoint \( S^\lambda_k, k \in \mathbb{N}_0 \).

First we need some notation. We set

\[
S^\lambda := \{ (\varphi, \tau, p) \in \mathscr{C} \times \mathbb{R}_+ \times \mathbb{R}_+; \ x(t) \text{ is a non-constant} \\
p\text{-periodic solution of (1) with } \lambda = F'(0) \text{ and } x_{[0]} = \varphi \}
\]

\[
\bar{S}^\lambda := \text{the closure of } S^\lambda \text{ in } \mathscr{C} \times \mathbb{R}_+ \times \mathbb{R}_+
\]

Now let \( p(\lambda) = 2\pi \tau_2(\lambda)/b_2(\lambda) = 2\pi / \sqrt{\lambda^2 - 1} \) with \( \lambda = F'(0); \tau_2(\lambda) \) and \( b_2(\lambda) \) as in Section 2, \( k \in \mathbb{N}_0 \). We define \( S^\lambda_k \) to be the maximal connected component of \( \bar{S}^\lambda \) in \( \mathscr{C} \times \mathbb{R}_+ \times \mathbb{R}_+ \) containing \((0, \tau_2(\lambda), p(\lambda)); k \in \mathbb{N}_0\), where \( \mathscr{C} = C([-\tau, 0], \mathbb{R}) \). Throughout this section we need the following assumptions:

(\text{GH} 1) \( F \in C^2, F(0) = 0, \) and \( |F'(0)| > 1 \).

(\text{GH} 2) \( \text{If } F(\bar{x}) = \bar{x}, \bar{x} \in \mathbb{R}, \text{ then } F'(\bar{x}) \neq 1. \)

Theorem 3.1. Assume that (\text{GH} 1) and (\text{GH} 2) hold. Then the maximal connected component \( S^\lambda_k \subset \mathscr{C} \times \mathbb{R}_+ \times \mathbb{R}_+ \) containing \((0, \tau_2(\lambda), p(\lambda)); \) where \( \lambda = F'(0) \), is unbounded, and \((\bar{x}, \bar{\tau}, \bar{p}) \in S^\lambda_k \) with \( \bar{x} \in \mathbb{R} \) if and only if \( \bar{x} = 0, \bar{\tau} = \tau_2(\lambda), \text{ and } \bar{p} = p(\lambda) = 2\pi \tau_2(\lambda)/b_2(\lambda) \), where \( b_2(\lambda) \) and \( \tau_2(\lambda) \) are given by (2) and (3), respectively.

Furthermore, if \( (\varphi, \tau, p) \in S^\lambda_k \) and \( \varphi \neq 0 \), then the unique existing solution \( x(t) \) of (1) with \( x_{[0]} = \varphi \) is periodic with minimal period \( p \) satisfying

1. \( p > 2\tau \) provided \( \lambda < -1 \) and \( k = 0; \)
2. \( p \in ]\tau/(k + 1), \tau/(k + \frac{1}{2}] \) provided \( \lambda < -1 \) and \( k \geq 1; \)
3. \( p \in ]\tau/(k + 1), \tau/(k + \frac{3}{4}] \) provided \( \lambda > 1 \) and \( k \in \mathbb{N}_0 \).
Proof. To prove that $S^k_\xi$ is unbounded we apply Theorem 3.3 in [11] (see also Theorem 2.5 in [32]). The theorem mentioned above provides that, under certain assumptions, either $S^k_\xi$ is unbounded or $S^k_\xi$ is bounded and additionally

$$S^k_\xi \cap \{ (\tilde{x}, \tilde{\tau}, \tilde{p}) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+ : \tilde{x} \in \mathbb{R}, \tilde{x} = F(\tilde{x}) \} = \{ (x_i, \tilde{\tau}_i, \tilde{p}_i) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+ : i = 0, 1, 2, \ldots, n \}$$

and

$$\sum_{i=0}^{n} \gamma(x_i, \tilde{\tau}_i, \tilde{p}_i) = 0. \quad (11)$$

where $\gamma(x_i, \tilde{\tau}_i, \tilde{p}_i)$ is the so-called crossing number of $(x_i, \tilde{\tau}_i, \tilde{p}_i)$ whose definition will be given below. Consequently, in order to prove that $S^k_\xi$ is unbounded, it suffices to show that Eq. (11) cannot be fulfilled. First we need to give the definition of $\gamma(x_i, \tilde{\tau}_i, \tilde{p}_i)$ (see [11, 32]).

Let $(\tilde{x}, \tilde{\tau}, \tilde{p})$ be an isolated center of (1); i.e., $\tilde{x} = F(\tilde{x})$, $\Delta(i(2\pi\tilde{\tau}/\tilde{p}), \tilde{\tau}, F'(\tilde{x})) = 0$, and there exists a neighborhood $U(x, \tilde{\tau}, \tilde{p})$ of $(x, \tilde{\tau}, \tilde{p})$ such that $(\tilde{x}, \tilde{\tau}, \tilde{p})$ is the only center in $U(x, \tilde{\tau}, \tilde{p})$.

Using the definition of $\Delta$ (see Eq. (6)) and Lemma 2.3 one can show that $\Delta(z; \tilde{\tau}, \tilde{\lambda})$ is analytic in $z \in \mathbb{C}$ and continuous in $\tau \in [\tilde{\tau} - \epsilon_0, \tilde{\tau} + \epsilon_0]$, where $\epsilon_0 > 0$ is an appropriate positive constant and $\tilde{\lambda} = F'(\tilde{x})$ (see Condition (A3) in [32]). Moreover, it follows from Lemma 2.1 and 2.3 that there exist small positive constants $\delta, \epsilon \in [0, \epsilon_0]$ such that on $\partial \Omega_{x, p} \times [\tilde{\tau} - \delta, \tilde{\tau} + \delta]$

$$\Delta \left( a + i \frac{2\pi \tau}{p}, \tau, \tilde{\lambda} \right) = 0 \quad \text{if and only if} \quad \tau = \tilde{\tau}, a = 0, \text{and} \quad p = \tilde{p},$$

where

$$\Omega_{x, p} := \{ (a, p) : 0 < a < \epsilon, p \in [\tilde{p} - \epsilon, \tilde{p} + \epsilon] \}$$

(see Condition (A4) in [32]).

Notice that $z = \pm i(2\pi/\tilde{p})$ are the only zeroes of $\Delta(z; \tilde{\tau}, \tilde{\lambda})$ on the imaginary axis (see Lemma 2.1). We define

$$H_x(a, p) := \Delta \left( a + i \frac{2\pi \tau}{p}, \tau, \tilde{\lambda} \right).$$

$H_x : \Omega_{x, p} \to \mathbb{R}^2 \equiv \mathbb{C}$ is a differentiable function of $(a, p) \in \Omega_{x, p}$. Now the crossing number $\gamma(x, \tilde{\tau}, \tilde{p})$ can be given by

$$\gamma(x, \tilde{\tau}, \tilde{p}) := \deg(H_{x-\delta}, \Omega_{x, p}) - \deg(H_{x+\delta}, \Omega_{x, p}).$$
where \( \text{deg} \) denotes the Brouwer degree with respect to \((0,0) \in \mathbb{R}^2 \cong \mathbb{C} \) and \( \Omega_{x_0,p} \) (see [11, 32]). This implies that the crossing number \( \gamma(x, \tau, p) \) is well defined. A calculation shows

\[
\det(DH_x(a, p)_{(a, p) \in H_{x_0}^{-1}(0, 0)}) = -\frac{2\pi \tau}{p^2} \left((1 + a + \tau)^2 + \frac{4\pi^2 \tau^2}{p^2}\right) < 0.
\]

Using the definition of the degree (see [6, p. 65]) we obtain

\[
\text{deg}(H_x, \Omega_{x_0,p}) = \sum_{(a, p) \in H_{x_0}^{-1}(0, 0)} (-1) = -\#H_{x_0}^{-1}(0, 0),
\]

where \( \#H_{x_0}^{-1}(0, 0) \) denotes the number of elements in the set \( H_{x_0}^{-1}(0, 0) \), provided \( H_{x_0}^{-1}(0, 0) \) consists of only a finite number of points in \( \Omega_{x_0,p} \).

Lemma 2.4 provides that, for sufficiently small \( \delta \) and \( \epsilon > 0 \), \( H_{x_0}^{-1}(0, 0) = \emptyset \) and \( H_{x_0}^{-1}(0, 0) \) consists of only one point in \( \Omega_{x_0,p} \). This implies

\[
\gamma(x, \tau, p) = -1,
\]

provided \((x, \tau, p)\) is an isolated center. If \((x, \tau, p)\) is not a center, we can easily prove that

\[
\gamma(x, \tau, p) = 0.
\]

Now, in order to be able to apply Theorem 3.3 in [11] (see also Theorem 2.5 in [32]), we still have to prove that all centers of (1) are isolated. Lemma 2.2 provides that if \( x \in \mathbb{R} \) is a stationary solution, all centers \((x, \tau_k, p_k)\), \( k \in \mathbb{N}_0 \), corresponding to \( x \), are isolated. By assumption (GH 2) this yields that all the centers are isolated, and thus we have proved that \( S_{x_0}^k \) is unbounded and \((x, \tau, p) \in S_{x_0}^k \) if and only if \( x = 0, \tau = \tau_k(\lambda) \), and \( p = p(\lambda) \) with \( \lambda = F'(0) \).

By applying Lemma 4.1 in [8] to (4), we get that Eq. (1) has no non-constant periodic solutions of period \( \frac{2\pi}{m} \) for any \( m \in \mathbb{N} \). From Theorem 2.1 and Lemma 2.1 we know

\[
p(\lambda) = \frac{2\pi}{b_k(\lambda)} \tau_k(\lambda), k \in \mathbb{N}_0,
\]

where

\[
b_k(\lambda) \in \left[(2k + \frac{3}{2})\pi, (2k + 2)\pi\right] \quad \text{for} \quad k \in \mathbb{N}_0, \lambda > 1
\]

and

\[
b_k(\lambda) \in \left[(2k + \frac{1}{2})\pi, (2k + 1)\pi\right] \quad \text{for} \quad k \in \mathbb{N}_0, \lambda < -1.
\]
This provides
\[ p(\lambda) > 2\tau_0(\lambda) \quad \text{for } k = 0, \text{if } \lambda < -1, \]
\[ p(\lambda) \in \left[ \frac{\tau_k(\lambda)}{k + \frac{1}{2}}, \frac{\tau_k(\lambda)}{k + \frac{3}{2}} \right] \quad \text{for any } k \in \mathbb{N}, \text{if } \lambda < -1 \]
and
\[ p(\lambda) \in \left[ \frac{\tau_k(\lambda)}{k + \frac{1}{2}}, \frac{\tau_k(\lambda)}{k + \frac{3}{2}} \right] \quad \text{for any } k \in \mathbb{N}_0, \text{if } \lambda > 1, \]
and thus we have shown that if \((\varphi, \tau, p) \in S_k^A\) and \(\varphi \neq 0\), the minimal period \(p\) of the periodic solution \(x(t)\) of (1) with \(x_{[-\tau, 0]} = \varphi\) satisfies one of the properties 1, 2, or 3.

This completes the proof of the theorem. 

Remark 3.1. 1. The proof of Theorem 3.1 is motivated by the proof of Theorem 4.7 and Remark 4.8 in [32].

2. If \(x = 0\) is the only equilibrium of (1) Theorem 3.1 can also be proved with the aid of Theorem 4 in [29] or by using the Fuller index theory (see Theorem 2.2 and Sect. 3 in [8]).

We conclude this section with a corollary providing conditions which guarantee the unboundness of the \(\tau\)-component (i.e., the projection of \(S_k^A\) onto \(\tau\)-line is unbounded).

In addition to (GH1) and (GH2) we need the next assumption.

(GH3) There exists a compact interval \(I \subset \mathbb{R}\) with \(0 \in \text{int}(I)\) such that \(F(I) \subset I\).

**Corollary 3.1.** Assume (GH1), (GH2), and (GH3) are satisfied. Then

1. For \(\lambda = F'(0) < -1\) there exists \(\tau_k(\lambda) \in [0, \tau_k(\lambda)]\) such that for any \(\tau \geq \tau_k(\lambda)\) there are \(\varphi \in \mathcal{C}^\circ\) and \(p \in ]\tau/(k + \frac{1}{2}), \tau/(k + \frac{3}{2})[\) such that \((\varphi, \tau, p) \in S_k^A, k \in \mathbb{N}\). Furthermore, if \((\varphi, \tau, p) \in S_k^A, k \in \mathbb{N}\), and \(x(t)\) is the unique existing \(p\)-periodic solution of (1) with \(x_{[-\tau, 0]} = \varphi\), then \(x(t) \in \text{int}(I)\) for all \(t \geq -\tau\).

2. For \(\lambda = F'(0) > 1\) there exists \(\tau_k(\lambda) \in [0, \tau_k(\lambda)]\) such that for any \(\tau \geq \tau_k(\lambda)\) there are \(\varphi \in \mathcal{C}^\circ\) and \(p \in ]\tau/(k + 1), \tau/(k + \frac{3}{2})[\) such that \((\varphi, \tau, p) \in S_k^A, k \in \mathbb{N}_0\). Furthermore, if \((\varphi, \tau, p) \in S_k^A, k \in \mathbb{N}_0\), and \(x(t)\) is the unique existing \(p\)-periodic solution of (1) with \(x_{[-\tau, 0]} = \varphi\), then \(x(t) \in \text{int}(I)\) for all \(t \geq -\tau\).

**Proof.** Applying Theorem 3.1 to (1) we get that the maximal connected component \(S_k^A\) of the closure \(\bar{S}^A\) containing \((0, \tau_k(\lambda), p(\lambda))\) is unbounded.
From Corollary 1.1 in [23] we know, because of the invariance condition (GH3), that if \(x(t)\) is a non-constant periodic solution of (1) satisfying \(x(t) \in I\) for all \(t \geq -\tau\) then \(x(t) \in \text{int}(I)\) for all \(t \geq -\tau\). On the other hand, using the local Hopf bifurcation theorem 2.1, we can prove that for \(\tau > 0\) there exists a non-constant \(p\)-periodic solution \(x(t)\) with \(x_{[t,\tau]} = \varphi \in \mathcal{C}\) and \(x(t) \in \text{int}(I)\) for all \(t \geq -\tau\) such that \((\tau, \varphi, p) \in S^\lambda_0\). Since the solutions of (1) depend continuously on parameters and initial data and \(\mathcal{S}\) is connected, it follows that for any \((\tau, \varphi, p) \in S^\lambda_0\) the unique existing \(p\)-periodic solution \(x(t)\) of (1) with \(x_{[t,\tau]} = \varphi\) satisfies \(x(t) \in \text{int}(I)\) for all \(t \geq -\tau\). Then, properties 1, 2, and 3 in Theorem 3.1 provide that, for \(k \in \mathbb{N}\), if \(\lambda < -1\), or for \(k \in \mathbb{N}_0\), if \(\lambda > 1\), the projection of \(S^\lambda_0\) onto the \(\tau\)-line is unbounded (see proof of Theorem 4.7 in [32]). This implies the existence of a positive real constant \(\tilde{\tau}_0(\lambda) \in (0, \tau_0(\lambda)]\) (note that for \(\tau = 0\) Eq. (1) possesses no non-constant periodic solutions), such that for any \(\tau \geq \tilde{\tau}_0(\lambda)\) there exist \(\varphi \in \mathcal{C}\) and \(p \in ]\tau/(k + 1), \tau/(k + 1/2)[\) (\(p \in [\tau/(k + 1), \tau/(k + 1/2)]\)) if \(\lambda < -1\) (\(\lambda > 1\)) such that \((\varphi, \tau, p) \in S^\lambda_0\).

Remark 3.2. 1. In the case of \(F'(0) < -1\) the assertions of Corollary 3.1.1 are proved in [23, Theorem 2.1] under the following assumptions:

\begin{align*}
(GH1') & \quad F \text{ is continuous, satisfies } F(0) = 0, \text{ and is differentiable at } x = 0 \text{ with } F'(0) < -1,
(GH3) & \quad \text{as above, and}
(GH4) & \quad xF(x) < 0 \text{ for } x \in I \setminus \{0\}.
\end{align*}

Notice that in our proof we do not need the assumption (GH4). On the other hand, (GH1') is stronger than (GH1).

2. Results similar to Corollary 3.1 are presented in [32, Theorem 4.1, 4.2] under the additional assumption that \(F\) is a bounded and monotone function with maximum slope at \(x = 0\). Notice that the proof provided in [32, Remark 4.8] is only valid for \(F'(0) < 1\) (see Eq. (4.4) in [32]).

Remark 3.3. In order to be able to prove that the \(\tau\)-component of \(S^\lambda_0\), where \(\lambda < -1\), is unbounded, we need the additional assumption that \(F\) satisfies the negative feedback condition (GH4) (see Remark 3.2.1). More precisely, Theorem 1.1 in [23] provides: Let (GH1'), (GH3), and (GH4) be satisfied. Then for \(\lambda < -1\) there exists \(\tilde{\tau}_0(\lambda) \in ]0, \tau_0(\lambda)[\) such that for any \(\tau > \tilde{\tau}_0(\lambda)\) there are a \(\varphi \in C([-\tau, 0], I)\) and a \(p > 2\tau\) with \((\varphi, \tau, p) \in S^\lambda_0\). Furthermore, if \((\varphi, \tau, p) \in S^\lambda_0\) and \(x(t)\) is the unique solution of (1) with \(x_{[t,\tau]} = \varphi\), then \(x(t) \in \text{int}(I)\) for all \(t \geq 0\).
4. EXAMPLES

In this last section we present and discuss five examples which illustrate our local (Theorem 2.1 and Corollary 2.1) and global results (Theorem 3.1 and Corollary 3.1). Furthermore we provide bifurcation diagrams by using a numerical continuation method for delay differential equations (see [10]) where the periodic solutions are approximated by a Fourier polynomial.

The last two examples indicate what can happen if the invariance condition (GH3) or feedback condition (GH4) is violated.

1. In our first example we use the sigmoid nonlinearity

\[ F(x) = F_1(x) := \frac{3}{2} \frac{1}{1 + \exp(-4x)} - \frac{3}{4}. \]

It follows that \( F \) is bounded by \(-\frac{1}{2} < F(x) < \frac{1}{4}\) for all \( x \in \mathbb{R} \), is strictly increasing, and has exactly one turning point at \( 0 \) \( (F''(0) = 0, F'''(0) < 0) \). Moreover it holds \( F(0) = 0 \) and \( \lambda = F'(0) = \frac{1}{2} > 1 \). \( F \) has three fixed points: \( \bar{x} = 0, \bar{x} \equiv 0.644, \) and \( -\bar{x} \).

By Theorem 2.1 we get that a supercritical Hopf bifurcation occurs at \( (\bar{x}, \tau) = (0, \tau_0(\lambda)), k \in \mathbb{N}_0 \). Corollary 3.1 provides a \( \tau_k(\lambda) \in ]0, \tau_0(\lambda)], k \in \mathbb{N}_0 \), such that for all \( \tau > \tau_0(\lambda) \) there is a periodic solution \( x(t) \) belonging to \( S_\lambda^k \). Furthermore we know that the period \( p \) of \( x(t) \) satisfies \( p \in ]\tau/(k + 1), \tau/(k + \frac{1}{2})[ \). Figures 3 and 4 show the numerical results. One can observe that the amplitude of the solution is strictly increasing in \( \tau \).

2. Now we use the nonlinearity \( F(x) = -F_1(x) \) with \( F_1 \) as in the first example. \( \bar{x} = 0 \) is the only fixed point of \( F \) and it holds \( \lambda = F'(0) = -\frac{1}{2} < -1, F''(0) = 0, F'''(0) > 0 \). Theorem 2.1 yields that a supercritical Hopf bifurcation occurs at \( (\bar{x}, \tau) = (0, \tau_0(\lambda)), k \in \mathbb{N}_0 \). Since \( F \) satisfies (GH1), (GH2), (GH3), and (GH4) with \( I = [-\bar{x}, \bar{x}] \) with \( \bar{x} \) as in Example 1, from Corollary 3.1 and Remark 3.3, we obtain that the branches of periodic solutions \( S_\lambda^k \) are unbounded in \( \tau \). For the period \( p \) of the solution \( x(t) \) belonging to \( S_\lambda^k \), \( p > 2\tau \) for \( k = 0 \) and \( p \in ]\tau/(k + \frac{1}{2}), \tau/(k + \frac{1}{2})[ \) for \( k \geq 1 \). Numerical simulations indicate that the amplitude of the solution is strictly increasing in \( \tau \) and for every \( \tau > \tau_0(\lambda) \) there is a unique slowly oscillating periodic solution. These properties for the slowly oscillating periodic solutions can be proved analytically. Applying Corollary 2.1 in [5] one can prove the uniqueness of the slowly oscillating periodic solution for \( \tau > \tau_0(\lambda) \). For \( \tau \leq \tau_0(\lambda) \) there are no slowly oscillating periodic solutions. Moreover Corollary 2.1 in [5] provides the monotonicity property of the branch of slowly oscillating periodic solutions \( S_\lambda^0 \) as function of \( \tau > \tau_0(\lambda) \): The orbit \( \Gamma(\tau) \) of the unique existing slowly oscillating periodic solution in
Fig. 3. Branches of periodic solutions in the $(\tau, |x|)$-plane. All bifurcations are supercritical and the corresponding periodic solutions are bounded by $\bar{x}$ (see Example 1).

Fig. 4. Branches of periodic solutions in the $(\frac{P}{\tau}, \tau)$-plane (see Example 1).
the \((x, \dot{x})\)-plane possesses the following property. \(\Gamma(\tau_1)\) is in the closure of the exterior of the orbit \(\Gamma(\tau_2)\) whenever \(\tau_2 > \tau_1 > \tau_0(\lambda)\).

3. This example demonstrates the change of the Hopf bifurcation direction (see Corollary 2.1). We consider (1) with

\[
F(x) = -575 \arctan(x + 10\sqrt{5}) + 575 \arctan(10\sqrt{5}).
\]

It holds \(F(0) = 0, \lambda := F'(0) = -575/(1 + 500) < -1, F''(0) = (11500/251,001)\sqrt{5}, F'''(0) = -1,723,850/125,751,501\). So we get \(C = (F'''(0)F'(0)/F''(0)^2) = 1499/1000 = 1.4990\) (see proof of Corollary 2.1). It follows that

\[
1.49716 \equiv C_0(\lambda) < C < C_1(\lambda) \equiv 1.49952.
\]

Applying Theorem 2.1 we can show that a Hopf bifurcation occurs at \(\tau = \tau_k(\lambda), k \in \mathbb{N}_0\). The first Hopf bifurcation at \(\tau = \tau_0(\lambda)\) is subcritical; all other Hopf bifurcations are supercritical (see Figs. 5 and 6).

4. Here we discuss an example where the function \(F\) does not satisfy condition \((GH3)\). Consider

\[
F(x) = -15 \sinh(x).
\]

It holds that \(F(0) = 0, \lambda := F'(0) = -15, F''(0) = 0,\) and \(F'''(0) = -15 F\) is strictly decreasing and satisfies conditions \((GH1)\) and \((GH2)\) but does not satisfy condition \((GH3)\). \(\bar{\tau} = 0\) is the only fixed point of \(F\). Using Theorem 2.1 we get that a subcritical Hopf bifurcation occurs at \(\tau = \tau_1(\lambda)\).

![Figure 5](image)

**Fig. 5.** Subcritical Hopf bifurcation at \(\tau = \tau_0(\lambda) = 4.66712\) (see Example 3).
From Theorem 3.1 we know that the branches of periodic solutions $S_k^l$ bifurcating from $(x = 0, \tau = \tau_3(\lambda))$ are unbounded but we do not know which component of $S_k^l$ is unbounded. Actually the numerically computed bifurcation diagram (see Fig. 7) shows that the branches are bounded in $\tau$. The numerically computed $p$-periodic solutions $x_k^l(t; \tau)$ belonging to $S_k^l$ exist only for $0 < \tau < \tau_3(\lambda)$ and $|x_k^l| \to \infty$ as $\tau \to 0$. Furthermore for the period $p$ of $x_k^l(t; \tau)$ we get $\frac{p}{\tau} \to 4$ as $\tau \to 0$. 

FIG. 6. Supercritical Hopf bifurcation at $\tau = \tau_1(\lambda) = 15.82280$ (see Example 3).

FIG. 7. Branches of periodic solutions for $F(x) = -15 \sinh(x)$ (see Example 4).
The last example deals with the case in which $F$ does not satisfy the negative feedback condition (GH4). We consider Eq. (1) with

$$F(x) = 2 \frac{x + 1}{1 + (x + 1)^{10}} - 2;$$

see [22]. This equation was introduced by Mackey and Glass as a mathematical model for the generation of white blood cells. $F$ is a bounded function with $F(0) = 0$, $\lambda = F'(0) = -4 < -1$, $F''(0) = -5$, and $F'''(0) = 255$. Using Theorem 2.1 we get that a supercritical Hopf bifurcation takes place at $(\xi = 0, \tau = \tau_0(\lambda))$, $k \in \mathbb{N}_0$. From Corollary 3.1 we know that for $k \geq 1$ the $\tau$-component of $S_k$ is unbounded. Since (GH4) is not satisfied we cannot investigate $S_k$ by applying Theorem 1.1 in [23] (see Remark 3.3). In [14, 18, 22] the Mackey-Glass equation is studied numerically. Numerical simulations show that the branch $S_0$ of slowly oscillating periodic solutions undergoes a cascade of period doubling bifurcations. Moreover numerical calculations indicate the existence of chaotic attractors for sufficiently large delays $\tau$.

**Appendix: Proof of Lemma 2.5**

Let $\tau = \tau_0(\lambda)$ and $b = b_0(\lambda)$, $k \in \mathbb{N}_0$. We compute the factor $K$ (see Eq. (10)) in several steps.

1. First, notice that the following equalities hold:

$$\cos(b) = \frac{1}{\lambda} \quad \cos(2b) = \frac{2 - \lambda^2}{\lambda^2} \quad \cos(3b) = \frac{4 - 3\lambda^2}{\lambda^3} \quad \sin(b) = \frac{\sqrt{\lambda^2 - 1}}{|\lambda|} \quad \sin(2b) = 2 \frac{\sqrt{\lambda^2 - 1}}{\lambda|\lambda|} \quad \sin(3b) = \frac{\sqrt{\lambda^2 - 1}}{\lambda^2|\lambda|}$$

$$b = \tau \sqrt{\lambda^2 - 1}.$$

2. We get $\psi_1(0) = \frac{1}{\beta}$ with $\beta := 1 + \tau \lambda e^{-ib}$. This provides

$$|\beta|^2 = \beta \overline{\beta} = (1 + \tau \lambda e^{-ib})(1 + \tau \lambda e^{ib})$$

$$= 1 + 2\tau \lambda \cos(b) + \tau^2 \lambda^2$$

$$= 1 + 2\tau + \tau^3 \lambda^2.$$
3. We set $C_1 := \text{Re}(e^{-ib}\psi_t(0))$. It holds that
\[
C_1 = \text{Re}\frac{e^{-ib\beta}}{|\beta|^2}
= \text{Re}\frac{e^{-ib} + \tau\lambda}{1 + 2\tau + \tau^2\lambda^2}
= \frac{1 + \tau\lambda^2}{(1 + 2\tau + \tau^2\lambda^2)\lambda}.
\]

4. 
\[
-\frac{\text{Re}(e^{-ib}\psi_t(0))}{L(1)} = -\frac{1 + \tau\lambda^2}{(1 + 2\tau + \tau^2\lambda^2)\lambda(\lambda - 1)\tau} =: T_2.
\]

5. 
\[
\alpha := 2ib - \tau(-1 + \lambda e^{-2ib}) = \tau - \lambda\tau \cos(2b) + i(2b + \lambda\tau \sin(2b))
= \tau\left(1 - \lambda - \frac{2 - \lambda^2}{\lambda^2}\right) + i\left(2b + 2\lambda\tau \frac{\sqrt{\lambda^2 - 1}}{|\lambda|}\right)
= \frac{\tau}{\lambda}(\lambda + 2)(\lambda - 1) + i\frac{2\tau\sqrt{\lambda^2 - 1}}{|\lambda|}(|\lambda| + 1).
\]
Therefore
\[
\bar{\alpha} = \frac{\tau}{\lambda}(\lambda + 2)(\lambda - 1) - i\frac{2\tau\sqrt{\lambda^2 - 1}}{|\lambda|}(|\lambda| + 1)
\]
and it follows that
\[
|\alpha|^2 = \alpha\bar{\alpha} = \frac{\tau^2}{\lambda^2}(\lambda + 2)^2(\lambda - 1)^2 + \frac{4\tau^2(\lambda^2 - 1)}{|\lambda|^2}(|\lambda| + 1)^2
= \frac{\tau^2(\lambda - 1)}{\lambda^2}((\lambda - 1)(\lambda + 2)^2 + 4(\lambda + 1)(|\lambda| + 1)^2).
\]
We distinguish between $\lambda > 1$ and $\lambda < -1$.
(a) The case $\lambda > 1$:
\[
|\alpha|^2 = \frac{\tau^2(\lambda - 1)}{\lambda^2}((\lambda - 1)(\lambda^2 + 4\lambda + 4) + 4(\lambda^3 + 3\lambda^2 + 3\lambda + 1))
= \frac{\tau^2(\lambda - 1)}{\lambda}(5\lambda^2 + 15\lambda + 12).
\]
(b) The case $\lambda < -1$:

$$|\alpha|^2 = \frac{\tau^2(\lambda - 1)^2}{\lambda^2} (\lambda^2 + 4\lambda + 4(\lambda^2 - 1))$$

$$= \frac{\tau^2(\lambda - 1)^2}{\lambda} (5\lambda + 4).$$

6. We compute two auxiliary terms $A$ and $B$.

$$A := \cos(3b) + \tau\lambda \cos(2b) = \frac{4 - 3\lambda^2}{\lambda^3} + \tau\lambda \frac{2 - \lambda^2}{\lambda^2}$$

$$= \frac{4 - 3\lambda^2 + \tau\lambda^2 (2 - \lambda^2)}{\lambda^3}$$

$$B := \sin(3b) + \tau\lambda \sin(2b) = \frac{\sqrt{\lambda^2 - 1}(4 - \lambda^2)}{\lambda^2 |\lambda|} + 2\tau\lambda \frac{\sqrt{\lambda^2 - 1}}{\lambda |\lambda|}$$

$$= \frac{\sqrt{\lambda^2 - 1}}{\lambda^2 |\lambda|} (4 - \lambda^2 + 2\tau\lambda^2).$$

7. It holds that

$$\text{Re}(e^{-3i\theta/B^2})$$

$$= \text{Re} \left( e^{-3ib}(1 + \lambda \tau e^{ib}) \left( \frac{\tau}{\lambda} (\lambda + 2)(\lambda - 1) - i \frac{2\tau\sqrt{\lambda^2 - 1}}{|\lambda|} (|\lambda| + 1) \right) \right)$$

$$= \text{Re} \left( (A - iB) \left( \frac{\tau}{\lambda} (\lambda + 2)(\lambda - 1) - i \frac{2\tau\sqrt{\lambda^2 - 1}}{|\lambda|} (|\lambda| + 1) \right) \right)$$

$$= \frac{\tau A}{\lambda} (\lambda - 1)(\lambda + 2) - \frac{2\tau\sqrt{\lambda^2 - 1}}{|\lambda|} B(|\lambda| + 1)$$

$$= \frac{\tau(\lambda - 1)}{\lambda^4} ((\lambda + 2)(4 - 3\lambda^2) - 2(\lambda + 1)(|\lambda| + 1)(4 - \lambda^2)$$

$$+ \tau\lambda^2((2 - \lambda^2)(\lambda + 2) - 4(\lambda + 1)(|\lambda| + 1))).$$
8. We set

\[ T_3 := \text{Re} \frac{e^{-3i\psi_2(0)}}{L(e^{2i\theta})} = \frac{\text{Re}(e^{-3i\beta\alpha})}{|\alpha|^2 |\beta|^2}. \]

(a) The case \( \lambda > 1 \):

\[ T_3 = \frac{(\lambda + 2)(4 - 3\lambda^2 - 2(\lambda + 1)^2)(2 - \lambda) + \tau \lambda^2 (2 - \lambda^2)(\lambda + 2) - 4(\lambda + 1)^2)}{\tau \lambda^3(5\lambda^2 + 15\lambda + 12)(1 + 2\tau + \tau^2\lambda^2)}. \]

(b) The case \( \lambda < -1 \):

\[ T_3 = \frac{(\lambda + 2)(4 - 3\lambda^2 - 2(\lambda + 1)^2)(2 - \lambda) + \tau \lambda^2 (2 - \lambda^2)(\lambda + 2) + 4(\lambda^2 - 1)}{\tau \lambda^3(\lambda - 1)(5\lambda + 4)(1 + 2\tau + \tau^2\lambda^2)}. \]

9. Now we calculate \( C_2 := T_2 + \frac{1}{2}T_3 \):

(a) The case \( \lambda > 1 \):

\[ C_2 = \frac{-2(1 + \tau \lambda^2)\lambda^2(5\lambda^2 + 15\lambda + 12)}{2\tau \lambda^3(\lambda - 1)(5\lambda^2 + 15\lambda + 12)(1 + 2\tau + \tau^2\lambda^2)} \]

\[ \times \frac{(\lambda - 1)((\lambda + 2)(4 - 3\lambda^2 - 2(\lambda + 1)^2)(2 - \lambda)) + \tau \lambda^2 (2 - \lambda^2)(\lambda + 2) - 4(\lambda + 1)^2)}{2\tau \lambda^3(\lambda - 1)(5\lambda^2 + 15\lambda + 12)(1 + 2\tau + \tau^2\lambda^2)}. \]

\[ = \frac{\tau \lambda^2(-11\lambda^3 - 35\lambda^2 - 24\lambda + 6) + (2\lambda^4 - 11\lambda^3 - 43\lambda^2 - 24\lambda + 12)}{2\tau \lambda^3(\lambda - 1)(5\lambda^2 + 15\lambda + 12)(1 + 2\tau + \tau^2\lambda^2)}. \]

(b) The case \( \lambda < -1 \):

\[ C_2 = \frac{-2(1 + \tau \lambda^2)\lambda^2(5\lambda + 4) + (\lambda + 2)(4 - 3\lambda^2 - 2(\lambda + 1)^2)(2 - \lambda))}{2\tau \lambda^3(\lambda - 1)(5\lambda + 4)(1 + 2\tau + \tau^2\lambda^2)} \]

\[ + \frac{\tau \lambda^2 (2 - \lambda^2)(\lambda + 2) + 4(\lambda^2 - 1)}{2\tau \lambda^3(\lambda - 1)(5\lambda + 4)(1 + 2\tau + \tau^2\lambda^2)}. \]

\[ = \frac{\tau \lambda^2(-11\lambda^3 - 6\lambda + 2) + (-2\lambda^3 - 13\lambda^2 - 4\lambda + 4)}{2\tau \lambda^3(\lambda - 1)(5\lambda + 4)(1 + 2\tau + \tau^2\lambda^2)}. \]

10. Finally we compute \( K = (F''(0)\tau/2)C_1 + F''(0)\tau^2 C_2 \) and the proof is finished.
REFERENCES


