Sinc numerical solution for solitons and solitary waves

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Abstract

A numerical scheme using Sinc–Galerkin method is developed to approximate the solution for the Korteweg–de Vries model equation. Sinc approximation to both derivatives and indefinite integral reduce the integral equation to an explicit system of algebraic equations, then using various properties of Sinc functions, it is shown that the Sinc solution produce an error of order $O(\exp(-c/h))$ for some positive constants $c, h$. The method is applied to a few test examples to illustrate the accuracy and the implementation of the method. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The theory of nonlinear dispersive wave motion has recently undergone much study, especially by Whitham [14]. It can be shown [14] that the theory of water waves for the case of shallow water and waves of small amplitude can be approximately described by the Korteweg–de Vries equation

$$u_t + (c + u)u_x + \beta u_{xxx} = 0, \quad (x, t) \in \mathbb{R} \times (0, T_0),$$  (1.1)

where $c$ and $\beta$ are given constants, and $u$ gives the height of a wave above some equilibrium level. Since the amplitude of these waves is assumed to be small, it can serve as a perturbation parameter. These problems have been studied by many authors [2,7–9,11,15]. However, they used a formal perturbation technique. Sometimes called multiscale expansion, or, using evens functions techniques, as in [11]. Also finite difference method for the Korteweg–de Vries equation have been analyzed in [5].

One aspect that has been investigated is the linearized form of Eq. (1.1):

$$u_t + cu_x + \beta u_{xxx} = 0$$  (1.2)
which has traveling wave solutions \( u(x,t) = a \cos(kx - \omega t) \), where \( a \) is constant and \( \omega = \omega(k) = ck - \beta k^3 \). The existence of traveling wave solutions to (1.2) already has been studied in [4,12].

If we drop the third derivative term in (1.1), we have

\[
u_t + (c + u)u_x = 0
\]

which is a quasi-linear first-order wave equation whose wave speed depends on the amplitude and has the implicit solutions \( u(x,t) = a \cos[kx - k(c + u)t] \).

If \( c = 0, \beta = 1 \) in Eq. (1.1) we get another form of Korteweg–de Vries equation

\[
u_t + uu_x + u_{xxx} = 0. \tag{1.3}
\]

This nonlinear equation admits traveling wave solutions of different types. One particular type of traveling wave that arises from the Korteweg–de Vries equation is the soliton, or solitary wave [10, p. 38]. The same equation (1.3) has also come up in the theory of plasma and several other branches of physics.

In general, it is difficult to specify all the terms in a perturbation series for problems involving partial differential equations. Thus only the first few terms in the series are determined, which is not enough sometimes. Also more that one type of expansion may be necessary to completely describe the perturbation or asymptotic solution of a given problem. For example, an expansion may break down in some region or may be insufficient to satisfy the data for the problem. These difficulties signify that the given expansion is not uniformly valid over the entire region of interest.

There exist relatively few types of procedures for obtaining approximate solutions of ordinary or partial differential equations. While these methods are referred to by a variety of names, such as Rayleigh’s method, Galerkin’s method, or the collocation method, they are all, in effect, variants of the same method. One first selects a suitable basis \( \{S_j\}_{j=1}^n \), and then attempts an approximate solution of the forms

\[
\sum_{j=1}^{n} a_j S_j \tag{1.4}
\]

in which the coefficients \( a_j \) are unknown. Form (1.4) is substituted into the differential equation to be solved, and the coefficients \( a_j \) are then determined in one of the variety of ways, depending on which of the above procedures is used. We rely heavily on collocation method in the sequel.

In this paper the method of solving (1.1) subject to the initial condition

\[
u(x,0) = u^0(x), \quad x \in \mathbb{R} \tag{1.5}
\]

is based on using the Sinc method, which builds an approximate solution valid on the entire spatial domain and on a small interval in the time domain. The main idea is to replace differential and integral equations by their Sinc approximations. The ease of implementation coupled with the exponential convergence rate have demonstrated the viability of the method. One avenue that deserves attention is, approximation by Sinc functions handles singularities in the problem.

The paper has been organized into four sections. Section 2 contains, definitions and some results of the Sinc functions theory. In Section 3 Sinc solution is developed for the Korteweg–de Vries equation, along with a detailed description of the associated errors. The resulting discrete system for (1.1) is then complied, the matrix structure is examined in detail and appropriate bounds are given also in Section 3. In Section 4 numerical examples are presented which illustrate the exponential convergence of the method.
2. Sinc function preliminaries

The goal of this section is to recall notation and definitions of Sinc function, state some known results, and derive useful formulas that are important for this paper. First, we denote the set of all integers, the set of all real numbers, the set of all complex numbers by \( \mathbb{Z} \), \( \mathbb{R} \) and \( \mathbb{C} \), respectively.

Let \( f \) be a function defined on \( \mathbb{R} \) and \( h > 0 \) a stepsize. Then the Whittaker cardinal function is defined by the series

\[
C(f, h, x) = \sum_{k=-\infty}^{\infty} f(kh) S(k, h)(x),
\]

whenever this series converges and where

\[
S(k, h)(x) = \frac{\sin[\pi(x - kh)/h]}{\pi(x - kh)/h}
\]

is known as the \( k \)th Sinc function.

Also for positive integer \( N \), define

\[
C_N(f, h, x) = \sum_{k=-N}^{N} f(kh) S(k, h)(x). \tag{2.1}
\]

**Definition 1.** Let \( d > 0 \), and let \( D_d \) denote the region \( \{ z = x + iy : |y| < d \} \) in the complex plane \( \mathbb{C} \), and \( \phi \) the conformal map of a simply connected domain \( D \) in the complex domain onto \( D_d \) such that \( \phi(a) = -\infty \) and \( \phi(b) = \infty \), where \( a \) and \( b \) are boundary points of \( D \), i.e., \( a, b \in \partial D \). Let \( \psi \) denote the inverse map of \( \phi \), and let the arc \( \Gamma \), with endpoints \( a \) and \( b \) \( (a, b \not\in \Gamma) \), given by \( \Gamma = \psi(\mathbb{R}) \). For \( h > 0 \), let the points \( x_k \) on \( \Gamma \) be given by \( x_k = \psi(kh) \), \( k \in \mathbb{Z} \), \( \rho(z) = \exp(\phi(z)) \). For \( 1 \leq p < \infty \), let \( H(D) \) denote the family of all functions \( f \) that are analytic in \( D \), such that

\[
\int_{\partial D} |f(z)| \, dz < \infty.
\]

Now if \( x \) is on the arc \( \Gamma \), then by introducing the conformal map \( \phi \), and a “nullifier” function \( g \) the following theorem gives a formula for approximating \( f^{(m)} \) on \( \Gamma \). Let \( g \) be an analytic function on \( \mathcal{D} \), and for \( k \in \mathbb{Z} \) set

\[
S_j(z) = g(z) S_m \left( \frac{\phi(z) - jh}{h} \right) = g(z) S(j, h) \circ \phi(z), \quad z \in D.
\]

**Theorem 2.1** (Stenger [13, p. 208]). Let \( \phi'f/g \in H(D) \), and let

\[
\sup_{-\pi/h \leq x \leq \pi/h} \left| \left( \frac{d}{dx} \right)^n g(x) \exp(it\phi(x)) \right| \leq C_2 h^{-n}, \quad x \in \Gamma
\]
for \( n = 0,1,2,\ldots,m \), with \( C_2 \) a constant depending only on \( m, \phi, \) and \( g \). If \( f/g \in L_2(\mathcal{D}) \), \( \alpha \) a positive constant, then taking \( h = \sqrt{\pi d} / \pi N \) it follows that

\[
\sup_{x \in \mathcal{T}} \left| f^{(n)}(x) - \sum_{j=-N}^{N} \frac{f(x_j)}{g(x_j)} S_j^{(n)}(x) \right| \leq C_3 N^{(n+1)/2} \exp\left(-\sqrt{\pi d} x N\right)
\]

for \( n = 0,1,2,\ldots,m \), with \( C_3 \) a constant depending only on \( m, \phi, g, \) and \( d \).

The approximation of the \( m \)th derivative of \( f \) in Theorem 2.1 is simply an \( m \)th derivative of \( C_N(f/g, h, x)/g \) in (2.1). The weight function \( g \) is chosen relative to the order of the derivative that is to be approximated. For instance, to approximate the \( m \)th derivative the choice \( g(x) = 1/(\phi'(x))^m \) often suffice, [13]. So the approximation of \( f' \) by Sinc expansion is given by

\[
f'(x) \approx \sum_{j=-N}^{N} \frac{f(x_j)}{g(x_j)} S'_j(x).
\]

(2.2)

The derivatives of Sinc functions evaluated at the nodes will also be needed and these quantities are delineated by

\[
\delta^{(q)}_{jk} \equiv h^q \frac{d^q}{d\phi^q} [S_j \circ \phi(x)]|_{x=x_k}.
\]

In particular, the following convenient notation will be useful in formulating the discrete system:

\[
\delta^{(0)}_{jk} = [S(j, h) \circ \phi(x)]|_{x=x_k} = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases}
\]

\[
\delta^{(1)}_{jk} = h \frac{d}{d\phi} [S(j, h) \circ \phi(x)]|_{x=x_k} = \begin{cases} 0, & j = k, \\ (-1)^{j-k} / (j-k), & j \neq k \end{cases}
\]

and

\[
\delta^{(3)}_{jk} = h^3 \frac{d^3}{d\phi^3} [S(j, h) \circ \phi(x)]|_{x=x_k} = \begin{cases} 0, & j = k, \\ (-1)^{j-k} / (j-k)^3 [6 - \pi^2 (j-k)^2], & j \neq k \end{cases}
\]

So that the approximation in (2.2) at the Sinc nodes \( x_k \) takes the form

\[
f'(x_k) \approx \sum_{j=-N}^{N} \left( \frac{\delta^{(1)}_{jk}}{h} + \delta^{(0)}_{jk} g'(x_j) \right) \frac{f(x_j)}{g(x_j)}.
\]

(2.3)

System (2.3) is more conveniently recorded by defining the vector \( f = (f_{-N}, \ldots, f_0, \ldots, f_N)^T \). Then define the \( m \times m, (m = 2N + 1) \) Toeplitz matrices \( I^{(q)}_m = [\delta^{(q)}_{jk}], \) \( q = 0,1,2, \) whose \( jk \)th entry is given by \( \delta^{(q)}_{jk}, \) \( q = 0,1,2. \) Also define the diagonal matrix \( D(g) = \text{diag}[g(x_{-N}), \ldots, g(x_0), \ldots, g(x_N)]^T. \)
The matrix $I_m^{(0)}$ is the identity matrix. The matrix $I_m^{(1)}$ takes the form

$$I_m^{(1)} = \begin{pmatrix}
0 & -1 & \cdots & (-1)^{m-1} \\
1 & & & \\
& \ddots & \ddots & -1 \\
(\ -1)^{m} & \cdots & \cdots & 0
\end{pmatrix}_{m \times m}$$

and

$$I_m^{(3)} = \begin{pmatrix}
0 & -(6 - \pi^2) & \cdots & (-1)^{m-1}(6 - (m-1)^2\pi^2) \\
6 - \pi^2 & & & \\
& \ddots & \ddots & - (6 - \pi^2) \\
(\ -1)^m(6 - (m-1)^2\pi^2) & \cdots & \cdots & (6 - \pi^2)
\end{pmatrix}_{m \times m}.$$  

System (2.3) takes the form

$$f' \approx \left( -\frac{1}{h} I_m^{(1)} D(1/g) + I_m^{(0)} D(g'/g) \right) f \equiv A_1 f.$$  

For the present paper the interval $I$ in Theorem 2.1 is $(-\infty, \infty)$. Therefore, to approximate the first derivative we take $\phi(x) = x$, and $g(x) = 1/\phi'(x)$, the square matrix (2.6) becomes $A_1 = (-1/h)I_m^{(1)}$.

Then the approximation of the first derivative evaluated at the vector nodes $x_i$ can be written as

$$f'(x_i) \approx A_1 f(x_i).$$

The same way we can approximate the third derivative by

$$f'''(x_i) \approx A_3 f(x_i),$$

where the matrix $A_3$ is defined by $A_3 = (-1/h^3)I_m^{(3)}$.

Let us describe Sinc definite integration over an interval. At the outset, we define the numbers

$$\delta_k^{(-1)} = \frac{1}{2} + \int_{0}^{k} \sin(\pi t)/\pi t \, dt, \quad k \in \mathbb{Z},$$

and we define Toeplitz matrix $\delta_{k-j}^{(-1)}$ of order $m$ by

$$I^{(-1)} = [\delta_{k-j}^{(-1)}]_{m \times m}$$

with $\delta_{k-j}^{(-1)}$ denoting the $k$th element of $I^{(-1)}$. For $h = \sqrt{\pi d/(\pi N)}$, define the matrix $B = h I^{(-1)} D(1/T')$.

Let $F \in H(D)$, set $\tilde{F} = (F(z_{-N}), \ldots, F(z_N))^T$, define $(\mathcal{F}_N F)(x) = \int_{a}^{x} F(t) \, dt$, set $\tilde{G} = (G_{-N}, \ldots, G_0, \ldots, G_N)^T = B \tilde{F}$, then define $\mathcal{F}_N F$ by

$$(\mathcal{F}_N F)(x) = \frac{\rho(x) G_N}{1 + \rho(x)} + \sum_{k=-N}^{N} \left( G_k - \frac{e^{hk} G_N}{1 + e^{hk}} \right) S(k, h) \circ T(x)$$

then for $F/T' \in L_{\infty}(D)$ we have [13, p. 219]

$$\sup_{x \in I} |(\mathcal{F}_N F)(x) - (\mathcal{F}_N F)(x)| \leq C_4 \exp(-\sqrt{\pi d z N}),$$

where $C_4$ is a constant that is independent of $N$. 
3. Implementation of the method

We determine the Sinc approximation for Korteweg–de Vries equation under the assumption that the initial condition in (1.5) belongs to the class of functions $L_2(D)$. Integrating Eq. (1.1) with respect to $t$, we get

\begin{equation}
 u(x, t) = - \int_0^t [\beta u_{xxx}(x, \tau) + (c + u(x, \tau)) u_x(x, \tau)] d\tau + u^0(x).
\end{equation}

(3.1)

To obtain a direct discretization of Eq. (3.1), and since the domain is $\mathbb{R} \times (0, T_0)$, the relevant maps are defined as follows: In the space direction, choose the map $\phi(x) = x$ which maps the infinite strip $D_d = \{ \xi = \zeta + i\eta; |\eta| < d \}$ onto $D_d$. In the time direction, choose the map $\gamma(t) = \log(t/T_0 - t)$ which carries the eye-shaped region $D_s = \{ t = x + iy; |\arg(t/T_0 - t)| < d \leq \pi/2 \}$ onto the infinite strip $D_d$. The compositions $S(m, h_x) \circ \phi(x)$, $m = -N_x, \ldots, N_x$ and $S(k, h_t) \circ \gamma(t)$, $k = -N_t, \ldots, N_t$ define the basis elements for $(-\infty, \infty)$ and $(0, T_0)$, respectively, the mesh sizes $h_x$ and $h_t$ represent the mesh sizes in the infinite strip $D_d$ for the uniform grid $\{ih_x\}$, $-\infty < i < \infty$, and $\{jh_t\}$, $-\infty < j < \infty$. The Sinc grid points $x_i \in (-\infty, \infty)$ in $D_d$ and $t_j \in (0, T_0)$ in $D_e$ are the inverse images of the equispaced grid points; that is,

\begin{equation}
 x_i = \phi^{-1}(ih_x) = ih_x \quad \text{and} \quad t_j = \gamma^{-1}(jh_t) = (T_0 \exp(jh_t))/(1 + \exp(jh_t)).
\end{equation}

In Eq. (3.1) let us carry out the Sinc approximation of $u_t$ and $u_{xxx}$. To proceed use Eqs. (2.7), (2.8) and replace $u_t$ by $-I_{m_t}^{(1)} u(x_t, t)/h_t$, and $u_{xxx}$ by $-I_{m_x}^{(3)} u(x, t)/h_x^3$ where $m_t = 2N_t + 1$ and the skew-symmetric matrices $I_{m_t}^{(1)}$ and $I_{m_x}^{(3)}$ as defined in (2.4), (2.5), respectively.

Next in Eq. (3.1) evaluate $u(x, t)$ at the $x$-nodes, and replace $u_x(x, t)$, $u_{xxx}(x, t)$ by their approximations, we get the Volterra integral equation

\begin{equation}
 u(t) = - \int_0^t [\beta A_3 u(t) + (c + u(t)) A_1 u(t)] d\tau + u^0,
\end{equation}

(3.2)

where the square matrices $A_1$, $A_3$ are given by $A_1 \approx -I_{m_x}^{(1)}/h_x$ and $A_3 \approx -I_{m_t}^{(3)}/h_x^3$, with $u(t) = [u_{-N_t}(t), \ldots, u_{N_t}(t)]^T$, where $u_i(t) = u(x_i, t)$ and $u^0 = [u^0(z_{-N_t}), \ldots, u^0(z_{N_t})]^T$.

We now collocate with respect to the $t$-variable via the use of the definite integration formula, (see Eq. (2.10)) with the conformal map $\gamma(t) = \log(t/T_0 - t)$. Thus, define the matrix $B$ by $B = h_t I_{m_t}^{(-1)} D(1/\gamma^0)$ with the nodes $t_i = \gamma^{-1}(jh_t)$ for $j = -N_t, \ldots, N_t$, where $h_t = \sqrt{\pi d/(2N_t)}$, and $I_{m_t}^{(-1)}$ as defined in (2.9), with $m_t = 2N_t + 1$. Define the matrix $U^0$ by $U^0 = [u^0(x_0, \ldots)]^T$. Then the solution of Eq. (3.2) in matrix form is given by the rectangular $m_x \times m_t$ matrix $U = [u_{i,t}]$:

\begin{equation}
 U = \left[\beta A_3 U + (C + U) \circ A_1 U\right] B^T + U^0,
\end{equation}

(3.3)

where the notation “$\circ$” denotes the Hadamard matrix multiplication. Note that in our discretization we are taking the time nodes as rows and the space nodes as columns, so the matrix $(\beta A_3 U + (C + U) \circ A_1 U)$ forms the vector nodes for the integral in (3.2). In (3.3) the vector $U^0$ has the same dimensions as the vector $U$ and every column of $U^0$ consists of the same vector $u^0$. Also the $m_x \times m_t$ matrix $C$ has each entry as the number $c$.

We use the notation $U = [u(x_i, t_j)]$ to denote the $m_x \times m_t$ matrix of node values of the function $u$ where $m_t = 2N_t + 1$, $m_x = 2N_x + 1$. If $u(x, t)$ is the exact solution of the integral equation (3.1). Then the approximation of $u_x(x, t)$ in matrix form has an exponential error (see Theorem 2.1)

\begin{equation}
 \|u_x(x, t) - A_1 U\| \leq C_3 N_x \exp(-\sqrt{\pi d/2N_t}).
\end{equation}

(3.4)
Here we assumed that $C_3$ is bounded with respect to $t$, and the matrix $A_1$ as defined above. Also we require that the error in approximating the third derivative $u_{xxx}(x,t)$ is exponentially small, i.e., with $A_3$ as defined above we have

$$
\| [u_{xxx}(x,t)] - A_3 U \| \leq C_3 N^2 \exp(-\sqrt{\pi d x N}).
$$

We also require that the error in approximating the integral $K(x,t) = - \int_0^t G(\tau) d\tau$, where $G(t) = [\beta u_{xxx}(x,t) + (c + u(x,t))u_x(x,t)]$ is exponentially small. This means, the approximation of the integral in matrix form takes the form

$$
\| [K(x_i,t_j)] - BU \| \leq C_4 \exp(-\sqrt{\pi d x N}),
$$

where again $C_4$ is bounded with respect to $x$, and the matrix $B$ as defined above.

We shall obtain a small error in our approximation to Eq. (1.1), as noted in the following two theorems. The proof resembles the proof of Theorems 4.1 and 4.2 in [1].

**Theorem 3.1.** Let $u^0 \in L_\infty(D)$, let the function $u(x,t)$ be as in Eq. (3.1), and let the matrix $U$ be defined as in (3.3). Then for $N_x, N_t > 16/\pi d x$ there exists a constant $C_5$ independent of $N_x, N_t$ such that

$$
\sup_{(x_i,t_j)} \| [u(x_i,t_j)] - U \| \leq C_5 N^2 \exp(-\sqrt{\pi d x N}),
$$

where $N = \min\{N_x, N_t\}$.

**Theorem 3.2.** Given a constant $R > 0$, there is a constant $T_0 > 0$ such that if $\| U^{n+1} - U^0 \| < R/2$, then the solution $U = G(U) + U^0$ has a unique solution. Moreover, the iteration scheme $U^{n+1} = G(U^n) + U^0$ converges to this unique solution.

4. Numerical examples

There are many numerical simulation results which seem to support the stability of pulses (for detailed simulations see [8,9]). However, one should notice that all these numerical simulations are performed in an interval of finite length. One might expect that the stability/instability problem of pulses can be numerically realized by taking sufficient calculation length. But still another difficulty remains. If one wants to approximate a pulse solution numerically in a wide interval, then the instability will grow [12, p. 490]. We therefore consider it is difficult to realize the stability/instability problem for pulses numerically. However, in this section, at least, suggest what happens in the idealized infinite interval situation. The model problem (1.1) was tested on four examples. For the four examples sequences of runs with $N_x = N_t = 8, 16, 24, 32$ are reported. The problem included here illustrate various features of the method, demonstrating the ease of implementation and assembly of the discrete system, the choice of parameters and the exponential convergence rate.

**Example 4.1.** We solve the model equation

$$
u_t + 6uu_x + u_{xxx} = 0, \quad x \in \mathbb{R}
$$

(4.1)
with the initial data of a single soliton, say
\[ u(x, 0) = 2 \sech^2 x. \]
Here the exact solution is given by \( u(x, t) = 2 \sech^2(x - 4t) \). We solve this model equation using our approach when \( d = \pi/2, \alpha = 1 \). Table 1 shows that the method converges for \( T_0 = 3 \). The second column reports the supremum norm of the error between the exact solution \( u(x_i, t_j) \) and the approximate solution \( u_{ij} \). Again the exponential convergence rate decreases the error that are initially present for small \( N_x, N_t \):

**Example 4.2.** Let us consider the same problem in the last example, and solve the Eq. (4.1) with the initial condition \( u(x, 0) = 6 \sech^2 x \). In this case the exact solution of Eq. (4.1) is given by

\[
 u(x,t) = \frac{12[3 + 4 \cosh(2x - 8t) + \cosh(4x - 64t)]}{[3 \cosh(x - 28t) + \cosh(3x - 36t)]^2}.
\]

We compared the performance of the method presented here by the finite element scheme. Finite element methods for the Korteweg–de Vries equation have been analyzed by Winther [15], where he proved convergence for a class of these equations. Tables 2 and 3 refer to a single-soliton solution and correspond, respectively, to the Sinc method and to the finite element method. The tables display information at \( T_0 = 3 \). We recall that, even though the tables correspond to \( T_0 = 3 \), the integration was followed up to \( T_0 = 20 \) and there was no problems of divergence of our scheme. Comparing the errors for the two schemes, we see that for the finite element method we do have an error of order \( O(h^2) \), while for the Sinc method it is \( O(\exp(-c/h)) \) where \( h \) is the step size and \( c > 0 \). And as predicted, the method using Sinc basis functions is more accurate for soliton propagation.

**Example 4.3.** The known exact one-soliton solutions of Eq. (4.1) exhibit exponential decay for large \( x \). The inverse scattering transform [3] tells us that the solution of (4.1) with generic initial
condition \( u(x,0) = u^0(x) \) (with \( u^0(x) \to 0 \) sufficiently rapidly as \( |x| \to \infty \)) consists of a train of solitons moving to the right, along with a dispersive wave travelling to the left. As in the above two examples, we can argue that when \( u^0(x) \) has compact support, the solution emerge from the rightmost soliton decaying as \( \exp(-\sqrt{c} x) \), where \( c \) is the relevant speed, but for sufficiently large \( x \) the presence of the solitons will not yet be felt, and the behavior of the solution will be determined by the Green’s function of the linearized equation \( u_t + u_{xxx} = 0 \), i.e., \( u \) will decay roughly as \( \exp(-2x^{5/2}/3\sqrt{3}t) \). Thus there is a transition in the nature of the decay. If there is more than one soliton in the soliton train, say two, with speeds \( c_1, c_2 \) \((c_2 > c_1)\), the a problem arises. The solution emerges from the faster-moving soliton decaying as \( \exp(-\sqrt{c_2} x) \), but because the tail of the slower-moving soliton falls slower, it is possible that it will return to dominate, i.e., the decay will slow to \( \exp(-\sqrt{c_1} x) \). We note that the exact two-soliton solutions

\[
u(x,t) = \frac{(c_2 - c_1)(c_2 \cosh^2 \alpha_1 + c_1 \sinh^2 \alpha_2)}{2 (\sqrt{c_2} \cosh h_1 \cos g_2 - \sqrt{c_1} \sin h_1 \sin h_2)^2},
\]

where \( \alpha_1 = \sqrt{c_1} (x - c_1 t)/2 \) and \( \alpha_2 = \sqrt{c_2} (x - c_2 t)/2 \) exhibit exactly this phenomenon: when \( x \) is large \( u \sim \exp(-\sqrt{c_1} x) \), i.e., the tail of the slow soliton dominates the decay. Table 4 refers to two-soliton solution with \( c_1 = 6, c_2 = 10 \). The table displays information at \( T_0 = 3 \), with \( \alpha = 1, d = \pi/2 \).

**Example 4.4.** Vanden-Broeck [16] considered the time-independent surface water of an incompressible and inviscid fluid flow over a bump in a two-dimensional channel. The model equation is the forced Korteweg–de Vries equation (fKdV):

\[
u_t + \lambda u_x + 2\alpha u u_x + \beta u_{xxx} = f_s(x), \quad x \in \mathbb{R}, \ t > 0.
\]

The forcing represented by the function \( f_s(x) \) in the fKdV equation is due to the bump on the bottom of the channel. Solitary wave means that the free-surface elevation \( u(x) \) has the property

\[
u(\pm \infty) = u_0(\pm \infty) = u_{xx}(\pm \infty) = 0.
\]
Table 5
Results for Example 4.4 using Sinc approach

<table>
<thead>
<tr>
<th>( N_x = N_t )</th>
<th>( h_x = h_t )</th>
<th>( \sup | u_{ij} - u(x_i, t_j) | )</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>0.785</td>
<td>( 4.1314 \times 10^{-2} )</td>
</tr>
<tr>
<td>16</td>
<td>0.555</td>
<td>( 1.6575 \times 10^{-2} )</td>
</tr>
<tr>
<td>24</td>
<td>0.453</td>
<td>( 3.5789 \times 10^{-3} )</td>
</tr>
<tr>
<td>32</td>
<td>0.392</td>
<td>( 2.4625 \times 10^{-4} )</td>
</tr>
</tbody>
</table>

To illustrate our numerical results, let us take \( \alpha = -\frac{3}{4}, \beta = -\frac{1}{6}, \lambda = 3 \), and let the forcing function \( f \) be defined by the equation

\[
f(x) = \begin{cases} 
\sin(\pi x), & |x| < 1, \\
0, & |x| \geq 1.
\end{cases}
\]

We use our approach and compare our result with the solution in [6]. Table 5 displays some results, where the solution \( u(x_i, t_j) \) is assumed to be the solution in [6], and \( u_{ij} \) is our Sinc solution.

References