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Stochastic optimal control and BSDEs with logarithmic growth ☆

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Abstract

We study the existence of an optimal strategy for the stochastic control of diffusion in general case and a saddle-point for zero-sum stochastic differential games. The problem is formulated as an extended BSDE with logarithmic growth in the z-variable ($|z|\sqrt{|\ln|z||}$) and an L^p-integrable terminal value, for a suitable $p > 2$. We also show the existence and uniqueness of solution for this BSDE.

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1. Introduction

We consider a backward stochastic differential equation (BSDE) with generator φ and terminal condition ξ

$$Y_t = \xi + \int_t^T \varphi(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dB_s, \quad t \in [0, T] \tag{1.1}$$

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where $(B_t)_{t\geqslant 0}$ is a standard Brownian motion. Such equations have been extensively studied since the paper of E. Pardoux and S. Peng [11]. We will consider the case when φ is allowed to have logarithmic growth ($|z|\sqrt{|\ln|z||}$) in the z-variable. Moreover, the terminal value is assumed to be merely L^p-integrable, with some $p > 2$.

The main objective in stochastic control is to show the existence of an optimal strategy for the stochastic control of diffusion. The main idea consists to characterize the value function as the unique solution of this BSDE (1.1). This problem has been previously studied by Elliott [7] and Davis and Elliott [6], who applied the martingale methods. Similar result on stochastic control has been obtained in [8] in the case where the drift term of the equation which defines the controlled system is bounded, and in [9] in the case where the running reward function is bounded. It should be noted that in our control problem, these boundedness conditions are neither satisfied by the drift term nor by the running reward function.

It is worth noting that the comparison methods are not used in the proof of the existence of solutions. Our technique is based on a localization method introduced in [1,2] and more developed and extended in [3,4].

The paper is organized as follows: in Section 2, we present the assumptions and we formulate the problem. In Section 3, we give the main result on existence and uniqueness of the solution of BSDE (1.1). In Section 4 is devoted to the proofs of the existence and of solutions for the considered BSDE. In Section 5, we introduce the optimal stochastic control problem and we give the connection between optimal stochastic control problem and the zero-sum stochastic differential games and the BSDE (1.1). We show the value function as a solution of BSDE (1.1).

2. Assumptions and formulation of the problem

Let $(\Omega, \mathcal{F}, P)$ be a fixed probability space on which is defined a standard d-dimensional Brownian motion $B = (B_t)_{0\leqslant t\leqslant T}$ whose natural filtration is $(\mathcal{F}_t^0 := \sigma\{B_s, s \leqslant t\})_{0\leqslant t\leqslant T}$. Let $\mathbf{F} = (\mathcal{F}_t)_{0\leqslant t\leqslant T}$ be the completed filtration of $(\mathcal{F}_t^0)_{0\leqslant t\leqslant T}$ with the P-null sets of $\mathcal{F}$. We consider the following assumptions,

(H.1) There exists a positive constant C such that $\mathbb{E}[|\xi|^{\ln(CT+2)+2}] < +\infty$.

(H.2) (i) φ is continuous in (y, z) for almost all (t, w);

(ii) There exist a positive constant c_0 and a process η_t satisfying

$$\mathbb{E}\left[\int_0^T \eta_s^{\ln(Cs+2)+2}\, ds\right] < +\infty,$$

and such that for every t, ω, y, z:

$$\left|\varphi(t, w, y, z)\right| \leqslant \eta_t + c_0|z|\sqrt{\left|\ln\left(|z|\right)\right|}.$$

(H.3) There exist $v \in \mathbb{L}^{q'}(\Omega \times [0, T]; \mathbb{R}_+)$ (for some $q' > 0$) and a real-valued sequence $(A_N)_{N>1}$ and constants $M_2 \in \mathbb{R}_+$, $r > 0$ such that:

i) $\forall N > 1$, $1 < A_N \leqslant N^r$.

ii) $\lim_{N\to\infty} A_N = \infty$.

iii) For every $N \in \mathbb{N}$, and every $y, y'z, z'$ such that $|y|, |y'|, |z|, |z'| \leqslant N$, we have

$$\big(y-y'\big)\big(\varphi(t,\omega,y,z)-\varphi\big(t,\omega,y',z'\big)\big)\mathbb{1}_{\{v_t(\omega)\leqslant N\}}$$
$$\leqslant M_2\big|y-y'\big|^2\ln A_N+M_2\big|y-y'\big|\big|z-z'\big|\sqrt{\ln A_N}+M_2\frac{\ln A_N}{A_N}.$$

3. Existence and uniqueness of solutions

The main objective of this paper is to focus on the existence and uniqueness of the solution of Eq. (1.1) under the previous assumptions.

We denote by $\mathbb{E}$ the set of $\mathbb{R}\times\mathbb{R}^d$-valued processes (Y,Z) defined on $\mathbb{R}_+\times\Omega$ which are $\mathcal{F}_t$-adapted and such that: $\|(Y,Z)\|^2=\mathbb{E}(\sup_{0\leqslant t\leqslant T}|Y_t|^2+\int_0^T|Z_s|^2\,ds)<+\infty$. The couple $(\mathbb{E},\|.\|)$ is then a Banach space.

For $N\in\mathbb{N}^*$, we define

$$\rho_N(\varphi)=E\int_0^T\sup_{|y|,|z|\leqslant N}\big|\varphi(s,y,z)\big|\,ds. \tag{3.1}$$

Definition 3.1. A solution of Eq. (1.1) is a couple (Y,Z) which belongs to the space $(\mathbb{E},\|.\|)$ and satisfies Eq. (1.1).

The main results of this section are the following two theorems.

Theorem 3.1. *Assume that* (H.1), (H.2) *and* (H.3) *are satisfied. Then, Eq.* (1.1) *has a unique solution.*

4. Proofs

To prove Theorem 3.1 we need the following lemmas.

4.1. Estimates of solutions

Lemma 4.1. *Let* (Y,Z) *be a solution of the above BSDE, where* (ξ,φ) *satisfies the assumptions* (H.1) *and* (H.2). *Then there exists a constant* C_T*, such that*:

$$\mathbb{E}\sup_{t\in[0,T]}|Y_t|^{\ln(Ct+2)+2}\leqslant C_T\mathbb{E}\left[|\xi|^{\ln(CT+2)+2}+\int_0^T\eta_s^{\ln(Cs+2)+2}\,ds\right].$$

Proof. Without loss of generality we assume that the y-variable is sufficiently large. For some constant C large enough, let us consider the function from $[0,T]\times\mathbb{R}$ into $\mathbb{R}^+$ defined by

$$u(t,x)=|x|^{\ln(Ct+2)+2}.$$

Then

$$u_t=\frac{C}{Ct+2}\ln\big(|x|\big)|x|^{\ln(Ct+2)+2},\qquad u_x=\big(\ln(Ct+2)+2\big)|x|^{\ln(Ct+2)+1}\operatorname{sgn}(x)$$

and $u_{xx}=(\ln(Ct+2)+2)(\ln(Ct+2)+1)|x|^{\ln(Ct+2)}$, with the notation $\operatorname{sgn}(x)=-\mathbf{1}_{x\leqslant 0}+\mathbf{1}_{x>0}$. For $k\geqslant 0$, let τ_k be the stopping time defined as follows:

$$\tau_k = \inf\left\{ t \geqslant 0,\ \int_0^T \big(\ln(Cs+2)+2\big)^2 |Y_s|^{2\ln(Cs+2)+2} |Z_s|^2\, ds \geqslant k \right\} \wedge T.$$

By Itô's formula, we have:

$$\begin{aligned}
|Y_{t\wedge\tau_k}|^{\ln(Ct+2)+2} &= |Y_{\tau_k}|^{\ln(Ct+2)+2} - \int_{t\wedge\tau_k}^{\tau_k} \frac{C}{Cs+2} \ln\big(|Y_s|\big)|Y_s|^{\ln(Cs+2)+2}\, ds \\
&\quad - \frac{1}{2}\int_{t\wedge\tau_k}^{\tau_k} |Z_s|^2 \big(\ln(Cs+2)+2\big)\big(\ln(Cs+2)+1\big)|Y_s|^{\ln(Cs+2)}\, ds \\
&\quad + \int_{t\wedge\tau_k}^{\tau_k} \big(\ln(Cs+2)+2\big)|Y_s|^{\ln(Cs+2)+1} \operatorname{sgn}(Y_s) f(s, Y_s, Z_s)\, ds \\
&\quad - \int_{t\wedge\tau_k}^{\tau_k} \big(\ln(Cs+2)+2\big)|Y_s|^{\ln(Cs+2)+1} \operatorname{sgn}(Y_s) Z_s\, dB_s \\
&\leqslant |Y_{\tau_k}|^{\ln(Ct+2)+2} - \int_{t\wedge\tau_k}^{\tau_k} \frac{C}{Cs+2} \ln\big(|Y_s|\big)|Y_s|^{\ln(Cs+2)+2}\, ds \\
&\quad - \frac{1}{2}\int_{t\wedge\tau_k}^{\tau_k} |Z_s|^2 \big(\ln(Cs+2)+2\big)\big(\ln(Cs+2)+1\big)|Y_s|^{\ln(Cs+2)}\, ds \\
&\quad + \int_{t\wedge\tau_k}^{\tau_k} \big(\ln(Cs+2)+2\big)|Y_s|^{\ln(Cs+2)+1} \Big(\eta_s + c_0|Z_s|\sqrt{\big|\ln\big(|Z_s|\big)\big|}\Big)\, ds \\
&\quad - \int_{t\wedge\tau_k}^{\tau_k} \big(\ln(Cs+2)+2\big)|Y_s|^{\ln(Cs+2)+1} \operatorname{sgn}(Y_s) Z_s\, dB_s.
\end{aligned}$$

By Young's inequality it hold:

$$\begin{aligned}
\big(\ln(Cs+2)+2\big)|Y_s|^{\ln(Cs+2)+1}\eta_s &\leqslant |Y_s|^{\ln(Cs+2)+2} \\
&\quad + \big(\ln(Cs+2)+2\big)^{\ln(Cs+2)+1} \eta_s^{\ln(Cs+2)+2}.
\end{aligned}$$

For $|y|$ large enough and the last inequality there exists C_1 such that:

$$\begin{aligned}
|Y_{t\wedge\tau_k}|^{\ln(Ct+2)+2} &= |Y_{\tau_k}|^{\ln(Ct+2)+2} - \int_{t\wedge\tau_k}^{\tau_k} C_1 \ln\big(|Y_s|\big)|Y_s|^{\ln(Cs+2)+2}\, ds \\
&\quad - \frac{1}{2}\int_{t\wedge\tau_k}^{\tau_k} |Z_s|^2 \big(\ln(Cs+2)+2\big)\big(\ln(Cs+2)+1\big)|Y_s|^{\ln(Cs+2)}\, ds
\end{aligned}$$

$$+\int_{t\wedge\tau_k}^{\tau_k}\big(\ln(Cs+2)+2\big)|Y_s|^{\ln(Cs+2)+1}c_0|Z_s|\sqrt{\big|\ln\big(|Z_s|\big)\big|}\,ds$$
$$+\int_{t\wedge\tau_k}^{\tau_k}\big(\ln(Cs+2)+2\big)^{\ln(Cs+2)+1}\eta_s^{\ln(Cs+2)+2}\,ds$$
$$-\int_{t\wedge\tau_k}^{\tau_k}\big(\ln(Cs+2)+2\big)|Y_s|^{\ln(Cs+2)+1}\operatorname{sgn}(Y_s)Z_s\,dB_s$$
$$\leqslant|Y_{\tau_k}|^{\ln(Ct+2)+2}-\int_{t\wedge\tau_k}^{\tau_k}\big(\ln(Cs+2)+2\big)\big(\ln(Cs+2)+1\big)|Y_s|^{\ln(Cs+2)}$$
$$\times\bigg[\frac{C_1\ln(|Y_s|)|Y_s|^2}{(\ln(Cs+2)+2)(\ln(Cs+2)+1)}+\frac{|Z_s|^2}{2}$$
$$-\frac{(\ln(Cs+2)+2)|Y_s|c_0|Z_s|\sqrt{|\ln(|Z_s|)|}}{(\ln(Cs+2)+2)(\ln(Cs+2)+1)}\bigg]ds$$
$$+\int_{t\wedge\tau_k}^{\tau_k}\big(\ln(Cs+2)+2\big)^{\ln(Cs+2)+1}\eta_s^{\ln(Cs+2)+2}\,ds$$
$$-\int_{t\wedge\tau_k}^{\tau_k}\big(\ln(Cs+2)+2\big)|Y_s|^{\ln(Cs+2)+1}\operatorname{sgn}(Y_s)Z_s\,dB_s.$$

There exist a constants C_2 and C_3 $(C_2>2C_3^2)$

$$|Y_{t\wedge\tau_k}|^{\ln(Ct+2)+2}$$
$$\leqslant|Y_{\tau_k}|^{\ln(Ct+2)+2}-\int_{t\wedge\tau_k}^{\tau_k}\big(\ln(Cs+2)+2\big)\big(\ln(Cs+2)+1\big)$$
$$\times|Y_s|^{\ln(Cs+2)}\bigg[C_2\ln\big(|Y_s|\big)|Y_s|^2+\frac{|Z_s|^2}{2}-C_3|Y_s||Z_s|\sqrt{\big|\ln\big(|Z_s|\big)\big|}\bigg]ds$$
$$+\int_{t\wedge\tau_k}^{\tau_k}\big(\ln(Cs+2)+2\big)^{\ln(Cs+2)+1}\eta_s^{\ln(Cs+2)+2}\,ds$$
$$-\int_{t\wedge\tau_k}^{\tau_k}\big(\ln(Cs+2)+2\big)|Y_s|^{\ln(Cs+2)+1}\operatorname{sgn}(Y_s)Z_s\,dB_s.\tag{4.1}$$

We shall show that

$$C_3|Y_s||Z_s|\sqrt{\big|\ln\big(|Z_s|\big)\big|}\leqslant\frac{|Z_s|^2}{2}+C_2\ln\big(|Y_s|\big)|Y_s|^2.\tag{4.2}$$

If $|Z_s| \leqslant |Y_s|$, (4.2) is obvious to prove. Assume now that $|Z_s| > |Y_s|$ and put $a_s := \frac{|Z_s|}{|Y_s|}$. a_s is then strictly greater than 1. Since Y_s is assumed to be large enough and $|Z_s| > |Y_s|$, then Z_s is also large enough, and we have

$$C_3|Y_s||Z_s|\sqrt{\ln(|Z_s|)} \leqslant C_3 a_s Y_s^2\left[\sqrt{\ln(a_s)} + \sqrt{\ln(|Y_s|)}\,\right]$$

and

$$\frac{|Z_s|^2}{2} + C_2\ln(|Y_s|)|Y_s|^2 = \left[\frac{a_s^2}{2} + C_2\ln(|Y_s|)\right]|Y_s|^2.$$

Obviously

$$C_3 a_s\left[\sqrt{\ln(|Y_s|)}\,\right] \leqslant \frac{1}{2}\left[\frac{a_s^2}{2} + 2C_3^2\ln(|Y_s|)\right].$$

Let r be the constant such that $C_3\sqrt{\ln(r)} = \frac{r}{4}$.

If $a_s \geqslant r$, then $C_3 a_s\sqrt{\ln(a_s)} \leqslant \frac{a_s^2}{4}$.
If $a_s \leqslant r$, then since $|y|$ large enough, we have $C_3 a_s\sqrt{\ln(a_s)} \leqslant C_3 r\sqrt{\ln(r)} \leqslant \frac{C_2}{2}\ln(|Y_s|)$.

This prove inequality (4.2). To finish the proof, take the limit as $k \to \infty$ in the inequality (4.1). Lemma 4.1 is proved. □

Lemma 4.2. *Let (Y, Z) be a solution of the above BSDE. Then, for every $0 < p \leqslant \frac{2(\ln(CT+2)+2)}{\ln(2)+2}$, there exits a positive constant C_p depending only on p such that*:

$$\mathbb{E}\left[\left(\int_0^T |Z_s|^2\,ds\right)^{p/2}\right] \leqslant C_p\mathbb{E}\left[|\xi|^p + \sup_{t\in[0,T]}|Y_t|^{p\frac{2+\ln(2)}{2}} + \left(\int_0^T |\eta_s|^2\,ds\right)^{\frac{p}{2}}\right].$$

Proof. Applying Itô's formula to the process Y_t and the function $y \mapsto y^2$ yields:

$$|Y_0|^2 + \int_0^T |Z_s|^2\,ds = |\xi|^2 + 2\int_0^T Y_s\varphi(s, Y_s, Z_s)\,ds - 2\int_0^T Y_s Z_s\,dB_s$$
$$\leqslant |\xi|^2 + 2\int_0^T |Y_s|\left(|\eta_s| + c_0|Z_s|\sqrt{\left|\ln(|Z_s|)\right|}\right)ds - 2\int_0^T Y_s Z_s\,dB_s.$$

As we have

$$2|Y_s||\eta_s| \leqslant |Y_s|^2 + |\eta_s|^2,$$

and for any $\varepsilon > 0$ we have:

$$\sqrt{2\varepsilon\left|\ln(|Z_s|)\right|} = \sqrt{\left|\ln(|z|^{2\varepsilon})\right|} \leqslant |z|^{\varepsilon}.$$

Then plug the two last inequalities in the previous one to obtain:

$$|Y_0|^2 + \int_0^T |Z_s|^2\, ds$$
$$\leqslant |\xi|^2 + \sup_{s\leqslant T} |Y_s|^2 + \int_0^T |\eta_s|^2\, ds + \frac{2}{\sqrt{2\varepsilon}} \int_0^T |Y_s||Z_s|^{1+\varepsilon}\, ds - 2\int_0^T Y_s Z_s\, dB_s.$$

We now choose $0 < \varepsilon < 1$ and by Young's inequality it holds true that:

$$2\frac{|Y_s|}{\sqrt{2\varepsilon}}|Z_s|^{1+\varepsilon} \leqslant \frac{1-\varepsilon}{2}\left(\frac{2}{\sqrt{2\varepsilon}}\right)^{\frac{2}{1-\varepsilon}} |Y_s|^{\frac{2}{1-\varepsilon}} + \frac{1+\varepsilon}{2}|Z_s|^2.$$

Then, there exists a positive constant c_ε

$$|Y_0|^2 + \int_0^T |Z_s|^2\, ds \leqslant |\xi|^2 + \sup_{s\leqslant T} |Y_s|^2 + \int_0^T |\eta_s|^2\, ds + c_\varepsilon \sup_{s\leqslant T} |Y_s|^{\frac{2}{1-\varepsilon}}$$
$$+ \frac{1+\varepsilon}{2}\int_0^T |Z_s|^2\, ds - 2\int_0^T Y_s Z_s\, dB_s.$$

For $|y|$ large enough and $\varepsilon \leqslant \frac{\ln(2)}{2+\ln(2)}$ we have

$$|Y_0|^2 + \int_0^T |Z_s|^2\, ds \leqslant |\xi|^2 + c_\varepsilon \sup_{s\leqslant T} |Y_s|^{2+\ln(2)} + \int_0^T |\eta_s|^2\, ds$$
$$+ \frac{1+\varepsilon}{2}\int_0^T |Z_s|^2\, ds - 2\int_0^T Y_s Z_s\, dB_s.$$

Hence:

$$\mathbb{E}\left(\int_0^T |Z_s|^2\, ds\right)^{p/2} \leqslant C_p\mathbb{E}\left[|\xi|^p + c_\varepsilon \sup_{s\leqslant T} |Y_s|^{p\frac{2+\ln(2)}{2}} + \left(\int_0^T |\eta_s|^2\, ds\right)^{\frac{p}{2}}\right.$$
$$\left. + \left(\frac{1+\varepsilon}{2}\right)^{\frac{p}{2}}\left(\int_0^T |Z_s|^2\, ds\right)^{\frac{p}{2}}\right] + C_p\mathbb{E}\left[\left|\int_0^T Y_s Z_s\, dB_s\right|^{\frac{p}{2}}\right].$$

Thanks to BDG's inequality we have for any $\beta > 0$

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t Y_s Z_s dB_s\right|^{p/2}\right] \leqslant \bar{C}_p\mathbb{E}\left[\left(\int_0^T |Y_s|^2|Z_s|^2\, ds\right)^{p/4}\right]$$
$$\leqslant \bar{C}_p\mathbb{E}\left[\left(\sup_{t\in[0,T]} |Y_t|\right)^{p/2}\left(\int_0^T |Z_s|^2\, ds\right)^{p/4}\right]$$

$$\leqslant \frac{\bar{C}_p^2}{\beta}\mathbb{E}\Big[\sup_{t\in[0,T]}|Y_t|^p\Big] + \beta\mathbb{E}\left[\left(\int_0^T |Z_s|^2\,ds\right)^{p/2}\right].$$

We choose ε and β small enough to obtain the desired result. Lemma 4.2 is proved. □

Lemma 4.3. *If* (H.2) *holds then,*

$$\mathbb{E}\int_0^T \big|\varphi(s,Y_s,Z_s)\big|^{\bar{\alpha}}\,ds \leqslant K\left[1+\mathbb{E}\int_0^T \eta_s^2\,ds + \mathbb{E}\int_0^T |Z_s|^2\,ds\right]$$

where $\bar{\alpha} = \min(2, \frac{2}{\alpha})$ *and* K *is a positive constant which depends on* c_0 *and* T.

Proof. Observe that assumption (H.2) implies that there exist $c_1 > 0$ and $0 \leqslant \alpha < 2$ such that:

$$\big|\varphi(t,\omega,y,z)\big| \leqslant \eta_t + c_1|z|^{\alpha}. \tag{4.3}$$

We successively use assumption (H.3) and inequality (4.3) to show that

$$\begin{aligned}
\mathbb{E}\int_0^T \big|f(s,Y_s,Z_s)\big|^{\bar{\alpha}}\,ds &\leqslant \mathbb{E}\int_0^T \Big(\eta_s + c_0|z|\sqrt{\big|\ln\big(|Z_s|\big)\big|}\Big)^{\bar{\alpha}}\,ds \\
&\leqslant \mathbb{E}\int_0^T \big(\eta_s + c_1|Z_s|^{\alpha}\big)^{\bar{\alpha}}\,ds \\
&\leqslant \big(1+c_1^{\bar{\alpha}}\big)\mathbb{E}\int_0^T \big((\eta_s)^{\bar{\alpha}} + \big(|Z_s|\big)^{\alpha\bar{\alpha}}\big)\,ds \\
&\leqslant \big(1+c_1^{\bar{\alpha}}\big)\mathbb{E}\int_0^T \big((1+\eta_s)^{\bar{\alpha}} + \big(1+|Z_s|\big)^{\alpha\bar{\alpha}}\big)\,ds \\
&\leqslant \big(1+c_1^{\bar{\alpha}}\big)\mathbb{E}\int_0^T \big((1+\eta_s)^2 + \big(1+|Z_s|\big)^2\big)\,ds \\
&\leqslant \big(1+c_1^{\bar{\alpha}}\big)\left(4T + \mathbb{E}\int_0^T \big(\eta_s^2 + |Z_s|^2\big)\,ds\right).
\end{aligned}$$

Lemma 4.3 is proved. □

Lemma 4.4. *There exists a sequence of functions* (φ_n) *such that:*

(a) *For each* n, φ_n *is bounded and globally Lipschitz in* (y,z) *a.e.* t *and* P*-a.s.* ω.
(b) $\sup_n |\varphi_n(t,\omega,y,z)| \leqslant \eta_t + c_0|z|\sqrt{|\ln(|Z_s|)|}$, P*-a.s., a.e.* $t \in [0,T]$.
(c) *For every* N, $\rho_N(\varphi_n - \varphi) \to 0$ *as* $n \to \infty$.

Proof. Let $\varepsilon_n : \mathbb{R}^2 \to \mathbb{R}_+$ be a sequence of smooth functions with compact support which approximate the Dirac measure at 0 and which satisfy $\int \varepsilon_n(u)\,du = 1$. Let ψ_n from $\mathbb{R}^2$ to $\mathbb{R}_+$ be a sequence of smooth functions such that $0 \leqslant |\psi_n| \leqslant 1$, $\psi_n(u) = 1$ for $|u| \leqslant n$ and $\psi_n(u) = 0$ for $|u| \geqslant n+1$. We put, $\varepsilon_{q,n}(t,y,z) = \int \varphi(t,(y,z)-u)\alpha_q(u)\,du\,\psi_n(y,z)$. For $n \in \mathbb{N}^*$, let $q(n)$ be an integer such that $q(n) \geqslant n + n^\alpha$. It is not difficult to see that the sequence $\varphi_n := \varepsilon_{q(n),n}$ satisfies all the assertions (a)–(c). □

Using Lemmas 4.1–4.4 and standard arguments of BSDEs, one can prove the following estimates.

Lemma 4.5. *Let φ and ξ be as in Theorem 3.1. Let (φ_n) be the sequence of functions associated to φ by Lemma 4.4. Denote by $(Y^{\varphi_n}, Z^{\varphi_n})$ the solution of equation (E^{φ_n}). Then, there exit constants K_1, K_2, K_3 and a universal constant ℓ such that*

a) $\sup_n \mathbb{E}\int_0^T |Z_s^{\varphi_n}|^2\,ds \leqslant K_1$;
b) $\sup_n \mathbb{E}\sup_{0\leqslant t\leqslant T}(|Y_t^{\varphi_n}|^2) \leqslant \ell K_1 := K_2$;
c) $\sup_n \mathbb{E}\int_0^T |\varphi_n(s, Y_s^{\varphi_n}, Z_s^{\varphi_n})|^{\bar\alpha}\,ds \leqslant K_3$

where $\bar\alpha = \min(2, \frac{2}{\alpha})$.

After extracting a subsequence, if necessary, we have

Corollary 4.1. *There are $Y \in \mathbb{L}^2(\Omega, L^\infty[0,T])$, $Z \in \mathbb{L}^2(\Omega \times [0,T])$, $\Gamma \in \mathbb{L}^{\bar\alpha}(\Omega \times [0,T])$ such that*

$$\begin{aligned}
&Y^{\varphi_n} \rightharpoonup Y, \quad \textit{weakly star in } \mathbb{L}^2\big(\Omega, L^\infty[0,T]\big),\\
&Z^{\varphi_n} \rightharpoonup Z, \quad \textit{weakly in } \mathbb{L}^2\big(\Omega \times [0,T]\big),\\
&\varphi_n\big(., Y^{\varphi_n}, Z^{\varphi_n}\big) \rightharpoonup \Gamma, \quad \textit{weakly in } \mathbb{L}^{\bar\alpha}\big(\Omega \times [0,T]\big),
\end{aligned}$$

and moreover

$$Y_t = \xi + \int_t^T \Gamma_s\,ds - \int_t^T Z_s\,dW_s, \quad \forall t \in [0,T].$$

The following lemma, were established in [3], is a direct consequence of Hölder's and Schwarz's inequalities.

Lemma 4.6. *For every $\beta \in]1,2]$, $A > 0$, $(y)_{i=1\ldots d} \subset \mathbb{R}$, $(z)_{i=1\ldots d, j=1\ldots r} \subset \mathbb{R}$ we have,*

$$A|y||z| - \frac{1}{2}|z|^2 + \frac{2-\beta}{2}|y|^{-2}|yz|^2 \leqslant \frac{1}{\beta-1}A^2|y|^2 - \frac{\beta-1}{4}|z|^2.$$

This lemma remains valid in multidimensional case.

4.2. Estimate between two solutions

The key estimate is given by

Lemma 4.7. *For every $R \in \mathbb{N}$, $\beta \in]1, \min(3-\frac{2}{\bar{\alpha}}, 2)[$, $\delta' < (\beta-1)\min(\frac{1}{4M_2^2}, \frac{3-\frac{2}{\bar{\alpha}}-\beta}{2rM_2^2\beta})$ and $\varepsilon > 0$, there exists $N_0 > R$ such that for all $N > N_0$ and $T' \leqslant T$:*

$$\limsup_{n,m\to+\infty} E \sup_{(T'-\delta')^+\leqslant t\leqslant T'} \left|Y_t^{\varphi_n} - Y_t^{\varphi_m}\right|^\beta + E\int_{(T'-\delta')^+}^{T'} \frac{|Z_s^{\varphi_n} - Z_s^{\varphi_m}|^2}{(|Y_s^{\varphi_n} - Y_s^{\varphi_m}|^2 + \nu_R)^{\frac{2-\beta}{2}}}\,ds$$

$$\leqslant \varepsilon + \frac{\ell}{\beta-1} e^{C_N\delta'} \limsup_{n,m\to+\infty} E\left|Y_{T'}^{\varphi_n} - Y_{T'}^{\varphi_m}\right|^\beta,$$

where $\nu_R = \sup\{(A_N)^{-1},\ N \geqslant R\}$, $C_N = \frac{2M_2^2\beta}{(\beta-1)}\log A_N$ and ℓ is a universal positive constant.

Proof. To simplify the computations, we assume (without loss of generality) that assumption (H.3)-iii) holds without the multiplicative term $\mathbb{1}_{\{v_t(\omega)\leqslant N\}}$.

Let $0 < T' \leqslant T$. It follows from Itô's formula that for all $t \leqslant T'$,

$$\left|Y_t^{\varphi_n} - Y_t^{\varphi_m}\right|^2 + \int_t^{T'} \left|Z_s^{\varphi_n} - Z_s^{\varphi_m}\right|^2 ds$$

$$= \left|Y_{T'}^{\varphi_n} - Y_{T'}^{\varphi_m}\right|^2 + 2\int_t^{T'} \left(Y_s^{\varphi_n} - Y_s^{\varphi_m}\right)\left(\varphi_n\left(s, Y_s^{\varphi_n}, Z_s^{\varphi_n}\right) - \varphi_m\left(s, Y_s^{\varphi_m}, Z_s^{\varphi_m}\right)\right) ds$$

$$- 2\int_t^{T'} \left\langle Y_s^{\varphi_n} - Y_s^{\varphi_m}, \left(Z_s^{\varphi_n} - Z_s^{\varphi_m}\right) dW_s\right\rangle.$$

For $N \in \mathbb{N}^\star$ we set, $\Delta_t := |Y_t^{\varphi_n} - Y_t^{\varphi_m}|^2 + (A_N)^{-1}$.

Let $C > 0$ and $1 < \beta < \min\{(3-\frac{2}{\bar{\alpha}}), 2\}$. Itô's formula shows that,

$$e^{Ct}\Delta_t^{\frac{\beta}{2}} + C\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}}\,ds$$

$$= e^{CT'}\Delta_{T'}^{\frac{\beta}{2}} + \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\left(Y_s^{\varphi_n} - Y_s^{\varphi_m}\right)\left(\varphi_n\left(s, Y_s^{\varphi_n}, Z_s^{\varphi_n}\right) - \varphi_m\left(s, Y_s^{\varphi_m}, Z_s^{\varphi_m}\right)\right) ds$$

$$- \frac{\beta}{2}\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\left|Z_s^{\varphi_n} - Z_s^{\varphi_m}\right|^2 ds - \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\left\langle Y_s^{\varphi_n} - Y_s^{\varphi_m}, \left(Z_s^{\varphi_n} - Z_s^{\varphi_m}\right) dW_s\right\rangle$$

$$- \beta\left(\frac{\beta}{2}-1\right)\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-2}\left(\left(Y_s^{\varphi_n} - Y_s^{\varphi_m}\right)\left(Z_s^{\varphi_n} - Z_s^{\varphi_m}\right)\right)^2 ds.$$

Put $\Phi(s) = |Y_s^{\varphi_n}| + |Y_s^{\varphi_m}| + |Z_s^{\varphi_n}| + |Z_s^{\varphi_m}|$. Then

$$
\begin{aligned}
&e^{Ct}\Delta_t^{\frac{\beta}{2}} + C\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}}\,ds \\
&\quad = e^{CT'}\Delta_{T'}^{\frac{\beta}{2}} - \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\big\langle Y_s^{\varphi_n} - Y_s^{\varphi_m}, \big(Z_s^{\varphi_n} - Z_s^{\varphi_m}\big)\,dW_s\big\rangle \\
&\qquad - \frac{\beta}{2}\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\big|Z_s^{\varphi_n} - Z_s^{\varphi_m}\big|^2\,ds \\
&\qquad + \beta\frac{(2-\beta)}{2}\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-2}\big(\big(Y_s^{\varphi_n} - Y_s^{\varphi_m}\big)\big(Z_s^{\varphi_n} - Z_s^{\varphi_m}\big)\big)^2\,ds \\
&\qquad + J_1 + J_2 + J_3 + J_4,
\end{aligned}
$$

where

$$
\begin{aligned}
J_1 &:= \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\big(Y_s^{\varphi_n} - Y_s^{\varphi_m}\big)\big(\varphi_n\big(s, Y_s^{\varphi_n}, Z_s^{\varphi_n}\big) - \varphi_m\big(s, Y_s^{\varphi_m}, Z_s^{\varphi_m}\big)\big)\mathbb{1}_{\{\Phi(s)>N\}}\,ds, \\
J_2 &:= \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\big(Y_s^{\varphi_n} - Y_s^{\varphi_m}\big)\big(\varphi_n\big(s, Y_s^{\varphi_n}, Z_s^{\varphi_n}\big) - \varphi\big(s, Y_s^{\varphi_n}, Z_s^{\varphi_n}\big)\big)\mathbb{1}_{\{\Phi(s)\leqslant N\}}\,ds, \\
J_3 &:= \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\big(Y_s^{\varphi_n} - Y_s^{\varphi_m}\big)\big(\varphi\big(s, Y_s^{\varphi_n}, Z_s^{\varphi_n}\big) - \varphi\big(s, Y_s^{\varphi_m}, Z_s^{\varphi_m}\big)\big)\mathbb{1}_{\{\Phi(s)\leqslant N\}}\,ds, \\
J_4 &:= \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\big(Y_s^{\varphi_n} - Y_s^{\varphi_m}\big)\big(\varphi\big(s, Y_s^{\varphi_m}, Z_s^{\varphi_m}\big) - \varphi_m\big(s, Y_s^{\varphi_m}, Z_s^{\varphi_m}\big)\big)\mathbb{1}_{\{\Phi(s)\leqslant N\}}\,ds.
\end{aligned}
$$

We shall estimate J_1, J_2, J_3, J_4. Let $\kappa = 3 - \frac{2}{\bar\alpha} - \beta$. Since $\frac{(\beta-1)}{2} + \frac{\kappa}{2} + \frac{1}{\bar\alpha} = 1$, we use the Hölder inequality to obtain

$$
\begin{aligned}
J_1 &\leqslant \beta e^{CT'}\frac{1}{N^\kappa}\int_t^{T'} \Delta_s^{\frac{\beta-1}{2}}\Phi^\kappa(s)\big|\varphi_n\big(s, Y_s^{\varphi_n}, Z_s^{\varphi_n}\big) - \varphi_m\big(s, Y_s^{\varphi_m}, Z_s^{\varphi_m}\big)\big|\,ds \\
&\leqslant \beta e^{CT'}\frac{1}{N^\kappa}\left[\int_t^{T'}\Delta_s\,ds\right]^{\frac{\beta-1}{2}}\left[\int_t^{T'}\Phi(s)^2\,ds\right]^{\frac{\kappa}{2}} \\
&\quad \times\left[\int_t^{T'}\big|\varphi_n\big(s, Y_s^{\varphi_n}, Z_s^{\varphi_n}\big) - \varphi_m\big(s, Y_s^{\varphi_m}, Z_s^{\varphi_m}\big)\big|^{\bar\alpha}\,ds\right]^{\frac{1}{\bar\alpha}}.
\end{aligned}
$$

Since $|Y_s^{\varphi_n} - Y_s^{\varphi_m}| \leqslant \Delta_s^{\frac{1}{2}}$, it easy to see that

$$J_2 + J_4 \leqslant 2\beta e^{CT'}\big[2N^2 + \nu_1\big]^{\frac{\beta-1}{2}}\Bigg[\int_t^{T'} \sup_{|y|,|z|\leqslant N}\big|\varphi_n(s,y,z) - \varphi(s,y,z)\big|\,ds + \int_t^{T'} \sup_{|y|,|z|\leqslant N}\big|\varphi_m(s,y,z) - \varphi(s,y,z)\big|\,ds\Bigg].$$

Using assumption (H.3), we get

$$\begin{aligned} J_3 &\leqslant \beta M_2 \int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\Bigg[\big|Y_s^{\varphi_n} - Y_s^{\varphi_m}\big|^2 \ln A_N \\ &\quad + \frac{\ln A_N}{A_N} + \big|Y_s^{\varphi_n} - Y_s^{\varphi_m}\big|\big|Z_s^{\varphi_n} - Z_s^{\varphi_m}\big|\sqrt{\ln A_N}\Bigg]\mathbb{1}_{\{\Phi(s)<N\}}\,ds \\ &\leqslant \beta M_2 \int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\big[\Delta_s \ln A_N + \big|Y_s^{\varphi_n} - Y_s^{\varphi_m}\big|\big|Z_s^{\varphi_n} - Z_s^{\varphi_m}\big|\sqrt{\ln A_N}\big]\mathbb{1}_{\{\Phi(s)\leqslant N\}}\,ds. \end{aligned}$$

We choose $C = C_N = \frac{2M_2^2\beta}{\beta-1}\ln A_N$, then we use Lemma 4.6 to show that

$$\begin{aligned} &e^{C_N t}\Delta_t^{\frac{\beta}{2}} + \frac{\beta(\beta-1)}{4}\int_t^{T'} e^{C_N s}\Delta_s^{\frac{\beta}{2}-1}\big|Z_s^{\varphi_n} - Z_s^{\varphi_m}\big|^2\,ds \\ &\quad \leqslant e^{C_N T'}\Delta_{T'}^{\frac{\beta}{2}} - \beta\int_t^{T'} e^{C_N s}\Delta_s^{\frac{\beta}{2}-1}\big\langle Y_s^{\varphi_n} - Y_s^{\varphi_m}, \big(Z_s^{\varphi_n} - Z_s^{\varphi_m}\big)\,dW_s\big\rangle \\ &\qquad + \beta e^{C_N T'}\frac{1}{N^\kappa}\Bigg[\int_t^{T'}\Delta_s\,ds\Bigg]^{\frac{\beta-1}{2}} \times \Bigg[\int_t^{T'}\Phi(s)^2\,ds\Bigg]^{\frac{\kappa}{2}} \\ &\qquad \times \Bigg[\int_t^{T'}\big|\varphi_n\big(s, Y_s^{\varphi_n}, Z_s^{\varphi_n}\big) - \varphi_m\big(s, Y_s^{\varphi_m}, Z_s^{\varphi_m}\big)\big|^{\bar\alpha}\mathbb{1}_{\{\Phi(s)>N\}}\,ds\Bigg]^{\frac{1}{\bar\alpha}} \\ &\qquad + \beta e^{C_N T'}\big[2N^2 + \nu_1\big]^{\frac{\beta-1}{2}}\Bigg[\int_t^{T'}\sup_{|y|,|z|\leqslant N}\big|\varphi_n(s,y,z) - \varphi(s,y,z)\big|\,ds \\ &\qquad + \int_t^{T'}\sup_{|y|,|z|\leqslant N}\big|\varphi_m(s,y,z) - \varphi(s,y,z)\big|\,ds\Bigg]. \end{aligned}$$

Burkholder's inequality and Hölder's inequality (since $\frac{(\beta-1)}{2} + \frac{\kappa}{2} + \frac{1}{\bar\alpha} = 1$) allow us to show that there exists a universal constant $\ell > 0$ such that $\forall\delta' > 0$,

$$\mathbb{E}\sup_{(T'-\delta')^+\leqslant t\leqslant T'}\big[e^{C_N t}\Delta_t^{\frac{\beta}{2}}\big] + \mathbb{E}\int_{(T'-\delta')^+}^{T'} e^{C_N s}\Delta_s^{\frac{\beta}{2}-1}\big|Z_s^{\varphi_n} - Z_s^{\varphi_m}\big|^2\,ds$$

$$\leqslant \frac{\ell}{\beta-1} e^{C_N T'} \Bigg\{ \mathbb{E}\big[\Delta_{T'}^{\frac{\beta}{2}}\big] + \frac{\beta}{N^\kappa} \Bigg[\mathbb{E} \int_0^T \Delta_s \, ds \Bigg]^{\frac{\beta-1}{2}} \Bigg[\mathbb{E} \int_0^T \Phi(s)^2 \, ds \Bigg]^{\frac{\kappa}{2}}$$

$$\times \Bigg[\mathbb{E} \int_0^T \big|\varphi_n\big(s, Y_s^{\varphi_n}, Z_s^{\varphi_n}\big) - \varphi_m\big(s, Y_s^{\varphi_m}, Z_s^{\varphi_m}\big)\big|^{\bar{\alpha}} \, ds \Bigg]^{\frac{1}{\bar{\alpha}}}$$

$$+ \beta\big[2N^2 + \nu_1\big]^{\frac{\beta-1}{2}} \mathbb{E}\Bigg[\int_0^T \sup_{|y|,|z| \leqslant N} \big|\varphi_n(s,y,z) - \varphi(s,y,z)\big| \, ds$$

$$+ \int_0^T \sup_{|y|,|z| \leqslant N} \big|\varphi_m(s,y,z) - \varphi(s,y,z)\big| \, ds \Bigg] \Bigg\}.$$

We use Lemma 4.4 and Lemma 4.5 to obtain, $\forall N > R$,

$$\mathbb{E} \sup_{(T'-\delta')^+ \leqslant t \leqslant T'} \big|Y_t^{\varphi_n} - Y_t^{\varphi_m}\big|^\beta + \mathbb{E} \int_{(T'-\delta')^+}^{T'} \frac{|Z_s^{\varphi_n} - Z_s^{\varphi_m}|^2}{(|Y_s^{\varphi_n} - Y_s^{\varphi_m}|^2 + \nu_R)^{\frac{2-\beta}{2}}} \, ds$$

$$\leqslant \frac{\ell}{\beta-1} e^{C_N \delta'} \Bigg\{ (A_N)^{\frac{-\beta}{2}} + \beta \frac{2K_3^{\frac{1}{\bar{\alpha}}}}{N^\kappa} (4TK_2 + T\ell)^{\frac{\beta-1}{2}} (8TK_2 + 8K_1)^{\frac{\kappa}{2}}$$

$$+ \mathbb{E}\big|Y_{T'}^{\varphi_n} - Y_{T'}^{\varphi_m}\big|^\beta + \beta\big[2N^2 + \nu_1\big]^{\frac{\beta-1}{2}} \big[\rho_N(\varphi_n - \varphi) + \rho_N(\varphi_m - \varphi)\big] \Bigg\}$$

$$\leqslant \frac{\ell}{\beta-1} e^{C_N \delta'} \mathbb{E}\big|Y_{T'}^{\varphi_n} - Y_{T'}^{\varphi_m}\big|^\beta + \frac{\ell}{\beta-1} \frac{A_N^{\frac{2M_2^2 \delta' \beta}{\beta-1}}}{(A_N)^{\frac{\beta}{2}}}$$

$$+ \frac{2\ell}{\beta-1} \beta K_3^{\frac{1}{\bar{\alpha}}} (4TK_2 + T\ell)^{\frac{\beta-1}{2}} (8TK_2 + 8K_1)^{\frac{\kappa}{2}} \frac{A_N^{\frac{2M_2^2 \delta' \beta}{\beta-1}}}{(A_N)^{\frac{\kappa}{r}}}$$

$$+ \frac{2\ell}{\beta-1} e^{C_N \delta'} \beta\big[2N^2 + \nu_1\big]^{\frac{\beta-1}{2}} \big[\rho_N(\varphi_n - \varphi) + \rho_N(\varphi_m - \varphi)\big].$$

Hence for $\delta' < (\beta-1)\min(\frac{1}{4M_2^2}, \frac{\kappa}{2rM_2^2\beta})$ we derive

$$\frac{A_N^{\frac{2M_2^2 \delta' \beta}{\beta-1}}}{(A_N)^{\frac{\beta}{2}}} \underset{N\to\infty}{\longrightarrow} 0$$

and

$$\frac{A_N^{\frac{2M_2^2 \delta' \beta}{\beta-1}}}{(A_N)^{\frac{\kappa}{r}}} \underset{N\to\infty}{\longrightarrow} 0.$$

Passing to the limits first on n and next on N, and using assertion (c) of Lemma 4.4. □

Remark 4.1. To deal with the case which take account of the process v_t appearing in assumption (H.3), it suffices to take $\Phi(s) := |Y_s^1| + |Y_s^2| + |Z_s^1| + |Z_s^2| + v_s$ in the proof of Lemma 4.7.

Proof of Theorem 3.1. Taking successively $T' = T$, $T' = (T - \delta')^+$, $T' = (T - 2\delta')^+ \ldots$ in Lemma 4.7, we obtain, for every $\beta \in]1, \min(3 - \frac{2}{\bar{\alpha}}, 2)[$

$$\lim_{n,m\to+\infty}\left(\mathbb{E}\sup_{0\leqslant t\leqslant T}\left|Y_t^{\varphi_n} - Y_t^{\varphi_m}\right|^{\beta} + \mathbb{E}\int_0^T \frac{|Z_s^{\varphi_n} - Z_s^{\varphi_m}|^2}{(|Y_s^{\varphi_n} - Y_s^{\varphi_m}|^2 + \nu_R)^{\frac{2-\beta}{2}}}\,ds\right) = 0.$$

But by the Schwarz inequality we have

$$\mathbb{E}\int_0^T \left|Z_s^{\varphi_n} - Z_s^{\varphi_m}\right| ds \leqslant \left(\mathbb{E}\int_0^T \frac{|Z_s^{\varphi_n} - Z_s^{\varphi_m}|^2}{(|Y_s^{\varphi_n} - Y_s^{\varphi_m}|^2 + \nu_R)^{\frac{2-\beta}{2}}}\,ds\right)^{\frac{1}{2}} \times \left(\mathbb{E}\int_0^T \left(\left|Y_s^{\varphi_n} - Y_s^{\varphi_m}\right|^2 + \nu_R\right)^{\frac{2-\beta}{2}} ds\right)^{\frac{1}{2}}.$$

Since $\beta > 1$, Lemma 4.5 allows us to show that

$$\lim_{n\to+\infty}\left(\mathbb{E}\sup_{0\leqslant t\leqslant T}\left|Y_t^{\varphi_n} - Y_t\right|^{\beta} + \mathbb{E}\int_0^T \left|Z_s^{\varphi_n} - Z_s\right| ds\right) = 0.$$

In particular, there exists a subsequence, which we still denote $(Y^{\varphi_n}, Z^{\varphi_n})$, such that

$$\lim_{n\to+\infty}\left(\left|Y_t^{\varphi_n} - Y_t\right| + \left|Z_t^{\varphi_n} - Z_t\right|\right) = 0 \quad \text{a.e. } (t, \omega).$$

On the other hand

$$\begin{aligned}
&\mathbb{E}\int_0^T \left|\varphi_n\left(s, Y_s^{\varphi_n}, Z_s^{\varphi_n}\right) - \varphi\left(s, Y_s^{\varphi_n}, Z_s^{\varphi_n}\right)\right| ds \\
&\quad\leqslant \mathbb{E}\int_0^T \left|\varphi_n\left(s, Y_s^{\varphi_n}, Z_s^{\varphi_n}\right) - f\left(s, Y_s^{\varphi_n}, Z_s^{\varphi_n}\right)\right| \mathbb{1}_{\{|Y_s^{\varphi_n}|+|Z_s^{\varphi_n}|\leqslant N\}}\,ds \\
&\qquad + \mathbb{E}\int_0^T \left|\varphi_n\left(s, Y_s^{\varphi_n}, Z_s^{\varphi_n}\right) - f\left(s, Y_s^{\varphi_n}, Z_s^{\varphi_n}\right)\right| \frac{(|Y_s^{\varphi_n}| + |Z_s^{\varphi_n}|)^{(2-\frac{2}{\bar{\alpha}})}}{N^{(2-\frac{2}{\bar{\alpha}})}} \mathbb{1}_{\{|Y_s^{\varphi_n}|+|Z_s^{\varphi_n}|\geqslant N\}}\,ds \\
&\quad\leqslant \rho_N(\varphi_n - \varphi) + \frac{2K_3^{\frac{1}{\bar{\alpha}}}[TK_2 + K_1]^{1-\frac{1}{\bar{\alpha}}}}{N^{(2-\frac{2}{\bar{\alpha}})}}.
\end{aligned}$$

Passing to the limit first on n and next on N we obtain

$$\lim_n E\int_0^T \left|\varphi_n\left(s, Y_s^{\varphi_n}, Z_s^{\varphi_n}\right) - \varphi\left(s, Y_s^{\varphi_n}, Z_s^{\varphi_n}\right)\right| ds = 0.$$

Finally, we use (H.1), Lemma 4.4 and Lemma 4.5 to show that,

$$\lim_n E\int_0^T \left|\varphi_n\left(s, Y_s^{\varphi_n}, Z_s^{\varphi_n}\right) - \varphi(s, Y_s, Z_s)\right| ds = 0.$$

The existence is proved.

Uniqueness. Let (Y, Z) and (Y', Z') be two solutions of equation (E^f). Arguing as previously one can show that:

for every $R > 2$, $\beta \in]1, \min(3 - \frac{2}{\bar{\alpha}}, 2)[$, $\delta' < (\beta - 1)\min(\frac{1}{4M_2^2}, \frac{3-\frac{2}{\bar{\alpha}}-\beta}{2rM_2^2\beta})$ and $\varepsilon > 0$ there exists $N_0 > R$ such that for all $N > N_0$, $\forall T' \leqslant T$

$$\mathbb{E}\sup_{(T'-\delta')^+\leqslant t\leqslant T'}\left|Y_t - Y_t'\right|^\beta + \mathbb{E}\int_{(T'-\delta')^+}^{T'} \frac{|Z_s - Z_s'|^2}{(|Y_s - Y_s'|^2 + \nu_R)^{\frac{2-\beta}{2}}} ds$$
$$\leqslant \varepsilon + \frac{\ell}{\beta - 1} e^{C_N\delta'}\mathbb{E}\left|Y_{T'} - Y_{T'}'\right|^\beta.$$

Again, taking successively $T' = T$, $T' = (T - \delta')^+$, $T' = (T - 2\delta')^+ \ldots$, we establish the uniqueness of solution. Theorem 3.1 is proved. □

5. Application to stochastic and control and stochastic games

In the sequel $\Omega = \mathcal{C}([0, T], \mathbb{R}^m)$ is the space of continuous functions from $[0, T]$ to $\mathbb{R}^m$.

Let us consider a mapping $\sigma : (t, w) \in [0, T] \times \Omega \to \sigma(t, w) \in \mathbb{R}^m \otimes \mathbb{R}^m$ satisfying the following assumptions:

(1.1) σ is $\mathcal{P}$-measurable.

(1.2) There exists a constant C such that $|\sigma(t, w) - \sigma(t, w')| \leqslant C\|w - w'\|_t$ and $|\sigma(t, w)| \leqslant C(1 + \|w\|_t)$, where for any $w, w' \in \Omega^2$ and $t \leqslant T$, $\|w\|_t = \sup_{s\leqslant t} |w_s|$.

(1.3) For any $(t, w) \in [0, T] \times \Omega$, the matrix $\sigma(t, w)$ is invertible and $|\sigma^{-1}(t, w)| \leqslant C$ for some constants C.

Let $x_0 \in \mathbb{R}^m$ and $x = (x_t)_{t\leqslant T}$ be the solution of the following standard functional differential equation:

$$x_t = x_0 + \int_0^t \sigma(s, x)\, dB_s, \quad t \leqslant T; \tag{5.1}$$

the process $(x_t)_{t\leqslant T}$ exists, since σ satisfies (1.1)–(1.3) (see, e.g., [12, p. 375]). Moreover,

$$\mathbb{E}\left[\left(\|x\|_T\right)^n\right] < +\infty, \quad \forall n \in [1, +\infty[\ \left(\text{see [10, p. 306]}\right). \tag{5.2}$$

5.1. *Stochastic control of diffusions*

Let A be a compact metric space and $\mathcal{U}$ be the space of $\mathcal{P}$-measurable processes $u := (u_t)_{t\leqslant T}$ with value in A. Let $f : [0, T] \times \Omega \times A \to \mathbb{R}^m$ be such that:

(1.4) For each $a \in A$, the function $(t, w) \to f(t, w, a)$ is predictable.
(1.5) For each (t, w), the mapping $a \to f(t, w, a)$ is continuous.
(1.6) There exists a real constant $K > 0$ such that

$$\left|f(t, w, a)\right| \leqslant K\left(1 + \|w\|_t\right), \quad \forall 0 \leqslant t \leqslant T,\ w \in \Omega,\ a \in A. \tag{5.3}$$

Under the previous assumptions, for a given admissible control strategy $u \in \mathcal{U}$, the exponential process,

$$\Lambda_t^u := \exp\left\{\int_0^T \sigma^{-1}(s, x) f(s, x, u_s)\, dB_s - \frac{1}{2}\int_0^T \left|\sigma^{-1}(s, x) f(s, x, u_s)\right|^2 ds\right\},$$
$$0 \leqslant t \leqslant T,$$

is a martingale. Therefore, $\mathbb{E}[\Lambda_T^u] = 1$ (see Karatzas and Shreve [10, pp. 191 and 200] for this result). The Girsanov theorem guarantees then that the process

$$B_t^u = B_t - \int_0^t \sigma^{-1}(s, x) f(s, x, u_s)\, ds, \quad 0 \leqslant t \leqslant T, \tag{5.4}$$

is a Brownian motion with respect to the filtration $\mathcal{F}_t$, under the new probability measure

$$P^u(B) = \mathbb{E}\left[\Lambda_T^u.\mathbf{1}_B\right], \quad B \in \mathcal{F}_T,$$

which is equivalent to P. It is now clear from Eqs. (5.1) and (5.15) that

$$x_t = x_0 + \int_0^t f(s, x, u_s)\, ds + \int_0^t \sigma(s, x)\, dB_s^u, \quad 0 \leqslant t \leqslant T, \tag{5.5}$$

holds almost surely. This will be our model for a controlled stochastic functional differential equation, with the control appearing only in the drift term.

In order to specify the objective of our stochastic game of control and stopping. Let us now consider the following assumptions,

(1.7) $h : [0, T] \times \Omega \times A \to \mathbb{R}$ is measurable and for each (t, w) the mapping $a \to h(t, w, a)$ is continuous. In addition there exists a positive constant $K > 0$ such that

$$\left|h(t, w, a)\right| \leqslant K\left(1 + \|w\|_t\right), \quad \forall 0 \leqslant t \leqslant T,\ w \in \Omega,\ a \in A. \tag{5.6}$$

(1.8) $g_1 : [0, T] \times \Omega \to \mathbb{R}$ is continuous function and there exists a positive constant C such that:

$$\left|g_1(t, w)\right| \leqslant C\left(1 + \|w\|_t\right), \quad \forall (t, w) \in [0, T] \times \Omega. \tag{5.7}$$

We shall study a stochastic control with one player. The controller, who chooses an admissible control strategy $u \in \mathcal{U}$ to minimize this amount

$$\int_0^T h(s, x, u_s)\, ds + g_1(T, x_T). \tag{5.8}$$

It is thus in the interest of the controller to make the amount (5.8) as small as possible, at least on the average. We are then led to a stochastic control, with

$$J(u)=\mathbb{E}^u\left[\int_0^T h(s,x,u_s)\,ds+g_1(T,x_T)\right]. \tag{5.9}$$

The problem we are interested in is finding an intervention strategies u^*, for controller such that for any $u\in\mathcal{U}$, we have

$$J\big(u^*\big)\leqslant J(u).$$

Then u^* is called an optimal control for the problem. Now let us set

$$\begin{aligned}&H(t,x,z,u_t):=z\sigma^{-1}(t,x)f(t,x,u_t)+h(t,x,u_t),\\&\quad\forall(t,x,z,u_t)\in[0,T]\times\Omega\times\mathbb{R}^m\times A.\end{aligned} \tag{5.10}$$

The function H is called the Hamiltonian associated with stochastic control such that:

(2.1) $\forall z\in\mathbb{R}^m$, the process $(H(t,x,z,u_t))_{t\leqslant T}$ is $\mathcal{P}$-measurable.

Lemma 5.1. *The Hamiltonian H satisfies* (H.2) *and* (H.3).

Proof. For (H.2), it is not difficult to show that for every $(t,x,z,u_t)\in[0,T]\times\Omega\times\mathbb{R}^m\times A$ and $|z|$ large enough, there exist a constants C and c_0 such that:

$$\big|H(t,x,z,u_t)\big|\leqslant C\exp\big(\|x\|_t\big)+c_0|z|\sqrt{\big|\ln\big(|z|\big)\big|}. \tag{5.11}$$

To prove that H satisfies assumption (H.3), it is enough to take $v_t:=\exp(|f(t,x,u_t)|^2)$. Indeed, we have

$$\begin{aligned}&\big(y-y'\big)\big(H(t,x,z,u_t)-H\big(t,x,z',u_t\big)\big)\mathbb{1}_{\{e^{(|f(t,x,u_t)|^2)}\leqslant N\}}\\&\quad\leqslant\big|y-y'\big|\big|z-z'\big|\big|f(t,x,u_t)\big|\mathbb{1}_{\{|f(t,x,u_t)|^2\leqslant\ln N\}}\\&\quad\leqslant\big|y-y'\big|\big|z-z'\big|\sqrt{\ln A_N}.\end{aligned}$$

To complete the proof, we shall show that $\exp(|f(t,x,u_t)|^2)$ belongs to $L^q(\Omega\times[0,T];\mathbb{R}_+)$ for some $q>0$. We have,

$$\begin{aligned}\mathbb{E}\int_0^T\exp\big(q\big|f(s,x,u_s)\big|^2\big)\,ds&\leqslant\mathbb{E}\int_0^T\exp\Big(2qK^2\Big(1+\sup_{s\leqslant T}|x_s|^2\Big)\Big)\,ds\\&\leqslant\exp\big(2qK^2T\big)\mathbb{E}\int_0^T\exp\Big(2qK^2\sup_{s\leqslant T}|x_s|^2\Big)\,ds\\&\leqslant T\exp\big(2qK^2T\big)\mathbb{E}\exp\Big(2qK^2\sup_{s\leqslant T}|x_s|^2\Big)\end{aligned}$$

and, since σ is with linear growth, it is well known that $E\exp(2qK^2\sup_{s\leqslant T}|x_s|^2)<\infty$ for q small enough. □

Let us define the notion of solution of the reflected BSDE associated with the triple (H, g_2, g_1) which we consider throughout this paper.

In order to construct a stochastic control, we need to do is find an admissible control strategy $u^*() \in \mathcal{U}$ for our stochastic control.

The Hamiltonian function defined in (5.10) attains its infimum over the set A at some $u^* \equiv u^*(t,x,p) \in A$, for any given $(t,x,p) \in [0,T] \times \Omega \times \mathbb{R}^m$, namely,

$$\inf_{u \in A} H(t,x,u,p) = H\big(t,x,u^*(t,x,p),p\big). \tag{5.12}$$

(This is the case, for instance, if the set A is compact and the mapping $u \mapsto H(t,x,u,p)$ continuous.) Then it can be shown (see Lemma 1 in Benes [5]), that the mapping $u^* : [0;T] \times \Omega \times \mathbb{R}^m \mapsto A$ can be selected to be $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^m)$-measurable.

Now let $H^*(t,x,z) = \inf_{u \in A} H(t,x,u,z)$ where x is the solution of (5.1). Let $(Y_t)_{t \leqslant T}$ be the process constructed as in Theorem 3.1 with (H^*, g_1). Using once again Theorem 3.1, there exists a unique pair $(Y_t, Z_t)_{t \leqslant T}$ such that

$$\begin{cases} (Y,Z) \in \big(\mathbb{E}, \|.\|\big); \\ Y_t = g_1(T, x_T) + \displaystyle\int_t^T H^*(s,x,Z_s)\,ds - \int_t^T Z_s\,dB_s, \quad t \in [0,T]. \end{cases} \tag{5.13}$$

We are ready to give the main result of this section.

Theorem 5.1. *The admissible control u^* is optimal for the stochastic control*; *i.e., it satisfies*

$$J\big(u^*\big) = Y_0 \leqslant J(u), \quad \forall u \in \mathcal{U}.$$

Additionally, Y_0 is the value of the stochastic control, i.e.

$$Y_0 = \inf_{u \in \mathcal{U}} J(u).$$

Proof. Let us show that $Y_0 = J(u^*)$. It follows that

$$\begin{aligned} Y_0 &= g_1(T,x_T) + \int_0^T H^*(s,x,Z_s)\,ds - \int_0^T Z_s\,dB_s \\ &= g_1(T,x_T) + \int_0^T h\big(s,x,u^*(s,x,Z_s)\big)\,ds - \int_0^T Z_s\,dB_s^{u^*}. \end{aligned}$$

Since $(\int_0^t Z_s\,dB_s)_{t \leqslant T}$ is an $(\mathbb{F}_t, P^{u^*})$-martingale, then taking expectation we get

$$Y_0 = \mathbb{E}^{u^*}[Y_0] = \mathbb{E}^{u^*}\left[g_1(T,x_T) + \int_0^T h\big(s,x,u^*(s,x,Z_s)\big)\,ds\right],$$

because Y_0 is $\mathbb{F}_0$-measurable, and hence deterministic. Now P-a.s., and also P^{u^*}-a.s. (since they are equivalent probabilities), we have $Y_0 = J(u^*)$.

Let us show that $Y_0 \leqslant J(u)$ for $u \in \mathcal{U}$. We have,

$$\begin{aligned} Y_0 &= g_1(T, x_T) + \int_0^T H^*(s, x, Z_s)\,ds - \int_0^T Z_s\,dB_s \\ &\leqslant g_1(T, x_T) + \int_0^T H(s, x, u, Z_s)\,ds - \int_0^T Z_s\,dB_s \\ &= g_1(T, x_T) + \int_0^T h\big(s, x, u(s, x, Z_s)\big)\,ds - \int_0^T Z_s\,dB_s^u. \end{aligned}$$

Once more $(\int_0^t Z_s\,dB_s)_{t \leqslant T}$ is an $(\mathbb{F}_t, P^u)$-martingale, then taking the expectation with respect to P^u and taking into account the fact that Y_0 is deterministic, we obtain

$$Y_0 = \mathbb{E}^u[Y_0] \leqslant \mathbb{E}^u\left[g_1(T, x_T) + \int_0^T h\big(s, x, u(s, x, Z_s)\big)\,ds \right],$$

then $Y_0 \leqslant J(u)$. Theorem 5.1 is proved. □

5.2. *Stochastic zero-sum differential games*

Let A (resp. B) be a compact metric space and $\mathcal{U}$ (resp. $\mathcal{V}$) be the space of $\mathcal{P}$-measurable processes $u := (u_t)_{t \leqslant T}$ (resp. $v := (v_t)_{t \leqslant T}$) with value in A (resp. B). Let $f : [0, T] \times \Omega \times A \times B \to \mathbb{R}^m$ be such that:

(1.4′) For each $a \in A$ and $b \in B$, the function $(t, x) \to f(t, w, a, b)$ is predictable.
(1.5′) For each (t, w), the mapping $(a, b) \to f(t, w, a, b)$ is continuous.
(1.6′) There exists a real constant $K > 0$ such that

$$\big|f(t, w, a, b)\big| \leqslant K\big(1 + \|w\|_t\big), \quad \forall 0 \leqslant t \leqslant T,\ w \in \Omega,\ a \in A,\ b \in B. \tag{5.14}$$

For any given admissible control strategy $(u, v) \in \mathcal{U} \times \mathcal{V}$, the exponential process

$$\Lambda^{(u,v)} = \exp\left\{ \int_0^T \sigma^{-1}(s, x) f(s, x, u_s, v_s)\,dB_s - \frac{1}{2} \int_0^T \big|\sigma^{-1}(s, x) f(s, x, u_s, v_s)\big|^2\,ds \right\}, \quad 0 \leqslant t \leqslant T$$

is a martingale under all these assumptions, namely, $\mathbb{E}[\Lambda_T^{(u,v)}] = 1$ (see Karatzas and Shreve [10, pp. 191 and 200] for this result). Then the Girsanov theorem guarantees that the process

$$B^{(u_t, v_t)} = B_t - \int_0^t \sigma^{-1}(s, x) f(s, x, u_s, v_s)\,ds, \quad 0 \leqslant t \leqslant T, \tag{5.15}$$

is a Brownian motion with respect to the filtration $\mathcal{F}_t$, under the new probability measure $P^{(u,v)}(B) = \mathbb{E}[\Lambda_T^{(u,v)}.\mathbf{1}_B]$, $B \in \mathcal{F}_T$, which is equivalent to P. It is now clear from Eqs. (5.1) and (5.15) that

$$x_t = x_0 + \int_0^t f(s, x, u_s, v_s)\, ds + \int_0^t \sigma(s, x)\, dB^{(u_s, v_s)}, \quad 0 \leqslant t \leqslant T, \tag{5.16}$$

holds almost surely.

Let us now consider the following:

(1.7′) $h : [0, T] \times \Omega \times A \times B \to \mathbb{R}$ is measurable and for each (t, w) the mapping $(a, b) \to h(t, w, a, b)$ is continuous. In addition there exists a real constant $K > 0$ such that

$$\left|h(t, w, a, b)\right| \leqslant K\left(1 + \|w\|_t\right), \quad \forall 0 \leqslant t \leqslant T,\ w \in \Omega,\ a \in A,\ b \in B. \tag{5.17}$$

(1.8′) $g_1 : [0, T] \times \mathbb{R}^m \to \mathbb{R}$ is continuous function and there exists a real positive constant C such that

$$\left|g_1(t, w)\right| \leqslant C\left(1 + \|w\|_t\right), \quad \forall (t, w) \in [0, T] \times \mathbb{R}^m. \tag{5.18}$$

The payoff corresponding to $u \in \mathcal{U}$ and $v \in \mathcal{V}$ is given by

$$J(u, v) := \mathbb{E}^{u,v}\left[\int_0^T h(s, x, u_s, v_s)\, ds + g_1(T, x_T)\right], \tag{5.19}$$

where $u \in \mathcal{U}$ (resp. $v \in \mathcal{U}$) is the strategy of the first (resp. second) player. The first player looks for minimize $J(u, v)$, while the second looks for maximize the same $J(u, v)$. We are concerned by the problem of the existence of a saddle-point for this game, i.e. the existence of an admissible control (u^*, v^*) which satisfies:

$$J\left(u^*, v\right) \leqslant J\left(u^*, v^*\right) \leqslant J\left(u, v^*\right), \quad (u, v) \in \mathcal{U} \times \mathcal{V}.$$

The Hamiltonian function is defined, for $(t, x, z, u_t, v_t) \in [0, T] \times \Omega \times \mathbb{R}^m \times A \times B$, by

$$H(t, x, z, u_t, v_t) = z\sigma^{-1}(t, x) f(t, x, u_t, v_t) + h(t, x, u_t, v_t). \tag{5.20}$$

We moreover assume that the following Isaacs' condition is satisfied

(H) for every $(t, x, p) \in [0, T] \times \Omega \times \mathbb{R}^m$, $\sup_{v \in \mathcal{V}} \inf_{u \in \mathcal{U}} H(t, x, p, u, v) = \inf_{u \in \mathcal{U}} \sup_{v \in \mathcal{V}} H(t, x, p, u, v)$

and

(2.1′) for every $z \in \mathbb{R}^m$, the process $(H(t, x, z, u_t, v_t))_{t \leqslant T}$ is $\mathcal{P}$-measurable.

Using a selection theorem [5] we easily get the following

Lemma 5.2. (H) *is equivalent to the following assumption*:

There exists $u^*(t, x, p), v^*(t, x, p)$ $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^m)$-measurable valued respectively in $\mathcal{U}$ and $\mathcal{V}$ such that for every (u, v, t, x, p)

$$\begin{aligned} H\left(t, x, p, u^*(t, x, p), v(t, x, p)\right) &\leqslant H\left(t, x, p, u^*(t, x, p), v^*(t, x, p)\right) \\ &\leqslant H\left(t, x, p, u(t, x, p), v^*(t, x, p)\right). \end{aligned}$$

Moreover u^* and v^* satisfy

$$H\big(t,x,p,u^*(t,x,p),v^*(t,x,p)\big)=\sup_{v\in\mathcal{V}}\inf_{u\in\mathcal{U}}H(t,x,p,u,v)=\inf_{u\in\mathcal{U}}\sup_{v\in\mathcal{V}}H(t,x,p,u,v).$$

Lemma 5.3. *The Hamiltonian H satisfies* (H.2) *and* (H.3).

The proof of the following results are similar to those of Lemma 5.1.

Proposition 5.1. 1) *For all $(u,v)\in\mathcal{U}\times\mathcal{V}$, let $(Y^{u,v},Z^{u,v})$ be the solution of the BSDE with the generator $(H(t,x,p,u_t,v_t), g_1(T,x_T)$ then $J(u,v)=Y_0^{u,v}$.*

2) *Similarly, let (Y^*,Z^*) be the solution of BSDE with generator $(H(t,x,p,u^*(t,x,p), v^*(t,x,p)), g_1(T,x_T))$ and define $(\widetilde{u},\widetilde{v})\in\mathcal{U}\times\mathcal{V}$ by $(\widetilde{u},\widetilde{v})=(u^*(t,x,Z_t^*),v^*(t,x,Z_t^*))_{t\leqslant T}$, then $J(\widetilde{u},\widetilde{v})=Y_0^*$.*

Theorem 5.2. *The strategy $(\widetilde{u},\widetilde{v})$ is a saddle-point for the game.*

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