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On the connectivity of complex affine hypersurfaces, II

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Abstract

We obtain sharp estimates on the connectivity of complex affine hypersurfaces in terms of the decomposition of the defining equation as a sum of weighted homogeneous components relative to some weight system. © 2000 Elsevier Science Ltd. All rights reserved.

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Let $f \in \mathbb{C}[x_0, x_1, \dots, x_n]$ be a polynomial and $\mathbf{w} = (w_0, \dots, w_n)$ a system of positive integer weights. Consider $f = f_0 + \cdots + f_e + f_d$ the decomposition of f as a sum of weighted homogeneous components f_i where $deg(f_i) = i$ with respect to $\mathbf{w}$. Here f_d is the top degree non-zero component and we assume that $f_j = 0$ for $e < j < d$ for some integer $e > 0$.

For any polynomial g we denote by ∂g its gradient

$$\partial g = (g_{x_0}, \dots, g_{x_n}),$$

where g_{x_i} is the partial derivative of g with respect to x_i.

Define the subset $S(f)$ in $\mathbb{C}^{n+1}$ given by the equations $\partial f_d = 0, f_e = 0$. With this notation our main result is the following.

Theorem. (i) *Let* $\Sigma(f) = \{x \in \mathbb{C}^{n+1}; \partial f = 0\}$ *be the set of singular points of the polynomial* f. *Then* $\dim \Sigma(f) \leqslant \dim S(f)$.

(ii) *Any fiber of the polynomial* f *is* $(n - 1 - \dim S(f))$-*connected.*

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Improving previous results on the connectivity of affine hypersurfaces, the first author has proved part (ii) of this result in the case of the usual weights $\mathbf{w} = (1,1,\dots,1)$ in [5].

The proof there was based on the properties of tame and quasi-tame polynomials introduced by Broughton [4], Némethi [15,16] and Némethi-Zaharia [19]. More precisely, the general case was reduced to these special classes of polynomials by applying an affine Lefschetz hyperplane section theorem due to Hamm [10]. This proof cannot be applied for arbitrary weights since in general we do not have enough weighted hyperplanes to proceed by induction (even though Lemma 9 in [5] can be easily extended to show that the polynomial f is quasi-tame when $\dim S(f) = 0$).

On the other hand, Némethi and Sabbah [18] have recently studied tame polynomials defined on affine varieties, but their definition does not seem easily adaptable to keep track of the chosen weights.

Our proof below is based on a powerful general result by Hamm, namely Proposition 3 in [10] (which is the key step in his proof of the affine Lefschetz theorem on hyperplane sections and which we slightly correct in Lemma 3 below) and on basic properties of isolated complete intersection singularities, for which we refer the reader to Looijenga's book [12].

The following example shows the strength of the new result over the old.

Example 1. Let $f\colon \mathbb{C}^6 \to \mathbb{C}$ be the polynomial given by

$$f(x) = x_0^3x_1^3x_2^3 + (x_0 + x_1 + x_2)^7 + x_3^5 + x_4^4 + x_5^3 + (x_0 + x_1 + x_2 + x_3 + x_4 + x_5)^2.$$

If we consider the usual weights $\mathbf{w} = (1,1,\dots,1)$, then $\dim S(f) = 4$, and hence we get that the fibers of f are 0-connected, i.e. they are connected. In this case $d = 9$, $e = 7$.

Now, if we use the system of weights $\mathbf{w} = (20,20,20,36,45,60)$ chosen such that the top degree form f_d contains as many monomials as possible, then $\dim S(f) = 1$ hence the fibers of f are in fact 3-connected. In this case $d = 180$, $e = 140$.

It is interesting to notice that the topological result above has some useful algebraic consequences expressed in terms of various complexes of differential forms associated naturally to the polynomial f.

To state them, let $A^* = (\Omega^*, \mathrm{d})$ denote the De Rham complex of global regular differential forms on $\mathbb{C}^{n+1}$ with d the exterior differentiation acting on forms (not to be confused to the degree d which occurred above).

The first complex associated to f is the complex $K_f^* = (\Omega^*, \mathrm{d}f\wedge)$ which can be identified to the Koszul complex of the partial derivatives of f in the polynomial ring $\mathbb{C}[x_0,\dots,x_n] = \Omega^0$.

The De Rham complex A^* has a natural subcomplex $B_f^* = (\mathrm{d}f\wedge\Omega^*, \mathrm{d})$ and a natural quotient complex $C_f^* = A^*/B_f^*$ called the complex of global relative differential forms.

Finally, one can consider as in Dimca-Saito [7,8], the mixture of De Rham and Koszul complexes, namely the complex $D_f^* = (\Omega^*, \mathrm{d} - \mathrm{d}f\wedge)$.

Corollary. *Let f be a polynomial as above and assume that $k = \dim S(f) < n$. Then*

(i) $H^i(K_f^*) = 0$ *for all* $0 \leqslant i \leqslant n - k$;
(ii) $H^1(B_f^*) = \mathbb{C}[f]\mathrm{d}f$ *and* $H^i(B_f^*) = 0$ *for all* $i \neq 1, 0 \leqslant i \leqslant n - k$;
(iii) $H^0(C_f^*) = \mathbb{C}[f]$ *and* $H^i(C_f^*) = 0$ *for all* $0 < i \leqslant n - k - 1$;
(iv) $H^i(D_f^*) = 0$ *for all* $0 \leqslant i \leqslant n - k$.

Proof of the Theorem. Let N be any common multiple of the weights w_i and consider the integers $m_i = N/w_i$. Let $\phi: \mathbb{C}^{n+1} \to \mathbb{C}^{2n+2}$ be the embedding

$$\phi(x) = (x_0, \dots, x_n, x_0^{m_0}, \dots, x_n^{m_n}).$$

Let $X = f^{-1}(0)$. It is enough to prove that

(i′) $\dim X_{\mathrm{sing}} \leqslant \dim S(f)$;
(ii′) X is $(n-1-\dim S(f))$ connected.

Let $Y = \phi(X)$ and note that $\phi: X \to Y$ is an isomorphism. Let $k = \dim S(f)$ and let H_1: $l_1 = 0, \dots, H_k: l_k = 0$ be generic hyperplanes in $\mathbb{C}^{2n+2}$.

Using repeatedly Lefschetz hyperplane section theorem in [10] we get that the inclusion $Y_0 = Y \cap H_1 \cdots \cap H_k \to Y$ is an $(n-k)$-equivalence.

Let $X_0 = \phi^{-1}(Y_0)$ and $g_i(x) = l_i(\phi(x))$ be the corresponding pull-back polynomials on $\mathbb{C}^{n+1}$. Note that the weighted homogeneous component $g_{i,N}$ is a polynomial of Pham-Brieskorn type, i.e. a sum $\sum a_{ij} x_j^{m_j}$.

Lemma 1. (a) $A = \{x \in \mathbb{C}^{n+1}; g_1(x) = \cdots = g_k(x) = 0\}$ *is a smooth complete intersection of dimension* $n+1-k$;

(b) *the set* $\{x \in \mathbb{C}^{n+1}; x \in S(f), g_{1,N}(x) = \cdots = g_{k,N}(x) = 0\}$ *is just the origin*;

(c) *there exist weighted homogeneous polynomials* $h_1, \dots, h_n$ *such that the germ at the origin of the set* $\{x \in \mathbb{C}^{n+1}; g_{1,N}(x) = \cdots = g_{k,N}(x) = f_d(x) = h_1(x) = \cdots = h_j(x) = 0\}$ *is an isolated complete intersection singularity of dimension* $\max(n-k-j, 0)$ *for all* $j = 1, \dots, n$.

(d) $(Y_0)_{\mathrm{sing}} = Y_{\mathrm{sing}} \cap H_1 \cap \cdots \cap H_k$ *and* $\dim (Y_0)_{\mathrm{sing}} = \dim Y_{\mathrm{sing}} - k$.

Proof of Lemma 1. All these claims follow by general transversality arguments involving the parameter space of all the coefficients a_{ij}.

More precisely, to have (a) we take the hyperplanes H_i such that their intersection is transverse to the smooth variety $\phi(\mathbb{C}^{n+1})$.

For (b), consider the composition $\psi = p_2 \circ \phi$, where $p_2: \mathbb{C}^{2n+2} \to \mathbb{C}^{n+1}, (x, y) \mapsto y$ is the projection on the last $(n+1)$ coordinates.

Then $B = \psi(S(f))$ is an algebraic set in $\mathbb{C}^{n+1}$ of dimension at most k. It follows that the intersection of k generic hyperplanes H'_i, $i = 1, \dots, k$, cuts B only at the origin. We can use the coefficients of the hyperplane H'_i to construct our hyperplane H_i, i.e. the corresponding coefficients of the y-variables coincide.

The last claims (c) and (d) follow by similar easy arguments.

Note that (i′) in our Theorem follows from Lemma 1(d) in view of the next result.

Lemma 2. *For any* $a \in \mathbb{C}$ *the variety* $X_a = \{x \in A; f(x) = a\}$ *has at most isolated singularities.*

Proof. If X_a has non-isolated singularities, then an application of the curve selection lemma, see [14], would produce a path $p: (0, \varepsilon) \to (X_a)_{\mathrm{sing}}$ given by a Laurent power series

$$p(t) = c_0 t^s + c_1 t^{s+1} + \cdots,$$

where $c_i \in \mathbb{C}^{n+1}$, $c_0 \neq 0$ and the integer s is strictly negative.

Note that we can reparametrize this path by replacing t with t^N, i.e. we can suppose that all exponents in $p(t)$ with non-zero coefficients are divisible by our weights w_i.

There is a standard $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$ associated with these weights, namely

$$u*x = (u^{w_0}x_0, \ldots, u^{w_n}x_n).$$

We can rewrite our path using this action in the form

$$p(t) = t^s*c_0 + t^{s+1}*c_1 + \cdots ,$$

where the coefficients c_i's and the exponent s are different in general from those considered first above, but which enjoy the same properties, i.e. $c_0 \neq 0$ and $s < 0$.

Next, we have $g_i(p(t)) = g_{i,N}(c_0)t^{sN} + \cdots = 0$ where the dots represent higher-order terms, hence we have $g_{i,N}(c_0) = 0$. In a similar way, we get $\partial f_d(c_0) = 0$.

Due to the Euler relations $kf_k = \sum_{i=0,n} w_i f_{k,x_i}$ we have also $0 = \mathrm{d}f(p(t)) - \sum_{i=0,n} w_i f_{x_i}(p(t))p_i(t) = (d-e)f_e(p(t)) + \cdots = (d-e)f_e(c_0)t^{se} + \cdots$ which implies that $f_e(c_0) = 0$, a contradiction with the property (b) above.

This finishes the proof of the Lemma and of part (i) in our Theorem.

To continue the proof, we need the following result.

Lemma 3. *Let A be a smooth complete intersection in $\mathbb{C}^p$ with $\dim A = m$. Let $f_1, \ldots, f_p$ be polynomials in $\mathbb{C}[x_1, \ldots, x_p]$ such that the following conditions hold*:

(c0) *The set $\{z \in A; |f_1(z)| \leqslant a_1, \ldots, |f_p(z)| \leqslant a_N\}$ is compact for any positive numbers a_j, $j = 1, \ldots, p$.*

(c1) *The connected components of the critical set of the function $f_1 : A \to \mathbb{C}$ are compact.*

(cj) (*for $j = 2, \ldots, p$*) *Let F be the critical set of the mapping $(f_1, \ldots, f_j) : A \to \mathbb{C}^j$ for some j. Then $(f_1, \ldots, f_{j-1}) : F \to \mathbb{C}^{j-1}$ is a proper mapping.*

Then A has the homotopy type of a space obtained from $A_1 = \{x \in A; f_1(x) = 0\}$ by adjoining m-cells.

Proof. Apart from condition (c0) this lemma is a special case of Proposition 3 in Hamm [10]. The following example shows that in fact condition (c0) is necessary. An inspection of the proof in [10] shows that with this extra condition his Proposition 3 and hence our Lemma 3 are true. More precisely, condition (c0) allows one to use the Morse theory for the function $|f_1|^2$ to increase a_1 for some fixed (and very large) $a_2, \ldots, a_p$ as in Hamm's proof. □

Example 2. Let A be $\mathbb{C}^2, f_1 = x^2y + x$ and $f_2 = x$. Then all the assumptions in Proposition 3 in [10] are fulfilled, but not the conclusion, i.e. $\mathbb{C}^2$ cannot be obtained from the non-connected space $f_1^{-1}(0)$ by adjoining 2-cells.

We return to the proof of our Theorem. To prove claim (ii′) we use Lemma 3 for the following data:

The set A is the smooth complete intersection $A = \{x \in \mathbb{C}^{n+1}; g_1(x) = \cdots = g_k(x) = 0\}$, see property (a) above.

The polynomials f_i which appear in Lemma 3 are chosen as follows $f_1 = f$ and $f_j = h_{j-1}$ for $j = 2, \ldots, n+1 = p$, with the polynomials h_j from Lemma 1(c).

These polynomials satisfy the following conditions:

(c0) as follows easily from Lemma 1(c),
(c1) as follows from Lemma 2 and
(cj) as we explain now. In our case, a critical point for $(f_1, \dots, f_j): A \to \mathbb{C}^j$ is just a singular point of the variety Z given by the equations

$$g_1(x) = \cdots = g_k(x) = 0, f(x) = a_1, h_1(x) = a_2, \dots, h_{j-1}(x) = a_j.$$

This variety corresponds to a fiber in a deformation of the isolated complete intersection singularity

$$g_{1,N}(x) = \cdots = g_{k,N}(x) = f_d(x) = h_1(x) = \cdots = h_{j-1}(x) = 0.$$

Indeed, we have the following general localization result, see also [6, pp. 157, 161] (the case discussed there is for the usual weights $\mathbf{w} = (1, \dots, 1)$).

Lemma 4. *Let Z be any affine variety in $\mathbb{C}^{n+1}$ given as the zero set of the polynomials $P_j(x) = Q_j(x) + R_j(x)$ for $j = 1, \dots, m$ where $\deg P_j(x) = \deg R_j(x) = d_j$, $\deg Q_j(x) < d_j$ and the polynomials R_j are weighted homogeneous with respect to the weights $\mathbf{w}$ (the degrees are relative to these weights as well).*

Then Z is homeomorphic to a fiber in a deformation of the weighted homogeneous singularity $(Z_0, 0)$: $R_1(x) = \cdots = R_m(0) = 0$.

Proof. For any real number $r > 0$ we set $y = x/r$ and define

$$Z_r = \{y \in \mathbb{C}^{n+1}; P_j(r * y) = 0 \quad \text{for } j = 1, \dots, m\}.$$

Multiplication by r induces a homeomorphism $x \mapsto r * x$ between $Z_r \cap B_e$ and $Z \cap B_{er}$ where $B_a = \{x \in \mathbb{C}^N; |x_0|^{2m/w_0} + \cdots + |x_n|^{2m/w_n} < a\}$ is a weighted open ball for any $a > 0$.

Note that $Z_r \cap B_e$ is the zero set of the polynomials $P_{j,r}(y) = R_j(y) + Q_j(r * y) r^{-d_j}$. For $r \gg 1/e$ large enough and e small, the following two claims hold.

(a1) $P_{j,r}$ is a small deformation of the polynomial P_j, i.e. $Z_r \cap B_e$ is a fiber in a deformation of the singularity $(Z_0, 0)$.

(a2) Z is homeomorphic to $Z \cap B_{er}$, see [6, p. 26] for a similar result.

This finishes the proof of Lemma 4. □

Back to checking conditions (cj), we see by Lemma 4 (or rather by its proof) that any such variety Z is either smooth or has isolated singularities. Projecting on the space $\mathbb{C}^{j-1}$ means that now we consider a family of fibers as above corresponding to a line in the base of the deformation of our singularity. If h_j is chosen general enough, this line is not contained in the discriminant Δ of this deformation, i.e. there is a finite number of singular fibers when we vary a_j. This follows from the basic fact that a non-constant regular function on a variety with isolated singularities has finitely many critical values.

It follows that the map $(f_1, \dots, f_{j-1}): F \to \mathbb{C}^{j-1}$ is the composition of two finite maps, the first one from F to the discriminant Δ of the deformation and the second one the projection $\Delta \to \mathbb{C}^{j-1}$. This explains why condition (cj) holds as well in our setting for $j > 1$.

By Lemma 3 we have that the inclusion $X_0 \to A$ in an $(n-k)$-equivalence. As above, we can identify A with a smooth fiber (Milnor fiber) in the deformation of the corresponding isolated complete intersection, hence A is $(n-k)$-connected being a bouquet of $(n-k+1)$-spheres.

Combining this with the $(n-k)$-equivalence $X_0 \to X$ obtained at the beginning, we have that X is $(n-k-1)$-connected.

This ends the proof of our theorem. $\square$

Proof of the Corollary. The first claim (i) depends only on part (i) of our theorem. Indeed, the cohomology groups of the Koszul complex are finitely generated $\mathbb{C}[x_0, \dots, x_n]$-modules and to prove that one of them is trivial it is enough to show that all its localizations at maximal ideals are trivial.

To check this local property we can use GAGA and replace algebraic localization by analytic localization. At this level the result follows from Greuel's generalized version of the de Rham Lemma, see [9], (1.7) or from a general result in Looijenga's book [12], namely Corollary (8.16), p. 157 (take X a smooth germ and $k = 1$ in that statement).

To prove (ii) and (iii), we consider the exact sequence of complexes

$$0 \to B_f^* \to A^* \to C_f^* \to 0.$$

This shows that it is enough to prove (ii). We have in fact the following more precise result. By convention, the dimension of the empty set is taken to be -1.

Lemma 5. *Let $f: \mathbb{C}^{n+1} \to \mathbb{C}$ be a non-constant polynomial function. Then the following statements are equivalent for any positive integer k:*

(i) $H^1(B_f) = \mathbb{C}[f]df$ *and* $H^i(B_f) = 0$ *for* $i \neq 1, i \leqslant n-k$;

(ii) *the reduced cohomology groups $\tilde{H}^i(F_t, \mathbb{C})$ are trivial for $0 \leqslant i \leqslant n-k-1$ for F_t the general fiber of the polynomial f.*

Proof. To simplify notation and make it similar to the notation in [7] for easy reference, in this proof we set $X = \mathbb{C}^{n+1}$ and $S = \mathbb{C}$. The algebraic Gauss–Manin system of $f: X \to S$ is the direct image $\mathscr{G}_f = f_+ \mathcal{O}_X[-n-1]$ of the D_X-module $\mathcal{O}_X$, see Borel [3]. We shift it by $(-n-1)$ to get a complex in positive degrees, as it is more usual in algebraic topology.

At the level of global sections on S, the algebraic Gauss–Manin system of f is represented by the complex of $A_1 = \mathbb{C}[t]\langle \partial \rangle$-modules $G_f^* = (\Omega^*[\partial], d_f)$ where the $\mathbb{C}$-linear differential d_f is defined by $d_f(\omega \partial^m) = \mathrm{d}\omega\, \partial^m - \mathrm{d}f \wedge \omega \partial^{m+1}$, see [7,8,20] for more on this complex.

The cohomology sheaves $\mathscr{G}_f^i = \mathscr{H}^i(\mathscr{G}_f)$ are regular holonomic D_S-modules and the Riemann–Hilbert correspondence [3] implies that $DR_S(\mathscr{G}_f^i) = {}^pR^i f_* \mathbb{C}_X$.

Here ${}^pR^i f_* \mathbb{C}_X = {}^p\mathscr{H}^i(f_* \mathbb{C}_X)$ and ${}^p\mathscr{H}^i$ denotes the perverse cohomology functor, see [2]. Basic properties of perverse sheaves gives the following isomorphism:

$${}^pR^i f_* \mathbb{C}_X = R^{i-1} f_* \mathbb{C}_X[1] \tag{$*$}$$

see [20] or [8] for a proof.

Let $R^i = R^i f_* \mathbb{C}_X$. Then the fiber R_y^i is isomorphic to the cohomology group (with $\mathbb{C}$ coefficients) $H^i(T(F_y))$ where $T(F_y)$ is the tube about the fiber $F_y = f^{-1}(y)$ defined by $T(F_y) = f^{-1}(D_y)$, with D_y a small open disc centered at y.

Let Λ be the finite bifurcation set for f, i.e. $f: X\backslash f^{-1}(\Lambda) \to S\backslash \Lambda$ is a locally trivial smooth fibration.

For $y \notin \Lambda$ we then have $R^i_y = H^i(F_y)$ (F_y is the general fiber of f by definition) so condition (ii) implies the vanishing of these stalks for $1 \leqslant i \leqslant n-k-1$. To obtain the vanising of the whole sheaf R^i we need the following vanishing result:

$$H^0_\Lambda(S, R^i) = 0$$

for all $i > 0$, i.e. there are no sections in these sheaves with supports only at the finite set Λ. To see this, note that the vector space $H^0_\Lambda(S, R^i)$ is a subspace of $H^0(S, R^i)$ which vanishes due to the E_2-term in the Leray Spectral sequence of $f: X \to S$. The only non-trivial fact here is that $E_2^{p,q} = H^p(S, R^q) = 0$ for $p \notin \{0,1\}$ by Artin vanishing theorem, see [23], XIV, 3.2 and XVI, 4.1.

In this way (ii) implies the following condition:

(ii′) $R^0 f_* \mathbb{C}_X = \mathbb{C}_S$ *and* $R^i f_* \mathbb{C}_X = 0$

for $0 < i < n-k$.

An alternative, more geometric way to prove implication (ii) ⇒ (ii′) is to use the same idea as in Lemma 1 in [1].

The converse implication is obvious, so (ii′) is equivalent to (ii).

Since the de Rham functor is faithful (i.e. $DR_S(M) = 0$ iff $M = 0$) it follows easily that condition (ii′) above is equivalent to

(ii″) $H^1(G^*_f) = \mathbb{C}[f]\mathrm{d}f$ *and* $H^i(G^*_f) = 0$ *for* $i \neq 1, i \leqslant n-k$.

The complex G^*_f comes equipped with a decreasing filtration given by

$$F^s G^m_f = \Omega^m[\partial]_{\leqslant m-s},$$

where the filtration on the right-hand side is by the degree with respect to ∂ (Sabbah prefers to work with a similar but increasing filtration in [10]).

The general theory of spectral sequences, see if necessary [13], associates to this decreasing, exhaustive and bounded below filtration a spectral sequence with $E_1^{s,t} = H^{s+t}(Gr^s_F G^*_f)$ converging to $H^{s+t}(G^*_f)$.

For $t > 0$, we have $E_1^{s,t} = H^{s+t}(K^*_f)$ and hence in particular in our case we have $E_1^{s,t} = 0$ for all $t > 0, s+t \leqslant n-k$ by our Corollary (i) above.

Moreover, the terms $E_1^{s,0}$ with the corresponding differential $d_1 : E_1^{s,0} \to E_1^{s+1,0}$ coming from the spectral sequence can be identified for $s < n-k$ (since we need again Corollary (i)) to the corresponding initial part in the complex B^*_f.

Since this part of the spectral sequence clearly degenerates at the E_2-term, i.e. $E_2^{s,0} = E_\infty^{s,0}$ for $s \leqslant n-k$, we obtain equivalence (i) ⇔ (ii″) which completes the proof of Lemma 2. □

To end the proof of our Corollary, we have just to use the main results in [7], saying that the cohomology of the complex D^*_f twisted by -1 is just the reduced cohomology of the general fiber of the polynomial f.

Note that we have shown in fact the equivalence of conditions (ii), (iii) and (iv) in our Corollary for any polynomial f.

Remark. (I) The condition $k = \dim \Sigma(f) < n$ is equivalent to saying that the fibers of f are all reduced. This condition is not needed for part (i) of our Corollary (which holds even when $k = n$), but it is necessary for parts (ii)–(iv) since it implies that the general fiber of f is connected.

Indeed, it is well known that any polynomial f can be written as a composition $f(x) = h(g(x))$ where $g : \mathbb{C}^{n+1} \to \mathbb{C}$ has a connected general fiber and $h : \mathbb{C} \to \mathbb{C}$, both g and h being polynomials. Since it is difficult to locate a reference for this fact, here is a short proof of it suggested to the first author by M. Zaidenberg, to whom we are grateful.

Assume that f is not a constant polynomial and let $\tilde{f} : X \to \mathbb{P}^1$ be a smooth compactification of f. Then the Stein Factorization Theorem, see [11, p. 280], gives a smooth curve C and morphisms $\tilde{g} : X \to C$ and $\tilde{h} : C \to \mathbb{P}^1$ such that $\tilde{h} \circ \tilde{g} = \tilde{f}$ and such that all the fibers of $\tilde{g}$ are connected.

A generic line L in $\mathbb{C}^{n+1}$ has the following properties:

(i) f is not constant when restricted to L;
(ii) the closure $\tilde{L}$ of L in X is a smooth rational curve which meets $\tilde{f}^{-1}(\infty)$ at exactly one point and this intersection is transverse.

Then $\tilde{L}$ is a rational curve and $\tilde{g}|\tilde{L}$ is a non-constant map. This implies that $C = \mathbb{P}^1$. Moreover, we have $\tilde{h}^{-1}(\infty) = \infty$, otherwise condition (ii) above is contradicted. This implies that $\tilde{h}$ gives by restriction to $\mathbb{C} = \mathbb{P}^1 \setminus \{\infty\}$ a morphism $h : \mathbb{C} \to \mathbb{C}$ such that $f = h \circ g$ where $g = \tilde{g}|\mathbb{C}^{n+1}$ has its general fiber connected.

(II) For a given polynomial $f : \mathbb{C}^{n+1} \to \mathbb{C}$ and a given system of weights $\mathbf{w}$ we define the connectivity order of f with respect to $\mathbf{w}$ to be the integer

$$c_{\mathbf{w}}(f) = n - 1 - k,$$

where $k = \dim S(f)$.

Then it is easy to see that for the sum $f(x) + g(y)$ of two polynomials $f \in \mathbb{C}[x_0, \dots, x_n]$ and $g \in \mathbb{C}[y_0, \dots, y_m]$ on two independent groups of variables and for any weights $\mathbf{w}'$ associated to g such that the corresponding degrees d for f and g coincide (by multiplying any weights $\mathbf{w}$ for f and $\mathbf{w}'$ for g by suitable positive integers this can be always achieved) we have

$$c_{\mathbf{w},\mathbf{w}'}(f + g) \geqslant c_{\mathbf{w}}(f) + c_{\mathbf{w}'}(g) + 1$$

in spite of the fact that in general the corresponding degrees e will be different.

Note that the general fiber of $f + g$ is the join of the general fibers of f and g, see [17], but nothing is known about the special fibers.

(III) Siersma and Tibăr have shown in [21] that in the case of the usual weights $\mathbf{w} = (1,1, \dots, 1)$ and if $e = d - 1$ and $\dim \Sigma(f) < \dim S(f) = k$, then the general fiber of f is $(n - k)$-connected, i.e. a better by 1 estimate than that given by our Theorem. They also give an example showing that this better estimate fails for the special fibers.

Improving the results in [21], Tibăr has recently obtained very general connectivity results for the fibers of polynomial mappings, see [22, Theorem 5.5]. However the effectiveness of his results depends on the explicit construction of fiber-compactifying extensions of the polynomial function f having a small critical set at infinity. It seems to us that such constructions are difficult to handle for general weights $\mathbf{w}$ or even for the usual weights when $e < d - 1$.

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