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There exists no tetravalent half-arc-transitive graph of order $2p^2$

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ABSTRACT

A graph is half-arc-transitive if its automorphism group acts transitively on its vertex set, edge set, but not arc set. In this paper, we show that there is no tetravalent half-arc-transitive graph of order $2p^2$.

1. Introduction

Throughout this paper graphs are assumed to be finite, simple and undirected, but with an implicit orientation of the edges when appropriate. For a graph X, we let $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}(X)$ be the vertex set, the edge set, the arc set and the automorphism group of X, respectively.

A graph X is said to be *vertex-transitive*, *edge-transitive* or *arc-transitive* if $\mathrm{Aut}(X)$ acts transitively on $V(X)$, $E(X)$, or $A(X)$, respectively. A graph is said to be 1/2-*arc-transitive* or *half-arc-transitive* provided that it is vertex-transitive and edge-transitive, but not arc-transitive. More generally, by a 1/2-arc-transitive or half-arc-transitive action of a subgroup G of $\mathrm{Aut}(X)$ on a graph X we shall mean a vertex-transitive and edge-transitive, but not arc-transitive action of G on X. In this case, we shall say that the graph X is $(G, 1/2)$-*arc-transitive*.

The investigation of half-arc-transitive graphs was initiated by Tutte [30] and he proved that a vertex- and edge-transitive graph with odd valency must be arc-transitive. In 1970, Bouwer [4] constructed the first family of half-arc-transitive graphs and later more such graphs were constructed (see for instance [2,10,15,16,29,31]). Let p be a prime. It is well known that there is no half-arc-transitive graph of order p or p^2. Xu [34] classified the tetravalent half-arc-transitive graphs of order p^3 and Feng et al. [12] classified the tetravalent half-arc-transitive graphs of order p^4. By Cheng and Oxley [6], there is no tetravalent half-arc-transitive graph of order $2p$, and a classification of tetravalent half-arc-transitive graphs of order $3p$ can be deduced from Alspach and Xu [2]. Feng et al. [14] recently classified the tetravalent half-arc-transitive graphs of order $4p$. In this paper we show that there is no tetravalent half-arc-transitive graph of order $2p^2$. For more results on tetravalent half-arc-transitive graphs, see [1,7,8,11,13,17–23,27,28,33,35].

For a finite group G and a subset S of G such that $1 \notin S$ and $S = S^{-1}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. Given a $g \in G$, define the permutation $R(g)$ on G by $x \mapsto xg$, $x \in G$. Then $R(G) = \{R(g) \mid g \in G\}$ is a permutation group isomorphic to G, which is called the *right regular representation* of G. The Cayley graph $\mathrm{Cay}(G, S)$ is vertex-transitive since it admits $R(G)$ as a regular subgroup of the automorphism group $\mathrm{Aut}(\mathrm{Cay}(G, S))$. Furthermore, the group $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))$. Actually, $\mathrm{Aut}(G, S)$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))_1$, the stabilizer of the vertex 1 in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. A graph X is isomorphic to a

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Cayley graph on a group G if and only if its automorphism group $\mathrm{Aut}(X)$ has a subgroup isomorphic to G, acting regularly on the vertex set of X (see [3, Lemma 16.3]). A Cayley graph $\mathrm{Cay}(G, S)$ is said to be *normal* if $\mathrm{Aut}(\mathrm{Cay}(G, S))$ contains $R(G)$ as a normal subgroup.

Let X and Y be two graphs. The *lexicographic product* $X[Y]$ is defined as the graph with vertex set $V(X[Y]) = V(X) \times V(Y)$ and two vertices $u = (x_1, y_1)$ and $v = (x_2, y_2)$ in $V(X[Y])$ being adjacent in $X[Y]$ whenever x_1 is adjacent to x_2, or $x_1 = x_2$ and y_1 is adjacent to y_2. Clearly, if both X and Y are arc-transitive then $X[Y]$ is arc-transitive.

To end the section we list some preliminary results that will be used later. Note that there is no half-arc-transitive graph of order p or $2p$ for a prime p (see [5,6]), and all vertex-transitive graphs with fewer than 22 vertices were listed in [24,25]. By the proof that there is no half-arc-transitive graph of order 24 given in [26], one may conclude that there is no half-arc-transitive graph with fewer than 27 vertices.

Proposition 1.1. *There is no half-arc-transitive graph with fewer than* 27 *vertices.*

Let $X = \mathrm{Cay}(G, S)$ be a Cayley graph on a group G with respect to S. If $s \in S$ is an involution then $R(s) \in \mathrm{Aut}(X)$ interchanges the two arcs $(1, s)$ and $(s, 1)$ in X. Moreover, if there exist $\alpha \in \mathrm{Aut}(G, S)$ and $t \in S$ such that $t^\alpha = t^{-1}$ then $\alpha R(t)$ interchanges the arcs $(1, t)$ and $(t, 1)$. This implies the following proposition.

Proposition 1.2. *Let $X = \mathrm{Cay}(G, S)$ be a half-arc-transitive graph. Then there is no involution in S and no $\alpha \in \mathrm{Aut}(G, S)$ such that $s^\alpha = s^{-1}$ for any given $s \in S$.*

Let $X = \mathrm{Cay}(G, S)$ be a Cayley graph on an abelian group G. Note that the mapping $\alpha : x \to x^{-1}, x \in G$, is an automorphism of G and so Proposition 1.2 implies the following proposition.

Proposition 1.3. *Every edge-transitive Cayley graph on an abelian group is also arc-transitive.*

The following is a fundamental result from permutation group theory.

Proposition 1.4 (*[32, Theorem 3.4]*). *Let G be a permutation group on Ω and $\alpha \in \Omega$. Let p be a prime number, p^m a divisor of $|\alpha^G|$, and P a Sylow p-subgroup of G. Then p^m is also a divisor of $|\alpha^P|$.*

It is well known that every transitive permutation group of prime degree p is either 2-transitive or solvable with a regular normal Sylow p-subgroup (for example, see [9, Corollary 3.5B]), which implies the following proposition.

Proposition 1.5. *Let X be a graph of prime order p which is neither the empty graph nor the complete graph. Then every vertex-transitive subgroup of $\mathrm{Aut}(X)$ has a normal Sylow p-subgroup.*

2. Main result

The main purpose of this paper is to prove the following theorem.

Theorem 2.1. *There is no tetravalent half-arc-transitive graph of order* $2p^2$.

Proof. Suppose to the contrary that X is a tetravalent half-arc-transitive graph of order $2p^2$. Then X is connected because there is no half-arc-transitive graph of order p, $2p$ or p^2. By Proposition 1.1, one may assume that $p \geq 5$. Let $A = \mathrm{Aut}(X)$.

Clearly, X is $(A, 1/2)$-arc-transitive graph. Then in the natural action of A on $V(X) \times V(X)$, the arc set of X is a union of two paired orbits of A, say A_1 and A_2, that is, $A_2 = \{(v, u) \mid (u, v) \in A_1\}$. Thus, one can obtain two oriented graphs having $V(X)$ as vertex set and A_1 or A_2 as arc set, respectively. Let $D_A(X)$ be one of the two oriented graphs. Then $D_A(X)$ has out-valency and in-valency equal to 2 and A acts arc-transitively on it. Since $D_A(X)$ has out-valency and in-valency equal to 2, the stabilizer A_u of $u \in V(X)$ in A is a 2-group. It follows that A is a $\{2, p\}$-group, implying that A is solvable. First, we prove the following claim.

Claim 1. *There is no tetravalent half-arc-transitive Cayley graph of order* $2p^2$.

By contradiction, let $X = \mathrm{Cay}(G, S)$ be a Cayley graph on a group G of order $2p^2$ with respect to S. Since X is connected, one has $|S| = 4$, $S^{-1} = S$ and $\langle S\rangle = G$. By Proposition 1.3, G is non-abelian. From the elementary group theory we know that up to isomorphism there are three non-abelian groups of order $2p^2$ for an odd prime p:

$$G_1(p) = \langle a, b \mid a^{p^2} = b^2 = 1, b^{-1}ab = a^{-1}\rangle,$$
$$G_2(p) = \langle a, b, c \mid a^p = b^p = c^2 = 1, c^{-1}ac = a^{-1}, c^{-1}bc = b^{-1}, [a, b] = 1\rangle,$$
$$G_3(p) = \langle a, b, c \mid a^p = b^p = c^2 = 1, [a, b] = [a, c] = 1, c^{-1}bc = b^{-1}\rangle.$$

It follows that G is isomorphic to one of $G_1(p)$, $G_2(p)$ or $G_3(p)$. Note that G has a normal Sylow p-subgroup. Suppose that G is isomorphic to $G_1(p)$ or $G_2(p)$. Since S generates G, S contains at least one involution, which contradicts Proposition 1.2.

Thus G is isomorphic to $G_3(p)$. Furthermore, S consists of either one element of order p, one element of order $2p$ and their inverses, or two elements of order $2p$ and their inverses.

Suppose first that S consists of one element of order p, one element of order $2p$ and their inverses. Then $S = \{a^ib^j, a^{-i}b^{-j}, ca^sb^t, ca^{-s}b^t\}$, where $js \neq 0 \pmod p$ because $\langle S\rangle = G$. Since the map $cb^t \to c$, $a^s \to a$, $b^j \to b$ induces an automorphism of G one may assume $S = \{a^ib, a^{-i}b^{-1}, ca, ca^{-1}\}$. In this case, the automorphism of G induced by $a \to a^{-1}$, $b \to b^{-1}$, $c \to c$ fixes S and maps s to s^{-1} for each $s \in S$, contrary to Proposition 1.2. Suppose now that S consists of two elements of order $2p$ and their inverses. By a similar argument as above, one may assume that $S = \{ca, ca^{-1}, ca^ib, ca^{-i}b\}$ where $i \neq 0 \pmod p$. Thus, the automorphism of G induced by $a \to a^{-1}$, $b \to b$, $c \to c$ fixes S and maps s to s^{-1} for each $s \in S$, contrary to Proposition 1.2. This completes the proof of Claim 1.

Let N be a minimal normal subgroup of A and let X_N be the quotient graph of X with respect to the orbits of N, that is the graph with the orbits of N as its vertex set such that two orbits are adjacent in X_N whenever there is an edge between the two orbits in X. Let K be the kernel of A acting on $V(X_N)$. Since A is solvable, N is an elementary abelian 2- or p-group. We now prove the following claim.

Claim 2. *A has a normal Sylow p-subgroup.*

Suppose that N is an elementary abelian 2-group. Then $|N| = 2^r$ for some integer r and $|V(X_N)| = p^2$. Sine $p \geq 5$ is odd, X_N has valency 2 or 4. If X_N has valency 2 then $X = C_{p^2}[2K_1]$, which is arc-transitive, a contradiction. If X_N has valency 4 then the stabilizer K_u of $u \in V(X)$ in K fixes the neighborhood of u in X pointwise because K fixes each orbit of N. The connectivity of X implies that $K_u = 1$ and hence $|K| = |N| = 2$. Thus, one may view A/N as a group of automorphisms of X_N, that is, $A/N \leq \mathrm{Aut}(X_N)$. By Proposition 1.4, a Sylow p-subgroup PN/N of A/N is transitive on X_N, where P is a Sylow p-subgroup of A. Since $p \geq 5$ and X_N has valency 4, PN/N is regular on $V(X_N)$, implying $|PN/N| = p^2$. It follows that PN is regular on $V(X)$ because $|PN| = 2p^2$, which means that X is a Cayley graph on PN, contrary to Claim 1.

Note that the above argument is true if N is replaced by any nontrivial normal 2-subgroup of A. Then $O_2(A) = 1$, where $O_2(A)$ is the largest normal 2-subgroup of A.

Thus, N is an elementary abelian p-group. Clearly, $|N| = p$ or p^2. If $|N| = p^2$ then N is a normal Sylow p-subgroup of A, as claimed. Let $|N| = p$. In this case, $|X_N| = 2p$ and the edge-transitivity of X implies that X_N has valency 2 or 4.

If X_N has valency 2 then $X_N \cong C_{2p}$, say $X_N = (B_0, B_1, \ldots, B_{2p-1})$. Thus, the induced subgraph $\langle B_i, B_{i+1}\rangle$ of $B_i \cup B_{i+1}$ in X is a cycle of length $2p$. This implies that $|K_u| = 1$ or 2 and hence $|K| = p$ or $2p$. On the other hand, $A/K \leq \mathrm{Aut}(X_N) \cong D_{4p}$. Then A/K has a normal Sylow p-subgroup, say PK/K, where P is a Sylow p-subgroup of A. Since $|PK : P| = 1$ or 2, P is characteristic in PK and hence normal in A, as claimed.

Suppose now that X_N has valency 4. In this case $K_u = 1$ for any $u \in V(X)$ and hence $|K| = |N| = p$, implying that $A/N \leq \mathrm{Aut}(X_N)$. Let M/N be a minimal normal subgroup of A/N. Then M/N is an elementary abelian p- or 2-group. For the former, M is a normal Sylow p-subgroup of A, as claimed. For the latter, M/N has orbits of length 2 because $|V(X_N)| = 2p$. Clearly, $M \lhd A$ has orbits of length $2p$. Let X_M be the quotient graph of X with respect to the orbits of M and let L be the kernel of A acting on $V(X_M)$. It follows that X_M has valency 4 or 2. If X_M has valency 4 then $|L| = |M| = 2p$ and $A/M \leq \mathrm{Aut}(X_M)$. Furthermore, X_M is $(A/M, 1/2)$-arc-transitive. Note that $|V(X_M)| = p$. By Proposition 1.5, A/M has a normal Sylow p-subgroup when $p \neq 5$. For $p = 5$, $X_M \cong K_5$ and it is easy to see that each half-arc-transitive subgroup of $\mathrm{Aut}(K_5) \cong S_5$ is isomorphic to D_{10} which has a normal Sylow 5-subgroup. It follows that A/M always has a normal Sylow p-subgroup, say PM/M, where P is a Sylow p-subgroup of A. Since $|PM| = 2p^2$, P is characteristic in PM and since $PM \lhd A$, one has $P \lhd A$, as claimed. Suppose now that X_M has valency 2, that is $X_M = (B_0, B_1, \ldots, B_{p-1})$ is a cycle of length p. It follows that $A/L \leq \mathrm{Aut}(X_M) \cong D_{2p}$ and so A/L has a normal Sylow p-subgroup, that is, $PL/L \lhd A/L$, where P is a Sylow p-subgroup of A. If $\langle B_0 \cup B_1\rangle$ is a union of two cycles of length $2p$ then N fixes the two cycles setwise and hence X_N has valency 2, contrary to the hypothesis that X_N has valency 4. Thus, $\langle B_0 \cup B_1\rangle$ is a cycle of length $4p$ or a union of p cycles of length 4. Let $u \in B_0$ and denote by L_u the stabilizer of u in L. Clearly, if $\langle B_0 \cup B_1\rangle$ is a cycle of length $4p$ then $|L_u| = 1$ or 2. If $\langle B_0 \cup B_1\rangle$ is a union of p cycles of length 4 then each of these 4-cycles contains precisely one vertex from each of the four N-orbits that constitute $B_0 \cup B_1$. Since X is half-arc-transitive, $X \ncong C_{p^2}[2K_1]$, which implies that L_u fixes at least three vertices in B_0, say u, v and w. It follows that $L_u = L_v = L_w$. Using the fact that N is a cyclic group of order p acting semiregularly on $V(X)$ one can easily show that L_u fixes all vertices in B_0. This means that L_u is the kernel of L acting on B_0 and hence $L_u \trianglelefteq L$. Since $O_2(A) = 1$, one has $O_2(L) = 1$ because $O_2(L) \trianglelefteq A$, implying $L_u = 1$. Thus, we always have $|L| = 2p$ or $4p$ and hence $|PL| = 2p^2$ or $4p^2$, forcing that P is characteristic in PL because $p \geq 5$. Normality of PL in A thus implies that P is normal in A, as claimed. This completes the proof of Claim 2.

By Claim 2, A has a normal Sylow p-subgroup, say P. Since $|P| = p^2$, P is abelian and since $|V(X)| = 2p^2$, X is a bipartite graph with the two orbits of P as its two bipartite sets. It is easy to see that P acts regularly on each of the two bipartite sets of X. Thus, one may identify $U(P) = \{U(n) \mid n \in P\}$ and $V(P) = \{V(n) \mid n \in P\}$ with the two bipartite sets of X in such a way that the action of $n \in P$ on $U(P)$ and on $V(P)$ is just the right multiplication by n, that is $U(g)^n = U(gn)$ and $V(g)^n = V(gn)$ for any $g \in P$. Let $V(n_1)$, $V(n_2)$, $V(n_3)$ and $V(n_4)$ be the neighbors of $U(1)$. Then $V(n_1n)$, $V(n_2n)$, $V(n_3n)$ and $V(n_4n)$ are the neighbors of $U(n)$ for each $n \in P$. Since P is abelian, $U(n_1^{-1}n)$, $U(n_2^{-1}n)$, $U(n_3^{-1}n)$ and $U(n_4^{-1}n)$ are the neighbors of $V(n)$ for each $n \in P$. Define the map α by $U(n) \to V(n^{-1})$ and $V(n) \to U(n^{-1})$. It is easy to show that $\alpha \in A$. Since $P \trianglelefteq A$, one has $|\langle P, \alpha\rangle| = 2p^2$ and $\langle P, \alpha\rangle$ acts regularly on $V(X)$. Thus, X is a Cayley graph, contrary to Claim 1. □

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