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A Generalization of Quasi-Frobenius Rings[1]

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1. INTRODUCTION

Let R be a ring with identity 1. We denote the category of unital right (respectively left) R-modules by $\mathfrak{M}_R({}_R\mathfrak{M})$. M_R will signify $M \in \mathfrak{M}_R({}_RM \in {}_R\mathfrak{M})$.

We will be concerned with certain categorical properties of modules. M_R is projective if and only if every epimorphism $N_R \to M_R$ splits. Dually, M_R is injective if and only if every monomorphism $M_R \to N_R$ splits. M_R is a generator in $\mathfrak{M}_R$ if and only if, for every $N_R \in \mathfrak{M}_R$, there is an epimorphism from a direct sum of copies of M_R to N_R. Dually, M_R is a cogenerator in $\mathfrak{M}_R$ if and only if, for every $N_R \in \mathfrak{M}_R$, there is a monomorphism from N_R into a direct product of copies of M_R. R_R is always a projective generator in $\mathfrak{M}_R$. This paper discusses rings for which R_R possesses the dual properties.

A ring R is quasi-Frobenius if R has descending chain condition (dcc) on right ideals, and R_R is injective. Such rings have been extensively studied (see, for example, Nakayama [*16*] and [*17*], Morita and Tachikawa [*15*], Ikeda and Nakayama [*10*], Morita [*14*], Dieudonné [*5*], Faith [*8*], and Faith and Walker [*9*]).

There are many properties equivalent to this definition (it is not the original definition of these rings). For example, the following are equivalent:

(i) R is quasi-Frobenius.

(ii) R_R is a cogenerator in $\mathfrak{M}_R$ and R has dcc on right ideals.

(iii) R_R has ascending chain condition (acc) on right ideals and R_R is injective.

(iv) R_R has acc on right ideals and R_R is a cogenerator in $\mathfrak{M}_R$.

(v)–(viii) The above with chain conditions on left rather than right ideals.

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In Section 2, we discuss what happens if we drop all chain conditions, but retain some of the other properties of quasi-Frobenius rings. There are rings with R_R injective but not a cogenerator (for example, the ring of all linear transformations on a vector space V_F of infinite dimension over the field F). R_R may be a cogenerator but not injective (example 2 in Section 2). Such rings have no chain conditions on any reasonable classes of right ideals. However, Theorem 1 states that if R_R is an injective cogenerator in $\mathfrak{M}_R$, then R_R is a finite direct sum of indecomposable right ideals, and R modulo its Jacobson radical is semi-simple Artin. Thus R_R has dcc and acc on direct summands. An example (example 1) is constructed to show that no other chain conditions are implied.

In Section 3, duality of modules is discussed. We assume we have rings S and T, and an $S - T$ bimodule U such that ${}_SU$ and U_T are injective cogenerators in ${}_S\mathfrak{M}$ and $\mathfrak{M}_T$ respectively. In this case, S and T both have semi-simple Artin quotients modulo their Jacobson radicals, and idempotents lift modulo the radical. Moreover, if S (or T) is left or right perfect, then S and T are two-sided Artin.

Section 4 contains a discussion of what happens if we only know that U_T is an injective cogenerator in $\mathfrak{M}_T$.

2. Ring Injective Cogenerators

In this section we study rings R for which R_R is an injective cogenerator in $\mathfrak{M}_R$.

A module M is called an essential extension of N_R (written $M_R \,'\!\supseteq N_R$ or $N_R \subseteq' M_R$) if $K_R \subseteq M_R$ and $K \cap N = 0 \Rightarrow K = 0$. Every M_R can be embedded in an injective module $E(M)$ which is an essential extension of M. Every injective module containing M contains an isomorphic copy of $E(M)$ (Eckmann and Schopf [6]). If $X \subseteq R$, let $(0:X) = \{r \in R \mid xr = 0 \text{ for all } x \in X\}$, and (X) = the ideal generated by X. In the sequel, J will denote the Jacobson radical of R, and π the natural map from R onto R/J.

We list a series of known lemmas.

LEMMA 1. *A module M_R is a cogenerator in $\mathfrak{M}_R \Leftrightarrow M$ contains a copy of the injective hull $E(U)$ of each simple right R-module U.*

Proof. Let M be a cogenerator in $\mathfrak{M}_R$, V a simple right R-module. $E(V)$ may be embedded in a direct product of copies of M, and the projection of V onto one of those copies is non-zero. Then the kernel of the projection of $E(V)$ onto that copy of M has zero intersection with V. We conclude that the kernel is zero, so the projection embeds $E(V)$ in M.

Now let M contain a copy of the injective hull $E(V)$ of each simple R-module V. Consider the map from the right R-module N to $\prod_{f \in \mathrm{Hom}_R(N,M)} M_f$ given by $x \to (f(x))$. Given $x \in N$, xR has a simple quotient module V. The map from xR to V extends to a map from N to $E(V)$ since $E(V)$ is injective. Since M contains a copy of $E(V)$, we get a map from N to M which is not zero on x. Thus N is embedded in $\prod_{f \in \mathrm{Hom}_R(N,M)} M_f$.

LEMMA 2. *$I_R \subseteq R_R$ is a direct summand of $R_R \Leftrightarrow I = eR$ for some $e = e^2 \in R$.*

Proof. $I = eR$ implies $R_R = eR \oplus (1 - e)R$. If I is a direct summand of R_R, say $R = I \oplus J$, then if $1 = e + f$, $e \in I$, $f \in J$, e is an idempotent generating I.

LEMMA 3. *Let U be a minimal right ideal of R such that $E(U) \subseteq R$. Then $E(U)/E(U)J$ is simple.*

Proof. Let $\Lambda = \mathrm{Hom}_R(E(U), E(U))$. Then $f \in \Lambda$ is a monomorphism if and only if kernel $f \cap U = 0$. Moreover, if f is a monomorphism, $f(E(U))$ is injective and hence a direct summand of $E(U)$. Since $E(U) \,'\!\supseteq U$ which is simple, the only direct summands of $E(U)$ are 0 and $E(U)$. Thus f is an isomorphism. If x, $y \in \Lambda$ are non-units, then kernel $x \cap$ kernel $y \supseteq U$, so $x + y$ is a non-unit. We conclude that Λ is a local ring.

Since $E(U) \subseteq R$, $E(U) = eR$ for some $e = e^2 \in R$ by Lemma 2. Then $\Lambda = eRe$, and $eRe/eJe \approx \Lambda/\mathrm{rad}\,\Lambda$ is a field. But eRe/eJe is the endomorphism ring of eR/eJ, a non-nilpotent right ideal of R/J. Hence eR/eJ is simple as an R/J-module. It is then simple as an R-module (see Jacobson [*11*], p. 65).

LEMMA 4. *If R_R is injective, then R/J is regular and finite sets of orthogonal idempotents in R/J lift to orthogonal idempotents in R. Moreover, $J = \{r \in R \mid R \,'\!\supseteq (0{:}r)\}$.*

Proof. Various portions of this lemma are due to different people, as Utumi [*22*] and Faith and Utumi [*23*].

LEMMA 5. *If $e = e^2 \in R$, $f = f^2 \in R$, then $eR/eJ \approx fR/fJ \Leftrightarrow eR \approx fR$.*

Proof. If $eR \approx fR$, clearly $eR/eJ \approx fR/fJ$. The proof in the other direction is Bass' proof of the uniqueness of projective covers (see [*3*], or Jacobson [*11*], Proposition 1, p. 53).

LEMMA 6. *Let A be a set of cardinality $\aleph \geqslant \aleph_0$. Then A can be decomposed into a class $\mathscr{K}$ of subsets of A with cardinality $\mathscr{K} > \aleph$, and for all X, $Y \in \mathscr{K}$, cardinality $X =$ cardinality $Y >$ cardinality $X \cap Y$* if $X \neq Y$.

Proof. See Tarski [*21*], page 191.

In the remainder of this section, R_R will be an injective cogenerator in $\mathfrak{M}_R$. Lemmas 1, 2, 3, and 5 enable us to establish a $1-1$ correspondence between isomorphism classes of simple R-modules and isomorphism classes of simple submodules of R/J. Our aim is to prove that every simple R-module appears in the socle of R/J by proving that there are only a finite number of non-isomorphic simple R-modules. Then R/J is semi-simple Artin since it is regular and no maximal ideal contains its socle. Lemma 4 enables us to lift idempotents and thus get the desired decomposition of R as a finite direct sum of indecomposable right ideals. As suggested by Lemma 6, a counting argument will be used.

Every isomorphism class of simple R-modules has been associated with a class of simple R/J modules—a representative U is associated with $E(U)/E(U)J$. We next extend this correspondence to one between homogeneous components of the socle of R and a set of central idempotents in R/J.

If U is an injective module, and M a submodule of U, U must contain at least one copy of $E(M)$. Although this copy is "unique up to isomorphism", it need *not* be a unique submodule of U; that is, U may contain more than one copy of $E(M)$ (see, for example, Osofsky [*18*]). Our next lemma basically states that homogeneous components of the socle of R_R do indeed have unique (modulo J) injective hulls in R_R. The symbol $E(M)$ will refer to any one of the copies of the injective hull in R_R of the right ideal M.

Let $C = \{U_\iota \mid \iota \in \mathscr{I}\}$ be a family of simple right R-modules, and let $S(C) = \sum U$, where the summation is taken over all minimal right ideals U of R isomorphic to U_ι for some $U_\iota \in C$. By Lemma 2, $E(S(C)) = e_C R$ for some idempotent e_C in R.

LEMMA 7. *$\pi(e_C R) \,'\!\supseteq \sum W$, where the summation is taken over all minimal right ideals W of R/J isomorphic to $E(U_\iota)/E(U_\iota)J$ for some U_ι in C.*

Proof. Let W be a minimal right ideal of R/J isomorphic to $E(U_\iota)/E(U_\iota)J$. Assume $W \nsubseteq \pi(e_C R)$. Then $W + \pi(e_C R) = \pi(f)R/J \oplus \pi(e_C)R/J$, where $\pi(e_C)$ and $\pi(f)$ are orthogonal idempotents in the regular ring R/J (see Von Neumann [*24*]). Since W is simple, $\pi(f)R/J \approx W$. By Lemma 4, $\pi(e_C)$ and $\pi(f)$ lift to orthogonal idempotents e and f in R. By Lemma 2, $E(U_\iota)$ is generated by an idempotent; by Lemma 5, $E(U_\iota) \approx fR$. Hence fR contains an isomorphic copy U' of U_ι. Since $\pi(e_C) = \pi(e)$, $e_C - e \in J$. Hence by Lemma 4, $(e_C - e)U' = 0$. But $e_C R \supseteq U'$, so $e_C U' = U'$. Then $eU' = U' = fU' = feU' = 0$, a contradiction. Thus $W \subseteq \pi(e_C R)$.

Now let $0 \neq \mu \in \pi(e_C R)$. Since R/J is regular, $\mu R/J = \epsilon R/J$ and $\pi(e_C R) = \epsilon R/J + \delta R/J$, where ϵ and δ are orthogonal idempotents in R/J. We lift ϵ and δ to orthogonal idempotents e and d in R by Lemma 4. Then $\pi(e + d)R = \pi(e_C)R$, so by Lemma 5, $(e + d)R \approx e_C R$. But then $eR \cap S(C) \neq 0$, so $eR \supseteq E(U)$ for some $U \subseteq S(C)$, and $\mu R/J = \pi(e)R/J \supseteq E(U)/E(U)J$.

Note added in proof. Since writing this paper, the author has found that Lemma 7 can be replaced by a result of Y. Utumi [On continuous rings and self injective rings. *Trans. Am. Math. Soc.* **118** (1965), 158-173]: If R is right self-injective, then so is R/J. This immediately gives an idempotent in R/J with the properties we need in Lemma 7.

For additional examples of ring injective cogenerators without chain conditions, see L. Levy [Commutative rings whose homomorphic images are self injective. *Pacific J. Math.* **18** (1966), 149-153].

LEMMA 8. *$\pi(e_C) = \epsilon$ is a central idempotent in R/J.*

Proof. Assume $\epsilon\mu(1 - \epsilon) \neq 0$ for some $\mu \in R/J$. By Lemma 7, there is a $\nu \in R/J$ such that $\epsilon\mu(1 - \epsilon)\nu$ is in W, a minimal right ideal of $\epsilon R/J$. Since W is generated by an idempotent in R/J, W is a projective R/J-module. Hence the R/J-homomorphism from $(1 - \epsilon)\nu R/J$ to W given by left multiplication by $\epsilon\mu$ must split. Thus $(1 - \epsilon)R/J$ contains a minimal right ideal isomorphic to W. But all such are contained in $\epsilon R/J$ by Lemma 7, a contradiction. We conclude that $\epsilon R/J(1 - \epsilon) = 0$.

Assume $(1 - \epsilon)\mu\epsilon \neq 0$ for some $\mu \in R/J$. Consider the map $\epsilon \rightarrow (1 - \epsilon)\mu\epsilon$ from $\epsilon R/J$ to $(1 - \epsilon)R/J$. Its image, a principal right ideal of R/J, is generated by an idempotent, and hence a projective R/J-module. Thus $\epsilon R/J$ contains a direct summand isomorphic to $(1 - \epsilon)\mu\epsilon R/J$. But the latter contains no submodule isomorphic to $E(U_\iota)/E(U_\iota)J$ for $U_\iota \in C$, so Lemma 7 is contradicted. Hence $(1 - \epsilon)R/J\epsilon = 0$.

We are now ready for our main result.

THEOREM 1. *Let R_R be an injective cogenerator in M_R. Then $R = \sum_{i=1}^{n} e_i R$, where $\{e_i \mid 1 \leqslant i \leqslant n\}$ is a set of orthogonal idempotents, and $e_i R/e_i J$ is simple for each i.*

Proof. Let $\{U_\iota \mid \iota \in \mathscr{I}\}$ be a complete set of representatives of the distinct isomorphism classes of simple R-modules. Assume $\mathscr{I}$ is infinite. We have a $1 - 1$ correspondence between $\mathscr{I}$, isomorphism classes of simple right ideals, and isomorphism classes of simple modules $E(U_\iota)/E(U_\iota)J$ in the socle of R/J. By Lemma 6, we may decompose $\mathscr{I}$ into a class $\mathscr{X}$ of subsets such that cardinality $\mathscr{X} >$ cardinality $\mathscr{I}$, and for all X, $Y \in \mathscr{X}$, cardinality $X =$ cardinality $Y >$ cardinality $X \cap Y$ if $X \neq Y$. For each $\mathscr{J} \subseteq \mathscr{I}$, Let $C(\mathscr{J}) = \{U_j \mid j \in \mathscr{J}\}$, and let $e_{\mathscr{J}}$ be the corresponding central idempotent generating $\pi(E(S(C(\mathscr{J}))))$. Let $I = \sum \pi(e_{\mathscr{J}} R)$, where cardinality $\mathscr{J} <$ cardinality X for all $X \in \mathscr{X}$. I is a two sided ideal of R/J. For $X \in \mathscr{X}$, set $I_X = I + \pi(1 - e_X)R/J$. This is also a two sided ideal of R/J since the e_X, $e_{\mathscr{J}}$ are central idempotents in R/J.

Since $S(X) = S(X - Y) \oplus S(X \cap Y)$, $E(S(X)) = E(S(X - Y)) \oplus E(S(X \cap Y)$, and $e_{X \cap Y} \in e_X R$. Similarly, $e_{X \cap Y} \in e_Y R$. Hence $\pi(e_{X \cap Y}) \in \pi(e_X R) \cap \pi(e_Y R) \subseteq \pi(e_X e_Y R) = \pi(e_Y e_X R)$. If $\mu \in \pi(e_X e_Y R)$, let W be a simple module of $\mu R/J$. Then $W \approx E(U_\varphi)/E(U_\varphi)J \approx E(U_\psi)/E(U_\psi)J$ for some $\varphi \in X$, $\psi \in Y$, since μ is in $\pi(e_X R)$ and in $\pi(e_Y R)$. Hence $\varphi = \psi \in X \cap Y$. Since $\pi(e_X R)$ is an essential extension of its socle, $\mu R/J$ is an essential extension of its socle. Moreover, $\pi(e_{X \cap Y} R)$ contains every essential extension of $\sum \{W \mid W \approx E(U_\varphi)/E(U_\varphi)J$ for some $\varphi \in X \cap Y\}$, so $\mu \in \pi(e_{X \cap Y} R)$. Thus $\pi(e_X e_Y R) \subseteq \pi(e_{X \cap Y} R)$, and equality must hold. Then $\pi(e_{X \cap Y}) = \pi(e_X e_Y)$ is the unique identity of the ring $\pi(e_X e_Y R)$.

Now assume $\pi(e_X) \in I_X$. Then $e_X = (1 - e_X)r_0 + \sum_{i=1}^n e_{X_i} r_i + j$ where $j \in J$ and cardinality $X_i <$ cardinality X for $1 \leqslant i \leqslant n$. Hence cardinality $\bigcup_{i=1}^n X_i <$ cardinality X since X is infinite. Thus there is a $\varphi \in X$, $\varphi \notin \bigcup_{i=1}^n X_i$, and $\pi(e_X R) = \pi(1 - e_X) r_0 R + \sum_{i=1}^n \pi(e_{X_i}) r_i R$ has no submodule isomorphic to $E(U_\varphi)/E(U_\varphi)J$, a contradiction to Lemma 7. We conclude that $\pi(e_X) \notin I_X$, so $I_X \neq R/J$. On the other hand, let $X \neq Y \in \mathscr{K}$. Then $\pi(e_Y) = \pi((1 - e_X)e_Y) + \pi(e_X e_Y) \in I_X$ since $\pi(e_X e_Y) = \pi(e_{X \cap Y})$, and cardinality $X \cap Y <$ cardinality X.

Now enlarge I_X to a maximal right ideal N_X of R/J. Then $R/(N_X + J)$ is a simple R-module annihilated by $\{e_Y \mid X \neq Y \in \mathscr{K}\}$, but not by e_X. Thus $X \neq Y$ implies $R/(N_X + J)$ is not isomorphic to $R/(N_Y + J)$. Hence we have found cardinality $\mathscr{K} >$ cardinality $\mathscr{I}$ non-isomorphic simple R-modules, a contradiction.

We conclude that $\mathscr{I}$ is finite. Hence our $1 - 1$ correspondence between simple R-modules and simple modules in the socle of R/J must be onto the classes of simple R-modules. Then every simple R/J-module appears in its socle, and so is projective. Then R/J equals its socle, and so is semi-simple Artin. Since idempotents lift modulo J by Lemma 4, we get the required structure on R.

We have shown that if R_R is an injective cogenerator in $\mathfrak{M}_R$, then R has acc and dcc on direct summands. The following example shows that no other chain conditions need hold.

Example 1. An injective cogenerator without chain conditions.

Let $Z_{(p)}$ denote the p-adic integers for some prime p.

Define a ring R by $(R, +) = Z_{(p)} \oplus Z_{p^\infty}$, and for (λ, x), $(\mu, y) \in R$, $(\lambda, x)(\mu, y) = (\lambda\mu, \lambda y + \mu x)$. This multiplication is associative and distributes over addition (the verification uses the facts that $Z_{(p)} = \mathrm{Hom}_Z(Z_{p^\infty}, Z_{p^\infty})$, and $Z_{(p)}$ is a commutative ring).

Let I be a proper ideal of R. I may be any additive subgroup of Z_{p^∞}. If not, let $(\lambda, x) \in I$, $\lambda \neq 0$. Then $(\lambda, x)Z_{p^\infty} = Z_{p^\infty} \subseteq I$, and I/Z_{p^∞} is an ideal of $Z_{(p)}$. Such an ideal is of the form (p^i) for some $i \geqslant 0$, so $I = ((p^i, 0))$.

Thus R is a local ring with maximal ideal $((p, 0))$, and R contains a copy of its only simple module, namely the subgroup of Z_{p^∞} of order p. Thus by Lemma 1, if R is injective, it will be a cogenerator in M_R.

Let f be a map from an ideal I of R into R.

If $I \subseteq Z_{p^\infty}$, f maps I into the torsion subgroup of $(R, +)$, namely Z_{p^∞}. Since Z_{p^∞} is an injective group, f extends to an element $\lambda \in \mathrm{Hom}_Z(Z_{p^\infty}, Z_{p^\infty}) = Z_{(p)}$. Then $f(0, x) = (\lambda, 0)(0, x)$ for all $(0, x)$ in I.

If $I = ((p^i, 0))$, and $f(p^i, 0) = (0, x)$, then there is a $y \in Z_{p^\infty}$ such that $p^i y = x$. Then $f(p^i, 0) = (0, y)(p^i, 0)$, so $f(\lambda, z) = (0, y)(\lambda, z)$ for all $(\lambda, z) \in ((p^i, 0)) = I$.

If $I = ((p^i, 0))$, and $f(p^i, 0) = (\lambda, x)$, then $(0\colon (p^i, 0))$, the additive subgroup of order p^i, is annihilated by λ. Hence p^i divides λ. Then $f(p^i, 0) = (\lambda/p^i, y)(p^i, 0)$, where y is defined above. Then $f(\mu, z) = (\lambda/p^i, y)(\mu, z)$ for all $(\mu, z) \in I$.

Hence, in all cases, f is given by left multiplication, and R_R is injective by Baer's criterion (see Baer [2]).

In our proof of Theorem 1, the injectivity of R was used to get cyclic essential extensions of homogeneous components in the socle of R/J, which enabled us to generate too many simples if R/J were not semi-simple Artin. The following example shows that this injectivity is necessary for the conclusion of the theorem.

Example 2. A non-injective cogenerator in M_R with no chain conditions on direct summands.

Let R be an algebra over a field F with basis $\{1\} \cup \{e_i \mid i = 0, 1, 2, \ldots\} \cup \{x_i \mid i = 0, 1, 2, \ldots\}$ such that:

(i) 1 is a two-sided identity of R.

(ii) For all i and for all j, $e_i x_j = \delta_{i,j} x_j$ and $x_j e_i = \delta_{i,j-1} x_j$, $e_i e_j = \delta_{i,j} e_i$ and $x_i x_j = 0$.

Here $\delta_{i,j}$ is the Kronecker δ.

One easily verifies that this multiplication of basis elements associates, and that $J = (\{x_i \mid i \geqslant 0\})$. Moreover, $(R/J, +) = \sum \pi(e_i)F + 1F$, and the simple R-modules are precisely $\{\pi(e_i)R\}$ and $R/\sum e_i R$. Since these are isomorphic to $\{x_{i+1}R\}$ and $x_0 R$ respectively, R will be a cogenerator if each $e_i R$ is injective.

Let I be a right ideal of R, $f\colon I \to e_i R$. We observe that $(e_i R, +) = e_i F + x_i F$, so that $f = 0$ on $I \cap R(1 - e_i - e_{i-1})$, where $e_{-1} = 0$. Hence f can be non-zero only on $e_i F + e_{i-1} F + x_i F + x_{i+1} F$. Moreover, $x_{i+1} R$ is simple, but not isomorphic to $x_i R$, so f must be 0 on it. We conclude that f can be extended to $I + (1 - [e_i + e_{i-1}])R$ by defining it to be 0 on the last summand. If $e_{i-1} \notin I$, define $f(e_{i-1}) = 0$. Then f is extended to $I + (1 - e_i)R$. If $e_i \in I$,

f is extended to a map from $R \to e_iR$; if $x_i \notin I$ or $f(x_i) = 0$, define $f(e_i) = 0$; if not, $f(x_i) = x_i\nu$ for some $\nu \in F$, and we define $f(e_i) = e_i\nu$. In all cases, we have extended f to R, so e_iR is injective by Baer's criterion.

3. Morita Duality

In this section, S and T will denote fixed rings with identities such that there exists an $S - T$ bimodule U with ${}_SU$ and U_T injective cogenerators in ${}_S\mathfrak{M}$ and $\mathfrak{M}_T$ respectively, and $S = \mathrm{Hom}_T(U, U)$, $T = \mathrm{Hom}_S(U, U)$.

For $M \in \mathfrak{M}_T$ ($N \in {}_S\mathfrak{M}$), set $M^* = \mathrm{Hom}_T(M, U)$ ($N^* = \mathrm{Hom}_S(N, U)$). Then $M^* \in {}_S\mathfrak{M}$ ($N^* \in \mathfrak{M}_T$). For M, $N \in \mathfrak{M}_T$ (or ${}_S\mathfrak{M}$), and $\nu : M \to N$, set $\nu^* = \mathrm{Hom}(\nu, 1_U) : N^* \to M^*$. Then $*$ is a functor from $\mathfrak{M}_T$ to ${}_S\mathfrak{M}$, and from ${}_S\mathfrak{M}$ to $\mathfrak{M}_T$. There is a natural homomorphism $\varphi_M : M \to M^{**}$ given by $\varphi_M(x)(f) = f(x)$ (or xf) for all $x \in M$, $f \in M^*$. M is called reflexive if φ_M is an isomorphism. $*$ is a category anti-isomorphism between the categories of reflexive modules, since there $**$ is naturally equivalent to the identity under the transformation φ.

Proposition. The category $\mathfrak{R}$ of reflexive right T-modules (or reflexive left S-modules) contains T and U_T (S and ${}_SU$) and is closed under taking finite direct sums, submodules, and quotient modules.

Proof. This is a portion of Morita's Theorem 2.4 [*14*].

This proposition is the major reason for studying the situation in this section.

A structure theorem for T and S comparable to Theorem 1 is readily obtained. A ring R is called semi-perfect if R/J is semi-simple Artin and idempotents lift modulo J. Bass [*3*] showed that R is semi-perfect if and only if every simple R-module is of the form eR/eJ for some $e = e^2 \in R$.

Theorem 2. *S and T are semi-perfect.*

Proof. Let M_T be a simple T-module. Since T is reflexive and M is a quotient of T, M is also reflexive, and M^* is a simple S-submodule of $T^* \approx {}_SU$. Since ${}_SU$ is injective, $U = E(M^*) \oplus N$, where N is some S-submodule of U. Then $U^* \approx T \approx E(M^*)^* \oplus N^*$, so $E(M^*)^*$ is projective. Since M^* is the unique simple submodule of $E(M^*)$, by our category anti-isomorphism, $M^{**} \approx M$ is the unique simple quotient of $E(M^*)^*$. Since $E(M^*)^*$ is a direct summand of T, it is generated by an idempotent $e = e^2 \in T$ (Lemma 2), and since M is its only simple image, the kernel of the epimorphism to M

must be $eT \cap J = eJ$. Then $M \approx eT/eJ$. Thus T is semi-perfect, and by a symmetrical argument, so is S.

If we impose additional finiteness conditions on S and T, we get stronger results. We first introduce some terminology.

A module $M_R \subseteq N_R$ is called small in N_R if $N = M + K \Rightarrow N = K$ for all submodules K of N. A projective module P_R is called a projective cover of M_R if there is an epimorphism $\mu : P \to M$ such that kernel μ is small in P. Semi-perfect rings are precisely those for which every finitely generated module has a projective cover. A ring R is called right (left) perfect if every right (left) R-module has a projective cover.

LEMMA 9. *The following are equivalent:*

(i) *R is right perfect. (R is left perfect.)*

(ii) *R/J is semi-simple Artin and every cyclic left (right) R-module has non-zero socle.*

(iii) *R/J is semi-simple Artin and every non-zero right (left) R-module has a non-zero simple epimorphic image.*

(iv) *R/J is semi-simple Artin and if $\{a_i \mid i = 0, 1, \ldots\} \subseteq J$, there is an n such that $a_n a_{n-1} \cdots a_0 = 0$ ($a_0 a_1 \cdots a_n = 0$).*

Proof. All implications are in Bass [3]. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv) is from Bass' proof of Theorem P. (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) is remark 2, p. 470.

It is an immediate consequence of Lemma 9 that if R is right perfect and $J^n \neq 0$, then $J^n J = J^{n+1} \neq J^n$ since J^n has a non-trivial epimorphic image.

Let $\{A_n \mid n = 0, 1, \ldots\}$ be a family of finite sets, and F a family of functions $\{f_n : A_n \to \text{power set of } A_{n+1}\}$. The pair $(\{A_n\}, F)$ is called a graph. A path in this graph is a set of elements $\{a_m\}$ such that $a_0 \in A_0$, and $a_m \in f_{m-1}(a_{m-1})$ for $m \geqslant 1$.

LEMMA 10. *If the graph $(\{A_n\}, F)$ has arbitrarily long paths, then it has a path of infinite length.*

Proof. This lemma is known as König's Graph Theorem. We call $a_n \in A_n$ "good" if there are paths of arbitrary length containing a_n. For convenience we set $A_{-1} = \{a\}$ and set $f_{-1}(a) = A_0$. Then a is good by hypothesis. Now assume a_n is good. If every element of the finite set $f_n(a_n)$ is not good, then there is an upper bound on the lengths of paths through each of these elements, and hence the maximum of these upper bounds is an upper bound on the length of a path through a_n. This contradicts the assumption that a_n is good. We conclude that some element of $f_n(a_n)$ is good. We then get an infinite path by selecting a_0 a good element in $f_{-1}(A_{-1})$, and a_n a good element in $f_{n-1}(a_{n-1})$ by induction.

LEMMA 11. *Let R be left (right) perfect, and $(J/J^2)_R$ finitely generated. Then J is nilpotent, and R has dcc on right ideals.*

Proof. Let $(J/J^2)_R = \sum_{i=1}^{n} x_i'R$, and let $x_i \to x_i'$ in the natural map from $J \to J/J^2$. Let A_n be the set of all products $x_{i_0} \cdots x_{i_n} \neq 0$. $x_{j_0} \cdots x_{j_{n+1}} \in f_n(x_{i_0} \cdots x_{i_n})$ if and only if $j_k = i_k$ $(i_k = j_{k+1})$ for $0 \leqslant k \leqslant n$, and $x_{j_0} \cdots x_{j_{n+1}} \neq 0$. Since R is left (right) perfect, by Lemma 9, $(\{A_n\}, \{f_n\})$ can have no paths of infinite length. Hence, by Lemma 10, there is an integer N such that no product of N x_i's is non-zero. Assume $J^N \neq 0$. Then $J^{N+1} \neq J^N$, so there is a $y \in J^N$, $y \notin J^{N+1}$. Then y is a sum of products of N elements of J, and at least one of these products, say z, is not in J^{N+1}. Then z is a product of N elements in J, none of which are in J^2. Thus $z = z' + z''$, where $z'' \in J^{N+1}$, and $z' = r_0 x_{i_1} r_1 x_{i_2} \cdots x_{i_N} r_N$, where the $r_j \in R$, $r_j \notin J$. Now J/J^2 is an $R - R$ bimodule, so $r_j x_i' = \sum_{k=1}^{n} x_k' r_{i,j,k}$. Then modulo J^{N+1}, we may pull each r_j appearing in z' past all of the x_{i_k}, and get z' is congruent modulo J^{N+1} to a sum of terms of the form $x_{j_1} \cdots x_{j_N} r$. But all of these terms are zero, so $z' \in J^{N+1}$, a contradiction. We conclude that $J^N = 0$.

We observe that the method of proof used shows that $(J^i/J^{i+1})_R$ is generated by images of products of i x_j's. Since there are only a finite number of such products, for each i, $(J^i/J^{i+1})_R$ is a finitely generated R/J module and so has a composition series. Since there are only a finite number of non-zero J^i/J^{i+1}, these composition series combine to yield a composition series for R_R, so R has dcc on right ideals.

We now return to our rings S and T, and investigate what occurs under the hypothesis that either is perfect on one side.

LEMMA 12. *If T is right perfect, then S is right perfect. If S is left perfect, then T is left perfect.*

Proof. Let T be right perfect, I a left ideal of S. Then $(S/I)^*$ is a right T-module. By Lemma 9, it has a non-zero epimorphic image M. Then M^* is a non-zero simple sub-module of $(S/I)^{**}$ by Morita's proposition. By Lemma 9, S is right perfect. The second part follows by symmetry.

LEMMA 13. *No infinite direct sum of T-modules (S-modules) is reflexive.*

Proof. Let $M = \sum \oplus M_i$ be an infinite direct sum of right T-modules. Then $M^* = \prod M_i^*$. Let $X \in M^*$, $X \notin \sum M_i^*$. Enlarge $\sum M_i^*$ to a maximal submodule of $SX + \sum M_i^*$, say N. The map $SX + \sum M_i^* \to (SX + \sum M_i^*)/N \to U$ exists since U contains a copy of every simple S-module, and extends to a map from $M^* \to {}_SU$ since ${}_SU$ is injective. Call this map λ. By definition, $\lambda \in M^{**}$, and $(\sum M_i^*)\lambda = 0$. But for all $x \neq 0 \in M$, there is an $f \in \sum M_i^*$ such that $f(x) \neq 0$. Hence $\lambda \notin$ image of φ_M, so M is not reflexive.

THEOREM 3. *If S is left or right perfect, or if T is left or right perfect, then S and T have dcc on left and right ideals.*

Proof. Assume T is left or right perfect. By Morita's proposition, $(J/J^2)_T$ is reflexive. Since it is a T/J-module, it is a direct sum of simple submodules. By Lemma 13, that sum must be finite. Then, by Lemma 11, T has dcc on right ideals, and J is nilpotent. Then T is two-sided perfect, so by Lemma 12, S is right perfect. A symmetrical argument shows that for S right or left perfect, S has dcc on left ideals, and T is left perfect.

By Morita [*14*], Theorem 6.3, U is a finitely generated T (and S) module. Hence every simple right T-module (simple left S-module) is embeddable in an injective module of finite length. Then as Rosenberg and Zelinsky remark in [*19*], p. 376, their Theorem 1 implies that T has dcc on left ideals (S has dcc on right ideals).

The category $\mathfrak{N}_T({}_S\mathfrak{N})$ of reflexive T-modules (S-modules) has the property that any module which is injective in that category is injective in $\mathfrak{M}_T({}_S\mathfrak{M})$, since R and all its submodules are reflexive and * is an anti-isomorphism. Thus the dual of a module which is projective in ${}_S\mathfrak{N}(\mathfrak{N}_T)$ must be injective in $\mathfrak{M}_T({}_S\mathfrak{M})$ (see Morita [*14*], Theorem 2.5). If T has dcc on right ideals, the dual of any module which is injective in $\mathfrak{N}_T$ must be projective in ${}_S\mathfrak{M}$. For injectivity in $\mathfrak{N}_T$ implies injectivity in $\mathfrak{M}_T$, and the socle of M must be a finite direct sum of simple right T-modules by Lemma 13. Then M is a finite direct sum of injective hulls of simple modules, since M is an essential extension of its socle. Hence M^* is a (finite) direct sum of projective modules by the proof of Theorem 2. M^* is thus projective in ${}_S\mathfrak{M}$. In general, however, the dual of an injective, reflexive module need not be projective.

Example 3. A reflexive, injective module whose dual is not projective.

Let $S = T = U$ be the ring R constructed in Example 1, and let $Q_R = E(R/Z_{p^\infty})$. Q is characterized by the properties:

(i) If Λ denotes the p-adic number field, then Q is a $\Lambda - R$ bimodule.
(ii) ${}_\Lambda Q$ is one-dimensional.

Since any map from an ideal of R into Q must be 0 on Z_{p^∞} (Q has no elements of finite additive order), Q is R-injective if and only if it is $R/Z_{p^\infty} = Z_{(p)}$-injective. Hence (i) and (ii) hold for Q. Conversely, if i) holds for an R-module M, then M has no elements of finite additive order, so $MZ_{p^\infty} = 0$. Moreover, M is a divisible $Z_{(p)}$-module by (i), so M is an injective $Z_{(p)}$-module (see Cartan and Eilenberg [*4*], p. 134). (ii) then implies that M is indecomposable, so M must be the injective hull of any of its cyclic submodules; in particular, $M = E(Z_{(p)})$, as a $Z_{(p)}$-module. But this is the same as $E(Z_{(p)})$ as an R-module, so $M = Q$.

We now show that Q^* possesses properties (i) and (ii). Since R is commutative, the side on which we multiply by elements of R is immaterial.

(i) By property (i) for Q, $\Lambda \subseteq \mathrm{Hom}_R(Q, Q)$. Hence $Q^* = \mathrm{Hom}_R(Q, R)$ is a $\Lambda - R$ bimodule. (See Cartan and Eilenberg [4], p. 22).

(ii) Let $f, g \in Q^*$, $f \neq 0$, $g \neq 0$. Since Q is a divisible $Z_{(p)}$-module, so are $f(Q)$ and $g(Q)$. The only such modules $\subseteq R$ are 0 and Z_{p^∞}. Hence range $f =$ range $g = Z_{p^\infty}$. Since $Q = \bigcup_{i=-\infty}^{\infty} p^i Z_{(p)}$, for some i', $f(p^{i'})$ is of order p. Then $f' = p^{(i'+1)}f$ takes p^{-i} to an element of additive order p^i for $i \geqslant 0$. We may similarly find a Λ-multiple g' of g with the same property. We define a sequence $\{\mu_i\}$ in $Z_{(p)}$ by requiring μ_i to extend the map $g'(p^{-i}) \to f'(p^{-i})$ to an element of $\mathrm{Hom}_Z(Z_{p^\infty}, Z_{p^\infty}) = Z_{(p)}$. Then $\mu_i - \mu_{i+1}$ sends $g'(p^{-i})$, and hence the entire subgroup of order p^i, to 0. Thus $\mu_i - \mu_{i+1}$ is divisible by p^i in $Z_{(p)}$. Then $\{\mu_i\}$ is a Cauchy sequence in the complete valuation ring $Z_{(p)}$, so it has a unique limit μ in $Z_{(p)}$. (See Schilling [20], p. 31.) Then for all $i \geqslant 0$, $\mu g'(p^{-i}) = \mu_i g'(p^{-i}) = f'(p^{-i})$. Hence $\mu g' = f'$, and f is a Λ-multiple of g. Thus ${}_\Lambda Q^*$ is one-dimensional.

We conclude that $Q^* \approx Q$; hence Q is reflexive, since Q and Q^{**} are one-dimensional Λ-spaces, and the natural map between them is not 0; hence it is an isomorphism. But Q is clearly not projective, as any free module is an essential extension of its socle, and Q has zero socle.

4. Discussion of Sides

In Section 3, we assumed ${}_S U_T$ was a two-sided injective cogenerator with appropriate centralizer conditions. Any ring R has an injective cogenerator $U_R =$ the injective hull of a sum of representatives of each isomorphism class of simple R-modules. Yet not every ring has a duality of the kind discussed in Section 3. If $\Lambda = \mathrm{Hom}_R(U_R, U_R)$, then, for the duality of Section 3 to fail, either i) $R \neq \mathrm{Hom}_\Lambda({}_\Lambda U, {}_\Lambda U)$; or ii) ${}_\Lambda U$ does not contain a copy of every simple Λ-module; or iii) ${}_\Lambda U$ is not injective. All of these cases are possible.

(i) If $R =$ the ring of integers, $\mathrm{Hom}_\Lambda(U, U)$ is a direct product of copies of $Z_{(p)}$ for each prime p.

If R contains a copy of every simple R-module, then $U = E(R)$. By a theorem of Lambek ([12], p. 364) $\mathrm{Hom}_\Lambda(U, U)$ may be identified with $\{x \in E(R) \mid \text{for all } \lambda \in \Lambda, \lambda(1) = 0 \Rightarrow \lambda(x) = 0\}$. Moreover, if $x \notin R$, $xR + R$ has a simple homomorphic image whose kernel contains R, so $x \notin \mathrm{Hom}_\Lambda$ (U, U). Hence $R = \mathrm{Hom}_\Lambda(U, U)$.

(ii) For the ring of Example 2, $R = \mathrm{Hom}_\Lambda(U, U)$ by the above remark. Moreover, the left socle of $U =$ the right socle of U by a theorem of Azumaya

([1] Theorems 4 and 1). Hence no element in the socle of ${}_{\Lambda}U$ is annihilated by extension of left multiplication by every e_i, so ${}_{\Lambda}U$ cannot contain a copy of every simple Λ-module.

(iii) Let R be the local ring with dcc on right but not left ideals constructed in Rosenberg and Zelinsky [19], p. 375. Then $R = \mathrm{Hom}_{\Lambda}(U, U)$ by Lambek's theorem, and the socle of ${}_{\Lambda}U \neq 0$ by Azumaya's theorem. Hence every simple Λ-module must be contained in ${}_{\Lambda}U$ since Λ is a local ring, and so has only one isomorphism class of simple modules. Then ${}_{\Lambda}U$ cannot be injective.

If $U_R = R_R$, then $\Lambda = R$ and $R = \mathrm{Hom}_{\Lambda}(U, U)$. By Theorem 1, R/J is semi-simple Artin. Hence the number of simple left R-modules = the number of simple right R-modules $= n < \infty$. By Azumaya [1], Theorems 4 and 1, each homogeneous component of the socle of R_R is a homogeneous component of the socle of ${}_RR$ and conversely. Thus there are n distinct homogeneous components of the socle of ${}_RR$, so every simple left R-module is isomorphic to a minimal left ideal. Then R not a two-sided injective cogenerator implies ${}_RR$ is not injective. It is an open question whether this can occur, even if R is right or left perfect. This problem turns out to be closely related to a problem about division subrings of full linear rings.

Let F be a simple Artin ring; that is, a complete matrix ring, and let V_F be a right F-module of dimension $\aleph$. Set $L = \mathrm{Hom}_F(V_F, V_F)$, I = the identity of L, and let $\epsilon = \epsilon^2 \in L$ be such that ϵV_F is one-dimensional. We note that ${}_LL\epsilon \approx {}_LV$ and $\epsilon L_L \approx (\mathrm{Hom}_F(V, \epsilon V))_L$.

Let (P) be the statement:

For all F and for all $\aleph \geqslant \aleph_0$, if D is a division subring of L such that $I \in D$, then the dimension of $\epsilon L_D > \aleph$.

We observe that, if D_n is a matrix ring over D, the dimension of a D_n-module is a finite multiple of its dimension as a D-module.

Assume $L \supset D_n \supseteq FI$; then the dimension of ϵL_D = the cardinality of $F^{\aleph}$. For L can be identified with all column finite $\aleph \times \aleph$ matrices with entries in F; ϵL with first rows, and $L\epsilon$ with first columns. Since every transformation in D is invertible, none can have all zeros in the first column. Hence the dimension of D over the division ring F' = the multiples of the identity in F, is at most $\aleph$ = the dimension of the module of first columns over F'. But the dimension of the module of first rows over F' = cardinality $F'^{\aleph}$ = cardinality $F^{\aleph}$; hence that must be the dimension of ϵL over D (see Jacobson [11], p. 68). Thus (P) is not unreasonable.

Observation. Statement $(P) \Rightarrow$ if R is a right or left perfect ring and R_R is an injective cogenerator in $\mathfrak{M}_R$, then R is quasi-Frobenius.

To prove this, one uses injectivity and the coincidence of the left and right socles of R to identify the $R - R$ bi-module $\mathrm{Hom}_R((J/J^2)_R, (0 : J)_R)$ with

$(0 : J^2)/(0 : J)$ and Hom $((0 : J^2)/(0 : J)_R, (0 : J)_R)$ with an $R - R$ quotient of J/J^2. If the largest dimension of a homogeneous component of $(J/J^2)_R$ is infinite, then for some primitive idempotent e in R, a homogeneous component of eJ/eJ^2 has this dimension. This component is a unital right module over some matrix ring ideal of R/J, and a unital left module over a matrix ring ideal of R/J. We may then use (P) to get a homogeneous component of $((0 : J^2)/(0 : J))_R$ of larger dimension, and use this component to get a component of $(J/J^2)_R$ of even larger dimension, a contradiction. We conclude that $(J/J^2)_R$ must be finitely generated, so R is quasi-Frobenius by Lemma 11.

On the other hand, if ${}_{\epsilon}L_D$ and ${}_DL_{\epsilon}$ are one-dimensional, and D contains a subfield with first row, first column entries taking all values in F' and all other entries in row 1 or column 1 $= 0$, then there is a perfect one-sided injective cogenerator R with $(R, +) = D + F' + {}_DD_{F'} + {}_{F'}D_D + F' + D$, where pairs of summands represent R/J, J/J^2, and $J^2 = (0 : J)$, respectively.

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