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On Liaison, Arithmetical Buchsbaum Curves and Monomial Curves in $\mathbb{P}^3$

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1. INTRODUCTION AND MAIN RESULTS

In the context of liaison, by [18] and [19], for curves in $\mathbb{P}^3$, the property of being the zero scheme of a section of a rank two bundle on $\mathbb{P}^3$ is connected to that of being ideally the intersection of three surfaces. An immediate consequence of this property is that the homogeneous ideal $I(C)$ of a curve $C \subset \mathbb{P}^3$ is generated by precisely three elements if and only if C is ideally the intersection of three suraces and C is arithmetically Cohen–Macaulay (non-complete intersection). Furthermore, the characterization of the equivalence classes, defined by liaison among curves in $\mathbb{P}^3$, yields the well-known statement that a curve C is in the liaison class of a complete intersection if and only if C is arithmetically Cohen–Macaulay (see, e.g., [2–4, 10, 13, or 14]). It seems that A. Cayley (see [7, p. 152]) in 1847 was the first who posed the problem to describe this liaison class of a complete intersection. Now, in this paper we will begin to investigate the next simple case; that is, the liaison classes which are characterized by a finite-dimensional vector space of dimension $\geqslant 1$. Using the theory of Buchsbaum rings (see Section 2) this means we will study liaison among arithmetical Buchsbaum curves in $\mathbb{P}^3$.

First, from the point of view of local algebra, we get the following more general result:

THEOREM 1. *Let R be a local Gorenstein ring of dimension $d \geqslant 1$ with*

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maximal ideal $\mathfrak{m}$. *Suppose that the ideals* $\mathfrak{a}$, $\mathfrak{b} \subset R$ *are linked. Put* $t := \dim R/\mathfrak{a} = \dim R/\mathfrak{b}$. *Then we have*:

(a) $R/\mathfrak{a}$ *is a Buchsbaum ring if and only if* $R/\mathfrak{b}$ *is a Buchsbaum ring.*

(b) *For the local cohomology modules* $H^i_{\mathfrak{m}}(R/\mathfrak{a}) = \operatorname{Hom}(H^{t-i}_{\mathfrak{m}}(R/\mathfrak{b}), E)$ *for* $i = 1,\dots, t-1$, *where* E *is the injective hull of the residue field of* R, *provided* $H^i_{\mathfrak{m}}(R/\mathfrak{a})$, $i \neq t$, *has finite length.*

For instance, applying (a) (the theory of residual intersections) of Theorem 1 to the special case of curves, we get a simple proof for the statement that the twisted quartic curve C in $\mathbb{P}^3$, given parametrically by $\{t^4, t^3u, tu^3, u^4\}$, is arithmetically Buchsbaum. (F. S. Macauly [16] in 1916 was the first who showed that C is not arithmetically Cohen–Macauly.) This property follows since C and the union X of two skew lines in $\mathbb{P}^3$ are linked. We have for the defining ideals $\mathfrak{a}$ and $\mathfrak{b} = (x_0, x_1) \cap (x_2, x_3)$ of C and X, resp.,

$$\mathfrak{a} \cap \mathfrak{b} = (x_0x_3 - x_1x_2, x_0x_2^2 - x_1^2x_3).$$

It is easy to see that X is arithmetically Buchsbaum (see, e.g., [20, Corollary 12]). In order to obtain this new property of the curve C it was hitherto necessary to perform some nasty calculations (see, e.g., [29]). The liaison of C and X was discovered by G. Salmon [23, p. 40] in 1848 and again a little later in 1857 by J. Steiner [27, p. 138]. Using the theory of liaison and the classification of curves of low degree given by M. Noether [17] in 1882, we can construct new arithmetically Buchsbaum curves in $\mathbb{P}^3$. For example, let C_5^1 be any irreducible and non-singular curve in $\mathbb{P}^3$ of degree 5 and of genus 1. Then C_5^1 is arithmetically Buchsbaum since C_5^1 and Macaulay's curve C are linked by two surfaces of degree 3 (see [17, p. 83, (a_1)]). Furthermore we also get new statements concerning the classification of algebraic space curves. For instance, let C_6^3 be an irreducible and nonsingular curve in $\mathbb{P}^3$ of degree 6 and genus 3 with defining ideal $I(C_6^3)$ in $S := K[x_0, x_1, x_2, x_3]$, K an algebraically closed field. It follows from M. Noether [17, p. 87, (a_3) and (a_3')] and Theorem 1 that either C_6^3 is arithmetically Cohen–Macaulay or C_6^3 is arithmetically Buchsbaum. We note that already in 1881 F. Schur [26] discovered a first difference between the curves C_6^3 in order to distinguish between them. Nowadays the resolution of the curve C_6^3 is well known if C_6^3 is arithmetically Cohen–Macaulay. It is (see, e.g., G. Ellingsrud [8] or L. Gruson and C. Peskine [13])

$$0 \to S^3(-4) \to S^4(-3) \to S \to S/I(C_6^3) \to 0.$$

If C_6^3 is arithmetically Buchsbaum then we have obtained, in addition, the resolution of C_6^3:

$$0 \to S(-6) \to S^4(-5) \to S(-2) \oplus S^3(-4) \to S \to S/I(C_6^3) \to 0.$$

(See also Section 4, Example 5.)

Secondly, we will study certain curves in $\mathbb{P}^3$ with the property that their homogeneous ideals are generated by precisely four elements. We have the following theorem.

THEOREM 2. *Let $C \subset \mathbb{P}^3$ be any curve with defining ideal $I(C)$. The following conditions are equivalent*:

(i) *C is arithmetically Buchsbaum (non-Cohen–Macaulay) and C is ideally the intersection of three hypersurfaces, say, $f_1 = f_2 = f_3 = 0$.*

(ii) *There are homogeneous elements f_1, f_2, f_3, f_4, which provide a minimal base for $I(C)$, and $x_i f_4 \in (f_1, f_2, f_3)$ for $i = 0, 1, 2, 3$.*

In Section 4, two examples shed some light on the case when the homogeneous ideal $I(C)$ of a curve C in $\mathbb{P}^3$ is generated by precisely four elements. The proof of Theorem 2 yields the following Corollary.

COROLLARY 2.1. *Every liaison equivalence class, corresponding to a finite-dimensional vector space of dimension >1, does not contain any curve that is ideally the intersection of three surfaces and therefore contains no curves coming from sections of rank two bundles.*

(See also Example 3 in Section 4.)

Thirdly, we want to illustrate our investigations by studying the class of monomial space curves in $\mathbb{P}_K^3$; that is, curves $C \subset \mathbb{P}_K^3$, given parametrically by

$$\{s^d, s^b t^{d-b}, s^a t^{d-a}, t^d\},$$

where $d > b > a \geqslant 1$ are integers with g.c.d.$(d, b, a) = 1$. We put $S := K[x_0, x_1, x_2, x_3]$, and let $I(C) \subset S$ be a defining ideal of C. We set $R := S/I(C)$. $\mu(M)$ denotes the minimum number of elements in a basis of a module M. We have the following theorem.

THEOREM 3. *Let C be a monomial curve in $\mathbb{P}_K^3$ over an algebraically closed field K. Then the following conditions are equivalent*:

(a) *C is an arithmetically non-Cohen–Macaulay Buchsbaum curve.*

(b) *$\mu(I(C)) = 4$ and C lies on a quadric.*

(c) *$I(C) = (x_0 x_3 - x_1 x_2, x_0^2 x_2^{2n-1} - x_1^{2n+1}, \quad x_0 x_2^{2n} - x_1^{2n} x_3, \quad x_2^{2n+1} - x_1^{2n-1} x_3^2)$ for an integer $n \geqq 1$.*

(d) *C is given parametrically by $\{s^{4n}, s^{2n+1} t^{2n-1}, s^{2n-1} t^{2n+1}, t^{4n}\}$ for an integer $n \geqq 1$.*

(e) *There is an integer $n \geqq 1$ such that the minimal finite resolution of $S/I(C)$ over S has form*

$$0 \to S(-2n-3) \to S^4(-2n-2) \to S(-2) \oplus S^3(-2n-1) \to S \to S/I(C) \to 0.$$

(f) $\oplus_\nu H^1(\mathbb{P}^3, \mathcal{J}_C(\nu)) \simeq K(-2n+1)$ *for an integer* $n \geqq 1$.

(g) *C and the union of the two skew lines* $x_0 = x_1 = 0 \cup x_2 = x_3 = 0$ *are linked.*

In Section 4 we conclude by studying some examples and add some remarks.

2. Notations and Preliminary Results

First we will recall some basic facts on Buchsbaum rings. Denote by $e_0(\mathbf{x}, A)$ the multiplicity of a local noetherian ring A with respect to a system of parameters $\mathbf{x} = \{x_1, ..., x_d\}$, $d := \dim(A)$ of A.

Definition. A local noetherian ring A with maximal ideal $\mathfrak{m}$ is said to be a Buchsbaum ring if the difference

$$1_A(A/\mathbf{x}A) - e_0(\mathbf{x}, A)$$

is an invariant $i(A)$ of A, which does not depend on the system of parameters $\mathbf{x}$ of A. This is equivalent to the condition, that every system of parameters $\mathbf{x} = \{x_1, ..., x_d\}$ of A is a weak A-sequence, i.e.,

$$(x_1, ..., x_{i-1}) : x_i \subseteqq (x_1, ..., x_{i-1}) : \mathfrak{m}$$

for every $i = 1, ..., d$.

The concept of the theory of Buchsbaum rings was given in [28, 29] and has its origin in an answer given in [31] to a problem of D. A. Buchsbaum [6]. For more specific information on Buchsbaum rings see, e.g., [24, 30] or the forthcoming monograph [25]. In particular we have for a Buchsbaum ring A of dimension $d \geqslant 1$: $\mathfrak{m}H^i_{\mathfrak{m}}(A) = 0$, $\dim_{A/\mathfrak{m}} H^i_{\mathfrak{m}}(A) < \infty$ for all $i \neq d$, and

$$i(A) = \sum_{i=0}^{d-1} \binom{d-1}{i} \dim_{A/\mathfrak{m}} H^i_{\mathfrak{m}}(A),$$

where $H^i_{\mathfrak{m}}$ denotes the ith derived local cohomology functor with support $\{\mathfrak{m}\}$. Let k be any field. By a graded k-algebra we understand a noetherian graded ring $R = \oplus_{n \geqslant 0} R_n$ with $R_0 = k$ and the irrelevant ideal $\mathfrak{M} = \oplus_{n \geqslant 1} R_n$ being generated by R_1. We say that the graded k-algebra R is a Buchsbaum ring if the local ring $R_{\mathfrak{M}}$ is a Buchsbaum ring.

Now, we recall a criterion for a local ring $R/\mathfrak{a}$ to be a Buchsbaum ring, where $R/\mathfrak{a}$ is the quotient of a local Gorenstein ring of dimension n by an ideal $\mathfrak{a} \subset R$.

Let $E_R^{\cdot}$ be the minimal injective resolution of R over itself; i.e.,

$$E_R^i \simeq \bigoplus_{\substack{P \in \operatorname{Spec} R \\ \dim R/P = n-1}} E_R(R/P), \qquad i \in \mathbb{Z},$$

where $E_R(R/P)$ denotes the injective hull of R/P. Then we define

$$I_{\mathfrak{a}}^{\cdot} = \operatorname{Hom}_R(R/\mathfrak{a}, E_R^{\cdot}).$$

It follows by standard arguments that

$$I_{\mathfrak{a}}^i \simeq \bigoplus_{\substack{\mathfrak{p} \in \operatorname{Spec} R/\mathfrak{a} \\ \dim R/\mathfrak{p} = n-i}} E_{R/\mathfrak{a}}(R/\mathfrak{p}).$$

Therefore we get $I_{\mathfrak{a}}^i = 0$ for $i < \dim R - \dim R/\mathfrak{a}$ and $i > \dim R$. In fact, the complex $I_{\mathfrak{a}}^{\cdot}$ is the dualizing complex of $R/\mathfrak{a}$ in the sense of R. Hartshorne [14]. Note that in the derived category the complex $I_{\mathfrak{a}}^{\cdot}$ is isomorphic to $\mathbf{R}\operatorname{Hom}_R(R/\mathfrak{a}, R)$. If we abbreviate $g := \dim R - \dim R/\mathfrak{a}$, then

$$K_{\mathfrak{a}} := H^g(I_{\mathfrak{a}}^{\cdot}) = \operatorname{Ext}_R^g(R/\mathfrak{a}, R) \neq 0$$

is called the canonical or dualizing module of $R/\mathfrak{a}$, see [15]. Factoring out the first non-vanishing cohomology module $K_{\mathfrak{a}}$ of $I_{\mathfrak{a}}^{\cdot}$, we get a short exact sequence

$$0 \to K_{\mathfrak{a}}[-g] \to I_{\mathfrak{a}}^{\cdot} \to J_{\mathfrak{a}}^{\cdot} \to 0,$$

where $J_{\mathfrak{a}}^{\cdot}$ is up to a shift in grading the truncated dualizing complex of $R/\mathfrak{a}$ (see, e.g., [14]) and is such that

$$H^i(J_{\mathfrak{a}}^{\cdot}) = \begin{cases} H^i(I_{\mathfrak{a}}^{\cdot}), & i \neq g, \\ 0, & i = g. \end{cases}$$

Furthermore, we call a homomorphism $\psi^{\cdot}: M^{\cdot} \to N^{\cdot}$ of two complexes $M^{\cdot}$ and $N^{\cdot}$ a quasi-isomorphism, if $\psi^{\cdot}$ induces an isomorphism in the cohomology. Now we can state our criterion from [24].

LEMMA 1. *Let R be a local Gorenstein ring and let $\mathfrak{a}$ be an ideal of R. The local ring $R/\mathfrak{a}$ is a Buchsbaum ring if and only if the truncated dualizing complex $J_{\mathfrak{a}}^{\cdot}$ is quasi-isomorphic to a complex of k-vector spaces, where k denotes the residue field of R.*

Since a complex of k-vector spaces is isomorphic to its cohomology complex, we get for a Buchsbaum ring $R/\mathfrak{a}$,

$$J_{\mathfrak{a}}^{\cdot} \simeq \operatorname{Hom}_k(C^{\cdot}(R/\mathfrak{a}), k),$$

where $C^{\cdot}(R/\mathfrak{a})$ denotes the complex

$$C^i(R/\mathfrak{a}) = \begin{cases} H^i_{\mathfrak{m}}(R/\mathfrak{a}), & 0 \leqslant i < \dim R/\mathfrak{a}, \\ 0, & \text{otherwise}, \end{cases}$$

and with trivial differentiation.

Now, let C be a curve in $\mathbb{P}^3_K$; that is, an equidimensional subscheme of $\mathbb{P}^3_K$ of codimension two, which is a generic complete intersection over an algebraically closed field K. Let A be the local ring of the vertex of the (affine) cone over C. The curve C is called arithmetically Buchsbaum if the local ring A is a Buchsbaum ring. This local ring A is a Buchsbaum ring if and only if the local cohomoloby module $H^1_{\mathfrak{m}}(A)$ is a vector space over the residue field of A. This assertion results immediately from Lemma 1 (see also [20, Theorem 3] or [30, Corollary 1.1]). Therefore we set for an arithmetical Buchsbaum curve C in $\mathbb{P}^3$:

$$i(C) := i(A) = \dim_{A/\mathfrak{m}} H^1_{\mathfrak{m}}(A).$$

Next we review the definition and basic results of liaison, see [18] and [19]. Let I, J be two ideals of the local Gorenstein ring R.

DEFINITION. The ideals I, J are algebraically linked by a complete intersection $\mathbf{x} = \{x_1, \ldots, x_g\} \subseteq I \cap J$ if

(i) I and J are ideals of pure height g, and

(ii) $J/\mathbf{x}R \simeq \operatorname{Hom}_R(R/I, R/\mathbf{x}R)$ and $I/\mathbf{x}R \simeq \operatorname{Hom}_R(R/J, R/\mathbf{x}R)$.

Furthermore, I and J are linked (geometrically) by a complete intersection $\mathbf{x} = \{x_1, \ldots, x_g\}$, if I and J have no components in common and $I \cap J = \mathbf{x}R$. The projective varieties $X, Y \subseteq \mathbb{P}^n_K$ are linked if the ideal sheaves of X and Y are linked.

Let $C \subset \mathbb{P}^3_K$ be a curve. A. P. Rao [19] studied the following invariant, due to R. Hartshorne:

$$M(C) := H^1_{\mathfrak{m}}(S/I(C)),$$

where $I(C) \subset S := K[x_0, x_1, x_2, x_3]$ is the defining ideal of C. Furthermore, if two curves X and Y in $\mathbb{P}^3_K$ are linked by two hypersurfaces $F_1 = 0$ and $F_2 = 0$, then we have that

$$M(Y) \simeq \operatorname{Hom}_K(M(X)(f+g-4), K) \qquad \text{as } S\text{-modules},$$

where $f =$ degree of F_1 and $g =$ degree of F_2. For curves in $\mathbb{P}^3$ the Introduction has illustrated already the usefulness of the property of being ideally the intersection of three surfaces. We want to recall this definition and to give a more precise statement of Proposition 3.1 of [19].

DEFINITION. A projective variety $V \subset \mathbb{P}_K^n$ is said to be ideally the intersection of d hypersurfaces, if there exists a surjection

$$\bigoplus_{i=1}^{d} \mathcal{O}_{\mathbb{P}}(-a_i) \to \mathcal{I}_V \to 0$$

for some integers a_i, $i = 1, \ldots, d$.

This definition is equivalent to saying that there are homogeneous elements $f_1, \ldots, f_d$ in the defining ideal $I(V)$ of V such that $I(V)/(f_1, \ldots, f_d)$ is a $K[x_0, \ldots, x_n]$-module of finite length; that is,

$$V = \operatorname{Proj}(K[x_0, \ldots, x_n]/(f_1, \ldots, f_d)), \qquad \text{or} \qquad I(V)|_{x_i=1} = (f_1, \ldots, f_d)|_{x_i=1}$$
$$\text{for} \quad i = 0, 1, \ldots, n.$$

The same argument of [19], suitably refined, can be used to prove the following statement.

PROPOSITION 1. *If a curve C in $\mathbb{P}_K^3$ is ideally the intersection of the three surfaces $f_1 = f_2 = f_3 = 0$ of degrees f, g, h, resp., then we have*

$$M(C) \simeq \operatorname{Hom}_K(I(C)/(f_1, f_2, f_3))(f + g + h - 4), K)$$

as S-modules.

3. PROOFS

Proof of Theorem 1. We will use the theory of derived functors and categories, see, e.g., [14]. If $\mathfrak{a}$ and $\mathfrak{b}$ are linked by a complete intersection $\mathbf{x} = \{x_1, \ldots, x_g\}$, then we have for the canonical module $K_\mathfrak{a}$ of $R/\mathfrak{a}$,

$$K_\mathfrak{a} \simeq \operatorname{Hom}_R(R/\mathfrak{a}, R/\mathbf{x}R) \simeq \mathfrak{b}/\mathbf{x}R.$$

Hence we have the following exact sequence:

$$0 \to K_\mathfrak{a} \to R/\mathbf{x}R \to R/\mathfrak{b} \to 0.$$

Using the quasi-isomorphism

$$R/\mathbf{x}R \simeq \mathbf{R}\operatorname{Hom}_R(R/\mathbf{x}R, R)[g]$$

and the canonical map

$$\mathbf{R}\operatorname{Hom}_R(R/\mathfrak{a}, R) \simeq \mathbf{R}\operatorname{Hom}_R(R/\mathbf{x}R, R),$$

induced by the canonical epimorphism

$$R/\mathbf{x}R \to R/\mathfrak{a} \to 0,$$

we get the following commutative diagram of complexes with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & K_{\mathfrak{a}}[-g] \arrow[r] & \mathbf{R}\operatorname{Hom}_R(R/\mathbf{x}R, R) \arrow[r] & R/\mathfrak{b}[-g] \arrow[r] & 0 \\ 0 \arrow[r] & K_{\mathfrak{a}}[-g] \arrow[r] \arrow[u, equal] & I^{\cdot}_{\mathfrak{a}} \arrow[r] \arrow[u] & J^{\cdot}_{\mathfrak{a}} \arrow[r] \arrow[u, "\phi"'] & 0, \end{tikzcd}$$

where ϕ is defined in an obvious manner. By applying the derived functor $\mathbf{R}\operatorname{Hom}_R(\ , R)$ and using the local duality theorem (see [14]) we obtain the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{R}\operatorname{Hom}_R(R/\mathfrak{b}, R)[g] \arrow[r] \arrow[d, "\phi"] & R/\mathbf{x}R \arrow[r] \arrow[d] & \mathbf{R}\operatorname{Hom}_R(K_{\mathfrak{a}}, R)[g] \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbf{R}\operatorname{Hom}_R(J^{\cdot}_{\mathfrak{a}}, R) \arrow[r] & R/\mathfrak{a} \arrow[r] & \mathbf{R}\operatorname{Hom}_R(K_{\mathfrak{a}}, R)[g] \arrow[r] & 0. \end{tikzcd}$$

Now we wil show that ϕ induces a quasi-isomorphism between $J^{\cdot}_{\mathfrak{b}}[g]$ and $\mathbf{R}\operatorname{Hom}_R(J^{\cdot}_{\mathfrak{a}}, R)$. Therefore we consider the induced homomorphisms on the homology modules. Hence we get the commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ext}^g_R(R/\mathfrak{b}, R) \arrow[r] \arrow[d] & R/\mathbf{x}R \arrow[r] \arrow[d] & \operatorname{Ext}^g_R(K_{\mathfrak{a}}, R) \arrow[r] \arrow[d, equal] & \operatorname{Ext}^{g+1}_R(R/\mathfrak{b}, R) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \operatorname{Hom}_R(J^{\cdot}_{\mathfrak{a}}, R) \arrow[r] & R/\mathfrak{a} \arrow[r] & \operatorname{Ext}^g_R(K_{\mathfrak{a}}, R) \arrow[r] & \operatorname{Ext}^1_R(J^{\cdot}_{\mathfrak{a}}, R) \arrow[r] & 0 \end{tikzcd}$$

and isomorphisms for $i \geqslant 1$

$$\begin{tikzcd} \operatorname{Ext}^{g+i}_R(K_{\mathfrak{a}}, R) \arrow[r, "\simeq"] \arrow[d, equal] & \operatorname{Ext}^{g+i+1}_R(R/\mathfrak{b}, R) \arrow[d] \\ \operatorname{Ext}^{g+i}(K_{\mathfrak{a}}, R) \arrow[r, "\simeq"] & \operatorname{Ext}^{i+1}(J^{\cdot}_{\mathfrak{a}}, R). \end{tikzcd}$$

Using $\operatorname{Ext}^g_R(R/\mathfrak{b}, R) \simeq \mathfrak{a}/\mathbf{x}R$ we get from the above diagram:

$$\operatorname{Hom}_R(J^{\cdot}_{\mathfrak{a}}, R) = 0 \qquad \text{and} \qquad \operatorname{Ext}^{g+1}(R/\mathfrak{b}, R) \simeq \operatorname{Ext}^1_R(J^{\cdot}_{\mathfrak{a}}, R).$$

Hence ϕ induces a quasi-isomorphism

$$J_{\mathfrak{b}}^{\cdot}[g] \simeq \mathbf{R}\,\mathrm{Hom}_R(J_{\mathfrak{a}}^{\cdot}, R).$$

Note that $J_{\mathfrak{a}}^{\cdot}$ is a new invariant under liaison. Assume now that $R/\mathfrak{a}$ is a Buchsbaum ring. Then $J_{\mathfrak{a}}^{\cdot}$ and also $\mathbf{R}\,\mathrm{Hom}_R(J_{\mathfrak{a}}^{\cdot}, R)$ are quasi-isomorphic to complexes of k-vector spaces (see Lemma 1). We again apply Lemma 1 and get that $R/\mathfrak{b}$ is a Buchsbaum ring. Replacing $\mathfrak{a}$ by $\mathfrak{b}$ we obtain the converse. For the proof of Theorem 1 the statement about the local cohomology of $R/\mathfrak{a}$ and $R/\mathfrak{b}$ remains to be shown. This follows immediately from the local duality theorem. Q.E.D.

Remark. Using the fact that $R/\mathfrak{a}$ is a Cohen–Macaulay ring if and only if $J_{\mathfrak{a}}^{\cdot} \simeq 0$ we get another proof that liaison respects the Cohen–Macaulay property.

Proof of Theorem 2. We need two lemmas.

LEMMA 2. *Let $C \subset \mathbb{P}_K^3$ be an arithmetically Buchsbaum curve with invariant $i(C) \geqslant 1$. Then we get for the defining ideal $I(C)$ of C,*

$$\mu(I(C)) \geqslant 3i(C) + 1.$$

LEMMA 3. *In addition to the hypothesis of Lemma 2, suppose that C is ideally the intersection of three surfaces. Then we have*

$$\mu(I(C)) = 4 \qquad \text{and} \qquad i(C) = 1.$$

Proof of Lemma 2. We set $S := K[x_0, x_1, x_2, x_3]$ and $i := i(C)$. Tensoring the Koszul resolution of K we get the following minimal free resolution of the Rao invariant $M(C) \simeq \bigoplus_i K$:

$$0 \to L_4 \to L_3 \to L_2 \to L_1 \to L_0 \to M(C) \to 0,$$

where L_j are free S-modules of rank $i \cdot \binom{4}{j}$, $0 \leqslant j \leqslant 4$. Aplying Theorem 2.5 of [19] we conclude that a minimal resolution of $S/I(C)$ is

$$0 \to L_4 \to L_3 \to \oplus \bigoplus_{1}^{r} S(-1_i) \to \bigoplus_{1}^{m} S(-e_i) \to S \to S/I(C) \to 0,$$

where $m = \mu(I(C))$. Therefore the Euler–Poincaré characteristic of the vector spaces of this resolution yields

$$\mu(I(C)) = 3i + 1 + r \geqslant 3i + 1, \qquad \text{Q.E.D.}$$

Proof of Lemma 3. It follows from Proposition 3.1 of [19] that $\mu(I(C)) \leqslant 3 + i(C)$. This statement and Lemma 2 provide the assertion. Q.E.D.

Now we prove the implication (i) $\Rightarrow$ (ii) of Theorem 2. Lemma 3 shows that $i(C) = 1$. Therefore we get from Proposition 3.1 of [19] that the S-module $I(C)/(f_1, f_2, f_3)$ has precisely one generator, which provides one basis element, say, f_4, of $I(C)$. Thus $I(C) = (f_1, f_2, f_3, f_4)$ and $\mu(I(C)) = 4$. Furthermore,

$$(x_0, x_1, x_2, x_3)\, I(C)/(f_1, f_2, f_3) = 0$$

since C is arithmetically Buchsbaum; that is, $x_i f_4 \in (f_1, f_2, f_3)$ for $i = 0, 1, 2, 3$.

(ii) $\Rightarrow$ (i) of Theorem 2. We have $(x_0, x_1, x_2, x_3) = (f_1, f_2, f_3){:}f_4$. It follows that (f_1, f_2, f_3) equals (f_1, f_2, f_3, f_4) up to a primary component belonging to (x_0, x_1, x_2, x_3); that is, C is ideally the intersection of the three hypersurfaces $f_1 = f_2 = f_3 = 0$. We get from Proposition 3.1 of [19] that $I(C)/(f_1, f_2, f_3)$ differs from the first local cohomology group $H^1_{\mathfrak{m}}(A)$ only by grading and duality, where A is the local ring of the (affine) cone over C at the vertex. The assumption shows that $(x_0, x_1, x_2, x_3)\, I(C)/(f_1, f_2, f_3) = 0$ and therefore we get that C is arithmetically Buchsbaum (non-Cohen–Macaulay). This concludes the proof of Theorem 2. Q.E.D.

Proof of Corollary 2.1. Lemma 3.

The proof of Theorem 2 yields the following.

Remark. Let $C \subset \mathbb{P}^3$ be a curve. Assume that C is ideally the intersection of three hypersurfaces, say, $f_1 = f_2 = f_3 = 0$, and $\mu(I(C)) = 4$. If C is arithmetically Buchsbaum, then there are homogeneous elements f_1, f_2, f_3, f_4 such that $I(C) = (f_1, f_2, f_3, f_4)$ and the first module of syzygies of (f_1, f_2, f_3, f_4) yields a linear second syzygy.

For the proof of Theorem 3 we need some preliminary results.

Remark. Assume C is our arbitrary monomial curve C given parametrically by

$$\{s^d, s^b t^{d-b}, s^a t^{d-a}, t^d\},$$

where $d > b > a \geqq 1$ are integers with g.c.d. $(a, b, d) = 1$. Then the forms in a minimal basis of the defining ideal $I(C)$ can be chosen to be of the following types:

1. There is exactly one form of the type

$$F_1 = x_1^{\alpha} - x_0^{\alpha - (\alpha_2 + \alpha_3)} x_2^{\alpha_2} x_3^{\alpha_3} \quad \text{resp.}$$

$$F_2 = x_2^{\beta} - x_0^{\beta_0} x_1^{\beta_1} x_3^{\beta - (\beta_0 + \beta_1)}.$$

2. There are forms of the type

$$G = x_0^{\gamma_0} x_3^{\gamma_3} - x_1^{\gamma_1} x_2^{\gamma_2}, \qquad \min \gamma_i > 0,$$
$$\gamma_1 < \alpha \qquad \text{and} \qquad \gamma_2 < \beta.$$

3. There are forms of the type

$$H = x_0^{\delta_0} x_2^{\delta_2} - x_1^{\delta_1} x_3^{\delta_3}, \qquad \min \delta_i > 0,$$
$$\delta_1 < \alpha \qquad \text{and} \qquad \delta_2 < \beta.$$

This follows easily by looking at the generic point of C (see, for example, [5, Lemma 2)]). (In [5] there is an algorithm for computing the base of our arbitrary monomial curve C.)

Next we want to relate the perfectness of C to the number of generators $\mu(I(C))$.

LEMMA 4. *Let C be a monomial curve in $\mathbb{P}_k^3$. Then C is arithmetically Cohen–Macaulay if and only if $\mu(I(C)) \leqq 3$.*

Proof. First of all let us assume C is perfect. Then we consider the homogeneous coordinate ring of C. In the case when C is perfect, we get that R is a two-dimensional Cohen–Macaulay ring. In particular, we have $R = R_{(x_0)} \cap R_{(x_3)}$, because $\{x_0, x_3\}$ is a homogeneous sytem of parameters of R, and therefore $R_{(x_0)} \cap R_{(x_3)}/R \simeq H_{\mathfrak{m}}^1(R) = 0$ in the Cohen–Macaulay case. That is, there is no quotient $\in R_{(x_0)} \cap R_{(x_3)}$ which is not contained in R. In other words, there does not exist any form of type H contained in $I(C)$. Now assume that there are at least two different forms $G = x_0^p x_3^q - x_1^m x_2^n$ and $G' = x_0^{p'} x_3^{q'} - x_1^{m'} x_2^{n'}$ of the second type. Without loss of generality we may assume $p > p'$ and $q < q'$. By multiplying with $x_3^{q'-q}$ resp. with $x_0^{p-p'}$ we get

$$x_1^m x_2^n x_3^{q'-q} - x_0^{p-p'} x_1^{m'} x_2^{n'} \in I(C).$$

Because $I(C)$ is a prime ideal it follows that there exists a form of type H. But this contradicts the Cohen–Macaulayness of R. For the converse we remark that $I(C)$ is in particular ideally the intersection of three hypersurfaces. By virtue of Proposition 1 we see that $M(C) = 0$, i.e., C is a perfect curve. Q.E.D.

As in Lemma 4 one may hope that there are similar results relating the Buchsbaum property to bigger numbers of generators. A result in this direction, which is known to us, is the following.

COROLLARY. *Let C be a monomial curve. Then $\mu(I(C)) = 4$ if and only if $\mu(H_{\mathfrak{m}}^1(R)) = 1$.*

Proof. Assume $\mu(I(C)) = 4$. Then first of all two forms of $I(C)$ are defined by F_1 and F_2, see the Remark. Now assume there exist two forms of type G. Then the discussion, given in the proof of Lemma 4, yields the existence of a form of type H in $I(C)$, which contradicts $\mu(I(C)) = 4$. Similarly the existence of two forms of type H results in a contradiction. Therefore there is exactly one form of type H and one form of type G. Because $H^1_{\mathfrak{m}}(R) \simeq R_{(x_0)} \cap R_{(x_3)}$, it follows readily that $\mu(H^1_{\mathfrak{m}}(R)) = 1$.

Now assume $\mu(H^1_{\mathfrak{m}}(R)) = 1$. It is clear that there is exactly one form of type H. If there are two forms of type G, then either there are two forms of type H or, as in Example 6, Section 4, $\mu(H^1_{\mathfrak{m}}(R)) \geqslant 2$. Thus, since there is exactly one form of each type, $\mu(I(C)) = 4$.

Under the additional assumption that C lies on a quadric, it will follow that C is arithmetically Buchsbaum (see Theorem 3).

LEMMA 5. *If C is arithmetically non-Cohen–Macaulay and C lies on a quadratic, then the defining equation of this quadric is given by* $x_0x_3 - x_1x_2 = 0$.

Proof. Using the above-mentioned remark and the parametric representation of C we get the following possibilities for the defining equation of the quadric:

(i) $x_0x_2 - x_1^2 = 0$

(ii) $x_0x_3 - x_1^2 = 0$

(iii) $x_0x_3 - x_2^2 = 0$

(iv) $x_1x_3 - x_2^2 = 0$

(v) $x_0x_3 - x_1x_2 = 0$.

We will show that C is arithmetically Cohen–Macaulay if the defining equation of the quadric has the form (i), (ii), (iii) or (iv). There are several possibilities to prove this assertion. For instance, we get from [5] that $\mu(I(C)) = 2$, if we have the cases (ii) and (iii). Having (i) or (iv) we obtain $\mu(I(C)) \leqq 3$. Therefore Lemma 4 yields that C is arithmetically Cohen–Macaulay. Another possibility is to apply the same methods used for proving Lemma 4 and (a) $\Rightarrow$ (c) of Theorem 3. We then get $H^1_{\mathfrak{m}}(S/I(C)) = 0$ if we have the cases (i),..., (iv). Q.E.D.

The proof of Lemma 5 yields the following interesting fact.

COROLLARY. *Let C be a monomial curve in $\mathbb{P}^3$. If C lies on a quadric cone, then C is arithmetically Cohen–Macaulay.*

Proof of Theorem 3. We will show the following implications:

$$\begin{tikzcd}[column sep=small] (\mathrm{a}) \arrow[r, Rightarrow] & (\mathrm{c}) \arrow[r, Rightarrow] & (\mathrm{d}) \arrow[r, Rightarrow] & (\mathrm{g}) \arrow[r, Rightarrow] & (\mathrm{e}) \arrow[r, Rightarrow] & (\mathrm{b}) \arrow[r, Rightarrow] & (\mathrm{a}) \\ & (\mathrm{f}) \arrow[ul, Rightarrow] \arrow[u, Rightarrow] & & & & & \end{tikzcd}$$

The difficulty lies in proving the implications (a) $\Rightarrow$ (c) and (b) $\Rightarrow$ (a). The other implications are more or less clear.

(a) $\Rightarrow$ (c): Because C is not an arithmetically Cohen–Macaulay curve, there exists at least one form $H = x_0^{\delta_0}x_2^{\delta_2} - x_1^{\delta_1}x_3^{\delta_3}$ contained in a minimal generating set of $I(C)$; see the discussion in the proof of Lemma 4. Therefore $\xi := x_1^{\delta_1}/x_0^{\delta_0} = x_2^{\delta_2}/x_3^{\delta_3} \in R_{(x_0)} \cap R_{(x_3)}/R \simeq H_{\mathfrak{m}}^1(R)$ defines a non-zero element of the first local cohomology group. Since R is a Buchsbaum ring, $H_{\mathfrak{m}}^1(R)$ is annihilated by $\mathfrak{m}$, i.e., $x_i\xi \in R$ for $i = 0, 1, 2, 3$. In particular, it yields

$$H = x_0x_2^{\gamma} - x_1^{\gamma}x_3, \qquad \gamma \geqq 1.$$

We choose γ minimal with respect to the property $H \in I(C)$. Then it follows easily that

$$F_1 = x_1^{\gamma+1} - x_0^{\gamma+1-(\alpha_2+\alpha_3)}x_2^{\alpha_2}x_3^{\alpha_3}, \qquad \gamma \geqq \alpha_2 + \alpha_3,$$

and

$$F_2 = x_2^{\gamma+1} - x_0^{\beta_0}x_1^{\beta_1}x_3^{\gamma+1-(\beta_0+\beta_1)}, \qquad \gamma \geqq \beta_0 + \beta_1.$$

Next we consider the quotients

$$\frac{x_1^{\gamma+1}}{x_0} = x_0^{\gamma-(\alpha_2+\alpha_3)}x_2^{\alpha_2}x_3^{\alpha_3} = \frac{x_1x_2^{\gamma}}{x_3}$$

and

$$\frac{x_2^{\gamma+1}}{x_3} = x_0^{\beta_0}x_1^{\beta_1}x_3^{\gamma-(\beta_0+\beta_1)} = \frac{x_1^{\gamma}x_2}{x_0}.$$

It yields two forms

$$\left.\begin{array}{l} x_0^{\gamma-(\alpha_2+\alpha_3)}x_3^{\alpha_3+1} - x_1x_2^{\gamma-\alpha_2} \\ x_0^{\beta_0+1}x_3^{\gamma-(\beta_0+\beta_1)} - x_1^{\gamma-\beta_1}x_2 \end{array}\right\} \qquad (*)$$

contained in $I(C)$. By comparing both x_1, x_2-terms we get a form

$$x_0^{\beta_0+1}x_2^{\gamma-\alpha_2-1}x_3^{\gamma-(\beta_0+\beta_1)} - x_0^{\gamma-(\alpha_2+\alpha_3)}x_1^{\gamma-\beta_1-1}x_3^{\alpha_3+1}$$

contained in the prime ideal $I(C)$. By factoring out irrelevant terms we get a form H' of type H, as can be easily seen. By the Buchsbaum property of R we see as above that if $H' \neq 0$, then $H' = x_0x_2^{\gamma'} - x_1^{\gamma'} - x_1^{\gamma'}x_3$ with $\gamma' < \gamma$. But this contradicts the minimality of γ. Therefore, both forms in $(*)$ have to be equal. This gives $\gamma-\beta_1=\gamma-\alpha_2=1$ and $\beta_0=\alpha_3=0$, i.e., the form is given by

$$x_0x_3-x_1x_2.$$

Because $x_0x_3-x_1x_2\in I(C)$, it follows that there is no further form of the type G. Also there is no further form of the type H contained in $I(C)$, because it would have to be equal to $x_0x_2^\alpha-x_1^\alpha x_3$ with $\alpha>\gamma$ by virtue of the Buchsbaum property of R. But $\alpha>\gamma$ is not possible, because this reduces modulo F_1 resp. F_2. Therefore we get the forms

$$F_1=x_1^{\gamma+1}-x_0^2x_2^{\gamma-1},\qquad F_2=x_2^{\gamma+1}-x_1^{\gamma-1}x_3^2,$$
$$G=x_0x_3-x_1x_2,\qquad \text{and}\qquad H=x_0x_2^\gamma-x_1^\gamma x_3$$

as a minimal generating set contained in $I(C)$. Because F_1 resp. F_2 must be irreducible, we get $\gamma=2n$. Therefore the proof of (a) $\Rightarrow$ (c) is complete.

(c) $\Rightarrow$ (d): Trivial.

(d) $\Rightarrow$ (g): Applying Bezout's theorem we get that

$$I(C)\cap(x_0,x_1)\cap(x_2,x_3)=(x_0x_3-x_1x_2,x_0x_2^{2n}-x_1^{2n}x_3)$$

for all $n\geqq 1$ since the degree of C is given by $4n$; that is, C and the skew lines $x_0=x_1=0$ and $x_2=x_3=0$ are linked.

(g) $\Rightarrow$ (e): From (g) follows (a) by applying Theorem 1. Hence from (d) we obtain that C and the union of the two skew lines are linked by two hypersurfaces of degree 2 and $2n+1$. Therefore the resolution of Example 5, Section 4, yields the assertion (e).

(e) $\Rightarrow$ (b): The assertion results immediately from the term

$$S(-2)\oplus S^3(-2n-1).$$

(b) $\Rightarrow$ (a): Lemma 4 and Lemma 5 show that C lies on the quadric with defining equation $x_0x_3-x_1x_2=0$. Our Example 7 in Section 4 provides the elements of a minimal base for $I(C)$. Now (a) follows from our Theorem 2.

(c) $\Rightarrow$ (f): It is easy to see that C is ideally the intersection of the three hypersurfaces $x_0x_3-x_1x_2=0$, $x_0^2x_2^{2n-1}-x_1^{2n+1}=0$, $x_2^{2n+1}-x_1^{2n-1}x_3^2=0$. Hence Proposition 1 proves the claim (f) about the Rao invariant of C.

(f) $\Rightarrow$ (a) is trivial, since $H^1_{\mathfrak{m}}(S/I(C))$ is a vector space of dimension 1. This concludes the proof of Theorem 3. Q.E.D.

Remark. We note that S. Gotto [11] proves a Buchsbaum criterion for an arbitrary affine semigroup ring using the main results of [12]. In fact, S.

Goto [11] computes the first local cohomology module $H^1_{\mathfrak{m}}$ of such a Buchsbaum ring in terms of the underlying semigroup. But his methods do not provide a proof for Theorem 3.

4. Examples and Applications

We start with two examples of curves in $\mathbb{P}^3$ with $\mu(I(C)) = 4$, which are related to Theorem 2.

1. Take the curve C, given parametrically by $\{s^7, s^5t^2, st^6, t^7\}$. The paper [5] again yields that $I(C) = (f_1, f_2, f_3, f_4)$, where

$$f_1 = x_0^2 x_2 - x_1^3, \qquad f_2 = x_0 x_3^2 - x_1 x_2^2,$$
$$f_3 = x_0 x_2^3 - x_1^2 x_3^2, \qquad \text{and} \qquad f_4 = x_1 x_3^4 - x_2^5.$$

Theorem 3 shows that C is not arithmetically Buchsbaum. It is easy to see that C is ideally the intersection of the three surfaces with defining equations

$$f_1 = f_2 = f_4 = 0.$$

This means that curves C in $\mathbb{P}^3$ with $\mu(I(C)) = 4$ and the property of being ideally the intersection of three surfaces are not in general arithmetically Buchsbaum.

2. We want to give an example of a curve in $\mathbb{P}^3$ such that $\mu(I(C)) = 4$ and C is arithmetically non-Cohen–Macaulay Buchsbaum but C is not ideally the intersection of three surfaces. The explicit description of such a curve, we believe, is not clear. The possibility of such a construction by using the theory of finite free resolutions was suggested to us by David Eisenbud. Let C be the curve in $\mathbb{P}^3$ with defining ideal

$$I(C) = (x_1 x_2 x_3 (x_0 x_2 - x_1 x_3), x_1^2 x_3 (x_3^2 - x_0^2), x_1^2 x_2 (x_0 x_1 - x_2 x_3),$$
$$x_3 (x_0^2 - x_1 x_3)(x_0 x_2 - x_3^2)) =: (f_1, f_2, f_3, f_4).$$

An easy computation now shows that the minimal free resolution of $S/I(C)$ $(S := k[x_0, x_1, x_2, x_3])$ has the form

$$0 \longrightarrow S(-8) \xrightarrow[\mathfrak{A}]{} S^4(-7) \xrightarrow[\mathfrak{B}]{} S^4(-5) \xrightarrow[\mathfrak{L}]{} S \longrightarrow S/I(C) \longrightarrow 0,$$

with $\mathfrak{A} = (x_0, x_1, x_2, x_3)$, $\mathfrak{L} = (f_1, f_2, f_3, f_4)$, and

$$\mathfrak{B} = \begin{pmatrix} x_0 x_1 & x_2^2 & x_3^2 & 0 \\ x_3^2 - x_0^2 & 0 & 0 & x_1 x_2 \\ 0 & x_1 x_3 - x_0 x_2 & 0 & -x_1^2 \\ -x_1 x_3 & -x_1 x_2 & -x_0 x_3 & 0 \end{pmatrix}.$$

Applying Theorem (2.5) of [19] and our remark after the proof of Corollary 2.1 we get from this explicit resolution that C is arithmetically Buchsbaum and C is not ideally the intersection of three surfaces.

3. Using the existence theorems of A. P. Rao [19] or P. A. Griffith and E. G. Evans [9] we can describe abstractly irreducible curves in $\mathbb{P}^3$, which lie in a liaison equivalence class, corresponding to any finite-dimensional vector space. More concretely we want to show that, for instance, there exist irreducible curves C_{42}^{145} in $\mathbb{P}_k^3$ of degree 42 and of genus 145, which belong to the liaison equivalence class corresponding to a vector space of dimension 3. In proving this, we want to mention that K. Rohn [22] studied the residual intersection of special classes of space curves lying on any surface of degree 4 in 1897. From these specific data (see [22, p. 660] we get that the rational twisted cubic curve C_3^0, counted with multiplicity 6, is linked to irreducible curves C_{42}^{145} by two hypersurfaces of degree 4 and 15. Let $\mathfrak{p}$ be the defining prime ideal of C_3^0; that is,

$$\mathfrak{p} = (x_0x_2 - x_1^2, x_0x_3 - x_1x_2, x_1x_3 - x_2^2) =: (f_1, f_2, f_3)$$

in $k[x_0, x_1, x_2, x_3]$. Put $R = k[x_0, x_1, x_2, x_3]_{(x_0, x_1, x_2, x_3)}$. Let $n \geqq 1$ be an integer. Then it is easy to see that $R/\mathfrak{p}^n$ is a (local) Buchsbaum ring if and only if $n \leqq 3$, in which case the invariants of the Buchsbaum rings are given by: $i(R/\mathfrak{p}) = 0$, $i(R/\mathfrak{p}^2) = 1$, and $i(R/\mathfrak{p}^3) = 3$. Furthermore, $\mathfrak{p}^2$ and $\mathfrak{p}^3$ are the defining ideals of C_3^0, counted with multiplicity 3 and 6, resp. In proving this we make use of the fact that $\mathfrak{p}^2$ and $\mathfrak{p}^3$ are primary (see, e.g., [4]).

Next we are looking for examples related to Theorem 1.

4. The following shows the usefulness of Theorem 1 in projective n-space $\mathbb{P}^n$ with $n \geqq 4$. Take the surface F in $\mathbb{P}^5$ with defining ideal

$$\begin{aligned}\mathfrak{a} = {} & (x_0, x_1, x_2) \cap (x_1, x_2, x_3) \cap (x_2, x_3, x_4) \cap (x_3, x_4, x_5) \\ & \cap (x_4, x_5, x_0) \cap (x_5, x_0, x_1).\end{aligned}$$

Using Theorem 1 we will prove that F is arithmetically Buchsbaum, which also follows from [30, Theorem 3]. For this let G be the surface in $\mathbb{P}^5$ with defining ideal

$$\mathfrak{b} = (x_0, x_2, x_4) \cap (x_1, x_3, x_5).$$

Then F is linked to G since $\mathfrak{a} \cap \mathfrak{b} = (x_0x_3, x_1x_4, x_2x_5)$. It follows immediately from [20, Corollary 12] that G is arithmetically Buchsbaum. Theorem 1 then establishes our claim.

5. Our discussion of the curve C_6^3 in the Introduction was a special case of the following example. Let V denote the union of two skew lines in $\mathbb{P}^3$. Let C be any curve in $\mathbb{P}^3$, which is linked to V. It follows from [19,

Remark (1.10)] that C is ideally the intersection of three hypersurfaces. Furthermore, it is not too difficult to give the resolution of $I(C)$ by applying the fact that C is arithmetically Buchsbaum with $i(C)=1$. If C and V are linked by two hypersurfaces of degree f and g, then we get the following free resolution:

$$0 \to S(-f-g) \to S^4(-f-g+1) \to S(-f) \oplus S(-g)$$
$$\oplus S^2(-f-g+2) \to S \to S/I(C) \to 0.$$

The specific data in the papers from the late 19th century yield many examples for such curves C (for instance: C_4^0, C_6^3, C_7^4, C_8^8 from the paper of M. Noether [17] or C_{10}^{11}, C_{13}^{21}, C_{16}^{34}, C_{19}^{50} from the paper of K. Rohn [21], where C_d^g is a curve of degree d and of genus g).

In order to demonstrate the Corollary of Lemma 4, we want to study the following example.

6. Let C be given parametrically by $\{s^8, s^7t, s^3t^5, t^8\}$. It follows from [5] that $I(C)=(x_1^5-x_0^4x_2,\ x_2^5-x_0x_1x_3^3,\ x_1^3x_2-x_0^3x_3,\ x_1x_2^3-x_0^2x_3^2,\ x_0x_2^2-x_1^2x_3)$. But $H_\mathfrak{m}^1(R) \simeq (x_1^4/x_0^2, x_1^2/x_0, 1)R/R$, where $R=S/I(C)$, that is, $\mu(H_\mathfrak{m}^1(R))=2$. As indicated previously in connection with Theorem 3(b) and Lemma 5, we need to consider the monomial curves lying on a nonsingular quadric.

7. We set $d=a+b$; that is, C is given parametrically by

$$\{s^d, s^bt^a, s^at^b, t^d\}, \qquad b>a.$$

Then we obain that $\mu(I(C))=b-a+2$ and $I(C)$ has the following minimal basis:

$$I(C)=(x_0x_3-x_1x_2, F_0, F_1, \dots, F_{b-a}),$$

where $F_i = x_0^{b-a-i}x_2^{a+i} - x_1^{b-i}x_3^i$, $0 \leqslant i \leqslant b-a$. This computation follows easily from [5]. Furthermore we note that $\mu(H_\mathfrak{m}^1(R))=b-a-1$ and the length of $H_\mathfrak{m}^1(R)$ is $\binom{b-a+1}{3}$.

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