A Matricial Description of Neville Elimination With Applications to Total Positivity

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ABSTRACT

The Neville elimination process, used by the authors in some previous papers in connection with totally positive matrices, is studied in detail in the case of nonsingular matrices. A wide class of matrices is found where Neville elimination has a lower computational cost than Gauss elimination. Finally some new characterizations are obtained for strictly totally positive and nonsingular totally positive matrices, in terms of their Neville elimination and that of their inverses.

1. INTRODUCTION

In several papers ([5, 10] among others) we have given a precise description of an elimination process which had been previously used by some authors in slightly different ways. We called it Neville elimination and showed its usefulness for the characterization of totally positive (strictly totally positive) matrices, that is, matrices whose minors are nonnegative (positive). These matrices play an important role in approximation theory and computer aided geometric design, as well as in statistics, economics, biology, etc. See for example [2, 6, and 9] in connection with approximation theory and [7, 8]

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for the applications to the *corner cutting* algorithms in CAGD. For other fields, see the references in [1] and [9].

The essence of Neville elimination (NE) is to make zeros in a column of a matrix by adding to each row a multiple of the previous one. Reorderings of the rows may be necessary in the process.

Section 2 starts by recalling how a square matrix can be transformed into diagonal form by complete Neville elimination (CNE). The particular case of nonsingular matrices A whose Neville elimination can be performed without row exchanges is studied in more detail. For brevity, these matrices will be referred to as matrices satisfying the WR (without row exchange) condition. Nonsingular totally positive matrices satisfy that condition. We prove that a nonsingular matrix A satisfies the WR condition if and only if it can be factorized in the form $A = LU$, with L lower triangular, unit diagonal (that is, with 1's as diagonal entries) and satisfying the WR condition and with U upper triangular. We also prove that if the nonsingular matrix A satisfies the WR condition, then the Neville elimination processes of A and L are the same. Moreover, that of L^{-1} (which is the same as that of $B = L^{-1}V$ with V nonsingular and upper triangular) can be carried out without row exchanges and with multipliers which are opposite in sign to those of L. As we shall see, this property does not hold for Gauss elimination, and so Neville elimination will sometimes provide a lower computational cost. In particular, the Neville elimination of a totally positive matrix $B = L^{-1}U$ will need less operations than Gauss elimination when L is a lower triangular band matrix.

In Section 3 similar results are given for complete Neville elimination, showing that certain factorizations of a matrix as a product of bidiagonal matrices are unique. Finally, in Section 4, we apply the results to nonsingular totally positive and strictly totally positive matrices to obtain some characterizations of those matrices in terms of their complete Neville elimination and that of their inverses.

2. NEVILLE ELIMINATION

First we recall the Neville elimination process [5] particularized to a square real matrix $A = (a_{ij})_{1 \leqslant i,j \leqslant n}$. The rectangular case is an obvious extension.

Let us define $\tilde{A}_1 = (\tilde{a}^1_{ij})_{1 \leqslant i,j \leqslant n}$ by $\tilde{a}^1_{ij} := a_{ij}$. If there are zeros in the first column of $\tilde{A}_1$, the corresponding rows are carried down to the bottom in such a way that the relative order among them is the same as in $\tilde{A}_1$. This new matrix is denoted by $A_1 = (a^1_{ij})_{1 \leqslant i,j \leqslant n}$. If no rows have been carried, then $A_1 := \tilde{A}_1$, and in both cases we define $z_1 := 1$.

The method consists in constructing a finite sequence of $n + 1$ matrices A_k such that the submatrix formed by the $k - 1$ initial columns of A_k is an upper echelon form (u.e.f.) matrix. Recall [5] that $V = (v_{ij})_{1 \leqslant i \leqslant n}^{1 \leqslant j \leqslant k-1}$ is a u.e.f. matrix if it satisfies the following conditions for any $i < n$:

(1) if the ith row of V is zero, then the rows below it are zero,

(2) if v_{ij} is the first nonzero entry in the ith row, then $v_{hj} = 0$ $\forall h > i$, and if $v_{i'j'}$ is the first nonzero entry in the i'th row ($i < i' \leqslant n$), the $j' > j$.

Starting as above with $\tilde{A}_1 := A$ and A_1, and continuing with the elimination process, we obtain $A_k = (a_{ij}^k)_{1 \leqslant i, j \leqslant n}$. In the next step we make zeros in its kth column below the (z_k, k) entry, thus forming

$$\tilde{A}_{k+1} = \left(\tilde{a}_{ij}^{k+1}\right)_{1 \leqslant i, j \leqslant n},$$

where for any j ($1 \leqslant j \leqslant n$) one defines

$$\tilde{a}_{ij}^{k+1} := \begin{cases} a_{ij}^k, & & 1 \leqslant i \leqslant z_k, \\ a_{ij}^k - \dfrac{a_{ik}^k}{a_{i-1,k}^k} a_{i-1,j}^k & \text{if } a_{i-1,k}^k \neq 0, & z_k < i \leqslant n, \\ a_{ij}^k & \text{if } a_{i-1,k}^k = 0, & z_k < i \leqslant n. \end{cases} \tag{2.1}$$

Observe that $a_{i-1,k}^k = 0$ implies $a_{ik}^k = 0$. The new value of z at this step of the elimination process is defined as

$$z_{k+1} := \begin{cases} z_k & \text{if } a_{z_k,k}^k \left(= \tilde{a}_{z_k,k}^{k+1}\right) = 0, \\ z_k + 1 & \text{if } a_{z_k,k}^k \left(= \tilde{a}_{z_k,k}^{k+1}\right) \neq 0. \end{cases} \tag{2.2}$$

If $\tilde{A}_{k+1}$ has some zeros in the $(k + 1)$th column, in any row starting from z_{k+1} or below, these rows are carried down as has been done with $\tilde{A}_1$, thus obtaining a matrix denoted by $A_{k+1} = (a_{ij}^{k+1})_{1 \leqslant i, j \leqslant n}$. Of course, if there are no row exchanges, then $A_{k+1} := \tilde{A}_{k+1}$.

After n steps (some of them may be obvious, because the corresponding column already has the necessary zeros) we get

$$\tilde{A}_{n+1} = U, \tag{2.3}$$

where U is an $n \times n$ u.e.f. matrix.

The element

$$p_{ij} := a_{ij}^j, \qquad 1 \leqslant j \leqslant n, \quad z_j \leqslant i \leqslant n, \tag{2.4}$$

is called the (i, j) pivot of the Neville elimination (NE) of A, and the number

$$m_{ij} := \begin{cases} a_{ij}^j / a_{i-1,j}^j & \text{if} \quad a_{i-1,j}^j \neq 0, \\ 0 & \text{if} \quad a_{i-1,j}^j = 0 \left(\Rightarrow a_{ij}^j = 0 \right), \end{cases}$$

$$1 \leqslant j \leqslant n, \quad z_j < i \leqslant n \tag{2.5}$$

the (i, j) multiplier of the NE of A. Observer that $m_{ij} = 0$ iff $a_{ij}^j = 0$.

We remark that, when A is nonsingular,

$$z_k = k \qquad \forall k \tag{2.6}$$

and $\tilde{A}_{n+1} = A_n$. Also observe that when no row exchanges are needed in the elimination process, we have

$$\tilde{A}_k = A_k \qquad \forall k \tag{2.7}$$

and

$$m_{ij} = 0 \left(\Leftrightarrow p_{ij} = 0 \right) \quad \Rightarrow \quad m_{tj} = 0 \qquad \forall t > i. \tag{2.8}$$

The complete Neville elimination (CNE) of a matrix A consists in performing the Neville elimination of A to obtain a u.e.f. matrix U and then proceeding with the NE of U^T (the transpose of U). The last part is equivalent to performing the Neville elimination of U by columns. When we say that the CNE of A is possible without row or column exchanges, we mean that there have not been any row exchanges in the NE of either A or U^T.

Let us now consider more in detail the case of a nonsingular matrix A whose Neville elimination can be performed without row exchanges. Since we are interested in these matrices, for the sake of brevity they will be referred to as matrices satisfying the WR condition. In this case, (2.6), (2.7), and (2.8) hold, and the Neville elimination process can be matricially described by elementary matrices without using permutation matrices.

To this end, we denote by $E_{ij}(\alpha)$ $(1 \leqslant i, j \leqslant n)$ the lower triangular matrix whose (r, s) entry $(1 \leqslant r, s \leqslant n)$ is given by

$$\begin{cases} 1 & \text{if} \quad r = s, \\ \alpha & \text{if} \quad (r, s) = (i, j), \\ 0 & \text{elsewhere}. \end{cases} \tag{2.9}$$

We are mainly interested in the matrices $E_{i,\,i-1}(\alpha)$, which for simplicity will be denoted by $E_i(\alpha)$. They are bidiagonal and lower triangular, and given explicitly by

$$E_i(\alpha) := \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & \ddots & & & & \\ & & & & 1 & & & \\ & & & & \alpha & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{bmatrix}. \tag{2.10}$$

The inverse of $E_{ij}(\alpha)$ is $E_{ij}(-\alpha)$, and also one has $E_{ij}(\alpha)E_{ij}(\beta) = E_{ij}(\alpha + \beta)$.

For a matrix A satisfying the WR condition, the Neville elimination process can be written

$$E_n(-m_{n,\,n-1}) \cdots \{E_3(-m_{32}) \cdots E_n(-m_{n2})\}$$

$$\times \{E_2(-m_{21}) \cdots E_{n-1}(-m_{n-1,1}) E_n(-m_{n1})\}\, A = U, \tag{2.11}$$

where U is a nonsingular upper triangular matrix, and the m_{ij}'s are the multipliers (2.5) satisfying (2.8). Equivalently, one has

$$F_{n-1}F_{n-2} \cdots F_1 A = U \tag{2.12}$$

with

$$F_i = \begin{bmatrix} 1 & & & & & & & & \\ 0 & 1 & & & & & & & \\ & 0 & \ddots & & & & & & \\ & & \ddots & 1 & & & & & \\ & & & -m_{i+1,i} & 1 & & & & \\ & & & & -m_{i+2,i} & 1 & & & \\ & & & & & \ddots & \ddots & & \\ & & & & & & -m_{ni} & 1 \end{bmatrix}. \tag{2.13}$$

We will say that the Neville elimination of A consists of K nontrivial steps if

$$K = \operatorname{card}\{(i,j) \mid i > j,\ m_{ij} \neq 0\}. \tag{2.14}$$

From (2.11) we get the factorization of A

$$A = \{E_n(m_{n1}) E_{n-1}(m_{n-1,1}) \cdots E_2(m_{21})\}\{E_n(m_{n2}) \cdots E_3(m_{32})\} \times \cdots E_n(m_{n,n-1}) U. \tag{2.15}$$

Let us discuss how these factorizations (with α_{ij} instead of m_{ij} for all i, j) work when the real numbers α_{ij} satisfy, for $1 \leqslant j \leqslant n-1$,

$$\alpha_{ij} = 0 \quad \Rightarrow \quad \alpha_{lj} = 0 \quad \forall l > i. \tag{2.16}$$

Denote for any $1 \leqslant j \leqslant n-1$

$$r_j := \begin{cases} j & \text{if} \quad \alpha_{ij} = 0\ \forall i, \\ \max\{i \mid \alpha_{ii} \neq 0\} & \text{otherwise}, \end{cases} \tag{2.17}$$

and, for a given k $(1 \leqslant k \leqslant n-1)$,

$$r_{k,j} := \begin{cases} j & \text{if} \quad j < k \\ \max\{r_k, r_{k+1}, \dots, r_j\} & \text{if} \quad j \geqslant k. \end{cases} \tag{2.18}$$

Observe that

$$r_{k,1} \leqslant r_{k,2} \leqslant \cdots \leqslant r_{k,k} = r_k \leqslant r_{k,k+1} \leqslant \cdots \leqslant r_{k,n-1}$$

and

$$E_l(\alpha_{lj}) = I \qquad \forall l \geqslant r_j + 1.$$

If we also denote by C_j the matrix

$$C_j := E_n(\alpha_{nj}) E_{n-1}(\alpha_{n-1,j}) \cdots E_{j+1}(\alpha_{j+1,j}), \tag{2.19}$$

then (2.15) can be written (replacing m_{ij} with α_{ij})

$$A = C_1 C_2 \cdots C_{n-1} U. \tag{2.20}$$

The following lemma shows the zero pattern of a matrix when its factorization (2.20) is known.

LEMMA 2.1. *Let U be a nonsingular upper triangular matrix of order n, and for $1 \leqslant k \leqslant n-1$, let $B^{(k)}$ be the matrix*

$$B^{(k)} = \left(b_{ij}^k\right)_{1 \leqslant i, j \leqslant n} := C_k C_{k+1} \cdots C_{n-1} U, \tag{2.21}$$

with C_j defined by (2.19). *Assume that the numbers α_{ij} which appear in the factors $C_k, C_{k+1}, \dots, C_{n-1}$ satisfy* (2.16).

Then for each $1 \leqslant j \leqslant n-1$, one has

$$b_{r_{k,j},j}^k \neq 0, \tag{2.22}$$

$$b_{ij}^k = 0 \qquad \forall i > r_{k,j}. \tag{2.23}$$

Proof. Let W be a matrix of order n, and $w_i^T \in \mathbb{R}^n$ its ith row. The rows of the matrix $E_h(\alpha)W$ are $w_1^T, \dots, w_{h-1}^T, w_h^T + \alpha w_{h-1}^T, w_{h+1}^T, \dots, w_n^T$. The proof of the lemma follows easily by induction on the number of factors C in (2.21) (starting, as usual, with one, i.e. with $k = n-1$), taking into account that with our notation

$$C_j = E_{r_j}(\alpha_{r_j,j}) E_{r_j-1}(\alpha_{r_j-1,j}) \cdots E_{j+1}(\alpha_{j+1,j})$$

and

$$r_{k,j} = \begin{cases} r_{k+1,j}(=j) & \text{if} \quad j < k, \\ \max\{r_k, r_{k+1,j}\} & \text{if} \quad j \geqslant k. \end{cases} \qquad \blacksquare$$

Now we easily prove

THEOREM 2.2. *A nonsingular $n \times n$ matrix A satisfies the WR condition if and only if it can be factorized in the form* (2.15) *with the m_{ij}'s satisfying* (2.8). *If A satisfies that condition, the factorization is unique and m_{ij} is the (i, j) multiplier of the Neville elimination of A.*

Proof. If A satisfies the WR condition, the Neville elimination process for A can be described as in (2.11) with the multipliers verifying (2.8). Thus we get (2.15).

Let us now prove the uniqueness of such a factorization. Suppose (2.15) holds, write it in the form (2.20) (with m_{ij} instead of α_{ij}), and assume there exists another decomposition

$$\begin{aligned} A &= \{E_n(m'_{n1}) \cdots E_2(m'_{21})\}\{E_n(m'_{n2}) \cdots E_3(m'_{32})\} \cdots E_n(m'_{n,n-1})U' \\ &= C'_1 C'_2 \cdots C'_{n-1} U' \end{aligned} \tag{2.24}$$

with the m'_{ij}'s satisfying a condition (2.8) and U' a nonsingular upper triangular matrix. Premultiplying (2.20) and (2.24) by C_1^{-1}, one has

$$C_2 \cdots C_{n-1} U = C_1^{-1} C'_1 C'_2 \cdots C'_{n-1} U'. \tag{2.25}$$

From the zero patterns of $C_2 \cdots C_{n-1}U$ and $C'_1 C'_2 \cdots C'_{n-1}U'$ given by Lemma 2.1, it easily follows that the unique possibility for C_1^{-1} to satisfy (2.25) is that $r'_1 = r_1$, and $m'_{i1} = m_{i1}$ for all $i \geqslant 2$. Observe that for $i > r'_1 = r_1$ one has $m'_{i1} = m_{i1} = 0$. In consequence, $C_1 = C'_1$. Proceeding similarly with C_2 in

$$C_2 \cdots C_{n-1} U = C'_2 \cdots C'_{n-1} U'$$

and so on, we get $C_i = C'_i$ $(2 \leqslant i \leqslant n-1)$, $U = U'$, and therefore the uniqueness of (2.20) is proved.

Conversely, if A can be factorized in the form (2.15) with the m_{ij}'s satisfying (2.8), from the uniqueness of such a factorization proved above, it follows that (2.11), obtained from (2.15), gives the Neville elimination of A.

■

An interesting property of the matrices E_{ij} defined in (2.9) is given by the following lemma.

LEMMA 2.3. *For any matrices $E_{ij}(\alpha)$, $E_{hk}(\beta)$, with $i > j$, $h > k$, $\alpha\beta \neq 0$, one has*

$$E_{ij}(\alpha)E_{hk}(\beta) = E_{hk}(\beta)E_{ij}(\alpha) \quad \Leftrightarrow \quad j \neq h.$$

Proof. Both sides of the equation above represent the matrix whose (r, s) entry is given by

$$\begin{cases} 1 & \text{if} \quad r = s, \\ \alpha & \text{if} \quad (r, s) = (i, j), \\ \beta & \text{if} \quad (r, s) = (h, k), \\ 0 & \text{elsewhere} \end{cases} \tag{2.26}$$

if and only if $j \neq h$. ■

REMARK 2.4. In particular one has $E_i(\alpha)E_j(\beta) = E_j(\beta)E_i(\alpha)$ except for $|i - j| = 1$ with $\alpha\beta \neq 0$.

Let us denote by $E_{ii}(\alpha)$ the elementary matrix of order n whose (r, s) entry is defined by

$$\begin{cases} 1 & \text{if} \quad r = s \neq i, \\ \alpha & \text{if} \quad r = s = i, \\ 0 & \text{elsewhere.} \end{cases} \tag{2.27}$$

Observe that, for $\alpha \neq 0$, $E_{ii}(\alpha)^{-1} = E_{ii}(1/\alpha)$. The following result is straightforward.

LEMMA 2.5. *For any matrices $E_{jj}(\alpha)$, $E_{i,i-1}(\beta)$ one has*

$$\begin{cases} E_{jj}(\alpha)E_{i,i-1}(\beta) = E_{i,i-1}(\beta)E_{jj}(\alpha) & \textit{if} \quad j \neq i, i-1, \\ E_{ii}(\alpha)E_{i,i-1}(\beta) = E_{i,i-1}(\alpha\beta)E_{ii}(\alpha), & \\ E_{i-1,i-1}(\alpha)E_{i,i-1}(\beta) = E_{i,i-1}(\beta/\alpha)E_{i-1,i-1}(\alpha) & (\alpha \neq 0). \end{cases} \tag{2.28}$$

Now we can prove

THEOREM 2.6. *Let L be a lower triangular, unit diagonal matrix. The following properties are equivalent:*

(i) *L satisfies the WR condition.*

(ii) *For any upper triangular matrix V, the matrix LV satisfies the WR condition.*

(iii) *L^{-1} satisfies the WR condition.*

(v) *For any upper triangular matrix V, the matrix $L^{-1}V$ satisfies the WR condition.*

Moreover, if these properties hold, the multipliers of the Neville elimination of LV are the same for any upper triangular matrix V. The multipliers of the Neville elimination of the matrices $L^{-1}V$ are opposite in sign to those of LV but, in general, occur in a different order.

Proof. According to Theorem 2.2, L satisfies the WR condition if and only if there exists a factorization

$$L = \{E_n(m_{n1}) \cdots E_2(m_{21})\}\{E_n(m_{n2}) \cdots E_3(m_{32})\} \cdots E_n(m_{n,n-1}) \quad (2.29)$$

such that (2.8) holds. Observe that $L = LI$ is a factorization (2.15) for L. Hence the equivalence of properties (i) and (ii) becomes apparent. On the other hand, by the second part of Theorem 2.2, the multipliers of the Neville elimination of any matrix LV are the same. The same reasoning proves the equivalence of (iii) and (iv). Therefore the theorem follows if we prove the equivalence of (i) and (iii) and see that the multipliers are opposite in sign.

If (2.29) and (2.8) hold, then one has

$$L^{-1} = E_n(-m_{n,n-1})\{E_{n-1}(-m_{n-1,n-2})E_n(-m_{n,n-2})\} \cdots$$
$$\times\{E_2(-m_{21}) \cdots E_n(-m_{n1})\}. \quad (2.30)$$

If all the m_{ij}'s are different from zero, then, using Lemma 2.3 to reorder the factors in (2.30), we get a factorization of L^{-1} as in (2.15):

$$L^{-1} = \{E_n(m'_{n1}) \cdots E_2(m'_{21})\} \cdots E_n(m'_{n,n-1})I \quad (2.31)$$

with multipliers

$$m'_{rs} = -m_{r,r-s} \qquad (r > s) \tag{2.32}$$

which also satisfy (2.8). Therefore, by Theorem 2.2, L^{-1} satisfies the WR condition.

Suppose now that $m_{ij} = 0$ in (2.29), with $m_{i-1,j} \neq 0$ if $i > j+1$, and $m_{rs} \neq 0$ for all $r > s$, $s < j$. Since (2.8) holds for (2.29), we have $m_{i+1,j} = \cdots = m_{nj} = 0$, and then $E_r(m_{rj}) = E_r(-m_{rj}) = I$ for $r \geqslant i$. As in the previous case, by Lemma 2.3 we can reorder the factors of (2.30) and write L^{-1} in the form

$$L^{-1} = H_1 H_2 \cdots H_{n-1} I \tag{2.33}$$

with

$$H_r = E_n(-m_{n,n-r}) \cdots E_{r+1}(-m_{r+1,1}). \tag{2.34}$$

Taking into account again Lemma 2.3, since $E_i(-m_{ij}) = I$, we know that the factors $E_{i+1}(-m_{i+1,j+1})$, $E_{i+2}(-m_{i+2,j+2}), \ldots, E_n(-m_{n,n-i+j})$ can be moved, just in that order, from H_{i-j} to H_{i-j+1}, where there was a factor $E_{i+1}(m_{i+1,j}) = I$. Then the factors $E_{i+2}(-m_{i+2,j+1}), \ldots, E_n(-m_{n,n-i+j-1})$ can be moved from H_{i-j+1} to H_{i-j+2}, and so on. Denoting by H'_{i-j}, $H'_{i-j+1}, \ldots, H'_{n-1}$ the matrices

$$H'_{i-j} = \begin{cases} E_{i-1}(-m_{i-1,j-1}) \cdots & \\ \quad \times E_{i-j+1}(-m_{i-j+1,1}) & \text{if} \quad i > 2, \\ I & \text{if} \quad i = 2\ (\Rightarrow j = 1), \end{cases} \tag{2.35}$$

$$\begin{aligned} H'_{i-j+k} = {} & E_n(-m_{n,n-i+j-k+1}) \cdots E_{i+k}(-m_{i+k,j+1}) E_{i+k-1}(-m_{i+k-1,j-1}) \\ & \times \cdots E_{i-j+k+1}(-m_{i-j+k+1,1}) \end{aligned} \tag{2.36}$$

for $k = 1, 2, \ldots, n-i+j-1$, one has

$$L^{-1} = H_1 H_2 \cdots H_{i-j-1} H'_{i-j} H'_{i-j+1} \cdots H'_{n-2} H'_{n-1} I. \tag{2.37}$$

If no more multipliers m_{rs} of (2.29) are zero, the conditions of Theorem 2.2 hold in (2.37). If more multipliers are zero, the above process can be repeated until the conditions of that theorem are satisfied. Therefore, L^{-1} satisfies the WR condition.

Since the roles of L and L^{-1} can be exchanged, the equivalence of (i) and (iii) is proved. In the above reasoning we have also proved that the multipliers of L and L^{-1} are opposite in sign, but in general occur in a different order [see for example (2.32)]. ■

REMARK 2.7. Any nonsingular lower triangular matrix L can be written in the form $L = TD$, where T is a lower triangular, unit diagonal and D is a diagonal matrix.

$$D = \operatorname{diag}(l_{11}, l_{22}, \dots, l_{nn}) = E_{11}(l_{11}) \cdots E_{nn}(l_{nn}).$$

By Theorem 2.6, L satisfies the WR condition if and only if T does. If that happens, T admits a factorization of the type (2.15), and so does T^{-1} with multipliers opposite in sign to those of T.

Writing

$$L^{-1} = E_{nn}(l_{nn}^{-1}) \cdots E_{11}(l_{11}^{-1}) T^{-1}, \tag{2.38}$$

by Lemma 2.5 the factors $E_{ii}(l_{ii}^{-1})$ can be moved to the right of T^{-1} if we modify [by product or quotient, according to (2.28)] some of the multipliers appearing in the factorization of T^{-1}. So we get

$$L^{-1} = T'D^{-1}, \tag{2.39}$$

where T' is a lower triangular, unit diagonal matrix satisfying the WR condition. It is clear that the number of nonzero multipliers of the Neville elimination of T' (and L^{-1}) is the same as that of T^{-1}, T, and L.

Therefore, we easily conclude that the equivalence of properties (i), (ii), (iii), and (iv) of Theorem 2.6 holds for any nonsingular lower triangular matrix L. For those matrices L the second part of the theorem does not hold, but the number of nonzero multipliers of the Neville elimination of LV is the same as that of $L^{-1}V$.

Theorem 2.6 and Remark 2.7 allow us to point out some matrices whose Neville elimination has a lower computational cost than Gauss elimination.

For instance, let L be a bidiagonal, lower triangular, nonsingular matrix

$$L = \begin{bmatrix} l_{11} & & & & \\ l_{21} & l_{22} & & & \\ & l_{32} & l_{33} & & \\ & & \ddots & \ddots & \\ & & & l_{n,n-1} & l_{nn} \end{bmatrix}. \tag{2.40}$$

Obviously, L satisfies the WR condition, and the number of nontrivial steps of its Neville elimination (that is, the number of nonzero multipliers) is

$$m = \operatorname{card}\{i \mid 2 \leqslant i \leqslant n \text{ and } l_{i,i-1} \neq 0\}.$$

In this case, the matrix L^{-1} is lower triangular, but not necessarily bidiagonal. However, the number of nontrivial steps in the Neville elimination of L^{-1} is m again.

This curious property does not hold in Gauss elimination. For example, the matrix

$$L = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -2 & 1 & \\ & & -3 & 1 \end{bmatrix}$$

requires three nontrivial steps to be transformed by Neville elimination (multipliers $1, 2, 3$) into the identity matrix I. The same happens (multipliers -3, -2, -1) with the inverse of L,

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 6 & 6 & 3 & 1 \end{bmatrix}.$$

On the contrary, the Gauss elimination process requires three steps for L and six steps for L^{-1}. In general, for matrices of order n of this type, the numbers are $n - 1$ for Neville elimination and $n(n - 1)/2$ for Gauss elimination.

By Theorem 2.6 and Remark 2.7 we get an analogous conclusion for the NE of any matrix $B = L^{-1}V$ with V an upper triangular matrix and L

bidiagonal lower triangular. In general, if L is a lower triangular band matrix, then the Neville elimination of $B = L^{-1}V$ will need less operations than for Gauss elimination. In fact, it can be deduced from Proposition 2 of [3] that the Gauss elimination of a totally positive matrix $B = L^{-1}V$ with all its elements nonzero has all the multipliers nonzero, even when L is a band matrix.

3. COMPLETE NEVILLE ELIMINATION

Taking into account that the complete Neville elimination consists of two simple Neville eliminations (which will be referred to as the lower and the upper Neville elimination respectively), the theorems of Section 2 can be easily modified. For brevity we will say that a nonsingular matrix A satisfies the WRC condition if the complete Neville elimination can be performed without row or column exchanges. Proceeding as with Theorem 2.2, Theorem 2.6, and Remark 2.7 we have

THEOREM 3.1. *A nonsingular matrix A satisfies the WRC condition if and only if it can be factorized in the form* (2.15) *with*

$$U = DE_n(m'_{n,n-1})^T \cdots \left\{E_3(m'_{32})^T \cdots E_n(m'_{n2})^T\right\}$$
$$\times\left\{E_2(m'_{21})^T \cdots E_n(m'_{n1})^T\right\}, \tag{3.1}$$

where the m_{ij}, m'_{ij} satisfy (2.8). *If A satisfies that condition, the factorization is unique, and m_{ij} (m'_{ij}) is the (i, j) multiplier of the lower (upper) Neville elimination of A.*

COROLLARY 3.2. *A nonsingular matrix A satisfies the WRC condition if and only if it can be factorized as $A = LDV$ with L (V) lower (upper) triangular and unit diagonal, D a diagonal matrix, and L, V^T satisfying the WR condition. If A satisfies that condition, the lower and upper Neville elimination processes of A coincide, respectively, with the Neville elimination of L and V^T.*

THEOREM 3.3. *Let A be a nonsingular matrix which can be factorized in the form $A = LDV$ with L (V) lower (upper) triangular and unit diagonal, and D a diagonal matrix. Then A satisfies the WRC condition if and only if the matrix $B = L^{-1}CV^{-1}$, with C a diagonal matrix, satisfies the same condition. In the affirmative case, the multipliers of the lower (or upper)*

Neville elimination of A are opposite in sign to those of B, but in general occur in a different order.

As we have seen in Section 2, a lower triangular, unit diagonal matrix L satisfies the WR condition iff it can be factorized in the form (2.29) with the m_{ij}'s satisfying (2.8). This is equivalent to the existence of a factorization (2.30) for L^{-1} with (2.8).

But (2.30) can be written in the form

$$L^{-1} = \begin{bmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ & \ddots & \ddots & & & \\ & & 0 & 1 & \\ & & & -m_{n,n-1} & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ & \ddots & & \ddots & & \\ & & -m_{n-1,n-2} & & 1 & \\ & & & & -m_{n,n-2} & 1 \end{bmatrix} \cdots$$

$$\times \begin{bmatrix} 1 & & & & \\ -m_{21} & 1 & & & \\ & \ddots & & & \\ & & \ddots & & \\ & & & -m_{n,1} & 1 \end{bmatrix}, \tag{3.2}$$

and so, taking into account (2.8) we can say that L satisfies the WR condition if and only if its inverse L^{-1} can be expressed as a product of lower triangular, bidiagonal, unit diagonal matrices

$$L^{-1} = S_{n-1}S_{n-2} \cdots S_1 \tag{3.3}$$

in such a way that if $s^{(i)}_{t+1,t}$ denotes the $(t + 1, t)$ entry of S_i one has

$$\text{for} \quad 2 \leqslant i \leqslant n - 1, \qquad s^{(i)}_{t+1,t} = 0 \quad \text{if } t < i, \tag{3.4}$$

and

$$\text{for} \quad 1 \leqslant i \leqslant n-1, \qquad \text{if } s_{t+1,t}^{(i)} = 0 \text{ and } t \geqslant i \text{ then}$$

$$s_{r+1,r}^{(i)} = 0 \text{ for } r > t. \quad (3.5)$$

If that decomposition of L^{-1} exists, then the entry $s_{t+1,t}^{(i)}$, with $t \geqslant i$, is opposite in sign to the $(t+1, i)$ multiplier of the Neville elimination of L.

Analogously, L^{-1} satisfies the WR condition if and only if L can be factorized as in (3.3):

$$L = J_{n-1} J_{n-2} \cdots J_1 \tag{3.6}$$

with the lower triangular, bidiagonal, unit diagonal matrices J_i satisfying the corresponding conditions (3.4), (3.5). The entry $j_{t+1,t}^{(i)}$ $(t \geqslant i)$ is opposite in sign to the $(t+1, i)$ multiplier of the Neville elimination of L^{-1}.

Thus, by Theorem 2.6 and Remark 2.7, we can summarize by saying that a matrix L satisfies the WR condition if and only if it can be factorized in the form (3.6) with conditions similar to (3.4), (3.5). The multipliers of the Neville elimination of L are opposite in sign to the subdiagonal entries of the matrices J_i of (3.6), but in general occur in a different order.

Finally, taking into account Theorem 3.1, a nonsingular matrix A satisfies the WRC condition if and only if A can be decomposed in the form

$$A = J_{n-1} J_{n-2} \cdots J_1 D K_1 K_2 \cdots K_{n-1}, \tag{3.7}$$

where J_i, K_i^T satisfy the same conditions as the matrices J_i in (3.6).

4. CHARACTERIZATIONS OF STP AND NONSINGULAR TP MATRICES BY THEIR COMPLETE NEVILLE ELIMINATION AND THAT OF THEIR INVERSES

In [5, Theorem 4.1] ([5, Corollary 5.5]), STP matrices (nonsingular TP matrices) were characterized by their complete Neville elimination. With our present terminology, those characterizations can be reformulated in the following theorem.

THEOREM 4.1. *A square matrix M is STP (is nonsingular TP) if and only if it satisfies the WRC condition, with positive (nonnegative) multipliers and positive diagonal pivots.*

To deal with complete Neville elimination of TP matrices, the concept of conversion of a matrix A of order n (see [1, p. 171]) is very useful. Recall that the *conversion* of the matrix $A = (a_{ij})_{1 \leqslant i, j \leqslant n}$ is the matrix $A^{\#}$ of order n whose (i, j) entry is $a_{n-i+1, n-j+1}$.

Given $k \leqslant n$, $Q_{k,n}$ will denote the totality of strictly increasing sequences of k natural numbers less than or equal to n:

$$\alpha = (\alpha_i)_{i=1}^{k} \in Q_{k,n} \qquad \text{if} \quad (1 \leqslant)\, \alpha_1 < \alpha_2 < \cdots < \alpha_k\, (\leqslant n).$$

For $\alpha, \beta \in Q_{k,n}$ $A[\alpha | \beta]$ will denote the $k \times k$ submatrix of A containing rows numbered by α and columns numbered by β. Then the following properties of the conversion of A are straightforward:

(1) for all $\alpha, \beta \in Q_{k,n}$ and $1 \leqslant k \leqslant n$, one has

$$\det A^{\#}[\alpha | \beta] = \det A\left[\alpha^{\#} | \beta^{\#}\right], \tag{4.1}$$

where $\alpha^{\#}$, $\beta^{\#}$ are the elements of $Q_{k,n}$ defined by

$$(\alpha^{\#})_i = n - \alpha_{k-i+1} + 1,$$

$$(\beta^{\#})_i = n - \beta_{k-i+1} + 1$$

$(1 \leqslant i \leqslant k)$;

(2) A is TP (STP) $\Leftrightarrow$ $A^{\#}$ is TP (STP);

(3) $(AB)^{\#} = A^{\#}B^{\#}$;

(4) $(A^{\#})^{-1} = (A^{-1})^{\#}$.

Now we can prove

THEOREM 4.2. *A nonsingular matrix M is totally positive if and only if M^{-1} satisfies the WRC condition with nonpositive multipliers and positive diagonal pivots.*

Proof. According to Theorem 4.1 and Corollary 3.2, M is totally positive if and only if it can be factorized as $M = LDV$, where L, V^T are products of elementary matrices E_i with nonnegative off-diagonal entries and D is a diagonal matrix with positive diagonal elements. Therefore L and V^T are nonsingular totally positive.

On the other hand, we have

$$M^{\#} = L^{\#}D^{\#}V^{\#} \tag{4.2}$$

and

$$(M^{\#})^{-1} = (M^{-1})^{\#} = (V^{\#})^{-1}(D^{\#})^{-1}(L^{\#})^{-1}. \tag{4.3}$$

$V^{\#}$ and $(L^{\#})^T$ are lower triangular nonsingular, and they are TP if and only if V, L are TP. By Theorem 4.1 this is equivalent to saying that $V^{\#}, (L^{\#})^T$ satisfy the WRC condition with nonnegative multipliers and positive diagonal pivots (in this case the diagonal pivots are 1). By Corollary 3.2 and Theorem 3.3 that happens if and only if $(M^{\#})^{-1}$ satisfies the WRC condition with nonpositive multipliers and positive diagonal elements. Since $M^{\#}$ is nonsingular TP if and only if M is TP, applying the above reasoning to $M^{\#}$ and taking into account that $[(M^{\#})^{\#}]^{-1} = M^{-1}$ proves the theorem. ∎

In the case of STP matrices we cannot use Theorem 4.1 as in the above proof, because in the decompositions (4.1) (4.2) the lower triangular matrices are not STP, but ΔSTP. Remember that a lower triangular matrix A is said to be ΔSTP if and only if all minors

$$\det A[\alpha | \beta], \qquad \alpha, \beta \in Q_{k,n},$$

with $\alpha_i \geqslant \beta_i$ $\forall i$ are positive (all the other minors are trivially zero).

From Lemma 2.6 of [5] and Theorem 3.1 of [4] we get directly the following theorem, which is similar to Theorem 4.1.

THEOREM 4.3. *A lower triangular matrix M is ΔSTP if and only if it satisfies the WR condition with positive multipliers and positive diagonal entries.*

Now it is easy to prove

THEOREM 4.4. *A square matrix M is strictly totally positive if and only if M^{-1} satisfies the WRC condition with negative multipliers and positive diagonal pivots.*

Proof. The proof of Theorem 4.2 applies with obvious changes: the matrices E_i have a positive $(i, i-1)$ entry, and the triangular matrices $L, V^T, L^{\#}, V^{\#}$ are now ΔSTP. Then we can use Theorem 4.3 instead of Theorem 4.1 and deduce that $(M^{\#})^{-1}$ satisfies the WRC condition with negative multipliers and positive diagonal entries. ∎

REMARK 4.5. The conversion of the product of matrices allows to provide UL factorization results by using arguments identical to those used for LU factorizations.

We can reformulate the theorems of this section in terms of factorizations by using the remarks at the end of Section 3.

THEOREM 4.1′. *A square matrix M of order n is STP if and only if it can be decomposed in the form*

$$M = J_{n-1} J_{n-2} \cdots J_1 D K_1 \cdots K_{n-1}, \tag{4.4}$$

where D is a diagonal matrix with positive diagonal entries, and for $i = 1, \dots, n-1$

$$J_i = \begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & 0 & 1 & & & \\ & & & j_{i+1}^{(i)} & 1 & & \\ & & & & \ddots & \ddots & \\ & & & & & j_n^{(i)} & 1 \end{bmatrix},$$

$$K_i = \begin{bmatrix} 1 & 0 & & & & & \\ & \ddots & \ddots & & & & \\ & & 1 & 0 & & & \\ & & & 1 & k_i^{(i)} & & \\ & & & & \ddots & \ddots & \\ & & & & & 1 & k_{n-1}^{(i)} \\ & & & & & & 1 \end{bmatrix} \tag{4.5}$$

with

$$j_r^{(i)} > 0 \quad \forall r > i, \qquad k_s^{(i)} > 0 \quad \forall s > i-1. \tag{4.6}$$

A nonsingular matrix M of order n is TP if and only if it can be decomposed in the form (4.4) *under the same conditions as above, but with* (4.6) *replaced by*

$$j_r^{(i)} \geqslant 0 \quad \forall r > i, \qquad k_s^{(i)} \geqslant 0 \quad \forall s > i-1 \tag{4.7}$$

and

$$j_r^{(i)} = 0\ (r > i) \quad \Rightarrow \quad j_t^{(i)} = 0 \quad \forall t > r,$$

$$k_s^{(i)} = 0\ (s \geqslant i) \quad \Rightarrow \quad k_t^{(i)} = 0 \quad \forall t > s. \tag{4.8}$$

Theorems 4.2 and 4.4 are combined in

THEOREM 4.2′. *A square matrix M of order n is STP if and only if M^{-1} can be decomposed in the form*

$$M^{-1} = J'_{n-1}J'_{n-2} \cdots J'_1 D' K'_1 \cdots K'_{n-1} \tag{4.9}$$

where D' is a diagonal matrix with positive diagonal entries, and for $i = 1, 2, \dots n-1$, J'_i, K'_i are of the form (4.5) *with off-diagonal entries $j_r'^{(i)}$, $k_s'^{(i)}$ satisfying*

$$j_r'^{(i)} < 0 \quad \forall r > i, \qquad k_s'^{(i)} < 0 \quad \forall s > i-1. \tag{4.10}$$

A nonsingular matrix M of order n is TP if and only if M^{-1} can be decomposed in the form (4.9) *under the same conditions as above, but with* (4.10) *replaced by*

$$j_r'^{(i)} \leqslant 0 \quad \forall r > i, \qquad k_s'^{(i)} \leqslant 0 \quad \forall s > i-1 \tag{4.11}$$

and

$$j_r'^{(i)} = 0\ (r > i) \quad \Rightarrow \quad j_t'^{(i)} = 0 \quad \forall t > r,$$

$$k_s'^{(i)} = 0\ (s \geqslant i) \quad \Rightarrow \quad k_t'^{(i)} = 0 \quad \forall t > s. \tag{4.12}$$

The particular case of nonsingular triangular TP matrices M is included in Theorems 4.1′ and 4.2′, but in that case $K_1 = \cdots = K_{n-1} = K'_1 = \cdots = K'_{n-1} = I$ in the lower triangular case and $J_1 = \cdots = J_{n-1} = J'_1 = \cdots = J'_{n-1} = I$ in the upper triangular case. The case of ΔSTP matrices is also included: for example, in the lower triangular case, $K_i = K'_i = I\ \forall i$ and

$$j_r^{(i)} > 0 \quad \forall r > i, \qquad j_r'^{(i)} < 0 \quad \forall r > i.$$

This result is equivalent to Theorem 4.3.

REMARK 4.6. The uniqueness of the factorizations given in Theorems 4.1′ and 4.2′ follows from Section 3.

REFERENCES

1 T. Ando, Totally positive matrices, *Linear Algebra Appl.* 90:165–219 (1987).
2 C. de Boor, Total positivity of the spline collocation matrix, *Indiana Univ. Math. J.* 25:541–551 (1976).
3 C. de Boor and A. Pinkus, Backward error analysis for totally positive linear systems, *Numer. Math.* 27:485–490 (1977).
4 C. Cryer, Some properties of totally positive matrices, *Linear Algebra Appl.* 15:1–25 (1976).
5 M. Gasca and J. M. Peña, Total positivity and Neville elimination, *Linear Algebra Appl.* 165:25–44 (1992).
6 M. Gasca, C. A. Micchelli, and J. M. Peña, Almost strictly totally positive matrices, *Numer. Algorithms* 2:225–236 (1992).
7 T. N. T. Goodman and C. A. Micchelli, Corner cutting algorithms for the Bezier representation of free forms, *Linear Algebra Appl.* 99:225–252.
8 T. N. T. Goodman, Shape preserving representations, in *Mathematical Methods in CAGD* (T. Lyche and L. Schumaker, Eds.), Academic, Boston, 1989, pp. 333–357.
9 S. Karlin, *Total Positivity*, Stanford, U.P., Stanford, Calif., 1968.
10 G. Mülbach and M. Gasca, A test for strict total positivity via Neville elimination, in *Current Trends in Matrix Theory* (F. Uhlig and R. Groue, Eds.), Elsevier Science, 1987, pp. 225–232.

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