Available online at www.sciencedirect.com
ScienceDirect
Computers and Mathematics with Applications 55 (2008) 776–787
An International Journal computers & mathematics with applications
www.elsevier.com/locate/camwa

Optimality conditions for differential system of Petrowsky type with infinite number of variables and boundary control

H.A. El-Saify

Mathematics Department, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt

Received 6 November 2005; received in revised form 5 April 2007; accepted 5 April 2007

Abstract

In this paper, we study the optimal control problem for an $n \times n$ coupled Petrowsky type system involving a 2ℓ-th order self-adjoint elliptic operator with an infinite number of variables and constrained boundary control acting through Neumann conditions. Also, we derived the necessary and sufficient conditions of optimality for two types of performance index (quadratic one, general integral form).

By using standard Lions's arguments [J.L. Lions, Optimal Control of Systems Governed by Partial Differential Equations, vol. 170, Springer-Verlag, 1971] we proved the existence of a solution to the $n \times n$ coupled Petrowsky system and we derived optimality conditions for the optimal control problem with a quadratic performance index. In the case of the general integral form of the performance index we applied Dubovitskii–Milyutin's formalism earlier used in Kotarski [W. Kotarski, Some problems of optimal and pareto optimal control for distributed parameter systems, Reports of Silesian University Katowice, Poland, 1997, no. 1668]. Finally, we provided some special cases.

Keywords: Differential system of Petrowsky type; Infinite number of variables; Neumann condition; Boundary control; Optimality conditions

1. Introduction

The controlled system arising in engineering practice, physics, medicine, etc., must often be considered with distributed parameters, being governed typically by partial differential equations. Tools used for the optimal control of distributed parameter systems vary from the purely theoretical to mathematical analysis and the theory of partial differential equations. A fundamental class of optimal controls and its mathematical approaches can be found in Lions [1].

In [3–7], we study the linear quadratic optimal control problem for systems described by different types of partial differential operator ($n \times n$ matrix operators) defined on spaces of functions of an infinite number of variables (understood here to be a vector in an infinite tensor product of one-dimensional spaces). To obtain optimality conditions, the arguments of Lions [1] have been applied.

Using the Dubovitskii–Milyutin theorem, Kotarski in [2] obtained the necessary and sufficient conditions of optimality for the single Petrowsky type equation with an infinite number of variables and performance index that was more general than the quadratic one and had an integral form.

The questions treated in this paper relate to the above results but in a different direction by taking the case of optimal boundary control of the $n \times n$ coupled Petrowsky type system involving a 2ℓth order operator with an infinite

E-mail address: haelsaify@yahoo.com.

0898-1221/$ - see front matter
doi:10.1016/j.camwa.2007.04.033

number of variables and boundary control through a Neumann condition. First, using Lions' theorems [1] we study the quadratic boundary control problem for this system, and also the application of the generalized Dubovitskii–Milyutin theorem demonstrated on an optimization problem for the same Petrowsky type system with the performance index in integral form. Finally, necessary and sufficient conditions for optimality of boundary control are given. A set of inequalities that characterize this optimal control is obtained and this set is studied in order to construct algorithms useful to numerical computations for the approximation of control.

The outline of this paper is as follows: in Section 2, we formulate the mixed Neumann problem for an $n \times n$ differential Petrowsky type system with an infinite number of variables. In Section 3, the quadratic boundary control problem of this system is formulated; then we give the necessary and sufficient conditions for the control to be optimal. In Section 4, we give special cases to derive optimality conditions. In Section 5, the boundary control problem with a general performance index and the optimality condition for this problem are formulated.

2. The Neumann problem for the differential Petrowsky type system

Below, we consider the functions of points $x \in R^\infty = R' \times R' \times \cdots$, the coordinate notation of such points being $x = (x_k)_{k=1}^\infty$, $x_k \in R'$. Let $(P_k)_{k=1}^\infty$ be a fixed sequence of positive continuously differentiable probability weights, $R^1 \in x_k \to P_k(x_k) \in (0, \infty)$. The weighted product measure on R^∞ given by, [8],

$$\begin{aligned} d\rho(x) &= (P_1(x_1)dx_1) \otimes (P_2(x_2)dx_2) \otimes \cdots \\ &= (d\rho_1(x_1)) \otimes (d\rho_2(x_2)) \otimes \cdots . \end{aligned}$$

Let Ω be a bounded open set in R^∞ with smooth boundary Γ and $(W^\ell(\Omega, R^\infty), d\rho(x))^n$ (briefly $(W^\ell(\Omega, R^\infty))^n$), $\ell = 1, 2, \ldots, n$-cartesian product of Sobolev space of vector function with infinitely many variables $\mathbf{y}(x) = \mathbf{y} = (y_1, y_2, \ldots, y_n) = (y_i)_{i=1}^n$ defined on Ω, i.e.

$$(W^\ell(\Omega, R^\infty))^n = \underbrace{(W^\ell(\Omega, R^\infty)) \times \cdots \times (W^\ell(\Omega, R^\infty))}_{n-\text{time}}.$$

This space is a Hilbert space endowed with the standard scalar product and is defined by

$$(\mathbf{y}, \varphi)_{(W^\ell(\Omega,R^\infty))^n} = \sum_{i=1}^n (y_i, \psi_i)_{W^\ell(\Omega,R^\infty)}, \quad \mathbf{y} = (y_i)_{i=1}^n, \varphi = (\varphi_i)_{i=1}^n \in (W^\ell(\Omega, R^\infty))^n.$$

We consider a family of the operator $A(t) \in L((W^\ell(\Omega, R^\infty))^n, (W^{-\ell}(\Omega, R^\infty))^n)$ such that

$$\begin{aligned} A(t)\mathbf{y} &= (y_1, y_2, \ldots, y_n) = (A_1(t)y_1, A_2(t)y_2, \ldots, A_n(t)y_n) \\ &\left(\sum_{|\alpha|\le\ell} \sum_{k=1}^\infty (1-)^{|\alpha|} D_k^{2\alpha} y_1(x) + q(x,t)y_1(x) + \sum_{j=1}^n a_{1j} y_j, \right. \\ &\sum_{|\alpha|\le\ell} \sum_{k=1}^\infty (1-)^{|\alpha|} D_k^{2\alpha} y_2(x) + q(x,t)y_2(x) + \sum_{j=1}^n a_{2j} y_j, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\left. \sum_{|\alpha|\le\ell} \sum_{k=1}^\infty (1-)^{|\alpha|} D_k^{2\alpha} y_n(x) + q(x,t)y_n(x) + \sum_{j=1}^n a_{nj} y_j \right) \\ &= \begin{pmatrix} \sum_{|\alpha|\le\ell} \sum_{k=1}^\infty (1-)^{|\alpha|} D_k^{2\alpha} + q + 1 & \cdots & -1 \\ 1 & \cdots & -1 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ 1 & \cdots & \sum_{|\alpha|\le\ell} \sum_{k=1}^\infty (1-)^{|\alpha|} D_k^{2\alpha} + q + 1 \end{pmatrix}_{n\times n} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_n \end{pmatrix}_{n\times 1} \end{aligned}$$

so that $A(t)$ is an $n \times n$ matrix operator with ith component

$$A_i(t)y_i(x) = \sum_{|\alpha|\le\ell}\sum_{k=1}^{\infty}(1-)^{|\alpha|}\cdot D_k^{2\alpha}y_i(x) + q(x,t)\varphi_i(x) + \sum_{j=1}^{n}a_{ij}y_j(x), \quad 1 \le i \le n$$

where $[\sum_{|\alpha|\le\ell}\sum_{k=1}^{\infty}(-1)^{|\alpha|}D_k^{2\alpha}y_i(x) + q(x,t)y_i(x)]$ is a bounded self-adjoint elliptic partial differential operator of the 2ℓth order with infinite variables,

$$D_k^{\alpha}y_i(x) = \frac{1}{\sqrt{P_k(x_k,t)}}\frac{\partial^{\alpha}}{\partial x_k^{\alpha}}\sqrt{P_k(x_k,t)}y_i(x),$$

the potential $q(x,t)$ is a real function in x which is bounded and measurable on Ω, such that $q(x,t) \ge C_0 > 0$, C_0 constant and a_{ij} is the coupled term defined by

$$a_{ij} = \begin{cases} 1 & \text{if } i \ge j \\ -1 & \text{if } i < j. \end{cases}$$

For each variable t which denotes the time, $t \in (0,T), T < \infty$ we define a family of bilinear form on $(W^{\ell}(\Omega, R^{\infty}))^n$ by

$$\pi : (W^{\ell}(\Omega,R^{\infty}))^n \times (W^{\ell}(\Omega,R^{\infty}))^n \to R^1,$$

$$\pi(t;\mathbf{y},\psi) = (A(t)\mathbf{y},\psi)_{(L_2(\Omega,R^{\infty}))^n} = \sum_{i=1}^{n}(A_i(t)y_i(x),\varphi_i(x))_{L_2(\Omega,R^{\infty})}.$$

Where $\mathbf{y} = (y_i)_{i=1}^n, \varphi = (\varphi_i)_{i=1}^n \in (W^{\ell}(\Omega,R^{\infty}))^n$ and $A(t)$ maps $(W^{\ell}(\Omega,R^{\infty}))^n$, onto $(W^{\ell}(\Omega,R^{\infty}))^n$ and takes the above form, so

$$\begin{aligned}\pi(t;\mathbf{y},\varphi) &= \sum_{i=1}^{n}\left(\sum_{|\alpha|\le\ell}\sum_{k=1}^{\infty}(-1)^{|\alpha|}D_k^{2\alpha}y_i(x) + q(x,t)y_i(x) + \sum_{j=1}^{n}a_{ij}y_j(x), \varphi_i(x)\right)_{L_2(\Omega,R^{\infty})} \\ &= \sum_{i=1}^{n}\int_{\Omega}\sum_{|\alpha|\le\ell}\sum_{k=1}^{\infty}D_k^{\alpha}y_i(x)D_k^{\alpha}\varphi_i(x)\mathrm{d}\rho + \sum_{i=1}^{n}\int_{\Omega}q(x,t)y_i(x)\varphi_i(x)\mathrm{d}\rho \\ &\quad + \sum_{i=1}^{n}\int_{\Omega}\sum_{j=1}^{n}a_{ij}y_j(x)\varphi_i(x)\mathrm{d}\rho. \end{aligned} \tag{1}$$

The above continuous bilinear form (1) is coercive on $(W^{\ell}(\Omega,R^{\infty}))^n$, that is, there exists $\lambda \in R^1, \lambda > 0$ such that

$$\pi(t;\mathbf{y},\mathbf{y}) \ge \lambda\|\mathbf{y}\|^2_{(W^{\ell}(\Omega,R^{\infty}))^n}. \tag{2}$$

Taking into account the form of a_{ij} we have

$$\begin{aligned}\pi(t;\mathbf{y},\varphi) &= \sum_{i=1}^{n}\int_{\Omega}\sum_{|\alpha|\le\ell}\sum_{k=1}^{\infty}D_k^{\alpha}y_i(x)D_k^{\alpha}\varphi_0(x)\mathrm{d}\rho + \sum_{i=1}^{n}\int_{\Omega}q(x,t)y_i(x)\varphi_i(x)\mathrm{d}\rho + \sum_{i=j=1}^{n}\int_{\Omega}y_i(x)\varphi_i(x)\mathrm{d}\rho \\ &\quad + \sum_{i>j}^{n}\int_{\Omega}y_i(x)\varphi_i(x)\mathrm{d}\rho - \sum_{i<j}^{n}\int_{\Omega}y_i(x)\varphi_i(x)\mathrm{d}\rho.\end{aligned}$$

Then

$$\begin{aligned}\pi(t;\mathbf{y},\mathbf{y}) &= \sum_{i=1}^{n}\left(\int_{\Omega}\sum_{|\alpha|\le\ell}\sum_{k=1}^{\infty}|D_k^{\alpha}y_i(x)|^2\mathrm{d}\rho + \int_{\Omega}q(x,t)|y_i(x)|^2\mathrm{d}\rho + \int_{\Omega}|y_i(x)|^2\mathrm{d}\rho\right) \\ &\ge \sum_{i=1}^{n}\left(\sum_{|\alpha|\le\ell}\sum_{k=1}^{\infty}\|D_k^{\alpha}y_i(x)\|^2_{L_2(\Omega,R^{\infty})} + C_0\|y_i(x)\|^2_{L_2(\Omega,R^{\infty})} + \|y_i(x)\|^2_{L_2(\Omega,R^{\infty})}\right)\end{aligned}$$

$$= \sum_{i=1}^{n} \|y_i(x)\|^2_{W^\ell(\Omega,R^\infty)} + C_0\|y_i(x)\|^2_{L_2(\Omega,R^\infty)}$$
$$\geq \sum_{i=1}^{n} \|y_i(x)\|^2_{W^\ell(\Omega,R^\infty)} = \|\mathbf{y}\|^2_{(W^\ell(\Omega,R^\infty))^n}.$$

For $\mathbf{y}, \varphi \in (W^\ell(\Omega, R^\infty))^n$ the function

$$t \to \pi(t; \mathbf{y}, \psi) \quad \text{is continuously differentiable with respect to } t \text{ in } (0, T). \tag{3}$$

In considerating the above in light of Lions and Magenes vol. 2. chapter 5 [9], we can formulate the following $n \times n$ coupled Petrowsky type system with mixed Neumann conditions which defines the state of our control problem.

Theorem 1. *Assume that (2) and (3) hold, then if given $f = f(x,t) \in (L_2(0,T; W^{-\ell}(\Omega, R^\infty)))^n$, $y_{i,0}(x) \in L_2(\Omega, R^\infty)$ and $y_{i,1}(x) \in W^{-\ell}(\Omega, R^\infty)$ there exists a unique element $\mathbf{y} = \mathbf{y}(u) \in (L_2(0,T; L_2(\Omega, R^\infty)))^n$ (briefly $(L_2(Q))^n$) such that $\forall 1 \leq i \leq n$*

$$\left.\begin{aligned}
&\frac{\partial^2 y_i(u)}{\partial t^2} + A_i(t)y_i(u) = f_i \quad \text{in } Q = \Omega\times]0,T[,\\
&\frac{\partial^m y_i(u)}{\partial \nu^m_{A_i}} = u_i \quad \text{on } S = \Gamma\times]0,T[,\\
&y_i(x,0) = y_{i,0}(x), \qquad \frac{\partial y_i(x,0)}{\partial t} = y_{i,1}(x) \quad \text{in } \Omega
\end{aligned}\right\} \tag{4}$$

where $\frac{\partial^m}{\partial \nu^m_{A_i}}$ derivatives of order m along the normal to S, $m = 0, 1, \ldots, \ell - 1$, S is the lateral boundary of Q and $(\frac{\partial^2}{\partial t^2} + A(t))$ is the $n \times n$ matrix operator well-positioned in the sense of Petrowsky type maps $(L_2(0,T; W^\ell(\Omega, R^\infty)))^n$ onto $(L_0(0,T; W^{-\ell}(\Omega, R^\infty)))^n$ and

$$\frac{\partial y_i}{\partial t} \in L_2(Q), \qquad \frac{\partial^2 y_i}{\partial t^2} \in L_2(0,T; W^{-\ell}(\Omega, R^\infty)).$$

Proof. Let $X = \{\varphi : \varphi = (\varphi_i)^n_{i=1}, \varphi_i \in L_2(0,T; W^\ell(\Omega, R^\infty)), \varphi_i' \in L_2(Q), \varphi_i'' + A_i^*(t)\varphi_i \in L_2(Q), \frac{\partial^m \varphi_i}{\partial \nu^m_{A_i^*}} = 0 \text{ on } S, \varphi_i(x,T) = 0, \varphi'(x,T) = 0\}$, the operator $\varphi \to \varphi'' + A(t)\varphi$ is an isomorphism of X onto $(L_2(Q))^n$, where $A_i^*(t)$ is the adjoint to $A_i(t)$.

By transposition: let $\varphi \to L(\varphi)$ be a continuous linear form on X; there exists a unique $\mathbf{y} = \mathbf{y}(u) \in (L_2(Q))^n$ such that

$$\sum_{i=1}^{n} \int_Q y_i(u)(\varphi_i'' + A_i^*(t)\varphi_i)\mathrm{d}\rho\mathrm{d}t = L(\varphi) \quad \forall \varphi \in X.$$

We define a continuous linear form on X by

$$L(\varphi) = \sum_{i=1}^{n} \left[\int_Q f_i\varphi_i \mathrm{d}\rho\mathrm{d}t + \int_S u_i\varphi_i \mathrm{d}S + \int_\Omega y_{i,1}\varphi_i(x,0)\mathrm{d}\rho - \int_\Omega y_{i,0}\frac{\partial \varphi_i(x,0)}{\partial t}\mathrm{d}\rho\right]$$

where $f_i \in L_2(0,T; W^{-\ell}(\Omega, R^\infty))$, $y_{i,0} \in L_2(\Omega, R^\infty)$, $y_{i,1} \in W^\ell(\Omega, R^\infty)$ and $u_i \in L_2(0,T; L_2(\Gamma))$ (briefly $L_2(S)$).

Then we have

$$\sum_{i=1}^{n} \int_Q y_i(u)(\varphi_i' + A_i^*(t)\varphi_i)\mathrm{d}\rho\mathrm{d}t$$
$$= \sum_{i=1}^{n} \left[\int_Q f_i\varphi_i \mathrm{d}\rho\mathrm{d}t + \int_S u_i\varphi_i \mathrm{d}S + \int_\Omega y_{i,1}\varphi_i(x,0)\mathrm{d}\rho - \int_\Omega y_{i,0}\frac{\partial \varphi_i(x,0)}{\partial t}\mathrm{d}\rho\right]. \tag{5}$$

Letting $\varphi(t) = (\varphi_i(t))_{i=1}^n$ with compact support in $]0, T[$, we deduce that

$$\frac{\mathrm{d}^2 y_i}{\mathrm{d}t^2} + A(t) y_i = f_i \quad \text{in }]0, T[. \tag{6}$$

Now, scalar multiplying (6) by $\varphi \in X$ and integrating by parts by applying Green's formula, we obtain

$$\sum_{i=1}^n \int_Q f_i \varphi_i \mathrm{d}\rho \mathrm{d}t = \sum_{i=1}^n \Bigg[-\int_\Omega \frac{\partial y_i(x,0)}{\partial t} \varphi_i(x,0) \mathrm{d}\rho + \int_\Omega y_i(x,0) \frac{\partial \varphi_i(x,0)}{\partial t} \mathrm{d}\rho + \int_Q y_i(\varphi_i'' + A^*(t)\varphi_i) \mathrm{d}\rho \mathrm{d}t - \int_S \frac{\partial^m y_i}{\partial \nu_{A_i}^m} \varphi_i \mathrm{d}S. \Bigg].$$

Comparing the latter equation with (5), we get

$$\sum_{i=1}^n \left[-\int_\Omega y_{i,1}(x) \varphi_i(x,0) \mathrm{d}\rho + \int_\Omega y_{i,0}(x) \frac{\partial \varphi_i(x,0)}{\partial t} \mathrm{d}\rho - \int_S u_i \varphi_i \mathrm{d}S \right]$$
$$= \sum_{i=1}^n \left[-\int_\Omega \frac{\partial y_i(x,0)}{\partial t} \varphi_i(x,0) \mathrm{d}\rho + \int_\Omega y_i(x,0) \frac{\partial \varphi_i(x,0)}{\partial t} \mathrm{d}\rho - \int_S \frac{\partial^m y_i}{\partial \nu_{A_i}^m} \varphi_i \mathrm{d}S \right].$$

From this we deduce that

$$\frac{\partial^m y_i}{\partial \nu_{A_i}^m} = u_i \quad \text{on } S,$$

$$y_i(x,0) = y_{i,0}(x), \qquad \frac{\partial y_i(x,0)}{\partial t} y_{i,1}(x) \quad \text{in } \Omega.$$

3. Quadratic boundary control problem

For the control $u = (u_i)_{i=1}^n \in (L_2(S))^n = U$ (space of controls) the state of the system $\mathbf{y}(u) \in (L_2(Q))^n$ is given by the solution of (4) with $y_i = y_i(u)$, so the control is being exercised through the boundary.

We observe $\mathbf{y}(u)$ on S, so $\mathbf{y}(u) \in (L_2(S))^n$ and mapping $u \to \mathbf{y}(u)|_S$ is a continuous affine map of $(L_2(S))^n$ onto itself, and the cost function is given by

$$J(u) = \sum_{i=1}^n \left[\|y_i(u) - z_{i,d}\|_{L_2(S)}^2 + (N_i u_i, u_i)_{L_2(S)} \right]$$
$$= \sum_{i=1}^n \int_S \left[(y_i(u) - z_{i,d})^2 + N_i u_i^2 \right] \mathrm{d}S$$

where $Z_d = (z_{i,d})_{i=1}^n \in (L_2(S))^n$ and $N = (N_i)_{i=1}^n \in L((L_2(S))^n, (L_2(S))^n)$ is a diagonal matrix of Hermitiun positive definite operators:

$$Nu = (N_i u_i)_{i=1}^n, \qquad (Nu, u)_{(L_2(S))^n} \geq \xi \|u\|_{(L_2(S))^n}^2, \quad \xi > 0.$$

If U_{ad} (set of admissible controls) is a closed convex subset of $(L_2(S))^n$, minimizing J over U_{ad}, i.e. we find u^0 (optimal control) such that

$$J(u^0) = \inf_{u \in U_{\mathrm{ad}}} J(u). \tag{7}$$

The solution to this problem is given in the following theorem.

Theorem 2. *Problem (7) admits a unique solution given by (4) and*

$$\sum_{i=1}^n \int_S (P_i(u^0) + N_i u_i^0)(u_i - u_i^0) \mathrm{d}S \geq 0 \quad \forall u = (u_i)_{i=1}^n \in U_{\mathrm{ad}}$$

where $P_i(u^0)$ is the adjoint state.

Proof. As in [1] the optimal control is characterized by

$$\sum_{j=1}^{n} J_i'(u^0)(u_i - u_i^0) \geq 0 \quad \forall u = (u_i)_{i=1}^n \in U_{\text{ad}}$$

that is

$$\sum_{i=1}^{n} \int_S \left[(y_i(u^0) - z_{id})(y_i(u) - y_i(u^0)) + (N_i u_i^0)(u_i - u_i^0) \right] \mathrm{d}S \geq 0. \tag{8}$$

For the control $u = (u_i)_{i=1}^n$ the adjoint state $P_i(u) \in L_2(Q)$ is given by

$$\left.\begin{aligned} &\frac{\partial^2 P_i(u)}{\partial t^2} + A_i^*(t) P_i(u) = 0 \quad \text{in } Q, \\ &\frac{\partial^m P_i(u)}{\partial \nu^m} = y_i(u) - z_{i,d} \quad \text{on } S, \\ &P_i(x, T; u) = 0, \qquad \frac{\partial P_i(x, T; u)}{\partial t} = 0 \quad \text{on } \Omega. \end{aligned}\right\} \tag{9}$$

From Theorem 1, this problem admits a unique solution $P_i(u) \in L_2(Q)$.

Using Green's formula we transform (8) as follows: formally, setting $u = u^0$ in (9) and multiplying the first equation in (9) by $(y_i(u) - y_i(u))$ and integrating by parts, we obtain

$$\begin{aligned} 0 &= -\int_S \frac{\partial^m P_i(u^0)}{\partial \nu_{A_i^*}^m} (y_i(u) - y_i(u^0)) \mathrm{d}S + \int_S P_i(u^0) \frac{\partial^m y_i(u^0)}{\partial \nu_{A_i}^m} - \frac{\partial^m y_i(u^0)}{\partial \nu_{A_i}^m} \mathrm{d}S \\ &= -\int_S (y_i(u^0) - z_{i,d})(y_i(u) - y_i(u^0)) \mathrm{d}S + \int_S P_i(u^0)(u_i - u_i^0) \mathrm{d}S; \end{aligned}$$

condition (8) then becomes

$$\sum_{i=1}^{n} \int_S (P_i(u^0) + N_i u_i^0)(u_i - u_i^0) \mathrm{d}S \geq 0 \quad \forall u_i \in U_{\text{ad}}.$$

4. Special cases

(1) If we take $n = 2$, then $U = L_2(S) \times L_2(S)$ and the optimality system is given by

$$\begin{aligned} &\frac{\partial^2 y_1(u^0)}{\partial t^2} + \left(\sum_{|\alpha| \leq \ell} \sum_{k=1}^{\infty} D_k^{2\alpha} + q(x, t) \right) y_1(u^0) + y_1(u^0) - y_2(u^0) = f_1 \quad \text{in } Q, \\ &\frac{\partial^2 y_2(u^0)}{\partial t^2} + \left(\sum_{|\alpha| \leq \ell} \sum_{k=1}^{\infty} D_k^{2\alpha} + q(x, t) \right) y_2(u^0) + y_2(u^0) + y_1(u^0) = f_2 \quad \text{in } Q, \\ &\frac{\partial^m y_1(u^0)}{\partial \nu^m} = u_1^0, \qquad \frac{\partial^m y_2(u^0)}{\partial \nu^m} = u_2^0 \quad \text{on } S, \\ &y_1(x, 0; u^0) = y_{1,0}(x), \qquad y_2(x, 0; u^0) = y_{2,0}(x) \quad \text{in } \Omega, \\ &\frac{\partial y_1(x, 0; u^0)}{\partial t} = y_{1,1}(x), \qquad \frac{\partial y_2(x, 0; u^0)}{\partial t} = y_{2,1}(x) \quad \text{in } \Omega, \\ &\frac{\partial^2 P_1(u^0)}{\partial t^2} + \left(\sum_{|\alpha| \leq \ell} \sum_{k=1}^{\infty} D_k^{2\alpha} + q(x, t) \right) P_1(u^0) + P_1(u^0) + P_2(u^0) = 0 \quad \text{in } Q, \\ &\frac{\partial^2 P_2(u^0)}{\partial t^2} + \left(\sum_{|\alpha| \leq \ell} \sum_{k=1}^{\infty} D_k^{2\alpha} + q(x, t) P_2(u^0) \right) + P_2(u^0) - P_1(u^0) = 0 \quad \text{in } Q, \end{aligned}$$

$$\frac{\partial^m P_1(u^0)}{\partial \nu^m_{A_1^*}} = y_1(u^0) - z_{1,d}, \qquad \frac{\partial^m P_2(u^0)}{\partial \nu^m_{A_2^*}} = y_2(u^0) - z_{2,d} \quad \text{on } S,$$
$$P_1(x, T; u^0) = 0, \qquad P_2(x, T; u^0) = 0 \quad \text{in } \Omega,$$
$$\frac{\partial P_1(x, T; u^0)}{\partial t} = 0, \qquad \frac{\partial P_2(x, T; u^0)}{\partial t} = 0 \quad \text{in } \Omega,$$
$$\int_S \left[(P_1(u^0) + N_1 u_1^0)(u_1 - u_1^0) + (P_2(u^0) + N_2 u_2^0)(u_2 - u_2^0) \right] \mathrm{d}S \geq 0 \tag{10}$$

for all $(u_1, u_2) \in U_{\text{ad}}$, where $u^0 = (u_1^0, u_2^0) \in U_{\text{ad}}$ and $P(u^0) = (P_1(u^0), P_2(u^0))$ is the adjoint state.

(2) If $n = 2$ and $U_{\text{ad}} = U$ (no constraints on controls) then the optimal $u^0 = (u_1^0, u_2^0)$ is obtained by solving the following system of partial differential equations:

$$\frac{\partial^2 y_1(u^0)}{\partial t^2} + \left(\sum_{|\alpha| \leq \ell} \sum_{k=1}^{\infty} D_k^{2\alpha} + q(x, t) \right) y_1(u^0) + y_1(u^0) - y_2(u^0) = f_1 \quad \text{in } Q,$$
$$\frac{\partial^2 y_2(u^0)}{\partial t^2} + \left(\sum_{|\alpha| \leq \ell} \sum_{k=1}^{\infty} D_k^{2\alpha} + q(x, t) y_2(u^0) \right) + y_2(u^0) + y_1(u^0) = f_2 \quad \text{in } Q,$$
$$\frac{\partial^2 P_1(u^0)}{\partial t^2} + \left(\sum_{|\alpha| \leq \ell} \sum_{k=1}^{\infty} D_k^{2\alpha} + q(x, t) \right) P_1(u^0) + P_1(u^0) + P_2(u^0) = 0 \quad \text{in } Q,$$
$$\frac{\partial^2 P_2(u^0)}{\partial t^2} + \left(\sum_{|\alpha| \leq \ell} \sum_{k=1}^{\infty} D_k^{2\alpha} + q(x, t) \right) P_2(u^0) + P_2(u^0) - P_1(u^0) = 0 \quad \text{in } Q,$$
$$\frac{\partial^m y_1(u^0)}{\partial \nu^m_{A_1}} + N^{-1} P_1(u^0) = 0, \qquad \frac{\partial^m y_2(u^0)}{\partial \nu^m_{A_2}} + N_2^{-1} P_2(u^0) = 0 \quad \text{on } S,$$
$$\frac{\partial^m P_1(u^0)}{\partial \nu^m_{A_1^*}} = y_1(u^0) - z_{1,d}, \qquad \frac{\partial^m P_2(u^0)}{\partial \nu^m_{A_2^*}} = y_2(u^0) - z_{2,d} \quad \text{on } S,$$
$$y_1(x, 0; u^0) = y_{1,0}(x), \qquad y_2(x, 0; u^0) = y_{2,0}(x) \quad \text{in } \Omega,$$
$$\frac{\partial y_1(x, 0; u^0)}{\partial t} = y_{1,1}(x), \qquad \frac{\partial y_2(x, 0; u^0)}{\partial t} = y_{2,1}(x) \quad \text{in } \Omega,$$
$$P_1(x, T; u^0) = 0, \qquad P_2(x, T; u^0) = 0 \quad \text{in } \Omega,$$
$$\frac{\partial P_1(x, T; u^0)}{\partial t} = 0, \qquad \frac{P_2(x, T; u^0)}{\partial t} = 0 \quad \text{in } \Omega.$$

Further

$$u_1^0 = -N_1^{-1} P(u^0), \qquad u_2^0 = -N_2^{-1} P_2(u^0).$$

(3) If we assume that

$$U_{\text{ad}} = \{u_i^0 | u_i^0 \geq 0 \text{ a.e on } S\}.$$

Then u_1^0, u_2^0 is obtained by solving the unilateral problem

$$\frac{\partial^2 y_1(u^0)}{\partial t^2} + \left(\sum_{|\alpha| \leq \ell} \sum_{k=1}^{\infty} D_k^{2\alpha} + q(x, t) \right) y_1(u^0) + y_1(u^0) - y_2(u^0) = f_1 \quad \text{in } Q,$$
$$\frac{\partial^2 y_2(u^0)}{\partial t^2} + \left(\sum_{|\alpha| \leq \ell} \sum_{k=1}^{\infty} D_k^{2\alpha} + q(x, t) \right) y_2(u^0) + y_2(u^0) + y_1(u^0) = f_2 \quad \text{in } Q,$$

$$\frac{\partial^2 P_1(u^0)}{\partial t^2} + \left(\sum_{|\alpha|\le \ell}\sum_{k=1}^{\infty} D_k^{2\alpha} + q(x,t)\right) P_1(u^0) + P_1(u^0) + P_2(u^0) = 0 \quad \text{in } Q,$$

$$\frac{\partial^2 P_2(u^0)}{\partial t^2} + \left(\sum_{|\alpha|\le \ell}\sum_{k=1}^{\infty} D_k^{2\alpha} + q(x,t)\right) P_2(u^0) + P_2(u^0) - P_1(u^0) = 0 \quad \text{in } Q,$$

$$\frac{\partial^m y_1(u^0)}{\partial \nu^m} \ge 0, \qquad P_1 + N_1 \frac{\partial^m y_1(u^0)}{\partial \nu^m} = 0 \quad \text{on } S,$$

$$\left(P_1 + N_1 \frac{\partial^m y_1(u^0)}{\partial \nu^m_{A_1}}\right)\left(\frac{\partial^m y_1(u^0)}{\partial \nu^m_{A_2}}\right) = 0, \qquad \frac{\partial^m P_1(u^0)}{\partial \nu^m_{A_1^*}} = y_1(u^0) - z_{1,d} \quad \text{on } S,$$

$$\frac{\partial^m y_2(u^0)}{\partial \nu^m_{A_1}} \ge 0, \qquad P_2 + N_2 \frac{\partial^m y_2(u^0)}{\partial \nu^m_{A_2}} = 0 \quad \text{on } S,$$

$$\left(P_2 + N_2 \frac{\partial^m y_2(u^0)}{\partial \nu^m}\right)\left(\frac{\partial^m y_2(u^0)}{\partial \nu^m}\right) = 0, \qquad \frac{\partial^m P_2(u^0)}{\partial \nu^m_{A_2^*}} = y_2(u^0) - z_{2,d} \quad \text{on } S,$$

$$y_1(x,0) = y_{1,0}(x), \qquad y_2(x,0) = y_{2,0}(x) \quad \text{in } \Omega,$$

$$\frac{\partial y_1(x,0)}{\partial t} = y_{1,1}(x), \qquad \frac{\partial y_2(x,0)}{\partial t} = y_{2,1}(x) \quad \text{in } \Omega,$$

$$P_1(x,T) = 0, \qquad P_2(x,T) = 0 \quad \text{in } \Omega,$$

$$\frac{\partial P_1(x,T)}{\partial t} = 0, \qquad \frac{\partial P_2(x,T)}{\partial t} = 0 \quad \text{in } \Omega,$$

then

$$u_1^0 = \frac{\partial^m y_1}{\partial \nu^m_{A_1}}, \qquad u_2^0 = \frac{\partial^m y_2}{\partial \nu^m_{A_2}}.$$

Note 1. We observe that the conditions of optimality derived above allow us to obtain an analytical formula for the optimal control in particular cases only (i.e. where there are no constraints on controls). These results are due to the determining of the function $P_i(\bar{u}^0)$ in the maximum condition from the adjoint equation if and only if we know $\bar{y}^0$ which corresponds to the control $\bar{u}^0$. These mutual connections make the practical use of the derived optimization formulas difficult. Therefore, we resign from the exact determining of the optimal control by using approximation methods. This requires further investigation and will form tasks for future research.

5. Boundary control problem with general performance functional

Let us denote by $\mathcal{U} = (L_2(S))^n$ the space of controls, by $Y = (L_2(Q))^n$ the space of state and for a control $\mathbf{u} = (u_i)_{i=1}^n \in (L_2(S))^n$ the state $\mathbf{y}(\mathbf{u}) = (y_i(\mathbf{u}))_{i=1}^n = (y_i(x,t;\mathbf{u}))_{i=1}^n$ of the system given by the solution of (3); the control time T is assumed to be fixed.

The performance functional is given by

$$I(\mathbf{y},\mathbf{u}) = \sum_{i=1}^n I_i(\mathbf{y},\mathbf{u}) = \sum_{i=1}^n \int_S F_i(x,t;\mathbf{y},\mathbf{u})\mathrm{d}S \to \min, \tag{11}$$

where for every $i = 1,\dots,n$, $F_i : \Omega \times (0,T) \times R^n \times R^n \to R^1$ that satisfies the following conditions:
(A_1) $F_i(x,t;\mathbf{y},\mathbf{u})$ is continuous with respect to $(x,t;\mathbf{y},\mathbf{u})$,
(A_2) there exists Fréchet derivatives $F_{iy}(x,t;\mathbf{y},\mathbf{u})$, $F_{iu}(x,t;\mathbf{y},\mathbf{u})$ which are continuous with respect to $(x,t;\mathbf{y},\mathbf{u})$.
(A_3) $F_i(x,t;\mathbf{y},\mathbf{u})$ is strictly convex with respect to the pair $(\mathbf{y},\mathbf{u})$, i.e.

$$F_i(x,t;\lambda\mathbf{y}^1 + (1-\lambda)\mathbf{y}^2, \lambda\mathbf{u}^1 + (1-\lambda)\mathbf{u}^2) < \lambda F_i(x,t;\mathbf{y}^1,\mathbf{u}^1) + (1-\lambda)F_i(x,t,\mathbf{y}^2,\mathbf{u}^2),$$
$$\forall \mathbf{y}^1\mathbf{y}^2, \mathbf{u}^1\mathbf{u}^2 \in R^n, (\mathbf{y}^1,\mathbf{u}^1) \ne (\mathbf{y}^2,\mathbf{u}^2), \lambda \in (0,1).$$

We assume the following constraints on controls:

Let $u \in \mathcal{U}_{\text{ad}}$ (set of admissible controls) be a closed convex and bounded subset of $\mathcal{U}$. (12)

The solution of the statement optimal control problem is equivalent to seeking a pair $(\mathbf{y}^0, \mathbf{u}^0) \in E$ where $Y \times \mathcal{U} = E = E_1 \times E_2 \times E_n$, that satisfies (3), and minimizes the performance functional (11) subject to the control constraints (12).

Using the extension of the Dubovitskii–Milyutin Theorem in the case of n equality constraints, [10], we derive the necessary and sufficient optimality condition for the optimal control problem (3), (11) and (12) in the following.

Theorem 3. *By the assumptions mentioned above, there exist a unique solution* $(\mathbf{y}^0, \mathbf{u}^0)$ *of the optimization problem* (3), (11) *and* (12) *which satisfies the maximum condition*

$$\sum_{i=1}^{n} \int_S (P_i + F_{iu})(u_i - u_i^0)\mathrm{d}S \geq 0 \quad \forall \mathbf{u} = (u_i)_{i=1}^n \in \mathcal{U}_{\text{ad}}, \tag{13}$$

where the superscript 0 *denotes the optimal element and* P_i *is the adjoint state.*

Proof. We apply the generalized Dubovitskii–Milyutin theorem. Therefore, denote by G_1, G_2 the following sets in the space $E = Y \times \mathcal{U}$, $E = E_1 \times E_2 \times \cdots \times E_n$,

$$G_1 = \bigcup_{1 \leq i \leq n} G_{1,i} = \bigcup_{1 \leq i \leq n} \left\{ (y_i, u_i) \in E_i; \quad \begin{array}{l} \dfrac{\partial^2 y_i}{\partial t^2} + A_i(t) y_i = f_i, \quad x \in \Omega, t \in (0,T), \\ y_i(x,0) = y_{i,1}(x), \quad x \in \Omega, \\ \dfrac{\partial y_i}{\partial t}(x,0) = y_{i,2}(x), \quad x \in \Omega, \\ \dfrac{\partial^m y_i(x,t)}{\partial \nu_{A_i}^m} = u_i, \quad x \in \Gamma, t \in (0,T). \end{array} \right\}$$

$$G_2 = \{(\mathbf{y}, \mathbf{u}) \in E; \mathbf{y} \in Y, \mathbf{u} \in \mathcal{U}_{\text{ad}}\}.$$

The problem (3)–(5) can then be formulated in the form

$$I(\mathbf{y}, \mathbf{u}) \to \min \quad \text{subject to } (\mathbf{y}, \mathbf{u}) \in G_1 \cap G_2.$$

We approximate the sets G_1 and G_2 by the regular tangent cone (*RTC*) and the performance functional by the regular improvement cone (*RFC*).

The tangent cone to the set G_1 at $(\mathbf{y}^0, \mathbf{u}^0)$ has the form

$$RTC(G_1(\mathbf{y}^0, \mathbf{u}^0)) = \{(\tilde{\mathbf{y}}, \tilde{\mathbf{u}}) \in E; B'(\mathbf{y}^0, \mathbf{u}^0)(\tilde{\mathbf{y}}, \tilde{\mathbf{u}}) = 0\}$$

so, for all $1 \leq i \leq n$ we have

$$RTC(G_1, (y_i^0, u_i^0)) = \left\{ (\tilde{y}_i, \tilde{u}_i) \in E; \quad \begin{array}{l} \dfrac{\partial^2 \tilde{y}_i}{\partial t^2} + A(t)\tilde{y}_i = 0, \quad x \in \Omega, t \in)0,T), \\ \tilde{y}_i(x,0) = 0, \quad x \in \Omega, \\ \dfrac{\partial \tilde{y}_i}{\partial t}(x,0) = 0, \quad x \in \Omega, \\ \dfrac{\partial^m \tilde{y}_i(x,t)}{\partial \nu_{A_i}^m} = \tilde{u}_i, \quad x \in \Gamma, t \in (0,T). \end{array} \right\}$$

where $B'(\mathbf{y}^0, \mathbf{u}^0)(\tilde{\mathbf{y}}, \tilde{\mathbf{u}})$ is the Fréchet differential of the operator B where,

$$B : (L_2(Q))^n \times (L_2(S))^n \to (L_2(0,T; W_0^{-\ell}(\Omega, R^\infty)))^n \times (W_0^\ell(\Omega, R^\infty))^n \times (L_2(\Omega, R^\infty))^n \times (L_2(S))^n,$$

$$B(\mathbf{y}, \mathbf{u}) = \left(\frac{\partial^2 \mathbf{y}}{\partial t^2} + A(t)\mathbf{y}, \mathbf{y}(x,0) - \mathbf{y}_1(x), \frac{\partial \mathbf{y}}{\partial t}(x,0) - \mathbf{y}_2(x), \frac{\partial^m y}{\partial \nu_A^m} - u \right).$$

The tangent cone to the set G_2 at $(\mathbf{y}^0, \mathbf{u}^0)$ has the form

$$RTC(G_2, (\mathbf{y}^0, \mathbf{u}^0)) = Y \times RTC(\mathcal{U}_{\text{ad}}, \mathbf{u}^0),$$

where $RTC(\mathcal{U}_{\text{ad}}, \mathbf{u}^0)$ is the tangent cone to the set $\mathcal{U}_{\text{ad}}$ at the point $\mathbf{u}^0$.

It is known that the tangent cones are closed and

$$RTC(G_1 \cap G_2, (\mathbf{y}^0, \mathbf{u}^0)) = RTC(G_1, (\mathbf{y}^0, \mathbf{u}^0)) \cap RTC(G_2, (\mathbf{y}^0, \mathbf{u}^0)),$$

further, $[RTC(G_1, (\mathbf{y}^0, \mathbf{u}^0))]^*$ and $[RTC(G_2, (\mathbf{y}^0, \mathbf{u}^0))]^*$ mean the same [2].

The regular improvement cone for the performance functional has the form

$$RTC(I, (\mathbf{y}^0, \mathbf{u}^0)) = \left\{ (\tilde{\mathbf{y}}, \tilde{\mathbf{u}}) \in E; \sum_{i=1}^{n} \int_S (F_{iy}\tilde{y}_i + F_{iu}\tilde{u}_i) \mathrm{d}S < 0 \right\}.$$

If $RFC(I, (\mathbf{y}^0, \mathbf{u}^0)) \neq \Phi$, then its adjoint cone consists of the elements of the form $g_3(\tilde{\mathbf{y}}, \tilde{\mathbf{u}}) = -\lambda_0 \sum_{i=1}^{n} \int_S (F_{iy}\tilde{y}_i + F_{iu}\tilde{u}_i)\mathrm{d}S$, where $\lambda_0 > 0$.

The functionals belonging to $[RTC(G_1, (\mathbf{y}^0, \mathbf{u}^0))]^*$ are

$$g_1(\tilde{\mathbf{y}}, \tilde{\mathbf{u}}) = 0 \quad \forall (\tilde{\mathbf{y}}, \tilde{\mathbf{u}}) \in RTC(G_1, (\mathbf{y}^0, \mathbf{u}^0)).$$

The functionals in $[RTC(G_2, (\mathbf{y}^0, \mathbf{u}^0))]^*$ can be expressed as

$$g_2(\tilde{\mathbf{y}}, \tilde{\mathbf{u}}) = g_2^1(\tilde{\mathbf{y}}) + g_2^2(\tilde{\mathbf{u}}),$$

where $g_2^1(\tilde{\mathbf{y}}) = 0 \ \forall \mathbf{y} \in Y$ and $g_2^2(\tilde{\mathbf{u}})$ is the support functional to the set U_{ad} at $\mathbf{u}^0$.

Now, we can write the Euler–Lagrange equation for our problem as

$$g_2^2(\tilde{\mathbf{u}}) = \sum_{i=1}^{n} \left[\lambda_0 \int_S F_{iy}\tilde{y}_i \mathrm{d}S + \lambda_0 \int_S F_{iu}\tilde{u}_i \mathrm{d}S \right]. \tag{14}$$

where $(\tilde{\mathbf{y}}, \tilde{\mathbf{u}}) \in RTC(G_1, (\mathbf{y}^0, \mathbf{u}^0))$.

Introducing the adjoint variable $\mathbf{P} = (P_i)_{i=1}^n$ by $\forall 1 \le i \le n$,

$$\begin{aligned}
&\frac{\partial^2 P_i}{\partial t^2} + A^*(t)P_i = 0, \quad x \in \Omega, t \in (0, T),\\
&P_i(x, T) = 0, \quad x \in \Omega,\\
&\frac{\partial P_i}{\partial t}(x, T) = 0, \quad x \in \Omega,\\
&\frac{\partial^m P_i(x, t)}{\partial \nu_{A_i^*}^m} = F_{iy}, \quad x \in \Gamma, t \in (0, T)
\end{aligned}$$

and taking into account that $\tilde{\mathbf{y}}$ is a solution of $B^1(\mathbf{y}^0, \mathbf{u}^0)(\tilde{\mathbf{y}}, \tilde{\mathbf{u}}) = 0$, for any fixed $\tilde{\mathbf{u}}$, we transform the first term of the right-hand side of (14) as

$$\begin{aligned}
0 &= \lambda_0 \int_Q \left(\frac{\partial^2 P_i}{\partial t^2} + A_i^*(t)P_i \right) \tilde{y}_i \mathrm{d}\rho \mathrm{d}t,\\
&= \lambda_0 \int_\Omega \frac{\partial P_i}{\partial t} \tilde{y}_i \,|_0^T \mathrm{d}\rho - \lambda_0 \int_Q \frac{\partial P_i}{\partial t}\frac{\partial \tilde{y}_i}{\partial t} \mathrm{d}\rho \mathrm{d}t + \lambda_0 \int_Q P_i A(t)\tilde{y}_i \mathrm{d}\rho \mathrm{d}t\\
&\quad + \lambda_0 \int_S P_i \frac{\partial^m \tilde{y}_i}{\partial \nu_{A_i}^m} \mathrm{d}S - \lambda_0 \int_S \frac{\partial^m P_i}{\partial \nu_{A_i^*}^m} \tilde{y}_i \mathrm{d}S,\\
&= \lambda_0 \int_\Omega \frac{\partial P_i}{\partial t} \tilde{y}_i \,|_0^T \mathrm{d}\rho - \lambda_0 \int_Q P_i \frac{\partial \tilde{y}_i}{\partial t} \,|_0^T \mathrm{d}\rho + \lambda_0 \int_Q P_i \frac{\partial^2 \tilde{y}_i}{\partial t^2} \mathrm{d}\rho \mathrm{d}t + \lambda_0 \int_Q P_i A(t)\tilde{y}_i \mathrm{d}\rho \mathrm{d}t
\end{aligned}$$

$$+\lambda_0\int_S P_i\tilde{u}_0\mathrm{d}S-\lambda_0\int_S F_{iy}\tilde{y}_i\mathrm{d}S,$$
$$=\lambda_0\int_Q P_i\left(\frac{\partial^2\tilde{y}_i}{\partial t^2}+A_i(t)\tilde{y}_i\right)\mathrm{d}S\mathrm{d}t+\lambda_0\int_S P_i\tilde{u}_i\mathrm{d}S-\lambda_0\int_S F_{iy}\tilde{y}_i\mathrm{d}S.$$

So

$$\lambda_0\int_S F_{iy}\tilde{y}_i\mathrm{d}S=\lambda_0\int_S P_i\tilde{u}_i\mathrm{d}S. \tag{15}$$

Substituting (15) into (14), we obtain

$$g_2^2(\tilde{\mathbf{u}})=\sum_{i=1}^{n}\lambda_0\int_S(P_i+F_{iu})\tilde{u}_i\mathrm{d}S,\quad \tilde{\mathbf{u}}\in\mathcal{U}_{\mathrm{ad}}. \tag{16}$$

The equality λ_0 in (16) cannot be equal to zero, because in this case all functionals in the Euler–Lagrange equation would be zero, which is impossible according to the Dubovitskii Milyutin theorem. Using the definition of the support functional and dividing both sides of the obtained inequality by λ_0, we finally obtain the maximum condition (13). If $RTC(I,(\mathbf{y}^0,\mathbf{u}^0))=\Phi$, then optimality conditions are fulfilled with equality in the maximum condition.

The uniqueness of the optimal control $\mathbf{u}^0$ follows from the strict convexity of the performance functional (assumption (A_3)). For the optimal control $\mathbf{u}^0$, there corresponds the optimal state $\mathbf{y}^0$ determined uniquely by the state equation. Therefore, the solution of the problem (3), (11) and (12) exists, is unique and is given by the pair $(\mathbf{y}^0,\mathbf{u}^0)$. This completes the proof of the theorem.

Note 2. If we take $n=2$, then $\mathcal{U}=L_2(S)\times L_2(S)$ and the optimality system is given by

$$\frac{\partial^2 y_1^0(\mathbf{u}^0)}{\partial t^2}+\left(\sum_{|\alpha|\le\ell}\sum_{k=1}^{\infty}(-1)^{|\alpha|}D_k^{2\alpha}+q(x,t)+1\right)y_1^0(\mathbf{u}^0)-y_2^0(\mathbf{u}^0)=f_1\quad\text{in } Q,$$
$$\frac{\partial^2 y_2^0(\mathbf{u}^0)}{\partial t^2}+\left(\sum_{|\alpha|\le\ell}\sum_{k=1}^{\infty}(-1)^{|\alpha|}D_k^{2\alpha}+q(x,t)+1\right)y_2^0(\mathbf{u}^0)+y_1^0(\mathbf{u}^0)=f_2\quad\text{in } Q,$$
$$y_1^0(x,0;\mathbf{u}^0)=y_{1,1}(x),\qquad y_2^0(x,0;\mathbf{u}^0)=y_{2,1}(x),\quad\text{in } \Omega,$$
$$\frac{\partial y_1^0}{\partial t}(x,0;\mathbf{u}^0)=y_{1,2}(x),\qquad \frac{\partial y_2^0}{\partial t}(x,0;\mathbf{u}^0)=y_{2,2}(x),\quad\text{in } \Omega,$$
$$\frac{\partial^m y_1^0(x,t;\mathbf{u}^0)}{\partial\nu_{A_1}^m}=u_1^0,\qquad \frac{\partial^m y_2^0(x,t;\mathbf{u}^0)}{\partial\nu_{A_2}^m}=u_2^0\quad\text{on } S,$$
$$\frac{\partial^2 P_1(\mathbf{u}^0)}{\partial t^2}+\left(\sum_{|\alpha|\le\ell}\sum_{k=1}^{\infty}(-1)^{|\alpha|}D_k^{2\alpha}+q(x,t)+1\right)P_1(\mathbf{u}^0)+P_2(\mathbf{u}^0)=0\quad\text{in } Q,$$
$$\frac{\partial^2 P_2(\mathbf{u}^0)}{\partial t^2}+\left(\sum_{|\alpha|\le\ell}\sum_{k=1}^{\infty}(-1)^{|\alpha|}D_k^{2\alpha}+q(x,t)+1\right)P_2(\mathbf{u}^0)-P_1(\mathbf{u}^0)=0\quad\text{in } Q,$$
$$P_1(x,T;\mathbf{u}^0)=0,\qquad P_2(x,T;\mathbf{u}^0)=0\quad\text{in } \Omega,$$
$$\frac{\partial P_1}{\partial t}(x,T;\mathbf{u}^0)=0,\qquad \frac{\partial P_2}{\partial t}(x,T;\mathbf{u}^0)=0\quad\text{in } \Omega,$$
$$\frac{\partial^m P_1(x,t;\mathbf{u}^0)}{\partial\nu_{A_2^*}^m}=F_{1y},\qquad \frac{\partial^m P_2(x,t;\mathbf{u}^0)}{\partial\nu_{A_2^*}^m}=F_{2y}\quad\text{on } \Sigma,$$
$$\int_S[(P_1(\mathbf{u}^0)+F_{1u}u_1^0)(u_1-u_1^0)+(P_2(\mathbf{u}^0)+F_{2u}u_2^0)(u_2-u_2^0)]\mathrm{d}S\ge 0,$$

for all $\mathbf{u}=(u_1,u_2)\in\mathcal{U}_{\mathrm{ad}}$ where $\mathbf{u}^0=(u_1^0,u_2^0)\in\mathcal{U}_{\mathrm{ad}}$ and $P(\mathbf{u}^0)=(P_1(\mathbf{u}^0),P_2(\mathbf{u}^0))$ is the adjoint state.

Acknowledgements

The author is grateful to the reviewers of the Journal of Computers and Mathematics with applications for their fruitful comments and for their valuable suggestions.

References

[1] J.L. Lions, Optimal Control of Systems Governed by Partial Differential Equations, vol. 170, Springer-Verlag, 1971.
[2] W. Kotarski, Some problems of optimal and pareto optimal control for distributed parameter systems, Reports of Silesian University Katowice, Poland, 1997, no. 1668.
[3] H.A. El-Saify, Boundary control problem with an infinite of variables, International Journal of Mathematics and Mathematical Science 28 (1) (2001) 57–62.
[4] H.A. El-Saify, On some control problems, Nonlinear Phenomena in Complex Systems 5 (3) (2002) 308–313.
[5] H.A. El-Saify, B.M. Gaber, Optimal control for $n \times n$ systems of hyperbolic types, Revista de Matemáticas Aplicadas 22 (1,2) (2001) 41–58.
[6] H.A. El-Saify, G.M. Bahaa, Optimal control for $n \times n$ coupled systems governed by Petrowsky type equations with control-constrained and infinite number of variables, Mathematica Slovaca 53 (3) (2003) 291–311.
[7] W. Kotarski, H.A. El-Saify, G.M. Bahaa, Optimal control of parabolic equations with inifinite number of variables for non-standard functions and time delay, IMA Journal of Control and Information 19 (4) (2002) 401–476.
[8] Ju.M. Berezanskii, Self-adjoint Operators in Spaces of Functions of Infinitely Many Variables, in: Transl. of Math. Monographs, vol 63, Amer. Math. Soc., Providence, RI, 1986.
[9] J.L. Lions, E. Magenes, Non Homogeneous Boundary Value Problem and Applications, I, II, Springer-Verlag, New York, 1972.
[10] U. Ledzewicz-Kowalewska, On the optimal control problems of parabolic equations with an infinite number of variables and with equality constractions, Differential and Integral Equations 4 (1991) 363–381.