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GRAPHS WHICH ARE LOCALLY A CUBE

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We prove that there are exactly two connected graphs which are locally a cube: a graph on 15 vertices which is the complement of the (3×5)-grid and a graph on 24 vertices which is the 1-skeleton of a certain 4-dimensional regular polytope called the 24-cell.

1. Introduction

Let $\{e_1, e_2, e_3, e_4\}$ be the standard basis of $\mathbb{R}^4$. The *24-cell* is a 4-dimensional regular polytope whose vertices are the 24 vectors $\pm e_i \pm e_j$ $(i \neq j)$ of $\mathbb{R}^4$, two vertices being adjacent iff the angle between the corresponding vectors is 60°. It is well known [2] and easy to check that the 1-skeleton (i.e. the graph consisting of the vertices and edges) of this polytope has the following property: for every vertex v, the neighbourhood of v (i.e. the subgraph induced by G on the set of vertices adjacent to v) is isomorphic to the 1-skeleton of a 3-dimensional cube. In other words, the 24-cell is locally a cube.

More generally [3], given a graph G', we shall say that a graph G is *locally* G' if, for evey vertex v of G, the neighbourhood $G(v)$ of v is isomorphic to G'. If G' is the 1-skeleton of a 3-dimensional cube and if G is locally G', we shall say that G is *locally a cube*.

It is natural to ask whether the 24-cell is the only connected graph which is locally a cube.

Theorem. *If a connected graph G is locally a cube, G is isomorphic either to the 1-skeleton of the 24-cell or to the complement of the (3×5)-grid.*

The $(p\times q)$-*grid* is the graph whose vertices are the pq ordered pairs (i, j) with $i = 1, \ldots, p$ and $j = 1, \ldots, q$, two vertices being adjacent iff they have one coordinate in common.

The adjacency relation in a graph G will be denoted by $\sim$ and the number of vertices of G by $|G|$.

2. Proof of the Theorem

Lemma 1. *For any two adjacent vertices of G, there are exactly 3 vertices of G adjacent to both of them.*

Proof. This follows immediately from the fact that the cube is a regular graph of degree 3.

Let v be a fixed vertex of G. We shall denote by v_i $(i = 1, \ldots, 8)$ the vertices of $G(v)$ and by G_i the subgraph induced by G on the set of vertices adjacent to v_i and at distance 2 from v. Since the neighbourhood of v_i is isomorphic to a cube, G_i is a 3-claw, that is G_i has 4 vertices w_i, i_1, i_2, i_3 such that $w_i \sim i_r$ for every $r = 1, 2, 3$ and $i_r \not\sim i_s$ for every $r \neq s$.

Lemma 2. *If $v_i \neq v_j$, then $G_i \neq G_j$ and the subgraph $G_i \cap G_j$ is not an edge.*

Proof. If $G_i = G_j$, the vertices v_i, v_j, i_1, i_2, i_3 are all in the neighbourhood of w_i, which is isomorphic to a cube. This is a contradiction because v_i and v_j are both adjacent to i_1, i_2, i_3 and the graph of a cube cannot contain 5 such vertices.

If the claws G_i and G_j have exactly one edge in common, we may assume without loss of generality that it is the edge $\{w_i, i_1\}$, so that $w_j = w_i$ or $w_j = i_1$. In any case, the neighbourhood $G(w_i)$ contains the vertices v_i, v_j, i_1, i_2, i_3 with $v_j \sim i_1$. Since $G(w_i)$ is isomorphic to a cube, v_j must also be adjacent to one of the vertices i_2 or i_3, and so G_i and G_j have at least two edges in common, contradicting the initial assumption.

Lemma 3. *If $v_i \sim v_j$, then $w_i \neq w_j$, $w_i \notin G_j$, $w_j \notin G_i$ and $|G_i \cap G_j| = 2$.*

Proof. Since $v_i \sim v_j$, there are exactly 3 vertices adjacent to v_i and v_j by Lemma 1. One of them is v. There is no vertex adjacent to v_i and v_j in $G(v)$. Therefore the two missing vertices are at distance 2 from v, and so $|G_i \cap G_j| = 2$.

If $w_i \in G_j$, then $w_i \sim v_j$. This contradicts Lemma 1 because v_i and w_i are both adjacent to i_1, i_2, i_3, v_j. Therefore $w_i \notin G_j$ and similarly $w_j \notin G_i$. In particular, $w_i \neq w_j$.

Let $d(v_i, v_j)$ denote the distance between v_i and v_j in the subgraph $G(v)$.

Lemma 4. *If $d(v_i, v_j) = 2$, then $G_i \cap G_j \neq \emptyset$, $\{w_i\}$ and $\{w_j\}$.*

Proof. If $G_i \cap G_j$ is equal to $\emptyset$, $\{w_i\}$ or $\{w_j\}$, then the vertices i_1, i_2, i_3, j_1, j_2, j_3 are pairwise distinct. Since $d(v_i, v_j) = 2$, there is a vertex $v_k \in G(v)$ adjacent to v_i and v_j. By Lemma 3, $w_k \notin G_i \cup G_j$ and we may assume without loss of generality that $G_i \cap G_k = \{i_1, i_2\}$ and $G_j \cap G_k = \{j_1, j_2\}$. It follows that $G(v_k)$ contains at least 9 vertices, a contradiction.

Lemma 5. *If $d(v_i, v_j) = 2$ and $w_i = w_j$, then the subgraph $G_i \cap G_j$ is a 2-claw (i.e. the union of two intersecting edges).*

Proof. This is a direct consequence of Lemmas 2 and 4.

Lemma 6. *If $d(v_i, v_j)=2$ and $w_i \neq w_j$, then $G_i \cap G_j = \{i_r\}$ for some $r \in \{1,2,3\}$. Moreover the 8 vertices of $G(i_r)$ are v_i, v_j, v_k, v_l, w_i, w_j, w_k, w_l where v_k and v_l are the vertices adjacent to v_i and v_j in the subgraph $G(v)$.*

Proof. By Lemmas 2 and 4, we already know that $|G_i \cap G_j| = 1, 2$ or 3. In view of Lemma 3, we may assume that $G_i \cap G_k = \{i_1, i_2\}$ and $G_j \cap G_k = \{j_1, j_2\}$.

By hypothesis, $w_i \neq w_j$. Moreover, $w_j \neq i_1, i_2$ because $w_j \notin G_k$ thanks to Lemma 3. If $w_j = i_3$, then $i_r \neq j_s$ for every $r, s \in \{1,2,3\}$ and so i_1, i_2, j_1, j_2 are 4 distinct vertices adjacent to v_k. It follows that $G(v_k)$ contains at least 9 vertices, a contradiction. Therefore $w_j \notin G_i$ and similarly $w_i \notin G_j$, so that $G_i \cap G_j \subseteq \{i_1, i_2, i_3\}$. Observe also that $\{i_1, i_2\} \not\subseteq G_i \cap G_j$, because otherwise i_1 and i_2 would be adjacent to v_i, v_j, w_k which is a contradiction since i_1, i_2, v_i, v_j, w_k are all in $G(v_k)$ and since the graph of a cube cannot contain 5 such vertices.

(i) If $|G_i \cap G_j| = 3$, then $G_i \cap G_j \supseteq \{i_1, i_2\}$, which is impossible as we have just seen before.

(ii) If $|G_i \cap G_j| = 2$, it is no loss of generality, thanks to the preceding observations, to assume that $G_i \cap G_j = \{i_1, i_3\}$ and $G_k = \{w_k, i_1, i_2, j_2\}$ with $i_1 = j_1$. The neighbourhood $G(i_1)$ contains v_i, v_j, v_k, w_i, w_j, w_k with $w_i \sim v_i \sim v_k \sim v_j \not\sim w_i$. Since $G(i_1)$ is isomorphic to a cube, there must be a vertex $x \in G(i_1)$ adjacent to v_i and v_j but not to w_i. Moreover $x \in G(v)$ because $x \sim v_i$, $x \neq v$, $x \neq w_i$ and $x \not\sim w_i$. Thus $x = v_l$ and so $v_l \sim i_1$. Using the fact that $G(v_i)$ is a cube, we get $v_l \sim i_3$. Now, in $G(v_l)$, v_i and v_j are both adjacent to v, i_1, i_3, a contradiction since $G(v_l)$ is a cube.

(iii) Therefore $|G_i \cap G_j| = 1$ and $G_i \cap G_j = \{i_r\}$ for some $r \in \{1,2,3\}$. Together with $G_i \cap G_k = \{i_1, i_2\}$ and $G_j \cap G_k = \{j_1, j_2\}$, this implies $r \neq 3$ and so, without loss of generality, $G_i \cap G_j = \{i_1\} = \{j_1\}$ and $G_k = \{w_k, i_1, i_2, j_2\}$. Using the same type of arguments as in (ii), we get $v_l \sim i_1$, and, because $G(i_1)$ is a cube, $v_l \not\sim w_k$. Thus $w_l \neq w_k$ and the vertices w_i, w_j, w_k, w_l are pairwise distinct.

Lemma 7. *If $d(v_i, v_j) = 3$, then $w_i \neq w_j$.*

Proof. Assume that $w_i = w_j$ and let $v_i \sim v_m \sim v_n \sim v_j$ be a path of length 3 joining v_i to v_j in $G(v)$.

If $|G_i \cap G_j| = 1$, then $G_i \cap G_j = \{w_i\}$ and $i_r \neq j_s$ for every $r, s \in \{1,2,3\}$. By Lemma 3, we may assume that $G_i \cap G_m = \{i_1, i_2\}$. Moreover, since $w_m \neq w_i = w_j$, we may assume, by Lemma 6, that $G_j = G_m = \{j_3\}$. Therefore $G_m = \{w_m, i_1, i_2, j_3\}$, which implies $j_3 \not\sim i_1$ and $j_3 \not\sim i_2$, a contradiction in the cube $G(w_i)$.

If $|G_i \cap G_j| = 2$, then $G_i \cap G_j$ is an edge, contradicting Lemma 2.

If $|G_i \cap G_j| = 3$, then the subgraph $G_i \cap G_j$ is a 2-claw and we may assume that $G_j = \{w_i, i_1, i_2, j_3\}$ with $i_1 = j_1$ and $i_2 = j_2$. By Lemma 3, $|G_i \cap G_m| = 2$ with $w_i \notin G_m$, and so G_m contains at least one of the two vertices i_1, i_2. Since $w_m \neq w_i = w_j$, Lemma 6 implies that $G_j \cap G_m = \{j_s\}$ for some $s \in \{1,2,3\}$, and so G_m contains at most one of the two vertices i_1, i_2. Therefore, without any loss of generality,

$G_j \cap G_m = \{i_2\}$. Now, by Lemma 6 again, $G(i_2)$ has exactly 4 vertices in common with the cube $G(v)$, namely v_j, v_m and the two vertices of $G(v)$ adjacent to v_j and v_m. On the other hand, $v_i \in G(i_2) \cap G(v)$. This is a contradiction since v_i is not adjacent to v_j.

If $|G_i \cap G_j| = 4$, then $G_i = G_j$, contradicting Lemma 2.

Proposition 1. *If a graph G is locally a cube and if, for some vertex v of G, there are two vertices $v_i, v_j \in G(v)$ such that $d(v_i, v_j) = 2$ and $w_i \neq w_j$, then G is isomorphic to the* 1*-skeleton of the* 24*-cell.*

Proof. It is easy to check that the 1-skeleton of the 24-cell satisfies the above hypothesis. Therefore, it suffices to prove that a graph G satisfying this hypothesis is uniquely determined up to isomorphism.

We shall denote the adjacencies in the cube $G(v)$ by

$$v_1 \sim v_2 \sim v_3 \sim v_4 \sim v_1, \qquad v_5 \sim v_6 \sim v_7 \sim v_8 \sim v_5$$

and

$$v_i \sim v_{i+4} \quad \text{for every } i = 1, 2, 3, 4.$$

Suppose that $w_1 \neq w_3$. Then, by Lemma 6, $G_1 \cap G_3 = \{1_1\}$ without any loss of generality and v_1, v_2, v_3, v_4, w_1, w_2, w_3, w_4 are the 8 vertices of the cube $G(i_1)$, with

$$w_1 \sim w_2 \sim w_3 \sim w_4 \sim w_1, \qquad w_1 \not\sim w_3, \quad w_2 \not\sim w_4$$

Since $v_5 \sim v_1$ and $v_5 \not\sim 1_1$, it follows from Lemma 3 that $G_1 \cap G_5 = \{1_2, 1_3\}$. Moreover, by Lemma 3 again, $G_1 \cap G_2 = \{1_1, 1_2\}$ without any loss of generality, and so $G_1 \cap G_4 = \{1_1, 1_3\}$ thanks to Lemma 6. The neighbourhood $G(i_1)$ contains the vertices v_1, v_2, v_5, w_1, w_2, w_5 with

$$w_5 \sim v_5 \sim v_1 \sim v_2 \sim w_2 \sim w_1 \sim v_1$$

and $w_5 \neq w_1$ by Lemma 3. Since $G(i_2)$ is a cube, we have $w_2 \neq w_5$ and so, by Lemma 6, $G_2 \cap G_5 = \{1_2\}$ and v_1, v_2, v_5, v_6, w_1, w_2, w_5, w_6 are the 8 vertices of the cube $G(1_2)$, with

$$w_1 \sim w_2 \sim w_6 \sim w_5 \sim w_1, \qquad w_1 \not\sim w_6, \quad w_2 \not\sim w_5$$

By similar arguments, $w_4 \neq w_5$, $G_4 \cap G_5 = \{1_3\}$ and v_1, v_4, v_5, v_8, w_1, w_4, w_5, w_8 are the 8 vertices of the cube $G(1_3)$, with

$$w_1 \sim w_4 \sim w_8 \sim w_5 \sim w_1, \qquad w_1 \not\sim w_8, \quad w_4 \not\sim w_5$$

The neighbourhood $G(w_1)$ contains the vertices v_1, 1_1, 1_2, 1_3, w_2, w_4, w_5. Since $G(w_1)$ is a cube, the missing vertex $w \in G(w_1)$ must be adjacent to w_2, w_4, w_5 and non adjacent to 1_1, 1_2, 1_3, so that $w \neq w_1$, w_2, w_3, w_4, w_5, w_6, w_8. Note that $w_2 \not\sim w_8$ (because w_2 is already adjacent to 8 vertices distinct from w_8) and also $w_4 \not\sim w_6$, $w_5 \not\sim w_3$.

Using similar arguments, it is now easy (but a little bit tedious) to show that the subgraph induced by G on the set of vertices w_i ($i = 1, \dots, 8$) is isomorphic to a cube which is precisely the neighbourhood $G(w)$. Moreover, given any 4 vertices v_i, v_j, v_k, v_l in a face of the cube $G(v)$, there is exactly one vertex f_t of G which is adjacent to v_i, v_j, v_k, v_l and to the vertices w_i, w_j, w_k, w_l of the corresponding face of the cube $G(w)$; for example, we have seen that 1_1, 1_2, 1_3 are three of the vertices f_1, f_2, f_3, f_4, f_5, f_6. This shows that the graph G has $1+8+6+8+1=24$ vertices and, being uniquely determined up to isomorphism by the preceding construction, it is isomorphic to the 1-skeleton of the 24-cell.

Proposition 2. *If a graph G is locally a cube and if, for some vertex v of G, $w_i = w_j$ whenever $v_i, v_j \in G(v)$ with $d(v_i, v_j) = 2$, then G is isomorphic to the complement of the* (3×5)*-grid.*

Proof. It is easy to check that the complement of the (3×5)-grid satisfies the above hypothesis. Therefore, it suffices to show that a graph G satisfying this hypothesis is uniquely determined up to isomorphism.

We use the same notations as in the proof of Proposition 1 to denote the adjacencies in $G(v)$.

The hypothesis implies that w_1 is adjacent to v_1, v_3, v_6, v_8 and that w_2 is adjacent to v_2, v_4, v_5, v_7, with $w_1 \neq w_2$ by Lemma 3. Using Lemma 5, we may assume without loss of generality that $G_1 = \{w_1, 1_1, 1_2, 1_3\}$ and $G_3 = \{w_1, 1_2, 1_3, 3_1\}$. Since $G(w_1)$ is a cube, it follows that 3_1 is adjacent to v_6 and v_8 and, without loss of generality, $G_6 = \{w_1, 1_1, 1_3, 3_1\}$ and $G_8 = \{w_1, 1_1, 1_2, 3_1\}$. By Lemmas 3 and 5, the subgraphs G_2, G_4, G_5, G_7 are then completely determined.

This construction shows that the graph G has 15 vertices and is uniquely determined up to isomorphism.

The proof of the Theorem follows immediately from Propositions 1 and 2.

3. Final comments

A. Brouwer [1] proved independently that there are exactly two connected graphs which are locally a cube. After some exchange of information, he could prove a more general result characterizing the graphs which are locally the complement of a $(p\times q)$-grid with $p \geq q \geq 2$ ($q > 2$ or $p > 3$). We shall say that these graphs are locally $\overline{p\times q}$.

Theorem (Brouwer [1]). *If G is a connected graph which is locally $\overline{p\times q}$ with $p \geq q \geq 2$ ($q > 2$ or $p > 3$), then G is the complement of a $((p+1)\times(q+1))$-grid or*

(i) $p = 4$, $q = 2$ *and G is the* 1*-skeleton of the* 24*-cell*

(ii) $p = q = 3$ *and G is the Johnson scheme* $\binom{6}{3}$ *on* 20 *vertices (that is the graph* *whose vertices are the* 3*-subsets of a* 6*-set, two vertices being adjacent iff the corresponding* 3*-subsets intersect in a* 2*-subset*).

The remaining cases $(p, q) = (3, 2)$, $(2, 2)$ or $(p, 1)$ with $p > 1$ allow infinitely many nonisomorphic solutions. K_2 is obviously the unique locally $\overline{1 \times 1}$ graph.

References

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