# Spectral Factorization via Hermitian Pencils

D. J. Clements
*School of Electrical Engineering and Computer Science*
*University of New South Wales*
*P.O. Box 1*
*Kensington, 2033, Australia*

and

K. Glover
*Information Engineering Division*
*Department of Engineering*
*University of Cambridge*
*Trumpington Street CB1 1PZ, England*

Submitted by Jan C. Willems

ABSTRACT

The spectral factorization problem is solved for state-space systems via results on the canonical forms and inertia of Hermitian matrix pencils. These algebraic results then give a deflation method for spectral factorization.

## 1. INTRODUCTION

A central algebraic problem in a variety of areas of system and control analysis and design is the spectral-factorization problem, which can be stated as follows. Let $\Phi(s)$ be a paraconjugate Hermitian rational matrix which is Hermitian positive semidefinite on the purely imaginary axis. By paraconjugate Hermitian, we mean

$$\Phi_*(s) = \Phi(s), \tag{1}$$



655 Avenue of the Americas, New York, NY 10010 0024-3795/89/$3.50

where

$$\Phi_*(s) \overset{\text{def}}{=} [\Phi(-\bar{s})]^*.$$

(Here, the * denotes Hermitian transpose and the overbar denotes complex conjugate.) From (1), it follows that $\Phi(i\omega)$ is Hermitian for $\omega$ real. The second requirement is

$$\Phi(i\omega) \geqslant 0$$

for all real $\omega$ whenever $\Phi(i\omega)$ is defined. The spectral-factorization problem is to find a rational matrix $G(s)$, not necessarily square, such that

$$\Phi(s) = G_*(s)G(s). \tag{2}$$

Any $G(s)$ satisfying (2) is called a (right) spectral factor of the spectral density $\Phi(s)$.

Alternatively, we could seek a (left) spectral factor $F(s)$ satisfying

$$\Phi(s) = F(s)F_*(s).$$

Observe that $G(s)$ is a right spectral factor of $\Phi(s)$ if and only if $F(s) = G_*(s)$ is a left spectral factor of $\Phi(s)$.

The $G(s)$ satisfying (2) are not unique. For example, if $V(s)$ is any matrix satisfying $V_*(s)V(s) = I$, then $V(s)G(s)$ is also a (right) spectral factor. For a complete discussion of the rational spectral-factorization problem, see [23]. As a first constraint we require

$G(s)$ has full normal row rank

which equals the normal rank of $\Phi(s)$. (The normal rank of a rational matrix is defined to be its rank as a matrix over the rational field.) Other constraints that might be imposed include:

1. $G(s)$ is analytic in the open right (or left) half plane.
2. $G(s)$ has full row rank everywhere in the open right (or left) half plane.

Spectral factorization has applications in Wiener filtering [19] (in fact, this is the origin of the term spectral factorization), $H^\infty$ control design [8], network synthesis [1], and linear-quadratic control design [22]. Here, we look at the last problem in some detail.

The standard linear-quadratic control problem is to minimize the cost

$$J(x(0); u(\cdot)) = \int_0^\infty [x^*(t)Qx(t) + u^*(t)Ru(t)]\, dt \tag{3}$$

with respect to the control $\{u(t), 0 \leqslant t < \infty\}$ and subject to the constraint

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with $x(0)$ given. In (3), the matrices $Q$ and $R$ are Hermitian. All other matrices and vectors are arbitrary but are required to have compatible dimensions.

It is well known that if

1. $R$ is positive definite Hermitian,
2. $Q$ is positive semidefinite Hermitian,
3. the pair $(A, B)$ is stabilizable, i.e., $[sI - A \;\; B]$ has full row rank for all complex $s$ in the closed right half plane, and
4. the pair $(A, Q)$ is detectable, i.e., $\begin{bmatrix} sI - A \\ Q \end{bmatrix}$ has full column rank for all complex $s$ in the closed right half plane,

then the minimum exists. Moreover, the value of this minimum cost is given by $x^*(0)Px(0)$ and is achieved by the feedback control

$$u(t) = -R^{-1}B^*Px(t), \tag{4}$$

where $P$ is the unique Hermitian matrix satisfying both the algebraic riccati equation (ARE)

$$PA + A^*P + Q - PBR^{-1}B^*P = 0 \tag{5}$$

and the requirement that

$$A_c \overset{\text{def}}{=} A - BR^{-1}B^*P \tag{6}$$

has all its eigenvalues in the open left half plane.

To each Hermitian solution $P$ of the ARE (5) there corresponds a spectral-factorization of the rational matrix

$$\Phi(s) = R + B^*(-sI - A^*)^{-1}Q(sI - A)^{-1}B. \tag{7}$$

Indeed, with

$$G(s) = R^{1/2} + R^{-1/2}B^*P(sI - A)^{-1}B,$$

(2) holds. Note incidentally that this shows that, if (5) has any Hermitian solution $P$, then $\Phi(s)$ must be a spectral density with one (right) spectral factor given by $G(s)$.

Moreover, $G(s)$ is invertible with

$$G(s)^{-1} = R^{-1/2} + R^{-1}B^*P(sI - A_c)^{-1}BR^{-1/2},$$

so the poles of $G(s)^{-1}$ are given by the eigenvalues of $A_c$. If, therefore, we choose the unique Hermitian solution $P$ of the ARE (5) subject to (6), then $G(s)^{-1}$ is analytic in the closed right half plane. Alternatively, we can say that $G(s)$ has all its zeros in the open left half plane. Such a $G(s)$ is called a minimum phase (right) spectral factor. With the optimal control (4), the closed-loop dynamics of the system become

$$\dot{x}(t) = A_c x(t)$$

which is an asymptotically stable system since $A_c$ has all its eigenvalues in the open left half plane. Thus, in the linear-quadratic control problem, there is a correspondence between the asymptotic stability of the closed-loop system and the choice of the minimum-phase spectral factor in the related spectral-factorization problem. This is one motivation for seeking a minimum-phase spectral factor.

It is also possible to associate an eigenvalue problem with the spectral-factorization problem. The spectral density $\Phi(s)$ in (7) can be rewritten as

$$\Phi(s) = R + \begin{bmatrix} 0 & B^* \end{bmatrix} \begin{bmatrix} sI - A & 0 \\ Q & sI + A^* \end{bmatrix}^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix}, \tag{8}$$

from which it follows that the inverse of $\Phi(s)$ can be written as

$$\Phi(s)^{-1} = R^{-1} - R^{-1}\begin{bmatrix} 0 & B^* \end{bmatrix}\begin{bmatrix} sI - A & BR^{-1}B^* \\ Q & sI + A^* \end{bmatrix}^{-1}\begin{bmatrix} B \\ 0 \end{bmatrix}R^{-1}.$$

First, the poles of the spectral density $\Phi(s)$ are given by the eigenvalues of the Hamiltonian matrix

$$H_p = \begin{bmatrix} A & 0 \\ -Q & -A^* \end{bmatrix},$$

and the poles of the inverse of the spectral density $\Phi(s)$ are given by the eigenvalues of another Hamiltonian matrix

$$H_z = \begin{bmatrix} A & -BR^{-1}B^* \\ -Q & -A^* \end{bmatrix}.$$

(A matrix of the form

$$\begin{bmatrix} X & Y \\ Z & -X^* \end{bmatrix}$$

with $Y^* = Y$ and $Z^* = Z$ is said to be Hamiltonian.) Next, observe that if $P$ is a Hermitian solution of the ARE (5) and if

$$T = \begin{bmatrix} I & 0 \\ P & I \end{bmatrix},$$

then, using $T$ as a similarity transformation on $H_z$, we obtain the Hamiltonian matrix

$$T^{-1}H_zT = \begin{bmatrix} A_c & -BR^{-1}B^* \\ 0 & -A_c^* \end{bmatrix}. \tag{9}$$

The block triangular structure of (9) indicates that the eigenvalues of $H_z$ are specified completely by the eigenvalues of $A_c$ and conversely. Moreover, $\begin{bmatrix} I \\ P \end{bmatrix}$ is a basis for the invariant subspace of $H_z$ corresponding to the eigenvalues of $A_c$.

In conjunction with the preceding material, this then suggests that a numerical approach to the spectral-factorization problem might be based on the Hamiltonian eigenvalue problem, at least when $R = \Phi(\infty)$ is invertible. Indeed, the generally accepted numerical technique for solving the ARE (5) subject to (6) and hence the spectral-factorization problem is to set up the Hamiltonian matrix $H_z$. Then, transform $H_z$ to upper Schur form, using an eigenvalue method such as the $QR$ algorithm, so that the upper left block only has eigenvalues in the open left half plane. The Hermitian solution $P$ is constructed from a basis of the invariant subspace of $H_z$ corresponding to these open-left-half-plane eigenvalues. To overcome numerical difficulties when $R$ may be close to a singular matrix, the method has been extended to a generalized eigenvalue problem, and the $QZ$ algorithm is used rather than the $QR$ algorithm. There is a substantial literature based on this approach. In particular, consult [2, 14, 20].

There are a number of objections to this approach. First, it is a Hamiltonian eigenvalue problem that is being solved, but this is not recognized in the above method. Papers which are concerned with numerical algorithms based on structural results include [5, 6, 15]. There has also been a large amount of work based on algebraic aspects of factorization problems in general, and spectral-factorization problems in particular. Papers of interest here include [4, 13], while the books [3, 11, 10] contain a wealth of relevant material.

In the spectral-factorization example that we looked at in detail viz. the linear-quadratic control problem, we specifically introduced assumptions which guaranteed that $H_z$ had no eigenvalues on the imaginary axis. In the context of the control problem, this is a natural requirement, but the spectral factorization itself imposes no such constraint. The ability to solve problems with purely imaginary eigenvalues of $H_z$ is closely related to the exploitation of the Hamiltonian structure.

Second, the technique discussed above fails when $R$ is singular even if the generalized eigenvalue approach is used. This case corresponds to $\Phi(s)$ having zeros at $s = \infty$. It is useful to think of this case as a limiting case of purely imaginary zeros of $\Phi(s)$.

The work reported in this paper arose out of an attempt to devise a numerical algorithm to solve spectral-factorization problems when $R$ is singular and/or the spectrum $\Phi(s)$ has purely imaginary zeros. In this paper, we report on the algebraic results of this investigation. The numerical work will be reported elsewhere. The final outcome is a general approach to the spectral factorization of an arbitrary proper, rational paraconjugate Hermitian matrix which is positive semidefinite on the purely imaginary axis.

The organization of this paper is as follows. In Section 2, some basic material on minimal realizations of rational matrices with structure is discussed; in Section 3, a canonical form for Hermitian pencils is discussed.

Section 4 presents a key result relating inertias of Hermitian matrices and canonical forms of Hermitian pencils. The main result of the paper is presented in Section 5. In Section 6, we introduce a deflation lemma which forms the basis of later numerical work. Section 7 contains an extension to the main result, while Section 8 contains concluding remarks.

## 2. REALIZATIONS

In the previous section, we showed that for the linear-quadratic control problem, there is an associated spectral-factorization problem with spectral density given by (7). Also, we observed that one realization (8) of this spectral density had the property that there is an associated Hamiltonian matrix. In this section, we look at the general implications of the paraconjugate property (1) on the properties of a realization of a proper rational matrix.

We assume that there is given a square rational matrix $\Phi(s)$ satisfying (1) with

$$\Phi(\infty) \text{ finite,} \tag{10}$$

that is, $\Phi(s)$ is proper and paraconjugate Hermitian. Further, let $\{\hat{A}, B, C^*, D\}$ be a realization of $\Phi(s)$ *in the control-system sense, that is,*

$$\Phi(s) = D + C^*(sI - \hat{A})^{-1}B. \tag{11}$$

The realization $\{\hat{A}, B, C^*, D\}$ is said to be minimal if there is no other such realization of $\Phi(s)$ having an $\hat{A}$ matrix with smaller dimension. Necessary and sufficient conditions for the minimality of the realization $\{\hat{A}, B, C^*, D\}$ are that $(\hat{A}, B)$ is controllable and $(\hat{A}, C^*)$ is observable. The most straightforward characterization of these conditions is that

$$[sI - \hat{A} \quad B] \quad \text{and} \quad \begin{bmatrix} sI - \hat{A} \\ C^* \end{bmatrix}$$

have full row and column ranks respectively for all complex $s$.

The following lemma shows that (10) and the structure (1) together with the minimality of the realization $\{\hat{A}, B, C^*, D\}$ imply certain relationships between $\hat{A}$, $B$, $C$, and $D$. The method of proof is not original; it has also been applied in other circumstances where the structure of a rational matrix is important [9, 16, 19].

LEMMA 2.1. *Let $\Phi(s)$ satisfy* (1) *and* (10) *together with a minimal realization* $\{\hat{A}, B, C^*, D\}$ *for* $\Phi(s)$ *as in* (11]). *Then* $D^* = D$, *and there exists a unique nonsingular matrix* $H$ *satisfying*

$$-\hat{A}^* = H^{-1}\hat{A}H, \tag{12}$$

$$-C = H^{-1}B, \tag{13}$$

$$B^* = C^*H, \tag{14}$$

$$-H^* = H. \tag{15}$$

*Further, if the realization* $\{\hat{A}, B^*, C^*, D\}$ *is real, then so is* $H$.

*Proof.* Noting that

$$\Phi_*(s) = D^* + B^*(-sI - \hat{A}^*)^{-1}C,$$

it follows from (1) that $\{-\hat{A}^*, -C, B^*, D^*\}$ is also a minimal realization of $\Phi(s)$. Thus, $D^* = D$, and there exists a unique nonsingular matrix $H$ such that (12), (13), and (14) hold. However, (12), (13), and (14) also hold with $H$ replaced by $-H^*$. Therefore, the uniqueness of $H$ implies (15).

Finally, noting that $H$ is given by the unique solution to

$$\mathscr{C}(\hat{A}, B) = H\mathscr{C}(-\hat{A}^*, -C),$$

where

$$\mathscr{C}(\hat{A}, B) = [B \quad \hat{A}B \quad \hat{A}^2B \quad \cdots \quad \hat{A}^{n-1}B]$$

with $n$ = dimension $\hat{A}$, it follows that $H$ is real if $\hat{A}$, $B$, and $C$ are all real. ■

Application of this lemma to (11) and setting $A = \hat{A}H$ yields

$$\Phi(s) = D + B^*(sH - A)^{-1}B,$$

where $D^* = D$, $H^* + H = 0$, and $A^* = A$. Thus, we can assume that for any $\Phi(s)$ satisfying (10) and (1), there exists a minimal realization

$$\Phi(s) = D + B^*(sH - A)^{-1}B$$

where $D^* = A$, $A^* = A$, and $-H^* = H$ with det $H \neq 0$.

Our test for minimality here is that $[sH - A \;\; B]$ has full row rank for all $s$. To see this, observe the following equivalences:

$$[sH - A \;\; B] \text{ has full row rank for all } s$$

$$\Leftrightarrow$$

$$[sI - AH^{-1} \;\; B] \text{ has full row rank for all } s$$

$$\Leftrightarrow$$

$$[sI - H^{-1}A \;\; H^{-1}B] \text{ has full row rank for all } s.$$

The first of these is our test for minimality, while the second and third are the tests for controllability and observability of the realization

$$\Phi(s) = B^*H^{-1}(sI - AH^{-1})^{-1}B$$

with $A^* = A$, and $-H^* = H$.

When considering factorizations of $\Phi(s)$, it is necessary to study both the poles and zeros of $\Phi(s)$. The poles of $\Phi(s)$ are the eigenvalues of the (regular) matrix pencil $sH - A$, where $H$ is skew-Hermitian and $A$ is Hermitian. Now, clockwise rotation of this eigenvalue problem by $\pi/2$ (that is, $s = i\lambda$) produces a Hermitian matrix pencil

$$\lambda(iH) - A$$

in the $\lambda$-plane, since $iH$ is Hermitian.

Similarly, the zeros of $\Phi(s)$ are defined as those values of $s$ for which $\Phi(s)$ has less than normal rank. Since

$$\text{dimension } A + \text{normal rank } \Phi(s) = \text{normal rank} \begin{bmatrix} A - sH & B \\ B^* & D \end{bmatrix},$$

then the zeros of $\Phi(s)$ are related to the eigenvalues of the matrix pencil

$$s\begin{bmatrix} H & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A & B \\ B^* & D \end{bmatrix},$$

where $\begin{bmatrix} H & 0 \\ 0 & 0 \end{bmatrix}$ is skew-Hermitian, and $\begin{bmatrix} A & B \\ B^* & D \end{bmatrix}$ is Hermitian. Exactly as for the poles, this leads to consideration of the Hermitian matrix pencil

$$\lambda\begin{bmatrix} iH & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}.$$

Hermitian matrix pencils have been studied in the mathematical literature. In particular, a (Jordan) canonical form, which we describe in the next section, is known.

Finally in this section, we look at the effects of the transformation $s = i\lambda$ on $\Phi(s)$. Define

$$\Psi(\lambda) \overset{\text{def}}{=} \Phi(i\lambda).$$

Then, instead of (1), we have

$$\Psi^*(\lambda) = \Psi(\lambda), \tag{16}$$

where

$$\Psi^*(\lambda) \overset{\text{def}}{=} [\Psi(\bar{\lambda})]^*,$$

and instead of the nonnegativity requirements on the $i\omega$ axis, we now have

$$\Psi(\lambda) \geqslant 0$$

for all real $\lambda$ whenever $\Psi(\lambda)$ is defined. A rational matrix satisfying (16) is said to be Hermitian.

This then leads to the following variation on Lemma 2.1.

LEMMA 2.2. *Let $\Psi(\lambda)$ satisfy* (16) *and* (10), *and let there be given a minimal realization $\{\hat{A}, B, C^*, D\}$ for $\Psi(\lambda)$:*

$$\Psi(\lambda) = D + C^*(\lambda I - \hat{A})^{-1}B.$$

*Then $D^* = D$, and there exists a unique nonsingular matrix $H$ satisfying*

$$\hat{A}^* = H^{-1}\hat{A}H,$$

$$C = H^{-1}B,$$

$$B^* = C^*H,$$

$$H^* = H.$$

Thus, we can assume that for any $\Psi(\lambda)$ satisfying (10) and (16) there exists a minimal realization

$$\Psi(\lambda) = D + B^*(\lambda H - A)^{-1}B$$

where $D^* = D$, $A^* = A$, and $H^* = H$ with $\det H \neq 0$.

We therefore consider a proper, Hermitian, rational matrix $\Psi(\lambda)$ which is nonnegative on the real $\lambda$-axis. Our spectral-factorization problem is then mathematically equivalent to seeking a proper rational matrix $G(\lambda)$ satisfying

$$\Psi(\lambda) = G^*(\lambda)G(\lambda).$$

For the remainder of this paper, we confine our attention to this formulation of the spectral-factorization problem.

## 3. HERMITIAN PENCILS

We have argued in the previous section that the pole-zero structure of a proper, Hermitian, rational matrix is closely related to the eigenvalue structure of the pole and zero pencils respectively. Since these pencils are both Hermitian, it is logical to investigate the underlying structure of Hermitian pencils. In this section, we state standard results on a (Jordan) canonical form for Hermitian pencils [18].

First, let us introduce some notation:

$$O_m \overset{\text{def}}{=} m \times m \text{ zero matrix},$$

$$I_m \overset{\text{def}}{=} m \times m \text{ matrix with entries } \delta_{i,j},$$

$$S_m \overset{\text{def}}{=} m \times m \text{ matrix with entries } \delta_{i,m+1-j},$$

$$N_m \overset{\text{def}}{=} m \times m \text{ matrix with entries } \delta_{i+1,j},$$

$$T_m \overset{\text{def}}{=} m \times m \text{ matrix with entries } \delta_{i,m+2-j}.$$

Let $H$ and $A$ be a pair of $n \times n$ Hermitian matrices. Then, there exists a nonsingular $n \times n$ matrix $T$ such that

$$T^*(\lambda H + \mu A)T = \bigoplus_{i=1}^{p} \mathscr{B}_i$$

where each $\mathscr{B}_i$ has one of four canonical forms which we classify as follows.

*Type 1:*

$$\mathscr{B}_i = \epsilon_i\left[(\lambda + \mu a_i)S_{m_i} + \mu T_{m_i}\right],$$

where $\epsilon_i = \pm 1$, $a_i$ is real, and $m_i$ is a positive integer. Thus, this block has the form

$$\epsilon_i \begin{bmatrix} & & & & & \lambda + \mu a_i \\ & & & & \lambda + \mu a_i & \mu \\ & & & \cdot & \mu & \\ & & \cdot & \cdot & & \\ & \lambda + \mu a_i & \mu & & & \\ \lambda + \mu a_i & \mu & & & & \end{bmatrix}.$$

We call $\epsilon_i$ the sign of the block $\mathscr{B}_i$.

*Type 2:*

$$\mathscr{B}_i = \begin{bmatrix} O_{m_i} & (\lambda + \mu a_i)S_{m_i} + \mu T_{m_i} \\ (\lambda + \mu \bar{a}_i)S_{m_i} + \mu T_{m_i} & O_{m_i} \end{bmatrix}$$

is a $2m_i \times 2m_i$ block, with $\bar{a}_i \neq a_i$. Notice that there is no sign associated with blocks of this type.

*Type 3:*

$$\mathscr{B}_i = \epsilon_i(\mu S_{m_i} + \lambda T_{m_i})$$

is an $m_i \times m_i$ block, with $\epsilon_i = \pm 1$. Thus, this block has the form

$$\epsilon_i \begin{bmatrix} & & & & & \mu \\ & & & & \mu & \lambda \\ & & & \cdot & \lambda & \\ & & \cdot & \cdot & & \\ & \cdot & \cdot & & & \\ & \mu & \lambda & & & \\ \mu & \lambda & & & & \end{bmatrix}.$$

Again, $\epsilon_i$ is called the sign of the block $\mathscr{B}_i$.

*Type 4:*

$$\mathscr{B}_i = \mu \hat{S}_{2m_i+1} + \lambda T_{2m_i+1},$$

where $\hat{S}_{2m_i+1}$ is simply $S_{2m_i+1}$ with the entry in the $(m_i+1, m_i+1)$ position changed from 1 to 0. Thus, this block is a type-3 block with odd size, $\epsilon_i = 1$, and the $\mu$ in the center position replaced by zero.

REMARKS.

1. Since blocks of type 1, 2, and 3 all have full normal rank, whereas each type-4 block has normal rank defect 1, it follows that the number of type-4 blocks in the canonical form equals the normal rank defect of the pencil $\lambda H + \mu A$. It is therefore clear that type-4 blocks are possible if and only if $\det(\lambda H + \mu A)$ is identically zero.

2. Similarly, from a study of the ranks at $\lambda \neq 0$, $\mu = 0$, it follows that the total number of blocks of types 3 and 4 equals the rank defect of $H$. Thus, the canonical form contains at least one block of type 3 or 4 if and only if $H$ is singular.

## 4. INERTIA RESULTS

We define the inertia $i\{H\}$ of a Hermitian matrix $H$ to be a triple of nonnegative integers $\alpha$, $\beta$, and $\gamma$ where

$$\alpha = \text{number of positive eigenvalues of } H,$$

$$\beta = \text{number of zero eigenvalues of } H,$$

$$\gamma = \text{number of negative eigenvalues of } H.$$

Our notation for the inertia will be

$$\begin{Bmatrix} \alpha \\ \beta \\ \gamma \end{Bmatrix}.$$

The inertia of a Hermitian matrix has two simple but useful properties:

1. For every nonsingular matrix $D$,

$$i\{D^*HD\} = i\{H\}.$$

2. If $H = \oplus_{j=1}^{n} H_j$, then

$$i\{H\} = \sum_{j=1}^{n} i\{H_j\},$$

where the inertias are summed as 3-dimensional vectors.

The inertia of a Hermitian pencil $\lambda H + \mu A$ is the set of inertias of the Hermitian matrices $\lambda H + \mu A$ parametrized by real $\lambda$ and $\mu$. It follows that the inertia of a Hermitian pencil $\lambda H + \mu A$ (for each real value of $\lambda$ and $\mu$) is specified by the sum of the inertias of the canonical blocks (for each real value of $\lambda$ and $\mu$). Thus, it is first necessary to determine the inertias for each canonical block.

We now describe the inertias of the fundamental blocks in the canonical form.

*Type 1:* Recall that each type-1 block has the form

$$\epsilon[(\lambda + a\mu)S_m + \mu T_m]$$

where $\epsilon = \pm 1$ and $a$ is real. The cases for $m$ even and $m$ odd are treated separately. (See Table 1.)

*Type 2:* From the form of a type-2 block, it is clear that for a block of size $2n$ the inertia for all $\lambda$ and $\mu$, not both zero, is always

$$\begin{Bmatrix} n \\ 0 \\ n \end{Bmatrix}.$$

TABLE 1
TYPE 1

| | Inertia | |
|---|---|---|
| $m$ | $\lambda \neq -a\mu$<br>$\mu$ arbitrary | $\lambda = -a\mu$<br>$\mu \neq 0$ |
| $2n$ | $\begin{Bmatrix} n \\ 0 \\ n \end{Bmatrix}$ | $i\{\epsilon\mu\} + \begin{Bmatrix} n-1 \\ 1 \\ n-1 \end{Bmatrix}$ |
| $2n+1$ | $i\{\epsilon(\lambda + a\mu)\} + \begin{Bmatrix} n \\ 0 \\ n \end{Bmatrix}$ | $\begin{Bmatrix} n \\ 1 \\ n \end{Bmatrix}$ |

This is a consequence of the fact that $\lambda + a\mu \neq 0$ for all real $\lambda$ and $\mu$, not both zero, since $\bar{a} \neq a$.

*Type 3:* Recall that each type-3 block has the form

$$\epsilon\left[\mu S_m + \lambda T_m\right]$$

where $\epsilon = \pm 1$. This is clearly identical to a type-1 block with $a = 0$ and the $\lambda$ and $\mu$ interchanged. (See Table 2.)

*Type 4:* From the form of a type-4 block, it is clear that for a block of size $2n+1$ the inertia for all $\lambda$ and $\mu$, not both zero, is always

$$\begin{Bmatrix} n \\ 1 \\ n \end{Bmatrix}.$$

TABLE 2
TYPE 3

| | Inertia | |
|---|---|---|
| $m$ | $\mu \neq 0$<br>$\lambda$ arbitrary | $\mu = 0$<br>$\lambda \neq 0$ |
| $2n$ | $\begin{Bmatrix} n \\ 0 \\ n \end{Bmatrix}$ | $i\{\epsilon\lambda\} + \begin{Bmatrix} n-1 \\ 1 \\ n-1 \end{Bmatrix}$ |
| $2n+1$ | $i\{\epsilon\mu\} + \begin{Bmatrix} n \\ 0 \\ n \end{Bmatrix}$ | $\begin{Bmatrix} n \\ 1 \\ n \end{Bmatrix}$ |

For a certain class of Hermitian pencils, the next lemma makes precise the connection between the inertia of a Hermitian pencil $\lambda H + \mu A$ as a function of real $\lambda$ and $\mu$ and the description of the canonical form. This lemma is central to a number of the subsequent results.

LEMMA 4.1. *The canonical form of the Hermitian pencil $\lambda H + \mu A$ has the property that*

1. *all type-1 blocks have even size and positive sign, and*
2. *all type-3 blocks have odd size and negative sign,*

*if and only if the inertia of $\lambda H - A$, for each real $\lambda$, has the form*

$$\begin{Bmatrix} N + c - a_\lambda \\ d + a_\lambda \\ N \end{Bmatrix}$$

*where $N$, $c$, and $d$ are nonnegative integer constants and $a_\lambda$ is a nonnegative integer which is zero except for a finite set of values of $\lambda$ for which $a_\lambda$ is positive, and the inertia of $H$ has the form*

$$\begin{Bmatrix} N \\ d + c \\ N \end{Bmatrix}.$$

*This latter inertia we call the inertia at $\lambda = \infty$, since it corresponds to $\mu = 0$ for $\lambda$ arbitrary but nonzero. Thus, this condition can be interpreted as the earlier condition with $\lambda = \infty$ and $a_\infty = c$.*

*Moreover, we can make the following correspondence:*

$a_\lambda =$ *the number of blocks of type 1 with even size and positive sign which correspond to the real eigenvalue $\lambda$,*
$c =$ *the number of blocks of type 3 with odd size and negative sign, and*
$d =$ *the number of blocks of type 4.*

*Proof.* First, we prove the necessity. Let $\lambda_0$ be a finite eigenvalue of the pencil $\lambda H + \mu A$. Thus, suppose that there are $a_0$ type-1 blocks with even size and positive sign corresponding to the eigenvalue $\lambda_0, a$ other type-1 blocks with even size and positive sign corresponding to the remaining real eigenvalues, $b$ type-2 blocks, $c$ type-3 blocks with odd size and negative sign, and $d$ type-4 blocks.

Then, for $\lambda$ not an eigenvalue and with $\mu = -1$, the direct sum of these blocks has inertia of the form

$$\begin{Bmatrix} N_{a_0} \\ 0 \\ N_{a_0} \end{Bmatrix} + \begin{Bmatrix} N_a \\ 0 \\ N_a \end{Bmatrix} + \begin{Bmatrix} N_b \\ 0 \\ N_b \end{Bmatrix} + \begin{Bmatrix} N_c + c \\ 0 \\ N_c \end{Bmatrix} + \begin{Bmatrix} N_d \\ d \\ N_d \end{Bmatrix},$$

which equals

$$\begin{Bmatrix} N + c \\ d \\ N \end{Bmatrix},$$

where $N = N_{a_0} + N_a + N_b + N_c + N_d$, and the $N_i$ represent the sizes of the various blocks.

Next, for $\lambda = \lambda_0$ and $\mu = -1$, the direct sum of these blocks has inertia of the form

$$\begin{Bmatrix} N_{a_0} - a_0 \\ a_0 \\ N_{a_0} \end{Bmatrix} + \begin{Bmatrix} N_a \\ 0 \\ N_a \end{Bmatrix} + \begin{Bmatrix} N_b \\ 0 \\ N_b \end{Bmatrix} + \begin{Bmatrix} N_c + c \\ 0 \\ N_c \end{Bmatrix} + \begin{Bmatrix} N_d \\ d \\ N_d \end{Bmatrix},$$

which equals

$$\begin{Bmatrix} N + c - a_0 \\ d + a_0 \\ N \end{Bmatrix}.$$

Finally, for $\lambda \neq 0$ and $\mu = 0$, the direct sum of these blocks has inertia of the form

$$\begin{Bmatrix} N_{a_0} \\ 0 \\ N_{a_0} \end{Bmatrix} + \begin{Bmatrix} N_a \\ 0 \\ N_a \end{Bmatrix} + \begin{Bmatrix} N_b \\ 0 \\ N_b \end{Bmatrix} + \begin{Bmatrix} N_c \\ c \\ N_c \end{Bmatrix} + \begin{Bmatrix} N_d \\ d \\ N_d \end{Bmatrix},$$

which equals

$$\begin{Bmatrix} N \\ c+d \\ N \end{Bmatrix}.$$

Since the same argument applies to all the finite real eigenvalues, this proves the necessity of the stated form.

The sufficiency proof is also straightforward, but messier than the above. Suppose the inertia of $\lambda H - A$ has the stated form, and again suppose there is a real eigenvalue of $\lambda H + \mu A$ at $\lambda = \lambda_0$. We need to show that certain type-1 blocks corresponding to $\lambda = \lambda_0$ and certain type-3 blocks cannot occur in the canonical form of $\lambda H + \mu A$. Thus, consider a further subdivision of block types as follows:

1. Type 1, even size, negative sign, $\lambda = \lambda_0$.
2. Type 1, even size, positive sign, $\lambda = \lambda_0$.
3. Type 1, odd size, negative sign, $\lambda = \lambda_0$.
4. Type 1, odd size, positive sign, $\lambda = \lambda_0$.
5. Type 1 blocks corresponding to all other finite real eigenvalues.
6. Type 3, even size.
7. Type 3, odd size, negative sign.
8. Type 3, odd size, positive sign.
9. Type 4.

Note that we have not included type-2 blocks, as it is clear from the fact that their inertia has the form

$$\begin{Bmatrix} N \\ 0 \\ N \end{Bmatrix},$$

independently of real $\lambda$ and $\mu$ not both zero, that we obtain no information about type-2 blocks from the inertia pattern.

Now, let the numbers of blocks of the above types be $n_1, n_2, \ldots, n_9$, and let the size information be specified by $N_1, \ldots, N_4, N_5^{(1)}, N_5^{(2)}, N_6, \ldots, N_9$. We will need the two size specifications $N_5^{(1)}$ and $N_5^{(2)}$, since class 5 possibly contains a mix of type-1 blocks. We show that $n_1$, $n_3$, $n_4$, $n_6$, and $n_8$ are all zero and that $n_7 = c$ and $n_9 = d$.

For the pencil $\lambda H + \mu A$, let $\lambda_1$ be the smallest finite real eigenvalue of $\lambda H + \mu A$ greater than $\lambda_0$, unless $\lambda_0$ is the largest real eigenvalue, in which case set $\lambda_1 = \infty$. Similarly, let $\lambda_{-1}$ be the largest finite real eigenvalue of

$\lambda H + \mu A$ less than $\lambda_0$, unless $\lambda_0$ is the smallest real eigenvalue, in which case set $\lambda_{-1} = -\infty$.

We compute the inertias of the pencil for $\lambda_{-1} < \lambda < \lambda_0$, $\lambda = \lambda_0$, and $\lambda_0 < \lambda < \lambda_1$. The computed inertias are than equated to the form given in the lemma statement. Note that for $\lambda_{-1} < \lambda < \lambda_1$ the inertia contribution from class 5 will be constant.

Let $\tilde{N} = N_1 + \cdots + N_4 + N_6 + \cdots + N_9$. Then, of $\lambda_{-1} < \lambda < \lambda_0$, we obtain

$$\tilde{N} + N_5^{(1)} + n_3 + n_7 = N + c, \tag{17}$$

$$n_9 = d, \tag{18}$$

$$\tilde{N} + N_5^{(2)} + n_4 + n_8 = N; \tag{19}$$

for $\lambda = \lambda_0$, we obtain

$$\tilde{N} + N_5^{(1)} - n_2 + n_7 = N + c - a_0,$$

$$n_1 + n_2 + n_3 + n_4 + n_9 = d + a_0, \tag{20}$$

$$\tilde{N} + N_5^{(2)} - n_1 + n_8 = N; \tag{21}$$

and for $\lambda_0 < \lambda < \lambda_1$, we obtain

$$\tilde{N} + N_5^{(1)} + n_4 + n_7 = N + c,$$

$$n_9 = d,$$

$$\tilde{N} + N_5^{(2)} + n_3 + n_8 = N. \tag{22}$$

First, from (18), we have $n_9 = d$. Then, subtracting (21) from (19), we obtain $n_1 + n_4 = 0$, from which we conclude $n_1 = 0$, and $n_4 = 0$, since $n_1$ and $n_4$ are nonnegative integers. Then from (19) and (22), it follows that $n_3 = 0$, and thus from (20) that $n_2 = a_0$. Thus, we have shown that for the eigenvalue at $\lambda = \lambda_0$, we have exactly $a_0$ type-1 blocks, which all have even size and positive sign. Clearly, we can repeat the above argument for each finite real eigenvalue. Thus, it follows that $N_5^{(1)} = N_5^{(2)} = N_5$, say.

Substituting the above in the equations (17) and (19), we obtain just two independent equations

$$\tilde{N} + N_5 + n_7 = N + c,$$

$$\tilde{N} + N_5 + n_8 = N,$$

which imply

$$n_7 - n_8 = c.$$

We still have the information about the inertia at $\lambda = \infty$ to use. Thus, it remains to consider the case $\lambda = 1$ and $\mu = 0$. Since all the type-1 blocks have even size, the contribution to the total inertia from these blocks has the form

$$\begin{Bmatrix} N_2 + N_5 \\ 0 \\ N_2 + N_5 \end{Bmatrix}.$$

Thus, we obtain

$$\begin{Bmatrix} N_2 + N_5 \\ 0 \\ N_2 + N_5 \end{Bmatrix} + \begin{Bmatrix} N_6^{(1)} \\ n_6 \\ N_6^{(2)} \end{Bmatrix} + \begin{Bmatrix} N_7 \\ n_7 \\ N_7 \end{Bmatrix} + \begin{Bmatrix} N_8 \\ n_8 \\ N_8 \end{Bmatrix} + \begin{Bmatrix} N_9 \\ d \\ N_9 \end{Bmatrix} = \begin{Bmatrix} N \\ c + d \\ N \end{Bmatrix}.$$

The two terms $N_6^{(1)}$ and $N_6^{(2)}$ are needed to take account of the possibility of having both positive- and negative-sign blocks among the even blocks of type 3. Then we have $n_6 + n_7 + n_8 = c$. Recall that $c = n_7 - n_8$, so we obtain $n_6 + 2n_8 = 0$, from which we conclude $n_6 = 0$ and $n_8 = 0$, since $n_6$ and $n_8$ are nonnegative integers. This completes the proof of the sufficiency. ■

This lemma may be of independent interest. It is clearly possible to explore other relationships between the canonical form of a Hermitian pencil $\lambda H + \mu A$ and the inertias of the Hermitian matrices for real $\lambda$ and $\mu$.

## 5. MAIN RESULT

In this section, we give various equivalent characterizations for the existence of spectral factors. This result should be seen as a generalization of the material in [13], where attention is restricted to the case when $\Psi(\infty)$ is nonsingular. There are also close connections with the work on linear-quadratic singular optimal-control problems [7], reducing subspaces [21], and the Hamiltonian Schur decomposition in [15].

THEOREM 5.1. *Suppose $\Psi(\lambda)$ is an $m \times m$ proper rational matrix which is Hermitian, and suppose that the normal rank of $\Psi(\lambda)$ is $p$. Further, let*

$$\Psi(\lambda) = R + \mathscr{B}^*(\lambda\mathscr{H} - \mathscr{A})^{-1}\mathscr{B}$$

*be a minimal realization where $\mathscr{H}^* = \mathscr{H}$, $\mathscr{A}^* = \mathscr{A}$, and $\mathscr{H}$ is nonsingular. Then the following statements are equivalent.*

1. $\Psi(\lambda) \geqslant 0$ *for all real $\lambda$ for which $\Psi(\lambda)$ is defined.*

2.[1] *The canonical form for $\lambda\mathscr{H} - \mathscr{A}$ contains blocks of types 1 and 2 only. Each block of type 1 has even size and positive sign. The canonical form for*

$$\begin{bmatrix} \mathscr{A} - \lambda\mathscr{H} & \mathscr{B} \\ \mathscr{B}^* & R \end{bmatrix}$$

*contains blocks of all types in general. Each block of type 1 has even size and positive sign. There are exactly $p$ blocks of type 3; each such block has odd size and negative sign. There are exactly $m - p$ blocks of type 4.*

3. *(A weaker version of the previous condition.) Same as the previous condition, but the sign of each type-1 block is unspecified.*

4. *There exists a minimal realization of $\Psi(\lambda)$ congruent to that given above and of the form*

$$\Psi(\lambda) = R + \begin{bmatrix} B^* & C^* \end{bmatrix} \begin{bmatrix} 0 & \lambda I - A \\ \lambda I - A^* & -Q \end{bmatrix}^{-1} \begin{bmatrix} B \\ C \end{bmatrix},$$

*and for any such realization there exists a skew-Hermitian matrix $P$ such that*

$$\begin{bmatrix} PA + A^*P^* + Q & PB + C \\ (PB + C)^* & R \end{bmatrix} \geqslant 0$$

[1]The matrix pencils are given for $\mu = -1$.

*with rank equal to p. This inequality is known as the linear matrix inequality.*

5. *There exists a factorization of* $\Psi(\lambda)$ *of the form* $\Psi(\lambda) = G^*(\lambda)G(\lambda)$ *where* $G(\lambda)$ *is a proper* $p \times m$ *rational matrix.*

*Proof.* Proof of $1 \to 2$: The majority of the detail needed for the proof of this implication is contained in the previous section and in Appendices A, B, and C.

From the material in Appendix A it follows that in the canonical form for $\lambda\mathscr{H} - \mathscr{A}$ each type-1 block has even size and positive sign.

From the material in Appendix B, together with the fact that the canonical form for $\lambda\mathscr{H} - \mathscr{A}$ only contains type-1 and type-2 blocks, with all the type-1 blocks having even size, it follows that there exists a nonsingular matrix $T$ such that

$$T^*\mathscr{H}T = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$

$$T^*\mathscr{A}T = \begin{bmatrix} 0 & A \\ A^* & Q \end{bmatrix},$$

where $Q = Q^*$. Indeed, $Q$ is positive semidefinite in this case.

Thus, using $T$ as state basis change, we assume a minimal realization of $\Psi(\lambda)$ with

$$\mathscr{H} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \qquad \mathscr{A} = \begin{bmatrix} 0 & A \\ A^* & Q \end{bmatrix}, \quad \text{and} \quad \mathscr{B} = \begin{bmatrix} B \\ C \end{bmatrix},$$

where $Q^* = Q$. Moreover, the zero pencil

$$\begin{bmatrix} \mathscr{A} - \lambda\mathscr{H} & \mathscr{B} \\ \mathscr{B}^* & R \end{bmatrix}$$

is congruent to

$$-\lambda \begin{bmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & A & B \\ A^* & Q & C \\ B^* & C^* & R \end{bmatrix}. \tag{23}$$

Finally, we also have

$$\Psi(\lambda) = R + B^*(\lambda I - A^*)^{-1}C + C^*(\lambda I - A)^{-1}B$$
$$+ B^*(\lambda I - A^*)^{-1}Q(\lambda I - A)^{-1}B.$$

First, consider the case when $\Psi(\lambda)$ has no real poles. The inertia of the zero pencil (23) is now shown to be of the form

$$\begin{Bmatrix} N + c - a_\lambda \\ d + a_\lambda \\ N \end{Bmatrix}$$

for each real $\lambda$. (Refer to Lemma 4.1 for the notation.)

If $\Psi(\lambda)$ has no real poles, then because we are considering a minimal realization of $\Psi(\lambda)$ it follows that $\mathscr{A}$ and consequently $A$ and $A^*$ have no real eigenvalues. Therefore, (23) is congruent to

$$\begin{bmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & \Psi(\lambda) \end{bmatrix}.$$

This can be checked using

$$\begin{bmatrix} -(\lambda I - A^*)^{-1}Q & \frac{1}{2}(\lambda I - A^*)^{-1}Q & (\lambda I - A^*)^{-1}\{Q(\lambda I - A)^{-1}B + C\} \\ 0 & I & (\lambda I - A)^{-1}B \\ 0 & 0 & I \end{bmatrix}$$

as a congruent transformation on the pencil (23). Since $\Psi(\lambda)$ is an $m \times m$ rational matrix, has normal rank $p$, and is nonnegative on the real axis, it follows that the inertia of the pencil (23) for each finite real $\lambda$ has the required form with $c = p$, $d = m - p$, and $N$ equal to the dimension of $A$. Furthermore, the inertia of

$$\begin{bmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is simply

$$\left\{\begin{matrix} N \\ m \\ N \end{matrix}\right\}.$$

Thus, application of Lemma 4.1 completes the proof when $\Psi(\lambda)$ has no finite real poles.

There now remains the case when $\Psi(\lambda)$ has poles on the real axis. The details are relegated to Appendix C.

Proof of $2 \to 3$: This is trivial.

Proof of $3 \to 4$: As in the proof of $1 \to 2$, we assume, without loss of generality, that

$$\mathscr{H} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \qquad \mathscr{A} = \begin{bmatrix} 0 & A \\ A^* & Q \end{bmatrix}, \qquad \mathscr{B} = \begin{bmatrix} B \\ C \end{bmatrix}$$

where $Q^* = Q$, so that the zero pencil

$$\begin{bmatrix} \mathscr{A} - \lambda\mathscr{H} & \mathscr{B} \\ \mathscr{B}^* & R \end{bmatrix}$$

is easily seen to be congruent to

$$-\lambda\begin{bmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix} + \begin{bmatrix} Q & C & A^* \\ C^* & R & B^* \\ A & B & 0 \end{bmatrix}.$$

Now, for Appendix B, the canonical form of this pencil implies that there exists a nonsingular matrix $D$ such that

$$D^*\begin{bmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix}D = \begin{bmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix} \tag{24}$$

and

$$D^*\begin{bmatrix} Q & C & A^* \\ C^* & R & B^* \\ A & B & 0 \end{bmatrix}D = \begin{bmatrix} S_1 & S_2 & S_4 \\ S_2^* & S_3 & S_5 \\ S_4^* & S_5^* & S_6 \end{bmatrix}, \tag{25}$$

where

$$\begin{bmatrix} S_1 & S_2 \\ S_2^* & S_3 \end{bmatrix} \geq 0$$

with rank $p$.

Now, consider the restrictions imposed on $D$ by equation (24). Let $Z = D^{-*}$, and write

$$D = \begin{bmatrix} D_1 & D_2 & D_3 \\ D_4 & D_5 & D_6 \\ D_7 & D_8 & D_9 \end{bmatrix}$$

and

$$Z = \begin{bmatrix} Z_1 & Z_2 & Z_3 \\ Z_4 & Z_5 & Z_6 \\ Z_7 & Z_8 & Z_9 \end{bmatrix}.$$

It follows from (24) that

$$\begin{bmatrix} D_7 & D_8 & D_9 \\ 0 & 0 & 0 \\ D_1 & D_2 & D_3 \end{bmatrix} = \begin{bmatrix} Z_3 & 0 & Z_1 \\ Z_6 & 0 & Z_4 \\ Z_9 & 0 & Z_7 \end{bmatrix},$$

from which we conclude that $D_2 = 0$, $D_8 = 0$ and $Z_6 = 0$, $Z_4 = 0$. Further, $Z_3 = D_7$, $Z_1 = D_9$, $Z_9 = D_1$, and $Z_7 = D_3$. Thus,

$$D = \begin{bmatrix} D_1 & 0 & D_3 \\ D_4 & D_5 & D_6 \\ D_7 & 0 & D_9 \end{bmatrix}.$$

Clearly, $D_5$ is nonsingular. Again, using congruence transformations, we force $D_4 = 0$, and $D_6 = 0$, and $D_5 = I$, without affecting the above form. In particular, (24) is left unchanged and the rank and positive semidefiniteness of

$$\begin{bmatrix} S_1 & S_2 \\ S_2^* & S_3 \end{bmatrix}$$

are preserved. This then implies $Z_1 = 0$, $Z_8 = 0$, and $Z_5 = I$, since $DZ^* = I$.

Altogether,

$$D = \begin{bmatrix} D_1 & 0 & D_3 \\ 0 & I & 0 \\ D_7 & 0 & D_9 \end{bmatrix},$$

$$Z = \begin{bmatrix} D_9 & 0 & D_7 \\ 0 & I & 0 \\ D_3 & 0 & D_1 \end{bmatrix},$$

which implies

$$D_1^* D_7 + D_7^* D_1 = 0,$$

$$D_1^* D_9 + D_7^* D_3 = I,$$

$$D_3^* D_9 + D_9^* D_3 = 0,$$

since $D^*Z = I$. From (25), we then obtain

$$\begin{bmatrix} Q & C & A^* \\ C^* & R & B^* \\ A & B & 0 \end{bmatrix} \begin{bmatrix} D_1 & 0 & D_3 \\ 0 & I & 0 \\ D_7 & 0 & D_9 \end{bmatrix} = \begin{bmatrix} D_9 & 0 & D_7 \\ 0 & I & 0 \\ D_3 & 0 & D_1 \end{bmatrix} \begin{bmatrix} S_1 & S_2 & S_4 \\ S_2^* & S_3 & S_5 \\ S_4^* & S_5^* & S_6 \end{bmatrix}, \quad (26)$$

where

$$\begin{bmatrix} S_1 & S_2 \\ S_2^* & S_3 \end{bmatrix} = \begin{bmatrix} N \\ L^* \end{bmatrix} [N^* \quad L]$$

with $\operatorname{rank}[N^* \quad L] = p$.

We show that $D_1$ is nonsingular. From the first column of (26), we have

$$QD_1 + A^*D_7 = D_9NN^* + D_7S_4^*,$$

$$C^*D_1 + B^*D_7 = L^*N^*,$$

$$AD_1 = D_3NN^* + D_1S_4^*.$$

Suppose that there exists $x \neq 0$ such that $D_1 x = 0$. Then,

$$A^* D_7 x = D_9 NN^* x + D_7 S_4^* x,$$

$$B^* D_7 x = L^* N^* x,$$

$$0 = D_3 NN^* x + D_1 S_4^* x,$$

and so

$$\begin{aligned} D_1^* A^* D_7 x &= D_1^* D_9 NN^* x + D_1^* D_7 S_4^* x \\ &= (I - D_7^* D_3) NN^* x - D_7^* D_1 S_4^* x \\ &= NN^* x - D_7^* (D_3 NN^* x + D_1 S_4^* x) \\ &= NN^* x. \end{aligned}$$

Thus, $x^* NN^* x = 0$, and so $N^* x = 0$.

From the above, this gives

$$A^* D_7 x = D_7 S_4^* x,$$

$$B^* D_7 x = 0,$$

$$0 = D_1 S_4^* x.$$

Using the last of these equations, we conclude that the null space of $D_1$ is an invariant subspace of $S_4^*$. Therefore, there exists $z \neq 0$ and $\lambda$ such that

$$S_4^* z = \lambda z,$$

$$D_1 z = 0.$$

Also, note that $y = D_7 z \neq 0$ (otherwise $D$ would be singular), but then we have

$$A^* y = \lambda y,$$

$$B^* y = 0,$$

which contradicts the controllability of the pair $(A, B)$. Thus, $D_1$ is nonsingular, and $P = D_1^{-*}D_7^*$ satisfies

$$P^* + P = 0$$

and

$$\begin{bmatrix} PA + A^*P^* + Q & PB + C \\ B^*P^* + C^* & R \end{bmatrix} = \begin{bmatrix} D_1^{-*}S_1D_1^{-1} & D_1^{-*}S_2 \\ S_2^*D_1^{-1} & S_3 \end{bmatrix} \geqslant 0$$

with rank $p$.

Proof of 4 → 5: Let $P$ be a skew-Hermitian matrix satisfying the linear matrix inequality

$$\begin{bmatrix} PA + A^*P^* + Q & PB + C \\ B^*P^* + C^* & R \end{bmatrix} = \begin{bmatrix} N \\ L^* \end{bmatrix}\begin{bmatrix} N^* & L \end{bmatrix}$$

with rank$[N^* \quad L] = p$. Then, with

$$G(\lambda) = L + N^*(\lambda I - A)^{-1}B,$$

it follows easily that

$$\Phi(\lambda) = G^*(\lambda)G(\lambda).$$

Proof of 5 → 1: This is trivial.

This completes the proof of the theorem. ■

This theorem assumes a minimal realization of $\Psi(\lambda)$. A less restrictive assumption is discussed in Theorem 7.1.

If $R = \Psi(\infty)$ is nonsingular, the existence of a solution to the linear matrix inequality with rank $p$ ($p = m$ in this case) is equivalent to a solution of the algebraic Riccati equation [22]

$$PA + A^*P^* + Q - (PB + C)R^{-1}(PB + C)^* = 0.$$

## 6. DEFLATION

In this section, we apply the inertia result of Lemma 4.1 to develop a symmetric deflation procedure for the class of Hermitian pencils of interest in this paper. The two deflation lemmas below are important in that they suggest a numerical technique which exploits the inherent structure of the Hermitian pencil. The proof of the first of these lemmas is long and not particularly interesting; the details can be found in Appendix D. The second lemma is trivial.

LEMMA 6.1. *If the canonical form of the Hermitian pencil*

$$\begin{bmatrix} 0 & 0 & \lambda h + \mu a \\ 0 & \lambda H + \mu A & X \\ \lambda \bar{h} + \mu \bar{a} & X & X \end{bmatrix}$$

*where X denotes don't care entries and h and a are not both zero, has the property that*

1. *all type-1 blocks have even size and positive sign, and*
2. *all type-3 blocks have odd size and negative sign,*

*then the canonical form of* $\lambda H + \mu A$ *has the same property, with the same numbers of type-3 and type-4 blocks.*

LEMMA 6.2. *If the canonical form of the Hermitian pencil*

$$\begin{bmatrix} 0 & 0 \\ 0 & \lambda H + \mu A \end{bmatrix}$$

*has the property that*

1. *all type-1 blocks have even size and positive sign, and*
2. *all type-3 blocks have odd size and negative sign,*

*then the canonical form of* $\lambda H + \mu A$ *is identical except for the removal of one type-4 block with size 1.*

*Proof.* The given pencil is clearly congruent to

$$[0] \oplus \lambda H + \mu H$$

but the first term is a type-4 block with size 1. This completes the proof. ■

We now show how these deflation lemmas can be applied to the spectral-factorization problem. By Theorem 5.1, it is enough to find a skew-Hermitian $P$ satisfying the linear matrix inequality with minimal rank.

Suppose therefore that we are given a Hermitian pencil $\lambda H + \mu A$ with the properties that (i) each type-1 block has even size and positive sign, and (ii) each type-3 block has odd size and negative sign. We study the various solutions $\lambda_0$, $\mu_0$, and $x$ of the equation

$$(\lambda_0 H + \mu_0 A)x = 0$$

such that $|\lambda_0| + |\mu_0| \neq 0$ and $x \neq 0$.

We are interested in the following cases.

1. *Rank deflation.* Suppose that $Hx = 0$ and $Ax = 0$. (Note that this is only possible if there are type-4 blocks in the canonical form of $\lambda H + \mu A$.) Construct a unitary matrix $D$ such that $D^*x$ is zero except for the first component. Then

$$D^*(\lambda H + \mu A)D = \begin{bmatrix} 0 & 0 \\ 0 & \lambda H_1 + \mu A_1 \end{bmatrix},$$

where, by Lemma 6.2, the Hermitian pencil $\lambda H_1 + \mu A_1$ has the same canonical structure as $\lambda H + \mu A$ except for a single type-4 block with size 1.

2. *Nonfinite deflation.* Suppose that $Hx = 0$, $Ax \neq 0$, and $x^*Ax = 0$. (Such an $x$ exists precisely when at least one type-3 or type-4 block has size greater than 1. The standing assumption that all type-3 blocks have odd size and the same sign is important here.) A slight extension of the construction above now produces a unitary matrix $D$ such that

$$D^*(\lambda H + \mu A)D = \begin{bmatrix} 0 & 0 & \mu a \\ 0 & \lambda H_1 + \mu A_1 & X \\ \mu \bar{a} & X & X \end{bmatrix}$$

where $a \neq 0$. Now, by Lemma 6.1, the Hermitian pencil $\lambda H_1 + \mu A_1$ has the same canonical structure as $\lambda H + \mu A$ with the same numbers of type-3 and type-4 blocks.

3. *Finite deflation.* Suppose that $Hx \neq 0$. It then follows that $\mu_0 \neq 0$. Thus, $-\lambda_0/\mu_0$ is a finite eigenvalue and $x$ a corresponding eigenvector provided the pencil $\lambda H + \mu A$ has full normal rank. This will be the case when no more rank and nonfinite deflations are possible. From the assumption that all type-1 blocks in the canonical form for $\lambda H + \mu A$ have even size and the same sign (positive), we conclude that $x^*Hx = 0$ and $x^*Ax = 0$.

Again, a construction similar to the above produces a unitary matrix $D$ such that

$$D^*(\lambda H+\mu A)D=\begin{bmatrix} 0 & 0 & \lambda h+\mu a \\ 0 & \lambda H_1+\mu A_1 & X \\ \lambda\bar{h}+\mu\bar{a} & X & X \end{bmatrix}$$

where $h\neq 0$.

If these deflations are applied repeatedly until no more deflations are possible, then accumulating the unitary transformations appropriately, we obtain a unitary $D$ such that

$$D^*(\lambda H+\mu A)D=\begin{bmatrix} 0 & 0 & \lambda H_1+\mu A_1 \\ 0 & \mu A_2 & \lambda H_3+\mu A_3 \\ \lambda H_1^*+\mu A_1^* & \lambda H_3^*+\mu A_3^* & \lambda H_4+\mu A_4 \end{bmatrix}$$

where (referring back to Lemma 4.1 for the definitions of $N$, $d$, and $c$)

1. $A_2$ is a $c\times c$ negative definite matrix.
2. $\begin{bmatrix} H_1 \\ H_3 \end{bmatrix}$ is an $(N+c+d)\times N$ matrix with full column rank $N$. This is a straightforward consequence of the fact that $H$ has rank $2N$.
3. $\lambda H_1+\mu A_1$ is an $(N+d)\times N$ pencil with full normal column rank. The $j$th column of $\lambda H_1+\mu A_1$ has the form

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \lambda h_j+\mu a_j \\ X \\ \vdots \\ X \end{bmatrix},$$

where the first nonzero entry $\lambda h_j+\mu a_j$ occurs in row $r_j$ (this means $h_j$ or $a_j$ is nonzero). There also holds

$$r_1>r_2>\cdots>r_N.$$

The leading nonzero entry $\lambda h_j + \mu a_j$ in the $j$th column of $\lambda H_1 + \mu A_1$ is the result of a nonfinite deflation step ($h_j = 0$) or a finite deflation step ($h_j \neq 0$); this latter case corresponding to a finite eigenvalue $-a_j/h_j$. From properties 1, 2, and 3 above, it follows that each finite eigenvalue of the pencil

$$\begin{bmatrix} 0 & \lambda H_1 + \mu A_1 \\ \mu A_2 & \lambda H_3 + \mu A_3 \end{bmatrix} \tag{27}$$

must be one of the values $-a_j/h_j$ with $h_j \neq 0$, i.e. one of the finite eigenvalues of the pencil $\lambda H + \mu A$ used in a finite deflation step. Thus, the choice of the finite eigenvalues in the finite deflation steps controls the locations of the finite eigenvalues of the pencil (27).

Now let us return to the specific pencil $\lambda H + \mu A$ of interest to us, i.e.

$$-\lambda \begin{bmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix} + \begin{bmatrix} Q & C & A^* \\ C^* & R & B^* \\ A & B & 0 \end{bmatrix},$$

so that

$$H = -\begin{bmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix}$$

and

$$A = -\begin{bmatrix} Q & C & A^* \\ C^* & R & B^* \\ A & B & 0 \end{bmatrix}$$

with an obvious abuse of notation. The above discussion implies that we have a unitary $D$ such that

$$-D^* \begin{bmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix} D = \begin{bmatrix} 0 & 0 & H_1 \\ 0 & 0 & H_3 \\ H_1^* & H_3^* & H_4 \end{bmatrix}$$

and

$$-D^*\begin{bmatrix} Q & C & A^* \\ C^* & R & B^* \\ A & B & 0 \end{bmatrix} D = \begin{bmatrix} 0 & 0 & A_1 \\ 0 & A_2 & A_3 \\ A_1^* & A_3^* & A_4 \end{bmatrix}.$$

Since $D$ is unitary, it is easy to show that $H_4$ is zero and $\begin{bmatrix} H_1 \\ H_3 \end{bmatrix}$ has orthogonal columns. Using only unitary transformations and incorporating these transformations into $D$, it follows that (24) and (25) hold, i.e.,

$$D^*\begin{bmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix} D = \begin{bmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix}$$

and

$$D^*\begin{bmatrix} Q & C & A^* \\ C^* & R & B^* \\ A & B & 0 \end{bmatrix} D = \begin{bmatrix} S_1 & S_2 & S_4 \\ S_2^* & S_3 & S_5 \\ S_4^* & S_5^* & S_6 \end{bmatrix},$$

where

$$0 \leqslant \begin{bmatrix} S_1 & S_2 \\ S_2^* & S_3 \end{bmatrix} = \begin{bmatrix} N \\ L^* \end{bmatrix} \begin{bmatrix} N^* & L \end{bmatrix}$$

with rank $p$. Note that $p$ corresponds to $c$ here, and that we have renamed the transformed $A$ matrix as $S$ with corresponding changes in the names of the component matrices. Further, the unitary matrix $D$ has the form

$$D = \begin{bmatrix} D_1 & 0 & D_3 \\ 0 & D_5 & 0 \\ D_7 & 0 & D_9 \end{bmatrix}.$$

Now, we continue as in the proof of Theorem 5.1. There is, however, a major difference between how we get to this stage in Theorem 5.1 and the approach above. In Theorem 5.1, we assumed we had the pencil in canonical form and knew the (almost certainly not unitary) transforming matrix $D$, whereas the above approach only assumes that we know the properties of the canonical form from which we construct a unitary transforming matrix $D$ and we end up with a triangular form rather than a (Jordan) canonical form. This is important numerically, as it is well known that the computation of the Jordan form is unreliable whereas the computation of a Schur upper triangular form using only unitary transformations is not.

Finally, in this section, we show that the zero structure of the adjoint of the right spectral factor $G(\lambda)$ is given by the structure of the pencil (27).

LEMMA 6.3. *The pencils*

$$\begin{bmatrix} 0 & \lambda H_1 + \mu A_1 \\ \mu A_2 & \lambda H_3 + \mu A_3 \end{bmatrix}$$

*and*

$$\begin{bmatrix} \mu D_1^{-*} N & \lambda I + \mu A^* \\ \mu L^* & \mu B^* \end{bmatrix}$$

*have the same structure.*

*Proof.* Let $\tilde{D}$ be the unitary matrix which satisfies

$$\tilde{D}^* \begin{bmatrix} H_1 \\ H_3 \end{bmatrix} = \begin{bmatrix} -I \\ 0 \end{bmatrix}.$$

It follows that

$$\tilde{D}^* \begin{bmatrix} 0 \\ -A_2 \end{bmatrix} = \begin{bmatrix} N \\ L^* \end{bmatrix} T$$

for some nonsingular $T$ satisfying $T^*T = -A_2$. It follows that

$$\begin{bmatrix} 0 & \lambda H_1 + \mu A_1 \\ \mu A_2 & \lambda H_3 + \mu A_3 \end{bmatrix}$$

has the same structure as

$$\begin{bmatrix} -\mu NT & -\lambda I - \mu S_4 \\ -\mu L^* T & -\mu S_6 \end{bmatrix},$$

which in turn has the same structure as

$$\begin{bmatrix} \mu N & \lambda I + \mu S_4 \\ \mu L^* & \mu S_5 \end{bmatrix}.$$

From the bottom row of (26), we obtain

$$S_4 = D_1^* A^* D_1^{-*} - NN^* D_3^* D_1^{-*}$$

$$S_5 = B^* D_1^{-*} - L^* N^* D_3^* D_1^{-*}.$$

Thus, the above pencil has the same structure as

$$\begin{bmatrix} \mu N & \lambda I + \mu D_1^* A^* D_1^{-*} \\ \mu L^* & \mu B^* D_1^{-*} \end{bmatrix},$$

which has the same structure as

$$\begin{bmatrix} \mu D_1^{-*} N & \lambda I + \mu A^* \\ \mu L^* & \mu B^* \end{bmatrix}.$$

This completes the proof. ■

We note that the right spectral factor $G(\lambda)$ computed in Theorem 5.1 is

$$G(\lambda) = L + N^* D_1^{-1}(\lambda I - A)^{-1} B.$$

## 7. EXTENSION

In this section we trade off the minimality assumptions imposed in Theorem 5.1 against a full-rank assumption on $\Psi(\lambda)$.

THEOREM 7.1. *Theorem* 5.1 *continues to hold provided we*

1. *replace the minimality assumption with the assumption that* $[\lambda \mathscr{H} - \mathscr{A}\ \mathscr{B}]$ *has full row rank for all* $\lambda$ *with nonnegative imaginary part,*
2. *add the assumption that* $\Psi(\lambda)$ *has full normal rank, i.e.* $p = m$, *and*
3. *delete the reference to minimality in condition* 4.

*Proof.* The proof goes through as in Theorem 5.1 essentially unchanged for $1 \to 2$, $2 \to 3$, $4 \to 5$, and $5 \to 1$.

The main difference is the proof $3 \to 4$. Construct the congruence matrix $D$ in such a way that the pencil

$$\begin{bmatrix} \mu N & \lambda I + \mu S_4 \\ \mu L^* & \mu S_5 \end{bmatrix}$$

has all its finite eigenvalues in the closed upper half plane. The discussion in the previous section indicates that this is possible, as does a reexamination of the proof of $3 \to 4$ in Theorem 5.1. The key step is showing the nonsingularity of $D_1$. The proof shows that singularity of $D_1$ implies the existence of a nonzero $z$ and a $\lambda$ satisfying

$$S_4^* z = \lambda z,$$

$$D_1^* z = 0,$$

$$N^* z = 0,$$

and of a nonzero $y$ satisfying

$$A^* y = \lambda y,$$

$$B^* y = 0.$$

Since $\Psi(\lambda)$ has full normal rank, so does the pencil above. Therefore, the first set of equations shows that $\bar{\lambda}$ is a finite eigenvalue, with left eigenvector $[z^* \;\; 0]$. Since the pencil has been constructed to have finite eigenvalues in the closed upper half plane only, it follows that $\lambda$ has nonpositive imaginary part and so $\bar{\lambda}$ has nonnegative imaginary part. Similarly, from the second set of equations and the relaxed minimality assumption, we conclude that $\bar{\lambda}$ has negative imaginary part. This is a contradiction. Thus, $D_1$ must be nonsingular.

This completes the proof. ∎

REMARK. The proof of Theorem 7.1 shows that if $[\lambda \mathscr{H} - \mathscr{A} \;\; \mathscr{B}]$ is full-rank in $\operatorname{Im} \lambda \geqslant 0$, then the spectral factor can be constructed to have no zeros on $\operatorname{Im} \lambda > 0$. A weaker result can be obtained under the assumption that $[\lambda\mathscr{H} - \mathscr{A}\ \mathscr{B}]$ is full-rank on $\operatorname{Im}\lambda = 0$ with $\Psi(\lambda)$ of full normal rank. Then Theorem 5.1(1,2,3,5) are still equivalent. The proof involves $1 \to 2 \to 3$ as above and then deflating the uncontrollable part as in Section 6 and finally applying $3 \to 4$ for the resulting minimal system.

## 8. CONCLUDING REMARKS

In this paper, we have studied in detail the algebraic structure of the spectral-factorization of proper rational Hermitian matrices with a view to producing a reliable but general-purpose computational algorithm. In this development, we have exploited the properties of Hermitian pencils. To the authors' knowledge, the only other work in this vein is [17, 12], where the connection between the algebraic Riccati equation and Hermitian pencils is studied.

The major restriction in this paper is that the spectral density $\Phi(s)$ or $\Psi(\lambda)$ must not have any poles at $\infty$. It is not clear how this restriction can be removed in the development in this paper.

## APPENDIX A

We show that if the proper, Hermitian rational matrix $\Psi(\lambda)$ has representation

$$\Psi(\lambda) = R + \mathscr{B}^*(\lambda\mathscr{H} - \mathscr{A})^{-1}\mathscr{B} \tag{28}$$

with $\det(\lambda_0\mathscr{H} - \mathscr{A}) = 0$ for some real value $\lambda_0, \Psi(\lambda)$ positive semidefinite for all real $\lambda$ in a neighborhood of $\lambda_0$, and $[\lambda_0\mathscr{H} - \mathscr{A}\ \mathscr{B}]$ with full row rank, then each type 1 block corresponding to $\lambda_0$ in the canonical form for the pencil $\lambda\mathscr{H} + \mu\mathscr{A}$ has even size and positive sign.

We need the following preliminary result.

RESULT A.1. *For $m \geqslant 1$,*

$$(I_m - \lambda N_m)^{-1}(\lambda S_m - T_m)(I_m - \lambda N_m^*)^{-1} = \lambda^m \oplus (\lambda T_{m-1} - S_{m-1})^{-1}.$$

*Proof.* Let $n = m-1$, $\Lambda_n^* = [\lambda,\dots,\lambda^n]$, and $e_n^* = [0,\dots,0,1]$. Then with

$$(I_m - \lambda N_m)^{-1} = \begin{bmatrix} 1 & \Lambda_n^* \\ O & (I_n - \lambda N_n)^{-1} \end{bmatrix},$$

$$(\lambda S_m - T_m) = \begin{bmatrix} O & \lambda e_n^* \\ \lambda e_n & (\lambda S_n T_n S_n - I_n) \end{bmatrix},$$

$$(I_m - \lambda N_m^*)^{-1} = \begin{bmatrix} 1 & O \\ \Lambda_n & (I_n - \lambda N_n^*)^{-1} \end{bmatrix},$$

the proof follows easily by carrying out the block matrix multiplication and simplifying the resulting terms. ■

Thus, we have the following corollary.

COROLLARY A.2. *There exist an $m\times m$ rational matrix $P_m(\lambda)$ and an $(m-1)\times(m-1)$ rational matrix $Q_{m-1}(\lambda)$, both invertible at $\lambda = 0$, such that*

$$P_m^*(\lambda)(\lambda S_m - T_m)P_m(\lambda) = \lambda^m \oplus Q_{m-1}(\lambda).$$

*Proof.* Observe that $I_m - \lambda N_m$ and $\lambda T_{m-1} - S_{m-1}$ are unimodular polynomial matrices. ■

Since $\det \mathscr{H} \neq 0$, only type-1 and type-2 blocks may appear in the canonical form for $\lambda\mathscr{H} - \mathscr{A}$. Since $\det(\lambda_0\mathscr{H} - \mathscr{A}) = 0$, suppose that $\lambda\mathscr{H} - \mathscr{A}$ has $r$ blocks of type 1 with sizes $m_1,\dots,m_r$ and signs $\epsilon_1,\dots,\epsilon_r$ corresponding to $\lambda_0$.

Applying Corollary A.2 to each of the type-1 blocks shifted to $\lambda = \lambda_0$, and noting that all other blocks not corresponding to $\lambda_0$ are invertible at $\lambda = \lambda_0$, it follows that there exist rational matrices $P(\lambda)$ and $Q(\lambda)$, both invertible at $\lambda = \lambda_0$, such that

$$P^*(\lambda)(\lambda\mathscr{H} - \mathscr{A})P(\lambda) = Q(\lambda) \oplus \bigoplus_{i=1}^{r} \epsilon_i(\lambda - \lambda_0)^{m_i}. \tag{29}$$

The assumed full row rank of $[\lambda_0\mathscr{H} - \mathscr{A} \;\; \mathscr{B}]$ implies that $[\lambda\mathscr{H} - \mathscr{A} \;\; \mathscr{B}]$, and hence

$$\left[P^*(\lambda)(\lambda\mathscr{H} - \mathscr{A})P(\lambda) \quad P^*(\lambda)\mathscr{B}\right]$$

have full row rank for all $\lambda$ in a neighborhood of $\lambda_0$. Partitioning

$$P^*(\lambda)\mathscr{B} = \begin{bmatrix} B_1(\lambda) \\ B_0(\lambda) \end{bmatrix}, \tag{30}$$

where $B_0(\lambda)$ has $r$ rows, it follows that $B_0(\lambda_0)$ has full row rank.

Substituting (29) and (30) into (28) gives

$$\Psi(\lambda) = \Psi_0(\lambda) + B_0^*(\lambda)\left[\bigoplus_{i=1}^{r} \epsilon_i(\lambda - \lambda_0)^{m_i}\right]^{-1} B_0(\lambda),$$

where

$$\Psi_0(\lambda) = R + B_1^*(\lambda)Q(\lambda)^{-1}B_1(\lambda).$$

Since $B_0(\lambda_0)$ has full row rank, there exists a rational matrix $\tilde{B}_0(\lambda)$, invertible at $\lambda = \lambda_0$, such that $B_0(\lambda)\tilde{B}_0(\lambda) = [I_r \;\; 0]$. Thus, altogether we have

$$\tilde{B}_0^*(\lambda)\Psi(\lambda)\tilde{B}_0(\lambda) = \tilde{B}_0^*(\lambda)\Psi_0(\lambda)\tilde{B}_0(\lambda) + \begin{bmatrix} \bigoplus_{i=1}^{r}\epsilon_i^{-1}(\lambda - \lambda_0)^{-m_i} & 0 \\ 0 & 0 \end{bmatrix}$$

where $\tilde{B}_0^*(\lambda)\Psi_0(\lambda)\tilde{B}_0(\lambda)$ is analytic at $\lambda = \lambda_0$. Since $\tilde{B}_0^*(\lambda)\Psi(\lambda)\tilde{B}_0(\lambda)$ is positive semidefinite near $\lambda = \lambda_0$, it follows that $\epsilon_i = 1$ for $i = 1, \ldots, r$ and $m_1, m_2, \ldots, m_r$ are all even.

## APPENDIX B

In this appendix, we examine some useful congruences for the various canonical blocks, at least in the cases of interest to us.

First, we introduce some notation:

$$E_n^{(i,j)} \overset{\text{def}}{=} n \times n \text{ matrix with entries } \delta_{k,i}\delta_{l,j},$$

$$e_n^{(i)} \overset{\text{def}}{=} n \text{ vector with entries } \delta_{k,i}.$$

*Type 1:* We only consider even-size blocks. Thus, with $m = 2n$ and $a$ real

$$\begin{aligned}\epsilon[(\lambda + a\mu)S_m + \mu T_m] &= \lambda\begin{bmatrix} O_n & \epsilon S_n \\ \epsilon S_n & O_n \end{bmatrix} + \mu\begin{bmatrix} O_n & \epsilon(aS_n + T_n) \\ \epsilon(aS_n + T_n) & \epsilon E_n^{(1,1)} \end{bmatrix} \\ &\sim \lambda\begin{bmatrix} O_n & I_n \\ I_n & O_n \end{bmatrix} + \mu\begin{bmatrix} O_n & aI_n + N_n \\ aI_n + N_n^* & \epsilon E_n^{(1,1)} \end{bmatrix}.\end{aligned}$$

Here, the congruent transformation can easily be seen to consist of permutations only when the sign $\epsilon$ is positive. Otherwise, row and column sign changes are also needed.

*Type 2:* A type-2 block with size $m = 2n$ is clearly congruent to

$$\lambda\begin{bmatrix} O_n & I_n \\ I_n & O_n \end{bmatrix} + \mu\begin{bmatrix} O_n & aI_n + N_n \\ \bar{a}I_n + N_n^* & O_n \end{bmatrix}.$$

*Type 3:* We only consider odd-size blocks. Then, with $m = 2n + 1$,

$$\begin{aligned}\epsilon[\lambda T_m + \mu S_m] &= \lambda\begin{bmatrix} O_n & O & \epsilon T_n \\ O & O & \epsilon e_n^{(1)*} \\ \epsilon T_n & \epsilon e_n^{(1)} & O_n \end{bmatrix} + \mu\begin{bmatrix} O_n & O & \epsilon S_n \\ O & \epsilon & O \\ \epsilon S_n & O & O_n \end{bmatrix} \\ &\sim \lambda\begin{bmatrix} O_n & O & N_n \\ O & O & e_n^{(1)*} \\ N_n^* & e_n^{(1)} & O_n \end{bmatrix} + \mu\begin{bmatrix} O_n & O & I_n \\ O & \epsilon & O \\ I_n & O & O_n \end{bmatrix} \\ &\sim \lambda\begin{bmatrix} O_n & O & I_n \\ O & O & O \\ I_n & O & O_n \end{bmatrix} + \mu\begin{bmatrix} \epsilon E_n^{(1,1)} & O & N_n^* \\ O & O & e_n^{(n)*} \\ N_n & e_n^{(n)} & O_n \end{bmatrix}.\end{aligned}$$

*Type 4:* From the type-3 material above, it follows that type-4 blocks are congruent to

$$\lambda \begin{bmatrix} O_n & O & I_n \\ O & O & O \\ I_n & O & O_n \end{bmatrix} + \mu \begin{bmatrix} O & O & N_n^* \\ O & O & e_n^{(n)*} \\ N_n & e_n^{(n)} & O_n \end{bmatrix}.$$

From these considerations, it follows that a canonical form containing only type-1 blocks with even size and type-2 blocks is congruent to

$$\lambda \begin{bmatrix} O_n & I_n \\ I_n & O_n \end{bmatrix} + \mu \begin{bmatrix} O_n & A \\ A^* & Q \end{bmatrix}.$$

If all the type-1 blocks have positive sign, then the congruent transformation can be chosen to be a permutation matrix, and $Q$ will then be positive semidefinite with rank equal to the number of type-1 blocks. More generally, the congruence transformation will also include row and column sign changes, and the inertia of $Q$ will correspond to the number of positive- and negative-signed blocks.

If blocks of all types are allowed with the restriction that type-1 blocks must have even size and type-3 blocks must have odd size and positive sign, then the above considerations show that the canonical form is congruent to

$$\lambda \begin{bmatrix} O_n & O & I_n \\ O & O & O \\ I_n & O & O_n \end{bmatrix} + \mu \begin{bmatrix} S_1 & S_2 & S_3 \\ S_2^* & S_4 & S_5 \\ S_3^* & S_5^* & S_6 \end{bmatrix},$$

where

$$\begin{bmatrix} S_1 & S_2 \\ S_2^* & S_4 \end{bmatrix}$$

is positive semidefinite with rank equal to the number of type-3 blocks. The congruent transformation is generally a permutation matrix with row and column sign changes.

## APPENDIX C

In this appendix, we complete the proof of the implication $1 \to 2$ in Theorem 5.1 for the case when $\Psi(\lambda)$ has poles on the real axis.

Let us write

$$\Psi(\lambda) = [B^*\Gamma^{-*} \quad I]\begin{bmatrix} Q & C \\ C^* & R \end{bmatrix}\begin{bmatrix} \Gamma^{-1}B \\ I \end{bmatrix}, \tag{31}$$

where $\Gamma = \lambda I - A$. With the definitions

$$A_K = A + BK,$$

$$Q_K = Q + CK + K^*C^* + K^*RK,$$

$$C_K = C + K^*R,$$

$$\Gamma_K = \lambda I - A_K,$$

$$T_K = (I - K\Gamma^{-1}B)^{-1},$$

we have

$$\begin{aligned}
\Psi_K(\lambda) &= T_K^*\Psi(\lambda)T_K \\
&= T_K^*[B^*\Gamma^{-*} \quad I]\begin{bmatrix} Q & C \\ C^* & R \end{bmatrix}\begin{bmatrix} \Gamma^{-1}B \\ I \end{bmatrix}T_K \\
&= [B^*\Gamma_K^{-*} \quad I]\begin{bmatrix} I & K^* \\ 0 & I \end{bmatrix}\begin{bmatrix} Q & C \\ C^* & R \end{bmatrix}\begin{bmatrix} I & 0 \\ K & I \end{bmatrix}\begin{bmatrix} \Gamma_K^{-1}B \\ I \end{bmatrix} \\
&= [B^*\Gamma_K^{-*} \quad I]\begin{bmatrix} Q_K & C_K \\ C_K^* & R \end{bmatrix}\begin{bmatrix} \Gamma_K^{-1}B \\ I \end{bmatrix}.
\end{aligned}$$

Here we have used the identities

$$\Gamma^{-1}BT_K = \Gamma_K^{-1}B,$$

$$T_K = I + K\Gamma_K^{-1}B.$$

With

$$\mathscr{H}_K = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \qquad \mathscr{A}_K = \begin{bmatrix} 0 & A_K \\ A_K^* & Q_K \end{bmatrix}, \quad \text{and} \quad \mathscr{B}_K = \begin{bmatrix} B \\ C_K \end{bmatrix}$$

we have the realization

$$\Psi_K(\lambda) = R + \mathscr{B}_K^*(\lambda \mathscr{H}_K - \mathscr{A}_K)^{-1}\mathscr{B}_K.$$

We choose $K$ so that this realization of $\Psi_K(\lambda)$ is minimal and so that $\Psi_K(\lambda)$ has no real poles. Such a $K$ always exists as follows. Since a minimal realization for $\Psi(\lambda)$ is given, we have $[A, B]$ forming a controllable pair. We choose $K$ so that $A_K$ has no real eigenvalues. It follows that $\Psi_K(\lambda)$ has no real poles. We have no guarantee, though, that $[\mathscr{A}_K - \lambda \mathscr{H}_K \;\; \mathscr{B}_K]$ has full row rank for all $\lambda$. Let $f(K)$ be a polynomial in the components of $K$ such that $f(K) = 0$ if and only if $[\mathscr{A}_K - \lambda \mathscr{H}_K \;\; \mathscr{B}_K]$ does not have full row rank for all $\lambda$. Such a polynomial can be constructed from the minors of the controllability matrix. Suppose $f(K) = 0$. Since $f(0) \neq 0$, there exists in every neighborhood of $K$, a $\tilde{K}$ such that $f(\tilde{K}) \neq 0$. For a small enough neighborhood of $K$, none of the poles of $\Psi_{\tilde{K}}(\lambda)$ will be real. Set $K = \tilde{K}$.

Now, clearly $\Psi_K(\lambda)$ is positive semidefinite everywhere on the real axis. Thus its zero pencil

$$\begin{bmatrix} 0 & A_K - \lambda I & B \\ A_K^* - \lambda I & Q_K & C_K \\ B^* & C_K^* & R \end{bmatrix}$$

has the required canonical form. However, this pencil is clearly congruent to

$$\begin{bmatrix} 0 & A - \lambda I & B \\ A^* - \lambda I & Q & C \\ B^* & C^* & R \end{bmatrix},$$

which is the zero pencil for $\Psi(\lambda)$. Thus, this pencil must also have the same canonical form. This completes the proof.

APPENDIX D

*Proof of Lemma 6.1.* First, we prove the result for $h$ not zero. Let $\lambda_0 = a/h$. The given pencil is then congruent to

$$\begin{bmatrix} 0 & 0 & \lambda + \mu\lambda_0 \\ 0 & \lambda H + \mu A & \mu b \\ \lambda + \mu\bar{\lambda}_0 & \mu b^* & \mu\theta \end{bmatrix} \tag{32}$$

for some vector $b$ and some real $\theta$.

From Lemma 4.1, the assumption of this lemma together with $h \neq 0$ is equivalent to the matrix

$$\begin{bmatrix} 0 & 0 & \lambda - \lambda_0 \\ 0 & \lambda H - A & -b \\ \lambda - \bar{\lambda}_0 & -b^* & -\theta \end{bmatrix} \tag{33}$$

having an inertia of the form

$$\begin{Bmatrix} N + c - a_\lambda \\ d + a_\lambda \\ N \end{Bmatrix} \tag{34}$$

for each real $\lambda$, including $\lambda = \infty$.

Clearly, for finite real $\lambda$ with $\lambda \neq \lambda_0$, the inertia of $\lambda H - A$ is, from (33) and (34), equal to

$$\begin{Bmatrix} \bar{N} + c - a_\lambda \\ d + a_\lambda \\ \bar{N} \end{Bmatrix}, \tag{35}$$

where $\bar{N} = N - 1$.

Now consider the case when $\lambda = \lambda_0$, i.e., $\lambda_0$ is real. Denote the inertia of $\lambda_0 H - A$ by

$$\begin{Bmatrix} \alpha \\ \beta \\ \gamma \end{Bmatrix}. \tag{36}$$

Then, by continuity of the eigenvalues of $\lambda H - A$ as a function of $\lambda$, it follows from (35) that

$$\begin{aligned} \alpha &\leqslant \overline{N} + c, \\ \beta &\geqslant d, \\ \gamma &\leqslant \overline{N}, \end{aligned} \tag{37}$$

since $a_{\lambda_0} = 0$ in a neighborhood of $\lambda = \lambda_0$.

From (33) and (34), it follows that the inertia of

$$\begin{bmatrix} \lambda_0 H - A & -b \\ -b^* & -\theta \end{bmatrix} \tag{38}$$

equals

$$\begin{Bmatrix} N + c - a_{\lambda_0} \\ d + a_{\lambda_0} - 1 \\ N \end{Bmatrix}. \tag{39}$$

With a little algebra, it also follows that the inertia of (38) has the form

$$\begin{Bmatrix} \alpha \\ \beta - 1 \\ \gamma \end{Bmatrix} + \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}, \tag{40}$$

where

$$x + y + z = 2, \tag{41}$$

$$x \leqslant 1, \tag{42}$$

$$z \leqslant 1. \tag{43}$$

Thus, combining (39) and (40), we obtain

$$N + c - a_{\lambda_0} = \alpha + x, \tag{44}$$

$$d + a_{\lambda_0} - 1 = \beta - 1 + y, \tag{45}$$

$$N = \gamma + z. \tag{46}$$

From (37), (43) and (46) we obtain

$$z = 1,$$
$$\gamma = \bar{N}.$$

From (41), (42), and (43), we have two remaining possibilities to consider, viz., $x = 0$, $y = 1$ and $x = 1$, $y = 0$. Suppose $x = 0$, $y = 1$. Then, from (44) and (45),

$$\alpha = (N-1) + c - (a_{\lambda_0} - 1),$$
$$\beta = d + (a_{\lambda_0} - 1).$$

This corresponds to the removal of one of the type-1 blocks corresponding to the real eigenvalue $\lambda_0$ and hence a reduction in the overall size of these type-1 blocks.

Next, suppose $x = 1$, $y = 0$. Then, again from (44) and (45),

$$\alpha = (N-1) + c - a_{\lambda_0},$$
$$\beta = d + a_{\lambda_0}.$$

This does not entail removal of one of the type-1 blocks corresponding to the real eigenvalue $\lambda_0$, but only a reduction in the overall size of these type-1 blocks.

Thus, altogether, the matrix $\lambda_0 H - A$ has inertia

$$\begin{Bmatrix} \bar{N} + c - \bar{a}_{\lambda_0} \\ d + \bar{a}_{\lambda_0} - 1 \\ \bar{N} \end{Bmatrix},$$

where $\bar{N} = N - 1$ and $\bar{a}_{\lambda_0} = a_{\lambda_0}$ or $a_{\lambda_0} - 1$.

Finally, we consider the case when $\lambda = \infty$, i.e., we are interested in the inertia of $H$. However, since this corresponds to $\lambda = 1$ and $\mu = 0$ in (32), it is clear that the inertia of $H$ is

$$\begin{Bmatrix} \bar{N} \\ d + c \\ \bar{N} \end{Bmatrix},$$

so that $\bar{a}_\infty = c$ also. From Lemma 4.1, this completes the proof when $h \neq 0$.

Next, we prove the result when $h=0$ but $a\neq 0$. In this case, it is clear that the given pencil is congruent to

$$\begin{bmatrix} 0 & 0 & \mu \\ 0 & \lambda H+\mu A & \lambda b \\ \mu & \lambda b^* & \lambda\theta \end{bmatrix}$$

for some vector $b$ and some real $\theta$.

This time, from Lemma 4.1, the assumption of the lemma is equivalent to the matrix

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & \lambda H-A & \lambda b \\ -1 & \lambda b^* & \lambda\theta \end{bmatrix} \tag{47}$$

having an inertia of the form

$$\begin{Bmatrix} N+c-a_\lambda \\ d+a_\lambda \\ N \end{Bmatrix}$$

for each real $\lambda$, including $\lambda=\infty$. From (47), it is clear that for all finite real $\lambda$, the inertia of $\lambda H - A$ is

$$\begin{Bmatrix} \bar{N}+c-a_\lambda \\ d+a_\lambda \\ \bar{N} \end{Bmatrix},$$

where $\bar{N}=N-1$. Thus, we need only consider the case $\lambda=\infty$ in detail. We show that the inertia of $H$ is

$$\begin{Bmatrix} \bar{N} \\ d+c \\ \bar{N} \end{Bmatrix}.$$

Denote the inertia of $H$ by

$$\begin{Bmatrix} \alpha \\ \beta \\ \gamma \end{Bmatrix}.$$

First, we observe that for all real $\mu < 0$ and sufficiently small in magnitude, the inertia of $H + \mu A$ equals the inertia of $\lambda H - A$, where $\lambda = 1/|\mu|$, which in turn equals

$$\begin{Bmatrix} \overline{N} + c \\ d \\ \overline{N} \end{Bmatrix}.$$

Thus, by continuity of the eigenvalues of $H + \mu A$ as a function of $\mu$, we obtain

$$\alpha \leqslant \overline{N} + c,$$

$$\beta \geqslant d,$$

$$\gamma \leqslant \overline{N}.$$

Treating the case $\mu > 0$ in a similar manner, we obtain

$$\alpha \leqslant \overline{N},$$

$$\beta \geqslant d,$$

$$\gamma \leqslant \overline{N} + c,$$

so that combining the above gives

$$\alpha \leqslant \overline{N}, \tag{48}$$

$$\beta \geqslant d,$$

$$\gamma \leqslant \overline{N}. \tag{49}$$

Next, an argument identical to that used in the first part of the proof yields

$$N = \alpha + x, \tag{50}$$

$$d + c - 1 = \beta - 1 + y, \tag{51}$$

$$N = \gamma + z, \tag{52}$$

where

$$x + y + z = 2, \tag{53}$$

$$x \leqslant 1, \tag{54}$$

$$z \leqslant 1. \tag{55}$$

It follows from (49), (52), and (55) that $z = 1$ and $\gamma = \bar{N}$; from (48), (50), and (54) that $x = 1$ and $\alpha = \bar{N}$; and finally from (53) and (51) that $y = 0$ and $\beta = d + c$. From Lemma 4.1, this completes the proof when $h = 0$ and $a \neq 0$.

Finally, it is clear from all the above that the numbers of type-3 and type-4 blocks in the canonical form of $\lambda H + \mu A$ equal those in the canonical form of the originally given pencil. ∎

*The first author would like to thank the Department of Engineering at the University of Cambridge for providing a pleasant environment during a study leave. He also acknowledges a conversation with Paul Van Dooren which eventually led to the use of the inertia ideas in the proofs. The original proofs were rather long, tortuous, and uninteresting.*

## REFERENCES

1 B. D. O. Anderson and S. Vongpanitlerd, *Network analysis and synthesis*, Lecture Notes in Control and Inform. Sci., Prentice-Hall, Englewood Cliffs, N.J., 1973.

2 W. F. Arnold and A. J. Laub, Generalized eigenproblem algorithms and software for algebraic Riccati equations, *Proc. IEEE* 72:1746–1754 (Dec. 1984).

3 H. Bart, I. Gohberg, and M. A. Kaashoek, *Minimal Factorization of Matrix and Operator Functions*, Oper. Theory: Adv. Appl., Vol. 1, Birkhäuser, Basel, 1979.

4 H. Bart, I. Gohberg, M. A. Kaashoek, and P. Van Dooren, Factorization of transfer functions, *SIAM J. Control Optim.* 18:675–696 (1980).

5 A. Bunse-Gerstner and V. Mehrmann, A symplectic *QR*-like algorithm for the solution of the real algebraic Riccati equation, *IEEE Trans. Automat. Control* AC-31(12):1104–1113 (Dec. 1986).

6 R. P. Byers, A Hamiltonian *QR* algorithm, *SIAM J. Sci. Statist. Comput.* 7:212–229 (1986).

7 D. J. Clements and B. D. O. Anderson, *Singular Optimal Control: The Linear-Quadratic Problem*, Lecture Notes in Control and Inform. Sci., Vol. 5, Springer, Berlin, 1978.

8 B. A. Francis and J. C. Doyle, Linear control theory with an $H^{\infty}$-optimality criterion, *SIAM J. Control Optim.* 25:815–844 (1987).

9 K. Glover, All optimal Hankel-norm approximations of linear multivariable systems and their $L^\infty$-error bounds, *Internat. J. Control* 39:1115–1193 (1984).
10 I. Gohberg and M. A. Kaashoek, (Eds.), *Constructive Methods of Wiener-Hopf Factorization*, Oper. Theory: Adv. Appl., Vol. 21, Birkhäuser, Basel, 1986.
11 I. Gohberg, P. Lancaster, and L. Rodman, *Matrix Polynomials*, Comput. Sci. Appl. Math., Academic, New York, 1982.
12 S. J. Hammarling and M. A. Singer, A canonical form for the algebraic Riccati equation, in *MTNS Proceedings*, 1983.
13 P. Lancaster and L. Rodman, Existence and uniqueness theorems for the algebraic Riccati equation, *Internat. J. Control* 32:285–309 (1980).
14 A. J. Laub, A Schur method for solving algebraic Riccati equations, *IEEE Trans. Automat. Control* AC-24(6):913–925 (Dec. 1979).
15 C. Paige and C. F. Van Loan, A Schur decomposition for Hamiltonian matrices, *Linear Algebra Appl.* 41:11–32 (1981).
16 A. C. M. Ran, Minimal factorization of self-adjoint rational matrix functions, *Integral Equations Operator Theory* 5:850–869 (1982).
17 M. A. Singer and S. J. Hammarling, The Algebraic Riccati Equation: A Summary of Some Available Results, NPL Report 23, 1983.
18 R. C. Thompson, The characteristic polynomial of a principal subpencil of a hermitian matrix pencil, *Linear Algebra Appl.* 14(2):135–177 (1976).
19 A. J. Van Der Schaft and J. C. Willems, A new procedure for stochastic realization of spectral density matrices, *SIAM J. Control Optim.* 22(6):845–855 (Nov. 1984).
20 P. Van Dooren, A generalized eigenvalue approach for solving Riccati equations, *SIAM J. Sci. Statist. Comput.* 2:121–135 (1981).
21 P. Van Dooren, Reducing subspaces: Definitions, properties and algorithms, in A. Ruhe and B. Kagstrom, (Eds.), *Matrix Pencils*, Springer, 1983, pp. 58–73.
22 J. C. Willems, Least squares stationary optimal control and the algebraic Riccati equations, *IEEE Trans. Automat. Control* AC-16(6):621–634 (Dec. 1971).
23 D. C. Youla, On the factorization of rational matrices, *IRE Trans. Inform. Theory* IT-7(3):172–189 (July 1961).

*Received July 1988; final manuscript accepted 26 January 1989*