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A selective bitopological version of the Reznichenko property in function spaces

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ABSTRACT

For a Tychonoff space X we consider the compact-open and the topology of pointwise convergence on the set of all continuous real-valued functions, define a selective version of the Reznichenko property connecting the two topologies and characterize it dually via a suitable covering property of the space X.

1. Introduction

The theory of function spaces is one of the most extensively studied areas of general topology, containing as a special part of it, the problem of finding closure type properties of the function spaces and various covering properties of the initial space dual to each other (see e.g. [1,4,6,9,12,14–17]). More recently, a discipline called *Selection principles* was developed and at the beginning it dealt primarily with different sorts of open covers and selection hypothesis concerning them (see [5,8,18]). Combining the two disciplines we find ourselves interested in trying to find closure type properties of function spaces dual to those described by these selection principles when applied to certain open covers (see e.g. [11,12,14,13]). Of course we could look at the problem the other way round. In this paper we are concerned with a selective variant of the property of Reznichenko in function spaces, but unlike papers [12,13], we will examine its behavior when it is considered as a bitopological property of function spaces endowed with the topology of pointwise convergence and the compact-open topology. This was already done in [11] but only in the special case when the compact-open topology was with countable tightness so Theorem 1.1 of this paper actually generalizes Theorem 4.3 of [11].

Before we proceed, a few words about the notation and the terminology which is mostly, up to some slight variations, taken from [3]. $a \subseteq b$ means that a is a subset of b whereas $a \subset b$ means that a is a proper subset of b. $\mathbb{N}$ is the set of positive integers. If τ is a topology on X and $x \in X$ then $\Omega_x(X,\tau) := \{A \subseteq X \setminus \{x\}\colon\ x \in Cl_\tau(A)\}$. *All spaces are assumed to be infinite Tychonoff.*

The Reznichenko property of a space X was introduced in 1996 (at a seminar at the Moscow State University) as follows: X has the Reznichenko property if for all $A \subseteq X$ and $x \in Cl(A) \setminus A$ there is a sequence $(B_n\colon\ n \in \mathbb{N})$ of pairwise disjoint finite

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subsets of A such that for every open $U \ni x$ the set $\{n \in \mathbb{N}\colon B_n \cap U = 0\}$ is finite. It has been considered in the general case in [10], in function spaces in [9,15,16], and in the context of hyperspaces in [7].

The definition of what could be called *a selective bitopological version of the Reznichenko property* was given in [11]:

we say that X is a *selectively* (τ_1, τ_2)*-Reznichenko space at* $x \in X$ **if** τ_1, τ_2 are two topologies on X, with $\tau_1 \supseteq \tau_2$ and **if** for each sequence $(A_n\colon n \in \mathbb{N})$ of elements of $\Omega_x(X, \tau_1)$ there is a sequence $(B_n\colon n \in \mathbb{N})$ of pairwise disjoint sets such that for each $n \in \mathbb{N}$, B_n is a finite subset of A_n and such that for each τ_2-neighborhood U of x, for all but finitely many $n \in \mathbb{N}$ we have that $U \cap B_n \neq 0$. If X is a selectively (τ_1, τ_2)-Reznichenko space at every $x \in X$ then we just call it a selectively (τ_1, τ_2)-Reznichenko space.

Now, we are interested (as it is very often practise) to characterize the selectively (τ_1, τ_2)-Reznichenko $C(X)$ spaces, for τ_1 being the compact-open and τ_2 the topology of pointwise convergence, using a suitable covering property of the space X. To find the corresponding dual property we first modify the definition of the notion of an *ω-shrinkable* open cover which was introduced it [15] for similar reasons:

we call a family $\mathcal{U}$ of open sets of X a *k-shrinkable* cover of X if there is a function f with $\mathrm{dom}(f) = \mathcal{U}$ such that for each $U \in \mathcal{U}$, $f(U)$ is a **closed** set with $f(U) \subseteq U$ and such that $\{f(U)\colon U \in \mathcal{U}\}$ is a k-cover of X. It is nontrivial if $X \notin \mathcal{U}$. As already pointed out, this definition originates from [15] and combines the notion of an ω-shrinkable cover with the notion of a k-cover (a family $\mathcal{A}$ of subsets of a space is a k-cover if each compact subset of the space in question is contained in a member of $\mathcal{A}$, see [2]).

It is convenient at this point to introduce one new sort of covers similar to the ones above (recall that *functionally closed* subsets of a space X are inverse images of the set $\{0\}$ under continuous real-valued functions defined on the space X):

we shall call a family $\mathcal{U}$ of open sets of X a *functionally k-shrinkable* cover of X if there is a function f with $\mathrm{dom}(f) = \mathcal{U}$ such that for each $U \in \mathcal{U}$, $f(U)$ is a **functionally closed** set with $f(U) \subseteq U$ and such that $\{f(U)\colon U \in \mathcal{U}\}$ is a k-cover of X. It is nontrivial if $X \notin \mathcal{U}$. The collection of all nontrivial functionally k-shrinkable covers of X will be denoted by $\mathcal{K}_{shr} \equiv \mathcal{K}_{shr}(X)$. It is pretty much obvious that for normal spaces, k-shrinkable and functionally k-shrinkable covers are the same thing.

For a space X by τ_k (τ_p) we will denote the compact-open (pointwise convergence) topology on $C(X)$. $\mathbf{o} \in C(X)$ is the constantly zero function. If we put $O(S, \varepsilon) := \{f \in C(X)\colon f[S] \subseteq (-\varepsilon, \varepsilon)\}$, for $S \subseteq X$ and $\varepsilon > 0$ a real number, then a standard local base at $\mathbf{o}$ for the compact-open (pointwise convergence) topology is exactly the family of sets $O(S, \varepsilon)$ where S ranges over the compact (finite) subsets of X and ε over the positive real numbers. $\Omega^k_{\mathbf{o}}$ stands for $\Omega_{\mathbf{o}}(C(X), \tau_k)$.

Now our result can be stated as follows:

Theorem 1.1. *For a space X the following are equivalent*:

(1) *$C(X)$ is a selectively (τ_k, τ_p)-Reznichenko space*;
(2) *for each sequence $(\mathcal{U}_n\colon n \in \mathbb{N})$ of nontrivial k-shrinkable covers of the space X there is a sequence $(\mathcal{V}_n\colon n \in \mathbb{N})$, such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, $n \neq m \Rightarrow \mathcal{V}_n \cap \mathcal{V}_m = 0$, and such that for each finite $F \subseteq X$, for all but finitely many $n \in \mathbb{N}$ there is a $U \in \mathcal{V}_n$ with $F \subseteq U$.*

We should prove this now, but before that we will reformulate the statement of the theorem in order to make it easier to prove.

Let us say that X has the property $\mathcal{P}$ **if** for each sequence $(\mathcal{U}_n\colon n \in \mathbb{N})$ of elements of $\mathcal{K}_{shr}$ there is a sequence $(\mathcal{V}_n\colon n \in \mathbb{N})$, such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, $n \neq m \Rightarrow \mathcal{V}_n \cap \mathcal{V}_m = 0$, and such that for each finite $F \subseteq X$, for all but finitely many $n \in \mathbb{N}$ there is a $U \in \mathcal{V}_n$ with $F \subseteq U$.

If a space X has the property $\mathcal{P}$ then it is clear that for each functionally k-shrinkable cover $\mathcal{A}$ there is a countable $\mathcal{B} \subseteq \mathcal{A}$ that is an ω-cover of X. In [11] it was shown that for each k-cover there is a functionally k-shrinkable cover refining it. Thus for spaces with the property $\mathcal{P}$ for each k-cover $\mathcal{A}$ there is a countable ω-cover $\mathcal{B} \subseteq \mathcal{A}$. Using this fact it is not difficult to verify that spaces with the property $\mathcal{P}$ are Lindelöf and therefore normal (recall that we assume all our spaces to be Tychonoff). So for such spaces the classes of k-shrinkable and functionally k-shrinkable covers coincide. Also, the property given by (2) of Theorem 1.1 is (formally) stronger than $\mathcal{P}$. That is why the statement of our theorem is actually:

Theorem 1.2. *For a space X the following are equivalent*:

(1) *$C(X)$ is a selectively (τ_k, τ_p)-Reznichenko space,*
(2) *X has the covering property $\mathcal{P}$,*

which is the formulation that we shall use when proving the result.

2. The proof

Let us now establish some terminology and notation that will be needed for the proof.

We write $x \perp y$ if neither $x \subseteq y$ nor $y \subseteq x$ holds. If a, b are finite sequences then $len(a)$ is the length of a and $a^\frown b$ is the finite sequence obtained by concatenation, that is, by adding b to a to the right.

For a set A and an $a \in \mathbb{N}$ we denote $A_y(a) := \{n \in \mathbb{N}\colon\ (a, n) \in A\}$. For a real-valued function f and a real number $x > 0$, $f * x$ and $f \underline{*} x$ denote, respectively, the inverse image under f of the subsets $(-x, x)$ and $[-x, x]$ of the real line. If f is a function then $f|A$ will denote its restriction to A. A family $(a_i\colon\ i \in I)$ is **inscribed** in a family $(b_s\colon\ s \in S)$ if $I = S$ and $a_i \subseteq b_i$ for each $i \in I$; $(a_i\colon\ i \in I)$ is a **disjoint** family if $i \neq j \Rightarrow a_i \cap a_j = 0$.

$(a_i\colon\ i \in I)$ is:

- a *cov*-family *if* each a_i is a finite set of subsets of X and for each finite $F \subseteq X$ there is a finite T with $i \in I \setminus T \Rightarrow \exists U \in a_i$ $(F \subseteq U)$;
- a *Rez*-family *if* each a_i is a finite subset of $C(X)$ and for each finite $F \subseteq X$ and each $\varepsilon > 0$ there is a finite T such that $i \in I \setminus T \Rightarrow a_i \cap O(F, \varepsilon) \neq 0$.

Notice that, formally, finite families are both *cov*- and *Rez*-families.

A set A is **bad** *if* $A \in \Omega^k_{\mathbf{o}}$ and for each $\varepsilon > 0$ there is an $f \in A$ with $X \subseteq f * \varepsilon$.

If $A \in \Omega^k_{\mathbf{o}}$ is not **bad** then there must be a real number $\delta(A)$ with $1 > \delta(A) > 0$ such that $X \notin \{f * \delta(A)\colon\ f \in A\}$. This way, for a fixed space X a function $\delta \equiv \delta_X$ is defined and we shall refer to it in further text as a **witnessing** function on X. Note that if $B \subseteq A$ and $B \in \Omega^k_{\mathbf{o}}$ then B is also not **bad**.

Obviously, a space X has the property $\mathcal{P}$ iff for each family $(\mathcal{U}_i\colon\ i \in I)$ of elements of $\mathcal{K}_{shr}$, with $\mathrm{card}(I) \leqslant \omega$, there is a disjoint *cov*-family inscribed in it.

Similarly, $C(X)$ is selectively (τ_k, τ_p)-Reznichenko iff for each family $(A_i\colon\ i \in I)$ of elements of $\Omega^k_{\mathbf{o}}$, with $\mathrm{card}(I) \leqslant \omega$, there is a disjoint *Rez*-family inscribed in it.

For clearer understanding of the proof of Theorem 1.2 (i.e. Theorem 1.1) we will extract one part of it and present it in form of a few preliminary lemmas.

The next lemma is a generalization of Lemma 3.3 of [11].

Lemma 2.1. *If $A \in \Omega^k_{\mathbf{o}}$ and ε is a positive real number then there is a $B \subseteq A$ and a function $s : B \to (0, \varepsilon)$ such that $\{f * s(f)\colon\ f \in B\}$ is a functionally k-shrinkable cover of X.*

Proof. Throughout the proof if $Y \subseteq C(X)$ then $\overline{Y}$ will denote the closure of Y with respect to the compact-open topology. Let $\mathbf{o} \in \overline{A} \setminus A \subseteq C_k(X)$ and let ε be a positive real number. Suppose there is no pair (B, s) such that $B \subseteq A$, $s : B \to (0, \varepsilon)$ and such that $\{f * s(f)\colon\ f \in B\}$ is a functionally k-shrinkable cover of X.

Let $A_1 := \{f \in A\colon\ \exists r_1, r_2 \in (\varepsilon/2, \varepsilon)\ (r_1 \neq r_2$ and $f * r_1 = f * r_2)\}$. Then for each $f \in A_1$ there is an $s(f) \in (\varepsilon/2, \varepsilon)$ such that $f * s(f)$ is closed. Put $\mathcal{U} := \{f * s(f)\colon\ f \in A_1\}$. If $\mathbf{o} \in \overline{A_1}$ then $\mathcal{U}$ k-covers X (because $\varepsilon/2 < s(f)$). Letting $g(U) := U$, $U \in \mathcal{U}$ we produce a function confirming that $\mathcal{U}$ is functionally k-shrinkable, which is impossible. Thus, if we put $A_2 := A \setminus A_1$, then $\mathbf{o} \in \overline{A_2}$. Note that if $f \in A_2$ and $\varepsilon/2 \leqslant r_1 < r_2 < \varepsilon$ then $f * r_1 \subset f * r_2$.

Call $((f_n, \varepsilon_n, \varepsilon'_n)\colon\ n \in \omega)$ an r-sequence if $f_n \in C(X)$, $\varepsilon_n, \varepsilon'_n$ are reals with $\varepsilon/2 \leqslant \varepsilon_n < \varepsilon'_n < \varepsilon$, $f_n * \varepsilon'_n \subseteq f_{n+1} \underline{*} \varepsilon_{n+1}$ and $n \neq m \Rightarrow f_n \neq f_m$.

If $M \subseteq C(X)$ we will call a family of triples $((f_i, \varepsilon_i, \varepsilon'_i)\colon\ i \in I)$ a *shr*-family on M if: $f_i \in M$, $\varepsilon_i, \varepsilon'_i$ are reals with $0 < \varepsilon_i < \varepsilon'_i < \varepsilon$, $i \neq j \Rightarrow (f_i \neq f_j$ and $f_i * \varepsilon'_i \neq f_j * \varepsilon'_j)$ and $\{f_i \underline{*} \varepsilon_i\colon\ i \in I\}$ k-covers X. Note that for each such family, if $L := \{f_i\colon\ i \in I\} \subseteq M$, the function $t : L \to (0, \varepsilon)$, $t(f_i) := \varepsilon'_i$ is correctly defined, as well as the function g such that $\mathrm{dom}(g) = \{f_i * \varepsilon'_i\colon\ i \in I\}$, $g(f_i * \varepsilon'_i) = f_i \underline{*} \varepsilon_i$. Then g witnesses that $\mathrm{dom}(g) \equiv \{f * t(f)\colon\ f \in L\}$ is a functionally k-shrinkable cover. Thus, by the assumption we made at the beginning of the proof of this lemma, **there are no** *shr*-families on A.

Claim 1. *For each $B \subseteq A_2$ with $\mathbf{o} \in \overline{B}$ and each compact $K \subseteq X$ there is an r-sequence $((f_n, \varepsilon_n, \varepsilon'_n)\colon\ n \in \omega)$ such that $f_n \in B$ and $K \subseteq f_0 \underline{*} \varepsilon_0$.*

Proof of Claim 1. Put: $B_3 := \{f \in B\colon$ for all $r \in (\varepsilon/2, \varepsilon)$ and for each finite set T there are $r_0 \in (r, \varepsilon)$, $r_1 \in (\varepsilon/2, \varepsilon)$, $g \in B \setminus T$ such that $f * r_0 = g * r_1\}$;

$B_1 := \{f \in B \setminus B_3\colon\ \exists r_0 \in (0, \varepsilon)\ \forall r \in (r_0, \varepsilon)\ \forall r_1 \in (\varepsilon/2, \varepsilon)\ \forall g \in B_3\ (f * r \neq g * r_1)\}$. $B_2 := B \setminus (B_1 \cup B_3)$.

Then $B = B_1 \cup B_2 \cup B_3$ and $B_i \cap B_j = 0 \Leftarrow i \neq j$. Note that:

(a) if $f \in B_1 \cup B_2$ then there is a finite $T(f)$ and an $\varepsilon^{(1)}(f) \in (\varepsilon/2, \varepsilon)$ such that $\forall r_0 \in (\varepsilon^{(1)}(f), \varepsilon)\ \forall r_1 \in (\varepsilon/2, \varepsilon)\ \forall g \in B$ $(f * r_0 = g * r_1 \Rightarrow g \in T(f))$;
(b) if $f \in B_1$ there is an $\varepsilon^{(2)}(f) \in (0, \varepsilon)$ such that $\forall r \in (\varepsilon^{(2)}(f), \varepsilon)\ \forall r_1 \in (\varepsilon/2, \varepsilon)\ \forall g \in B_3\ (f * r \neq g * r_1)$.

List injectively B_1 as $(f_\alpha\colon \alpha<\kappa)$. We define inductively $((\varepsilon_\alpha,\varepsilon'_\alpha)\colon \alpha<\kappa)$ so that for each $\alpha<\kappa$:

(D_α) $\max\{\varepsilon/2, \varepsilon^{(2)}(f_\alpha)\}\leqslant \varepsilon_\alpha<\varepsilon'_\alpha<\varepsilon$ and $f_\alpha*\varepsilon'_\alpha\notin\{f_\gamma*\varepsilon'_\gamma\colon \gamma<\alpha\}$

holds.

Let $\varepsilon_0:=\max\{\varepsilon/2,\varepsilon^{(2)}(f_0)\}$ and choose an $\varepsilon'_0\in(\varepsilon_0,\varepsilon)$. Let $\beta<\kappa$ and suppose we have constructed $(\varepsilon_\alpha,\varepsilon'_\alpha)$ for all $\alpha<\beta$ so that the conditions (D_α) hold for each $\alpha<\beta$. We define $\varepsilon_\beta,\varepsilon'_\beta$.

Let $S_\beta:=\{\alpha<\beta\colon f_\alpha\in T(f_\beta)\}$ and $\varepsilon_\beta:=\max\{\varepsilon/2,\varepsilon^{(2)}(f_\beta),\varepsilon^{(1)}(f_\beta)\}$. Then $R:=\{r$ is a real number: $\varepsilon_\beta<r<\varepsilon$ and $f_\beta*r=f_\alpha*\varepsilon'_\alpha$ for an $\alpha\in S_\beta\}$ is finite (because S_β is finite and $f_\beta\in A_2$) so there is an $\varepsilon'_\beta\in(\varepsilon_\beta,\varepsilon)\setminus R$. We verify (D_β):

suppose there is an $\alpha<\beta$ with $f_\alpha*\varepsilon'_\alpha=f_\beta*\varepsilon'_\beta$. As $\varepsilon'_\beta\in(\varepsilon^{(1)}(f_\beta),\varepsilon)$ and $\varepsilon'_\alpha\in(\varepsilon/2,\varepsilon)$ it follows from (a) that $f_\alpha\in T(f_\beta)$ so $\alpha\in S_\beta$. Since $\varepsilon_\beta<\varepsilon'_\beta<\varepsilon$, it must be that $\varepsilon'_\beta\in R$, a contradiction.

If $\{f_\alpha\underline{*}\varepsilon_\alpha\colon \alpha<\kappa\}$ were to k-cover X the conditions (D_α), $\alpha<\kappa$ would imply that $((f_\alpha,\varepsilon_\alpha,\varepsilon'_\alpha)\colon \alpha<\kappa)$ is a *shr*-family on A, which cannot be true as previously noted. Also, as $\varepsilon_\alpha\geqslant\varepsilon/2$ for all $\alpha<\kappa$ we have that $\{f_\alpha*\varepsilon/2\colon \alpha<\kappa\}$ does not k-cover X, so $\mathbf{o}\notin\overline{B_1}$. Therefore $\mathbf{o}\in\overline{B_2\cup B_3}$. Choose any compact $K'\subseteq X$ with $K'\subseteq f_\alpha\underline{*}\varepsilon_\alpha$ for no $\alpha<\kappa$.

We now define an r-sequence $((h_n,\delta_n,\delta'_n)\colon n\in\omega)$ so that $h_n\in B_2\cup B_3$ and $K'\cup K\subseteq h_0\underline{*}\delta_0$.

As $\mathbf{o}\in\overline{B_2\cup B_3}$ there is a $g\in B_2\cup B_3$ with $K'\cup K\subseteq g*\varepsilon/2$. Let $h_0:=g$, $\delta_0:=\varepsilon/2$ and choose a $\delta'_0\in(\delta_0,\varepsilon)$.

Given (h_k,δ_k,δ'_k), $k\in n+1$, so that

(L_n) for each $k,j\in n+1$: $\varepsilon/2\leqslant\delta_k<\delta'_k<\varepsilon$, $h_k\in B_2\cup B_3$ and $k\neq j\Rightarrow h_k\neq h_j$, and for each $k\in n$: $h_k*\delta'_k\subseteq h_{k+1}\underline{*}\delta_{k+1}$,

we define $h_{n+1},\delta_{n+1}\delta'_{n+1}$.

Case 1: $h_n\in B_3$. If so there are $r\in(\delta'_n,\varepsilon)$, $l\in(\varepsilon/2,\varepsilon)$ and a $q\in B\setminus\{h_i\colon i\in n+1\}$ such that $h_n*r=q*l$.

We first show that $q\notin B_1$. Suppose to the contrary that $q\in B_1$. Then $\exists\alpha<\kappa$ $(q=f_\alpha)$. $K'\subseteq h_0\underline{*}\delta_0\subseteq h_n\underline{*}\delta_n\subseteq h_n*r=q*l=f_\alpha*l$ and $\neg(K'\subseteq f_\alpha\underline{*}\varepsilon_\alpha)$ imply $\varepsilon_\alpha<l$. Also, $\varepsilon>l>\varepsilon_\alpha\geqslant\varepsilon^{(2)}(f_\alpha)$ so, by (b), $f_\alpha*l\neq t*x$ whenever $t\in B_3$, $x\in(\varepsilon/2,\varepsilon)$. But $f_\alpha*l=h_n*r$, $h_n\in B_3$ and $r\in(\varepsilon/2,\varepsilon)$, a contradiction.

Therefore $q\in B_2\cup B_3$. We put $h_{n+1}:=q$, $\delta_{n+1}:=l$ and pick a $\delta'_{n+1}\in(\delta_{n+1},\varepsilon)$. We have $h_{n+1}=q\notin\{h_i\colon i\in n+1\}$, by construction, and $h_n*\delta'_n\subseteq h_n*r=q*l=h_{n+1}*\delta_{n+1}\subseteq h_{n+1}\underline{*}\delta_{n+1}$. Thus, (L_{n+1}) is satisfied.

Case 2: $h_n\in B_2$. Now there are $c\in(\delta'_n,\varepsilon)$, $d\in(\varepsilon/2,\varepsilon)$ and a $w\in B_3$ such that $h_n*c=w*d$. As $w\in B_3$ there are $c'\in(d,\varepsilon)$, $d'\in(\varepsilon/2,\varepsilon)$ and a $p\in B\setminus\{h_i\colon i\in n+1\}$ such that $w*c'=p*d'$. As before: $K'\subseteq h_0\underline{*}\delta_0\subseteq h_n\underline{*}\delta_n\subseteq h_n*c=w*d\subseteq w*c'=p*d'$ and $p=f_\alpha$ for an $\alpha<\kappa$ imply $\varepsilon>d'>\varepsilon^{(2)}(f_\alpha)$, thus $p*d'\neq t*x$ for any $t\in B_3$, $x\in(\varepsilon/2,\varepsilon)$, contradicting the fact that $p*d'=w*c'$, $w\in B_3$, $c'\in(\varepsilon/2,\varepsilon)$. Therefore $p\in B_2\cup B_3$, so we let $h_{n+1}:=p$, $\delta_{n+1}:=d'$ and we take arbitrary $\delta'_{n+1}\in(\delta_{n+1},\varepsilon)$. Again, $h_{n+1}=p\notin\{h_i\colon i\in n+1\}$ and $h_n*\delta'_n\subseteq h_n*c\subseteq p*d'=h_{n+1}*\delta_{n+1}\subseteq h_{n+1}\underline{*}\delta_{n+1}$. (L_{n+1}) is satisfied.

Clearly $((h_n,\delta_n,\delta'_n)\colon n\in\omega)$ is the required r-sequence. □

We now get back to proving the lemma. Enumerate all compact subsets of X as $(K_\alpha\colon \alpha<\mu)$. We shall define inductively r-sequences $((f_{\alpha,n},\varepsilon_{\alpha,n},\varepsilon'_{\alpha,n})\colon n\in\omega)$ for $\alpha<\mu$, so that for each $\beta<\mu$:

(C_β) $\forall\alpha_1,\alpha_2\leqslant\beta$ $\forall n,m\in\omega$ $((\alpha_1,n)\neq(\alpha_2,m)\Rightarrow(f_{\alpha_1,n}\neq f_{\alpha_2,m}$ and $f_{\alpha_1,n}*\varepsilon'_{\alpha_1,n}\neq f_{\alpha_2,m}*\varepsilon'_{\alpha_2,m}))$ and $K_\beta\subseteq f_{\beta,0}\underline{*}\varepsilon_{\beta,0}$.

Let, in accordance with Claim 1, $((f_{0,n},\varepsilon_{0,n},\varepsilon'_{0,n})\colon n\in\omega)$ be an r-sequence with $K_0\subseteq f_{0,0}\underline{*}\varepsilon_{0,0}$, $f_{0,n}\in A_2$. $n\neq m\Rightarrow f_{0,n}\neq f_{0,m}$ by the very definition of r-sequences, and $f_{0,n}*\varepsilon'_{0,n}\subseteq f_{0,n+1}\underline{*}\varepsilon_{0,n+1}\subset f_{0,n+1}*\varepsilon'_{0,n+1}$ (because $f_{0,n+1}\in A_2$ and $\varepsilon/2\leqslant\varepsilon_{0,n+1}<\varepsilon'_{0,n+1}<\varepsilon$). Thus $f_{0,n}*\varepsilon'_{0,n}\subset f_{0,n+1}*\varepsilon'_{0,n+1}\subseteq f_{0,m}*\varepsilon'_{0,m}$ for $m>n$, so (C_0) is satisfied.

Let $\beta<\mu$ and suppose $((f_{\alpha,n},\varepsilon_{\alpha,n},\varepsilon'_{\alpha,n})\colon n\in\omega)$ have been constructed for all $\alpha<\beta$ so that the conditions (C_α), $\alpha<\beta$, are satisfied. $\{f_{\alpha,n}\underline{*}\varepsilon_{\alpha,n}\colon (\alpha,n)\in\beta\times\omega\}$ cannot k-cover X since otherwise, as (C_α) holds for all $\alpha<\beta$, $\{(f_{\alpha,n},\varepsilon_{\alpha,n},\varepsilon'_{\alpha,n})\colon (\alpha,n)\in\beta\times\omega\}$ would be a *shr*-family on A, which is impossible. Thus there is a compact $K'_\beta\subseteq X$ with $K'_\beta\subseteq f_{\alpha,n}\underline{*}\varepsilon_{\alpha,n}$ for no $(\alpha,n)\in\beta\times\omega$. Since $f_{\alpha,n}*\varepsilon'_{\alpha,n}\subseteq f_{\alpha,n+1}\underline{*}\varepsilon_{\alpha,n+1}$, we must also have that $K'_\beta\subseteq f_{\alpha,n}*\varepsilon'_{\alpha,n}$ for no $(\alpha,n)\in\beta\times\omega$. As $\varepsilon/2\leqslant\varepsilon_{\alpha,n}$ this also means that $\{f_{\alpha,n}\colon (\alpha,n)\in\beta\times\omega\}\cap O(K'_\beta,\varepsilon/2)=0$. Letting $B\equiv O(K'_\beta,\varepsilon/2)\cap A_2$ and $K\equiv K_\beta$ in Claim 1 we get an r-sequence $((f_{\beta,n},\varepsilon_{\beta,n},\varepsilon'_{\beta,n})\colon n\in\omega)$ with $f_{\beta,n}\in O(K'_\beta,\varepsilon/2)\cap A_2$, $K_\beta\subseteq f_{\beta,0}\underline{*}\varepsilon_{\beta,0}$. We check (C_β):

By the definition of r-sequence we have that $n\neq m\Rightarrow f_{\beta,n}\neq f_{\beta,m}$. If $\alpha<\beta$ then $f_{\alpha,n}\notin O(K'_\beta,\varepsilon/2)$ and $f_{\beta,m}\in O(K'_\beta,\varepsilon/2)$ so again $f_{\alpha,n}\neq f_{\beta,m}$.

Now, $f_{\beta,n}*\varepsilon'_{\beta,n}\subseteq f_{\beta,n+1}\underline{*}\varepsilon_{\beta,n+1}\subset f_{\beta,n+1}*\varepsilon'_{\beta,n+1}\subseteq f_{\beta,m}*\varepsilon'_{\beta,m}$ for $m>n$ (because $f_{\beta,n+1}\in A_2$ and $\varepsilon/2\leqslant\varepsilon_{\beta,n+1}<\varepsilon'_{\beta,n+1}<\varepsilon$), so $f_{\beta,n}*\varepsilon'_{\beta,n}\neq f_{\beta,m}*\varepsilon'_{\beta,m}$ when $n\neq m$. If $\alpha<\beta$ then $\neg(K'_\beta\subseteq f_{\alpha,n}*\varepsilon'_{\alpha,n})$. But $K'_\beta\subseteq f_{\beta,m}*\varepsilon/2\subseteq f_{\beta,m}*\varepsilon'_{\beta,m}$, so again, $f_{\alpha,n}*\varepsilon'_{\alpha,n}\neq f_{\beta,m}*\varepsilon'_{\beta,m}$.

This, along with (C_α), $\alpha<\beta$, guarantees (C_β).

Having finished the construction, we see that, by virtue of (C_α), $\alpha<\mu$, $\{(f_{\alpha,n},\varepsilon_{\alpha,n},\varepsilon'_{\alpha,n})\colon (\alpha,n)\in\mu\times\omega\}$ is a *shr*-family on A, a contradiction. □

It is easy to see that we could actually have $B = A$ in the lemma above since by adding functionally open sets to a functionally k-shrinkable cover of a Tychonoff space we end up also with a functionally k-shrinkable cover.

Also, let us note that if $A \in \Omega^k_{\mathbf{o}}$ is not **bad** and $\varepsilon \leqslant \delta(A)$ then, using the notation of the previous lemma, $\{f * s(f)\colon f \in B\}$ is a *nontrivial* functionally k-shrinkable cover of X.

Lemma 2.2. *Let X have the property $\mathcal{P}$, $E \subseteq \mathbb{N}$ and let $(Y_n\colon n \in E)$ be a family of* **not bad** *elements of $\Omega^k_{\mathbf{o}}$. If there exists a bijection $f : E \to L \times \mathbb{N}$, for an $L \subseteq \mathbb{N}$, such that*:

(i) *if $(k, n), (m, s) \in L \times \mathbb{N}$ and $k \neq m$ then $C_{k,n} \cap C_{m,s} = 0$;*
(ii) *if $k \in L$ and $n, m \in \mathbb{N}$ then $C_{k,m} = C_{k,n}$,*

where $C_{n,m} = Y_{f^{-1}(n,m)}$ for each $(n, m) \in L \times \mathbb{N}$, then there is a disjoint Rez-family inscribed in $(A_n\colon n \in E)$.

Proof. If a family $(B_{n,m}\colon (n, m) \in L \times \mathbb{N})$ of not **bad** elements of $\Omega^k_{\mathbf{o}}$ satisfies (i) and (ii) of this lemma we shall call it an L-**matrix**. Thus, we actually need to show that in each L-matrix we can inscribe a disjoint *Rez*-family.

Claim 1. *For each L-matrix $(A_{n,m}\colon (n, m) \in L \times \mathbb{N})$ and each $\varepsilon > 0$ there is an L-matrix $(B_{n,m}\colon (n, m) \in L \times \mathbb{N})$ inscribed in it and a family $(D_{n,m}\colon (n, m) \in L \times \mathbb{N})$ such that*:

$D_{n,m} \subseteq A_{n,m} \setminus B_{n,m}$ for each $(n, m) \in L \times \mathbb{N}$;
$D_{n,m} \cap D_{k,s} = 0$ whenever $(n, m), (k, s) \in L \times \mathbb{N}$ and $(n, m) \neq (k, s)$;

for each finite $F \subseteq X$ there is a finite T such that if $(n, m) \in (L \times \mathbb{N}) \setminus T$ then $D_{n,m} \cap O(F, \varepsilon) \neq 0$.

Proof of Claim 1. We can of course suppose that $0 < \varepsilon < 1$.

By Lemma 2.1, for each $A \in \Omega^k_{\mathbf{0}}$ and each $\varepsilon > 0$ there is a function $g(A, \varepsilon)$, with $\mathrm{dom}(g(A, \varepsilon)) \subseteq A$, $\mathrm{ran}(g(A, \varepsilon)) \subseteq (0, \varepsilon)$, such that $\mathcal{U}(A, \varepsilon) := \{f * g(A, \varepsilon)(f)\colon f \in \mathrm{dom}(g(A, \varepsilon))\}$ is a functionally k-shrinkable cover. Choose any **witnessing function** δ on X.

Apply $\mathcal{P}$ to the family $(\mathcal{U}(A_{n,m}, \varepsilon\delta(A_{n,m}))\colon (n, m) \in L \times \mathbb{N})$ so as to obtain a *cov*-family $(\mathcal{H}_{n,m}\colon (n, m) \in L \times \mathbb{N})$ inscribed in it. Find finite $P_{n,m} \subseteq A_{n,m}$ such that $\mathcal{H}_{n,m} = \{h * g(A_{n,m}, \varepsilon\delta(A_{n,m}))(h)\colon h \in P_{n,m}\}$. It is not difficult to see that if $(n, m) \neq (k, s)$ then $P_{n,m} \cap P_{k,s} = 0$. For each $n \in L$ put $Q_n := A_{n,1} \setminus \bigcup_{m\in\mathbb{N}} P_{n,m}$.

Fix an $n \in L$. $\mathbf{o}$ is in the τ_k-closure of at least one of the sets $R_{n,1} := Q_n \cup (\bigcup_{m\in\mathbb{N}} P_{n,2m-1})$ and $R_{n,2} := Q_n \cup (\bigcup_{m\in\mathbb{N}} P_{n,2m})$ because $A_{n,1} = R_{n,1} \cup R_{n,2}$. If $R_{n,1} \in \Omega^k_{\mathbf{o}}$ then let $B_{n,m} := R_{n,1}$ and $D_{n,m} := P_{n,2m}$ for each $m \in \mathbb{N}$. If this is not the case then $R_{n,2} \in \Omega^k_{\mathbf{o}}$ so let $B_{n,m} := R_{n,2}$ and $D_{n,m} := P_{n,2m-1}$ for each $m \in \mathbb{N}$. It is an easy task to see that the families $(B_{n,m}\colon (n, m) \in L \times \mathbb{N})$ and $(D_{n,m}\colon (n, m) \in L \times \mathbb{N})$ are as required. □

Given an L-matrix $(B_{n,m}\colon (n, m) \in L \times \mathbb{N})$, put $B^1_{n,m} := B_{n,m}$ and use Claim 1 to obtain an L-matrix $(B^2_{n,m}\colon (n, m) \in L \times \mathbb{N})$ inscribed in it and a family $(D^1_{n,m}\colon (n, m) \in L \times \mathbb{N})$ such that: $D^1_{n,m} \subseteq B^1_{n,m} \setminus B^2_{n,m}$; $D^1_{n,m} \cap D^1_{k,s} = 0$ whenever $(n, m) \neq (k, s)$; for each finite $F \subseteq X$ there is a finite $T(F, 1)$ such that if $(n, m) \in (L \times \mathbb{N}) \setminus T(F, 1)$ then $D^1_{n,m} \cap O(F, 1) \neq 0$.

Suppose L-matrices $(B^l_{n,m}\colon (n, m) \in L \times \mathbb{N})$, for $1 \leqslant l \leqslant l_0$, and families $(D^s_{n,m}\colon (n, m) \in L \times \mathbb{N})$, for $1 \leqslant s < l_0$, have been constructed so that the inductive hypothesis:

$D^s_{n,m} \subseteq B^s_{n,m} \setminus B^{s+1}_{n,m}$; $B^{s+1}_{n,m} \subseteq B^s_{n,m}$ for each $1 \leqslant s < l_0$; for each finite $F \subseteq X$ and each $1 \leqslant s < l_0$ there is a finite $T(F, s)$ with $\forall (n, m) \in (L \times \mathbb{N}) \setminus T(F, s)$ $(D^s_{n,m} \cap O(F, 1/s) \neq 0)$

holds. We apply Claim 1, with $1/l_0$ playing the role of ε, to $(B^{l_0}_{n,m}\colon (n, m) \in L \times \mathbb{N})$ so as to produce $B^{l_0+1}_{n,m}$, $D^{l_0}_{n,m}$ and finite sets $T(F, l_0)$, $F \in [\mathbb{N}]^{<\omega}$, in such a way that the inductive hypothesis is satisfied.

Having finished the construction we have that:

$\forall (n, m) \in (L \times \mathbb{N}) \setminus T(F, s)$ $(D^s_{n,m} \cap O(F, 1/s) \neq 0)$ for each finite $F \subseteq X$ and each $s \in \mathbb{N}$; if $(s, n, m) \neq (s', n', m')$ then $D^s_{n,m} \cap D^{s'}_{n',m'} = 0$; $D^s_{n,m} \subseteq B_{n,m}$, for each $s \in \mathbb{N}$.

Define finite sets $H_{n,m} := (\bigcup_{s=1}^n D^s_{n,m}) \cup (\bigcup\{D^j_{n,i}\colon i, j \in \mathbb{N},\ j > n \text{ and } (j - n) + i = m\})$. As $D^j_{n,i} \subseteq B_{n,i} = B_{n,m}$, each $H_{n,m}$ is a finite subset of $B_{n,m}$. We show that $(H_{n,m}\colon (n, m) \in L \times \mathbb{N})$ is the required disjoint *Rez*-family.

First for disjointness. Let $n_1, n_2 \in L$, $m_1, m_2 \in \mathbb{N}$. For $k = \overline{1, 2}$ put

$$a_k := \{(s, j, i) \in \mathbb{N} \times L \times \mathbb{N}\colon (j = n_k) \wedge \big((i = m_k \wedge 1 \leqslant s \leqslant n_k) \vee (s > n_k \wedge s + i = m_k + n_k)\big)\}.$$

Then $H_{n_k,m_k} = \bigcup\{D^s_{j,i}\colon (s, j, i) \in a_k\}$, $k = \overline{1, 2}$. If $(n_1, m_1) \neq (n_2, m_2)$ then $a_1 \cap a_2 = 0$ so $H_{n_1,m_1} \cap H_{n_2,m_2} = 0$, too.

To show that $(H_{n,m}\colon\ (n,m) \in L \times \mathbb{N})$ is a *Rez*-family consider a finite $F \subseteq X$ and an $\varepsilon > 0$. Take an $s_0 > 1/\varepsilon$. We have

$$(n,m) \in (L \times \mathbb{N}) \setminus T(F, s_0) \quad \Rightarrow \quad D^{s_0}_{n,m} \cap O(F, 1/s_0) \neq 0.$$

There is an $n_0 \in \mathbb{N}$, $n_0 \geqslant s_0$, such that if $m \in \mathbb{N}$ and $n \geqslant n_0$ then $(n,m) \notin T(F, s_0)$. For each $k \in \mathbb{N}$ with $1 \leqslant k < n_0$ there is an $p_k \in \mathbb{N}$ such that $(k,m) \notin T(F, s_0)$, for all $m \geqslant p_k$. Put $m_0 := \max \bigcup_{1 \leqslant k < n_0} \{p_k, p_k + s_0 - k\}$. Obviously $T(F, s_0) \cap (\mathbb{N} \times \mathbb{N}) \subseteq n_0 \times m_0$. Let $(n,m) \in (L \times \mathbb{N}) \setminus (n_0 \times m_0)$.

Case 1: $s_0 \leqslant n$. As obviously $(n,m) \notin T(F, s_0)$ we have $0 \neq D^{s_0}_{n,m} \cap O(F, 1/s_0)$. But $D^{s_0}_{n,m} \subseteq \bigcup_{j=1}^{n} D^{j}_{n,m} \subseteq H_{n,m}$ so $H_{n,m} \cap O(F, \varepsilon) \neq 0$.

Case 2: $n < s_0$. As now $n < n_0$, we have $m \geqslant m_0 \geqslant p_n, p_n + s_0 - n$. Thus, for $i := m - (s_0 - n)$, $i \geqslant p_n$ so $(n,i) \notin T(F, s_0)$ and $D^{s_0}_{n,i} \cap O(F, 1/s_0) \neq 0$. As $s_0 - n + i = m$ and $s_0 > n$ we have $D^{s_0}_{n,i} \subseteq \bigcup \{D^{s}_{n,j}\colon\ s > n \text{ and } s - n + j = m\} \subseteq H_{n,m}$. Thus, once again $H_{n,m} \cap O(F, \varepsilon) \neq 0$. □

A particular trivial consequence of the previous lemma that we will need at some point on is formulated as follows.

Lemma 2.3. *Let X have the property $\mathcal{P}$, $E \subseteq \mathbb{N}$ and let $(A_n\colon\ n \in E)$ be a family of* **not bad** *elements of $\Omega^k_{\mathbf{0}}$. If there exists a bijection $f : E \to L$, for an $L \subseteq \mathbb{N} \times \mathbb{N}$, such that*:

(i) *for each $n \in \mathbb{N}$ the set $L_y(n)$ is finite*;
(ii) *if $(k,n), (m,s) \in L$ and $k \neq m$ then $B_{k,n} \cap B_{m,s} = 0$*;
(iii) *if $(k,n), (k,m) \in L$ then $B_{k,m} = B_{k,n}$*,

where $B_{n,m} = A_{f^{-1}(n,m)}$ for each $(n,m) \in L$, then there is a disjoint Rez-family inscribed in $(A_n\colon\ n \in E)$.

Proof. If $m, n \in \mathbb{N}$ such that $L_y(n) \neq 0$, then choose a $k \in L_y(n)$ and put $B_{n,m} := B_{n,k}$. Denoting $L_0 := \{n \in \mathbb{N}\colon\ L_y(n) \neq 0\}$, we have just defined an L_0-**matrix**, in terms of Lemma 2.2, so by that the same lemma there is a disjoint *Rez*-family $(J_{n,m}\colon\ (n,m) \in L_0 \times \mathbb{N})$ inscribed in $(B_{n,m}\colon\ (n,m) \in L_0 \times \mathbb{N})$. Then $(J_{f(n)}\colon\ n \in E)$ is the desired disjoint *Rez*-family. □

Lemma 2.4. *Let X have the property $\mathcal{P}$, $E \subseteq \mathbb{N}$ and let $(A_n\colon\ n \in E)$ be a family of elements of $\Omega^k_{\mathbf{0}}$. If there exists a bijection $f : E \to L \times \mathbb{N}$, for an $L \subseteq \mathbb{N}$, such that*:

(i) *if $n \in L$ and $k, l \in \mathbb{N}$ then $k < l \Rightarrow B_{n,k} \supseteq B_{n,l}$*;
(ii) *if $n \in L$ then $\bigcap_{k \in \mathbb{N}} B_{n,k} = 0$*,

where $B_{n,m} = A_{f^{-1}(n,m)}$ for each $(n,m) \in L \times \mathbb{N}$, then there is a disjoint Rez-family inscribed in $(A_n\colon\ n \in E)$.

Proof. We prove there is a disjoint *Rez*-family inscribed in $(B_{n,m}\colon\ (n,m) \in L \times \mathbb{N})$.

$L_1 := \{n \in L\colon$ for each $m \in \mathbb{N}$ the set $B_{n,m}$ is **bad**$\}$; $L_2 := L \setminus L_1$. If $n \in L_2$ then there is an $s_n \in \mathbb{N}$ such that $\forall m \in \mathbb{N}$ ($m \geqslant s_n \Rightarrow B_{n,m}$ is not **bad**). For $n \in L_2$, $m \in \mathbb{N}$ put $F_{n,m} := B_{n,s_n+m} \subseteq B_{n,m}$. Then none of the sets $F_{n,m}$, $(n,m) \in L_2 \times \mathbb{N}$, is **bad**.

(A) Fix functions $g, \mathcal{U}$ and δ as in Lemma 2.2. For each $n \in \mathbb{N}$ apply the property $\mathcal{P}$ to the family $(\mathcal{U}(F_{m,k}, \delta(F_{m,k})/n) \times (m,k) \in L_2 \times \mathbb{N})$ so as to get a disjoint *cov*-family $(\mathcal{H}^n_{m,k}\colon\ (m,k) \in L_2 \times \mathbb{N})$ inscribed in it. Find finite $H^n_{m,k} \subseteq F_{m,k} \subseteq B_{m,k}$ with $\mathcal{H}^n_{m,k} = \{f * g(F_{m,k}, \delta(F_{m,k})/n)(f)\colon\ f \in H^n_{m,k}\}$.

Let $C_{m,k} := \bigcup_{i=1}^{m+k} H^i_{m,k} \subseteq B_{m,k}$ for each $(m,k) \in L_2 \times \mathbb{N}$.

We show that $(C_{m,k}(m,k) \in L_2 \times \mathbb{N})$ is a *Rez*-family. For a finite $M \subseteq X$ and an $\varepsilon > 0$ fix an $n_0 > 1/\varepsilon$ and a finite set T such that if $(m,k) \in (L_2 \times \mathbb{N}) \setminus T$ then $\exists U \in \mathcal{H}^{n_0}_{m,k}$ $(M \subseteq U)$, i.e. $H^{n_0}_{m,k} \cap O(M, \varepsilon) \neq 0$. Set $T_x := \{n \in L_2\colon\ \exists m \in \mathbb{N}\ ((n,m) \in T)\}$, $T_y := \{m \in \mathbb{N}\colon\ \exists n \in L_2\ ((n,m) \in T)\}$. As T_x and T_y are clearly finite, there is an $m_0 \in \mathbb{N}$, $m_0 > n_0$ such that $T_x \cup T_y \subseteq m_0$. Let $(m,k) \in (L_2 \times \mathbb{N}) \setminus (m_0 \times m_0)$.

If $k < m_0$ then $m \geqslant m_0$ so $m \notin T_x$ and $(m,k) \notin T$. Thus $H^{n_0}_{m,k} \cap O(M, \varepsilon) \neq 0$. But $m \geqslant m_0 > n_0$ so $H^{n_0}_{m,k} \subseteq \bigcup_{i=1}^{m+k} H^i_{m,k} = C_{m,k}$ and $C_{m,k} \cap O(M, \varepsilon) \neq 0$.

If $m < m_0$ then $k \geqslant m_0 > n_0$ so, again, $H^{n_0}_{m,k} \subseteq \bigcup_{i=1}^{m+k} H^i_{m,k} = C_{m,k}$, $(m,k) \notin T$ and consequently $H^{n_0}_{m,k} \cap O(M, \varepsilon) \neq 0$, i.e. $C_{m,k} \cap O(M, \varepsilon) \neq 0$.

(B) If $(n,m) \in L_1 \times \mathbb{N}$ choose a $h_{n,m} \in B_{n,m}$ with $X \subseteq h_{n,m} * (1/(n+m))$ and set $C_{n,m} := \{h_{n,m}\}$.

We show that $(C_{n,m}\colon\ (n,m) \in L_1 \times \mathbb{N})$ is a *Rez*-family. If $(n,m) \in (L_1 \times \mathbb{N}) \setminus (k \times k)$, for a $k \in \mathbb{N}$, then $m \geqslant k$ or $n \geqslant k$, so $h_{n,m}[X] \subseteq (-1/(m+n), 1/(m+n)) \subseteq (-1/k, 1/k)$, i.e. $h_{n,m} \in O(M, 1/k)$ for arbitrary finite $M \subseteq X$.

(C) Obviously, by what we have shown, $(C_{n,m}\colon\ (n,m) \in L \times \mathbb{N})$ is a *Rez*-family inscribed in $(B_{n,m}\colon\ (n,m) \in L \times \mathbb{N})$. We now proceed towards a disjoint *Rez*-family inscribed in it.

Fix a bijection $\nu : \mathbb{N} \to L \times \mathbb{N}$. Let $D_1 := C_{\nu(1)} \subseteq B_{\nu(1)}$ and $m_1, k_1 \in \mathbb{N}$ such that $(m_1, k_1) = \nu(1)$. Suppose finite sets $D_1, \ldots, D_l$ have been constructed along with $(m_i, k_i) \in L \times \mathbb{N}$, $i = \overline{1,l}$ so that $D_i = C_{m_i,k_i} \subseteq B_{\nu(i)}$ and so that $i \neq j \Rightarrow D_i \cap D_j = 0$. Let $\nu(l+1) = (a,b) \in L \times \mathbb{N}$. $D'_l := \bigcup_{i=1}^{l} D_i$ is a finite set, $\bigcap_{s \in \mathbb{N}} B_{a,s} = 0$, $B_{a,s} \supseteq B_{a,s+1}$ so there must be a $k_{l+1} \geqslant b$

with $D'_l \cap B_{a,k_{l+1}} = 0$. Put $m_{l+1} := a$ and $D_{l+1} := C_{m_{l+1},k_{l+1}} \subseteq B_{a,k_{l+1}} \subseteq B_{a,b} = B_{\nu(l+1)}$ (because $k_{l+1} \geqslant b$). Clearly $D_i \cap D_j = 0 \Leftarrow i \neq j$ for $i, j \leqslant l+1$.

Having finished this construction we let $G_{n,m} := D_{\nu^{-1}(n,m)} \subseteq B_{\nu(\nu^{-1}(n,m))} = B_{n,m}$. The $G_{n,m}$-s obviously form a disjoint family of finite sets because the sets D_n do. Fix a finite $M \subseteq X$ and an $\varepsilon > 0$. There is a finite set T such that if $(n,m) \in (L \times \mathbb{N}) \setminus T$ then $C_{n,m} \cap O(M,\varepsilon) \neq 0$. As $i \neq j \Rightarrow (m_i,k_i) \neq (m_j,k_j)$ (because $D_i \neq D_j$) there is an $i_0 \in \mathbb{N}$ such that if $i \geqslant i_0$ then $(m_i,k_i) \notin T$. Put $P := \{\nu(i)\colon\ 1 \leqslant i < i_0\}$. P is a finite set.

Let $(n,l) \in (L \times \mathbb{N}) \setminus P$. For $i := \nu^{-1}(n,l)$ we have $i \geqslant i_0$ (since $(n,l) \notin P$), thus $(m_i,k_i) \notin T$. Therefore $C_{m_i,k_i} \cap O(M,\varepsilon) \neq 0$. But $G_{n,l} = D_i = C_{m_i,k_i}$ so $G_{n,l} \cap O(M,\varepsilon) \neq 0$. Thus, $(G_{n,m}\colon\ (n,m) \in L \times \mathbb{N})$ is the required disjoint *Rez*-family inscribed in $(B_{n,m}\colon\ (n,m) \in L \times \mathbb{N})$. □

Now we are able to prove our result.

Proof of Theorem 1.2. (or Theorem 1.1) Let $\mathcal{F}$ be a family of subsets of an infinite set Z such that $0 \notin Z$, $Z \in \mathcal{F}$, $x \cup y \in \mathcal{F} \Rightarrow (x \in \mathcal{F} \vee y \in \mathcal{F})$, and let $(A_n\colon\ n \in \mathbb{N})$ be a sequence of elements of $\mathcal{F}$. Put $A_0 := Z$, $p(0) := 0$, $B(0) := Z$ and if $B(i) \in \mathcal{F}$, $p(i) \in \mathbb{N}^{<\omega}$, $i \in n+1$ have been defined define $p(n+1)$, $B(n+1)$ as follows (in the sequel $\underline{(x)}$ denotes the finite sequence $f : 1 \to \{x\}$ of length 1):

There is a (and for definiteness we could choose the greatest such) $i_{n+1} \in n+1$ such that $A_{n+1} \setminus \bigcup_{j=i_{n+1}+1}^{n} B(j) \in \mathcal{F}$, $A_{n+1} \setminus \bigcup_{j=i_{n+1}}^{n} B(j) \notin \mathcal{F}$. Let $S_{n+1} := \{k \in \mathbb{N}\colon\ p(i_{n+1})^\frown \underline{(k)} \in \{p(j)\colon\ j = \overline{i_{n+1},n}\}\}$, $k_{n+1} := \max S_{n+1} + 1$ (where we take $\max 0 = 0$) and set $p(n+1) := p(i_{n+1})^\frown \underline{(k_{n+1})}$, $B(n+1) := B(i_{n+1}) \cap (A_{n+1} \setminus \bigcup_{j=i_{n+1}+1}^{n} B(j))(\in \mathcal{F})$. When $n,m \in \mathbb{N}$, $m > n$ then by "$\bigcup_{i=m}^{n}(\cdots)$" we mean the empty set.

We show that:

(1.n) if $s \in \mathbb{N}^{<\omega}$ and $s \subset p(n)$ then there is a $m < n$ with $s = p(m)$;
(2.n) if $k,m \leqslant n$ then $p(k) \subseteq p(m) \Rightarrow B(k) \supseteq B(m)$;
(3.n) if $k < m \leqslant n$ then $p(k) \neq p(m)$;
(4.n) if $m,k \leqslant n$ then $p(m) \perp p(k) \Rightarrow B(m) \cap B(k) = 0$.

hold for each $n \in \omega$.

Obviously $(i.0)$ are satisfied. Suppose that (i,j) holds for all $j \leqslant n$ and $i = \overline{1,4}$.

$s \subset p(n+1) \Rightarrow s \subseteq p(i_{n+1})$, by construction. $i_{n+1} \leqslant n$ so $(1.i_{n+1})$ holds. Therefore: either $s = p(i_{n+1})$ or $s \subset p(i_{n+1})$ and $s = p(m)$ for an $m < i_{n+1}$. Thus, $(1.n+1)$ holds.

To check $(3.n+1)$ fix a $\nu \leqslant n$. If $\nu = 0$ then $p(\nu) = 0 \neq p(n+1)$ by construction. Let $\nu \geqslant 1$. Suppose $p(\nu) = p(n+1)$. Then $p(\nu) = p(i_{n+1})^\frown \underline{(k_{n+1})}$, $p(\nu) = p(i_\nu)^\frown \underline{(k_\nu)}$ so $p(i_\nu) = p(i_{n+1})$. But $i_{n+1} < n+1$, $i_\nu < \nu$ so $i_\nu, i_{n+1} \leqslant n$ and we have that $(3.n)$ holds. Hence $i_\nu = i_{n+1}$. Now, $i_{n+1} < \nu \leqslant n$ and $p(i_{n+1})^\frown \underline{(k_{n+1})} = p(\nu) \in \{p(j)\colon\ i_{n+1} \leqslant j \leqslant n\}$, so $k_{n+1} \in S_{n+1}$, which is impossible because $k_{n+1} = \max S_{n+1} + 1$. This means that $p(\nu) \neq p(n+1)$ in the first place, so, having in mind $(3.i)$, $i \leqslant n$, we are done.

Now for $(2.n+1)$. Let $\nu \leqslant n$.

Case 1: $p(\nu) \subseteq p(n+1)$. By $(3.n+1)$, $p(\nu) \subset p(n+1)$, so $p(\nu) \subseteq p(i_{n+1})$. Then by $(2.n)$, $B(\nu) \supseteq B(i_{n+1})$. But $B(i_{n+1}) \supseteq B(n+1)$, by construction, so finally $B(\nu) \supseteq B(n+1)$.

Case 2: $p(n+1) \subseteq p(\nu)$. By $(1.\nu)$ there is an $m \leqslant \nu$ with $p(n+1) = p(m)$, which contradicts $(3.n+1)$.

Finally, we show $(4.n+1)$. First note that if $0 < m < m_0$, $i_m = i_{m_0}$ then $B(m) \cap B(m_0) = 0$: as $m > i_m$ this follows from $B(m_0) \subseteq A_{m_0} \setminus \bigcup_{j=i_{m_0}+1}^{m_0-1} B(j)$ and $B(m) \subseteq \bigcup_{j=i_{m_0}+1}^{m_0-1} B(j)$. Now let $p(r_1) \perp p(r_2)$ for some $r_1, r_2 \leqslant n+1$. There are $s \in \mathbb{N}^{<\omega}$ and $l_1 \neq l_2$ such that $s^\frown \underline{(l_j)} \subseteq p(r_j)$, $j = \overline{1,2}$. By $(1.r_1)$ and $(1.r_2)$ there are $m_j \leqslant r_j$, $j = \overline{1,2}$ such that $p(m_j) = s^\frown \underline{(l_j)}$. Then, as clearly $m_1, m_2 > 0$, $s = p(i_{m_1}) = p(i_{m_2})$, so by $(3.n+1)$, $i_{m_1} = i_{m_2}$. $p(m_1) \neq p(m_2)$ implies $m_1 \neq m_2$. Hence, by our earlier observation, $B(m_1) \cap B(m_2) = 0$. By $(2.n+1)$ $B(r_j) \subseteq B(m_j)$ so, finally, $B(r_1) \cap B(r_2) = 0$.

Put $Tr := \{p(i)\colon\ i \in \mathbb{N}\}$ and $Br := \{l \in {}^{\omega}\mathbb{N}\colon\ \forall k \in \mathbb{N}\ (l|k \in Tr)\}$. Choose a well-order $<_0$ of $Tr \cup Br$. Fix an $n \in \mathbb{N}$.

If there is an $s \in Tr$ with $p(n) \subseteq s$, such that $s \subset p(m)$ for no $m \in \mathbb{N}$ then let $b(n)$ be the $<_0$-least such s. In this case we shall say that n is of type 1 and define $tp(n) := 1$. Set $C(n) := \cap\{B(k)\colon\ p(k) \subseteq b(n)\}(\equiv B(p^{-1}(b(n))))$.

If n is not of type 1 but there is an $l \in Br$ with $p(n) \subseteq l$ such that $\cap\{B(k)\colon\ p(k) \subseteq l\} \in \mathcal{F}$ then let $b(n)$ be the $<_0$-least such l. In this case we shall say that n is of type 2 and define $tp(n) := 2$. Set $C(n) := \cap\{B(k)\colon\ p(k) \subseteq b(n)\}$.

If n is neither of type 1 nor 2 then for all $l \in Br$ such that $p(n) \subseteq l$ we have $\cap\{B(k)\colon\ p(k) \subseteq l\} \notin \mathcal{F}$. Let $b(n)$ be the $<_0$-least such l. In this case we shall say that n is of type 3 and define $tp(n) := 3$. Set $C(n) := B(n) \setminus \bigcap\{B(k)\colon\ p(k) \subseteq b(n)\}$.

Note that $C(n) \in \mathcal{F}$, $C(n) \subseteq B(n)$ for all $n \in \mathbb{N}$.

We show:

(1) if $p(n) \subseteq p(m) \subseteq b(n)$ then $tp(n) = tp(m)$ and $b(n) = b(m)$.

Indeed, as clearly $p(k_1) \subseteq b(k_2) \Rightarrow tp(k_1) \leqslant tp(k_2)$, if $p(n) \subseteq p(m) \subseteq b(n)$ then both $p(n) \subseteq b(m)$ (because $p(m) \subseteq b(m)$) and $p(m) \subseteq b(n)$ hold, so $tp(n) = tp(m)$ and then trivially, $b(n) = b(m)$.

Put $P := \{n \in \mathbb{N}\colon len(p(n)) = \min\{len(p(m))\colon m \in \mathbb{N} \text{ and } b(m) = b(n)\}\}$. Fix an $n \in \mathbb{N}$, $n \in \{s \in \mathbb{N}\colon b(s) = b(n)\} \neq 0$, so there exists a positive integer $e_n = \min\{len(p(s)) : b(s) = b(n)\}$. If $b(m) = b(l) = b(n)$ and $len(p(m)) = len(p(l))$, then $p(m) = p(l)$ (because $p(m)$ and $p(l)$ are $\subseteq$-comparable), thus, by the conditions (3.i), $m = l$. Hence, there is exactly one $q(n) \in \mathbb{N}$ such that $b(q(n)) = b(n)$ and $len(p(q(n))) = e_n$. Clearly $q(n) \in P$. This way a function $q : \mathbb{N} \to P$ is defined so that:

(2) $b(q(n)) = b(n)$, $tp(q(n)) = tp(n)$, $p(q(n)) \subseteq p(n)$ and $n \in P \Rightarrow q(n) = n$.

which is not difficult to see.

Claim 1. *The following hold*:

(3) $b(n) = b(m) \Rightarrow q(m) = q(n)$;
(4) *if* $n, k \in P$ *and not both* n *and* k *are of type* 3 *then* $n \neq k \Rightarrow C(n) \cap C(k) = 0$;
(5) *if* $tp(n) \in \{1,2\}$ *and* $q(a) = q(d) = n$ *then* $C(a) = C(d)$;
(6) *if* $b(n) = b(m)$ *and* $p(n) \subseteq p(m)$ *then* $C(m) \subseteq C(n)$;
(7) $tp(n) \in \{1,2\} \Rightarrow C(q(n)) = C(n)$.

Proof of Claim 1.

(3) We have that $len(p(q(m))) = \min\{len(p(k)) : b(k) = b(m)\} = len(p(q(n)))$, because $b(m) = b(n)$. But $p(q(m))$ and $p(q(n))$ are $\subseteq$-comparable as they both are subsets of $b(q(m)) = b(m) = b(n) = b(q(n))$ (here we bear in mind (2)), so $p(q(m)) = p(q(n))$ and finally, $q(m) = q(n)$ since p is a one-to-one function.
(4) For definiteness, let $tp(n) \in \{1,2\}$. If $p(n) \perp p(k)$ then, by (4.i), $B(n) \cap B(k) = 0$ so also $C(n) \cap C(m) = 0$. Thus it remains to consider the following two cases:
Case 1: $p(n) \subseteq p(k)$. If $p(n) \subseteq p(k) \subseteq b(n)$ then, by (1), $b(n) = b(k)$, so $q(n) = q(k)$ by (3). But $n, k \in P$ and consequently $q(n) = n \neq k = q(k)$, a contradiction. Thus there must be an $s \in \mathbb{N}$ such that $p(n) \subseteq p(s) \subseteq b(n)$, $p(s) \perp p(k)$. Then $C(s) \cap C(k) = 0$. But $tp(n) \in \{1,2\}$, so $C(n) = \bigcap\{B(l) : p(l) \subseteq b(n)\} = \bigcap\{B(l) : p(l) \subseteq b(s)\} = C(s)$ (note that, by (1), $b(n) = b(s)$ and $tp(s) = tp(n) \in \{1,2\}$).
Case 2: $p(k) \subseteq p(n)$. Then $tp(k) \leqslant tp(n)$, so $tp(k) \in \{1,2\}$ and this is in fact Case 1.
(5) By (2) we have $b(a) = b(n) = b(d)$ and $tp(a) = tp(d) = tp(n) \in \{1,2\}$. Hence $C(a) = \bigcap\{B(l) : p(l) \subseteq b(a)\} = \bigcap\{B(l) : p(l) \subseteq b(d)\} = C(d)$.
(6) By (3) we have $q(n) = q(m) = k$ so, by (2) $tp(n) = tp(m) = tp(k)$, thus by (5), we only need to consider the case $tp(n) = tp(m) = 3$. But then:

$$C(n) = B(n) \setminus \bigcap\big\{B(l) : p(l) \subseteq b(n)\big\},$$

$$C(m) = B(m) \setminus \bigcap\big\{B(l) : p(l) \subseteq b(m)\big\} = B(m) \setminus \bigcap\big\{B(l) : p(l) \subseteq b(n)\big\},$$

$$B(m) \subseteq B(n) \quad \big(\text{by (2.i) because } p(n) \subseteq p(m)\big),$$

so $C(m) \subseteq C(n)$.
(7) $q(n) \in P$ so, by (2), $q(q(n)) = q(n)$. $tp(q(n)) = tp(n) \in \{1,2\}$, also by (2), so we can use (5) to conclude $C(q(n)) = C(n)$. □

Proof of (2) ⇒ (1). Let $(A_n\colon n \in \mathbb{N})$ be a sequence of elements of $\Omega^k_{\mathbf{o}}$. Letting $\mathcal{F} := \Omega^k_{\mathbf{o}}$, $Z := C(X) \setminus \{\mathbf{o}\}$ in the discussion above we obtain functions p, B, b, C, tp, q and a set P as described there.

Set:

$$T_0 := \big\{n \in \mathbb{N}\colon tp(n) \in \{1,2\} \text{ and } C(n) \text{ is } \textbf{bad}\big\};$$

$$T_i := \big\{n \in \mathbb{N}\colon tp(n) = i \text{ and } C(n) \text{ is not } \textbf{bad}\big\}, \quad i = \overline{1,2};$$

$$T_3 := \big\{n \in \mathbb{N}\colon tp(n) = 3\big\};$$

$$P_i := P \cap T_i, \quad i \in \{1,2,3\};$$

$$f(n) := \big(q(n), 1 + len\big(p(n)\big) - len\big(p\big(q(n)\big)\big)\big),$$

$$f : \mathbb{N} \to P \times \mathbb{N} \quad \text{and} \quad f_i := f|T_i, \quad i \in \{1,2,3\}.$$

Claim 2.

(8) f *is injective*;
(9) $\mathrm{ran}(f_i) \subseteq P_i \times \mathbb{N}$ *for* $i \in \{1,2,3\}$;
(10) $\mathrm{ran}(f_i) = P_i \times \mathbb{N}$ *for* $i \in \{2,3\}$.

Proof of Claim 2.

(8) If $f(n) = f(m)$ then $q(n) = q(m) =: k$ and $len(p(n)) - len(p(k)) + 1 = len(p(m)) - len(p(k)) + 1$ so $len(p(n)) = len(p(m))$. Also $b(n) = b(q(n)) = b(q(m)) = b(m) = b(k)$ and $p(n), p(m) \subseteq b(k)$. Thus, as $\subseteq$-comparable finite sequences of equal length, $p(n) = p(m)$, i.e. $n = m$.

(9) Let $i \in \{1, 2, 3\}$.
If $n \in T_i$ then $k := q(n) \in P$, $tp(q(n)) = tp(n) = i$. If $i = 3$ it follows that $k \in T_3$, so $k \in P_3$ and $f_3(n) \in P_3 \times \mathbb{N}$. If $i \in \{1, 2\}$ then $q(n) = k = q(k)$, $tp(k) \in \{1, 2\}$ so, by (5), $C(k) = C(n)$ is not **bad**. Again $k \in P_i$ and $f_i(n) \in P_i \times \mathbb{N}$. Therefore $\mathrm{ran}(f_i) \subseteq P_i \times \mathbb{N}$.

(10) By (9) we only need to prove $P_i \times \mathbb{N} \subseteq \mathrm{ran}(f_i)$. Let $i \in \{2, 3\}$ and fix a $(n, m) \in P_i \times \mathbb{N}$, $i \in \{2, 3\}$, implies $b(n) \in Br$ so $t := b(n)|(m - 1 + len(p(n)))) \in Tr$ and $len(t) = m - 1 + len(p(n))$. Thus there is an $s \in \mathbb{N}$ with $t = p(s)$. As $len(t) \geqslant len(p(n))$ and $t, p(n) \subseteq b(n)$ we have $p(n) \subseteq p(s) \subseteq b(n)$. Hence, by (1) and (3), $q(s) = q(n) = n$ (remember that $n \in P$). Finally, $len(p(s)) - len(p(q(s))) + 1 = len(t) - len(p(n)) + 1 = m$. Therefore $f_i(s) = (n, m)$. □

Put $R_{n,m} := C(f^{-1}(n, m))$, $(n, m) \in \mathrm{ran}(f)$.

(I) Let $\mathcal{F}_1 := \{C(n)\colon n \in T_1\}$, $L := \mathrm{ran}(f_1)$. As, by (8), $f_1 : T_1 \to L \subseteq P_1 \times \mathbb{N} \subseteq \mathbb{N} \times \mathbb{N}$ is bijective we check the conditions of Lemma 2.3.

(i) Let $(a, d) \in L$. There is an $n \in T_1$ such that $q(n) = a$ and $d = len(p(n)) - len(p(a)) + 1$. $tp(n) = 1$ so $b(a) = b(q(n)) = b(n) \in Tr$ and, since $p(n) \subseteq b(n)$, $len(p(n)) \leqslant len(b(n))$. Therefore $d \leqslant len(b(a)) - len(p(a)) + 1 =: l_a$. Thus $L_y(a) \subseteq l_a + 1$ so $L_y(a)$ is finite.

(ii) Let $(n, m), (k, l) \in L$, $n \neq k$. Denote $a := f^{-1}(n, m)$, $d := f^{-1}(k, l)$. $q(a) = n$ so by (2), $b(a) = b(n)$ and $p(a) \supseteq p(n)$. Thus, by (6), $C(a) \subseteq C(n)$. Similarly $C(d) \subseteq C(k)$. But $n, k \in P_1$, $n \neq k$ so, by (4), $C(n) \cap C(k) = 0$. Hence, finally, $R_{n,m} \cap R_{k,l} = 0$.

(iii) Let $(n, m), (n, l) \in L$. $a := f^{-1}(n, m)$, $d := f^{-1}(n, l)$. $q(a) = q(d) = n$ so, by (5), $C(a) = C(d)$, i.e. $R_{n,m} = R_{n,l}$.

Now, by Lemma 2.3 there is a disjoint *Rez*-family $(D(n)\colon n \in T_1)$ inscribed in $\mathcal{F}_1$.

(II) Let $\mathcal{F}_2 := \{C(n)\colon n \in T_2\}$. By (8) and (10), $f_2 : T_2 \to P_2 \times \mathbb{N}$ is bijective and the remaining conditions of Lemma 2.2 can be verified exactly as it was done for Lemma 2.3 in (I). Thus there is a disjoint *Rez*-family $(D(n)\colon n \in T_2)$ inscribed in $\mathcal{F}_2$.

(III) Let $\mathcal{F}_3 := \{C(n)\colon n \in T_3\}$. As, by (8) and (10), $f_3 : T_3 \to P_3 \times \mathbb{N}$ is bijective we check the conditions of Lemma 2.4.

Fix an $n \in P_3$. For each $m \in \mathbb{N}$ there is an $a_m \in T_3$ such that $f(a_m) = (n, m)$. As $R_{n,m} = C(a_m)$ we need to check whether $\bigcap_{m\in\mathbb{N}} C(a_m) = 0$ holds. $q(a_m) = n$ for all $m \in \mathbb{N}$, so by (2), $b(a_m) = b(n)$ and $p(n) \subseteq p(a_m) \subseteq b(n)$. Also, $m = len(p(a_m)) - len(p(n)) + 1$, so $\lim_{m\to\infty} len(p(a_m)) = +\infty$. Fix an l_0 with $p(l_0) \subseteq b(n)$. As $p(a_m) \subseteq b(n)$ and $\lim_{m\to\infty} len(p(a_m)) = +\infty$, there is an m_0 with $p(l_0) \subseteq p(a_{m_0})$. Therefore by (2.*i*), $B(a_{m_0}) \subseteq B(l_0)$. But then $\bigcap_{m\in\mathbb{N}} C(a_m) \subseteq C(a_{m_0}) = B(a_{m_0}) \setminus \bigcap\{B(s) : p(s) \subseteq b(n)\} \subseteq B(l_0) \setminus \bigcap\{B(s) : p(s) \subseteq b(n)\}$. As l_0 with $p(l_0) \subseteq b(n)$ was arbitrary we have that $\bigcap_{m\in\mathbb{N}} C(a_m) \subseteq \bigcap\{B(l) \setminus \bigcap\{B(s) : p(s) \subseteq b(n)\} : p(l) \subseteq b(n)\} = 0$.

Let now $n \in P_3$ and $k, l \in \mathbb{N}$, $k < l$. For $a := f_3^{-1}(n, k)$, $d := f_3^{-1}(n, l)$ we have $q(a) = q(d) = n$, $k = len(p(a)) - len(p(n)) + 1$ and $l = len(p(d)) - len(p(n)) + 1$. Thus, $len(p(a)) < len(p(d))$ and, by (2), $b(a) = b(d) = b(n)$. This implies $p(a), p(d) \subseteq b(n)$ and $p(a) \subseteq p(d)$, i.e. by the conditions (2.*i*), $B(a) \supseteq B(d)$. But $C(a) = B(a) \setminus \bigcap\{B(s) : p(s) \subseteq b(a)\}$, $C(d) = B(d) \setminus \bigcap\{B(s) : p(s) \subseteq b(d)\}$ and $b(a) = b(d)$, so $C(d) \subseteq C(a)$, i.e. $R_{n,l} \subseteq R_{n,k}$.

So, we can use Lemma 2.4 to find a disjoint *Rez*-family $(D(n)\colon n \in T_3)$ inscribed in $\mathcal{F}_3$.

(IV) To each pair (A, n), where A is **bad** and $n \in \mathbb{N}$ assign a $w(A, n) \in A$ with $w(A, n)[X] \subseteq (-1/n, 1/n)$, in such a way that $n \neq m \Rightarrow w(A, n) \neq w(A, m)$. If $n \in T_0$ put $h_n := w(C(n), n)$ and $D(n) := \{h_n\}$. Obviously $D(n) \subseteq C(n)$.

We show that $(D(n)\colon n \in T_0)$ is a disjoint family. Let $n, m \in T_0$, $n \neq m$. If $q(n) = q(m)$ then, by (2) $tp(q(n)) = tp(n) \in \{1, 2\}$, so by (5) $C(n) = C(m) =: G$. Thus $h_n = w(C(n), n) = w(G, n) \neq w(G, m) = w(C(m), m) = h_m$, by the definition of the function w, so $D(n) \cap D(m) = 0$. If $q(n) \neq q(m)$ then by (4) $C(q(n)) \cap C(q(m)) = 0$. But $D(n) \subseteq C(n) = C(q(n))$ and $D(m) \subseteq C(m) = C(q(m))$, by (7), so again $D(n) \cap D(m) = 0$.

We show that it is a *Rez*-family. Fix an $n_0 \in \mathbb{N}$. If $n \in T_0 \setminus n_0$ then $w(C(n), n)[X] \subseteq (-1/n, 1/n) \subseteq (-1/n_0, 1/n_0)$, i.e. $h_n \in O(F, \varepsilon)$ for any finite $F \subseteq X$ and any $\varepsilon \geqslant 1/n_0$.

(V) To see that $(D(n)\colon n \in \mathbb{N})$ is the required disjoint *Rez*-sequence inscribed in $(A_n\colon n \in \mathbb{N})$ (because $D(n) \subseteq C(n) \subseteq B(n) \subseteq A_n$) we only need to show that $D(n) \cap D(m) = 0$ for $n \in T_i$, $m \in T_j$, $i \neq j$.

As clearly not both n and m are of type 3 neither are both $q(n)$ and $q(m)$ of that type. Using (2) and (6) we get $C(n) \subseteq C(q(n))$, $C(m) \subseteq C(q(m))$. Thus if $q(n) \neq q(m)$ the assertion follows directly from (4) and $D(n) \subseteq C(n)$, $D(m) \subseteq C(m)$.

If on the other hand $q(n) = q(m)$ then $tp(n) = tp(m) =: a$. This implies $a \in \{1, 2\}$. But then $C(n) = C(m)$, by (7). Hence $i = j$, so this case is not possible.

Proof of (1) ⇒ (2). Throughout the proof by $h^{\leftarrow}S$ we will denote the inverse image of a set S under a function h.

Let $(A_n\colon n \in \mathbb{N})$ be a sequence of elements of $\mathcal{K}_{shr}$. If in the discussion at the beginning of the proof of this theorem we let $\mathcal{F} := \mathcal{K}_{shr}$ and let Z be the family of all nontrivial open subsets of X, we obtain p, B, b, C, tp, P, q as described there.

Let $\mathcal{U}$ be any nontrivial functionally k-shrinkable open cover of X and L a function such that for each $U \in \mathcal{U}$, $L(U) \subseteq U$, $L(U)$ is functionally closed and such that $\{L(V)\colon V \in \mathcal{U}\}$ is a k-cover of X. List injectively the compact subsets of X as

$(F_\alpha\colon \alpha < \mu)$. Choose a $U_0 \in \mathcal{U}$ with $F_0 \subseteq L(U_0)$ and a $f_0 \in C(X)$ with $f_0[L(U_0)] \subseteq \{0\}$, $f_0[X \setminus U_0] \subseteq \{1\}$. If (U_β, f_β) have been defined for all $\beta < \alpha$, so that $f_\beta \in C(X)$, $F_\beta \subseteq L(U_\beta) \subseteq f_\beta^{\leftarrow}\{0\}$, $X \setminus U_\beta \subseteq f_\beta^{\leftarrow}\{1\}$ and for all $\beta_1 < \beta_2 < \alpha$ $f_{\beta 1} \neq f_{\beta_2}$, $U_{\beta_1} \neq U_{\beta_2}$, proceed the recursive definition as follows:

if $\{f_\beta^{\leftarrow}\{0\}\colon \beta < \alpha\}$ k-covers X then we are over; if $\{f_\beta^{\leftarrow}\{0\}\colon \beta < \alpha\}$ does not k-cover X take a compact $T_\alpha \subseteq X$ with $T_\alpha \subseteq f_\beta^{\leftarrow}\{0\}$ for no $\beta < \alpha$, a $U_\alpha \in \mathcal{U}$ with $T_\alpha \cup F_\alpha \subseteq L(U_\alpha)$ and a $f_\alpha \in C(X)$ such that $L(U_\alpha) \subseteq f_\alpha^{\leftarrow}\{0\}$, $X \setminus U_\alpha \subseteq f_\alpha^{\leftarrow}\{1\}$; it is clear that $f_\alpha \neq f_\beta$ for all $\beta < \alpha$ and also, for each $\beta < \alpha$ we must have that $U_\alpha \neq U_\beta$ because otherwise $T_\alpha \subseteq L(U_\alpha) = L(U_\beta) \subseteq f_\beta^{\leftarrow}\{0\}$ for a $\beta < \alpha$, which is impossible. Having finished this recursive definition there is a $\beta_0 \leqslant \mu$ such that $\{f_\beta^{\leftarrow}\{0\}\colon \beta < \beta_0\}$ k-covers X and such that for each $\beta_1 < \beta_2 < \beta_0$ $U_{\beta_1} \neq U_{\beta_2}$, $f_{\beta 1} \neq f_{\beta_2}$. Therefore, the function $g_{\mathcal{U}}$ such that $\forall \beta < \beta_0$ $(g_{\mathcal{U}}(U_\beta) = f_\beta)$ and $\operatorname{dom}(g_{\mathcal{U}}) = \{U_\beta\colon \beta < \beta_0\} \subseteq \mathcal{U}$ is correctly defined. For $g_{\mathcal{U}}$ the following hold: $\operatorname{dom}(g_{\mathcal{U}}) \subseteq \mathcal{U}$, $\operatorname{ran}(g_{\mathcal{U}}) \subseteq C(X)$, $\{g_{\mathcal{U}}(U)^{\leftarrow}\{0\}\colon U \in \operatorname{dom}(g_{\mathcal{U}})\}$ k-covers X and $X \setminus U \subseteq g_{\mathcal{U}}(U)^{\leftarrow}\{1\}$ for all $U \in \operatorname{dom}(g_{\mathcal{U}})$. Clearly $\mathbf{o} \in \overline{\operatorname{ran}(g_{\mathcal{U}})}$ with respect to the compact-open topology and as $X \notin \mathcal{U}$, $\operatorname{ran}(g_{\mathcal{U}})$ does not contain the function $\mathbf{o}$.

For each $\mathcal{U} \in \mathcal{K}_{shr}$ choose a function $g_{\mathcal{U}}$ as described above. Apply the selectively (τ_k, τ_p)-Reznichenko property to the family $(\operatorname{ran}(g_{C(n)})\colon n \in \mathbb{N})$ so as to obtain a disjoint *Rez*-family $(\mathcal{D}_n\colon n \in \mathbb{N})$ inscribed in it. Find finite $D_n \subseteq C(n)$ with $\mathcal{D}_n = \{g_{C(n)}(U)\colon U \in D_n\}$. Define $H(n)$, $n \in \mathbb{N}$, recursively as follows.

Let $H(1) := D_1$.

If $tp(n) \in \{1, 2\}$ then $H(n) := D_n \subseteq C(n)$. Suppose now $tp(n) = 3$. Let $m, s \in \mathbb{N}$ be arbitrary with $p(n) \subseteq p(s) \subseteq p(m) \subseteq b(n)$. Then $tp(m) = tp(s) = tp(n) = 3$, $b(m) = b(s) = b(n)$ (by (1)) and $B(m) \subseteq B(s)$. So $C(m) = B(m) \setminus \bigcap\{B(k) : p(k) \subseteq b(m)\} \subseteq B(s) \setminus \bigcap\{B(k) : p(k) \subseteq b(s)\} = C(s)$. Further, $\bigcap\{C(i) : p(n) \subseteq p(i) \subseteq b(n)\} = \bigcap\{B(i) \setminus \bigcap\{B(k) : p(k) \subseteq b(i)\} : p(n) \subseteq p(i) \subseteq b(n)\} = \bigcap\{B(i) \setminus \bigcap\{B(k) : p(n) \subseteq p(k) \subseteq b(n)\} : p(n) \subseteq p(i) \subseteq b(n)\} = \bigcap\{B(i) : p(n) \subseteq p(i) \subseteq b(n)\} \setminus \bigcap\{B(k) : p(n) \subseteq p(k) \subseteq b(n)\} = 0$. To sum up, $\{C(i) : p(n) \subseteq p(i) \subseteq b(n)\}$ can be viewed as a decreasing sequence of sets with an empty intersection, so $\bigcup_{i=1}^{n-1} H(i)$ being finite (since all the sets $H(i)$, $1 \leqslant i < n$ are), there must be a $l_n \in \mathbb{N}$ with $p(n) \subseteq p(l_n) \subseteq b(n)$, such that $(\bigcup_{i=1}^{n-1} H(i)) \cap C(l_n) = 0$ (thus $(\bigcup_{i=1}^{n-1} H(i)) \cap D_{l_n} = 0$ also). Put $H(n) := D_{l_n} \subseteq C(l_n) \subseteq C(n)$. Notice that in this case $(tp(n) = 3)$ we have that $H(n) \cap H(i) = 0$, $1 \leqslant i < n$, by the construction. □

We check the disjointness of $(H(n)\colon n \in \mathbb{N})$. Let $n, m \in \mathbb{N}$, $n < m$. If $tp(m) = 3$ then by the construction $H(n) \cap H(m) = 0$. So let $tp(m) \in \{1, 2\}$.

If $q(n) \neq q(m)$ then, as by (2) $tp(q(m)) = tp(m)$, we can use (4) to deduce $C(q(m)) \cap C(q(n)) = 0$. But $H(m) \subseteq C(m) \subseteq C(q(m))$ and $H(n) \subseteq C(n) \subseteq C(q(n))$, thus $H(n) \cap H(m) = 0$.

Now, let $q(m) = q(n)$. Then by (2), $tp(n) = tp(q(n)) = tp(q(m)) = tp(m) \in \{1, 2\}$. Therefore $C(n) = C(m)$ (by (5)) and $H(n) = D_n$, $H(m) = D_m$ (by the construction of the $H(i)$-s as $\{tp(m), tp(n)\} \subseteq \{1, 2\}$). Suppose now there is a $U \in H(n) \cap H(m)$. Then $g_{C(n)}(U) = g_{C(m)}(U)$ would be in $\mathcal{D}_n \cap \mathcal{D}_m$, which is impossible.

That $(H(n)\colon n \in \mathbb{N})$ is a *cov*-family follows from the fact that there is an injective $v : \mathbb{N} \to \mathbb{N}$ with $H(n) = D_{v(n)}$ and the fact that $(D_n\colon n \in \mathbb{N})$ is a *cov*-family which can be established in the following way:

fix a finite $F \subseteq X$. As $(\mathcal{D}_n\colon n \in \mathbb{N})$ is a *Rez*-family there is an $n_0 \in \mathbb{N}$ with $\forall n \geqslant n_0$ $(\mathcal{D}_n \cap O(F, 1) \neq 0)$. Thus if $n \geqslant n_0$ there is a $U \in D_n$ with $g_{C(n)}(U) \in O(F, 1)$, which, in view of $X \setminus U \subseteq g_{C(n)}(U)^{\leftarrow}\{1\}$, implies $F \subseteq U$. □

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